Introduction to Inverse Scattering Transform

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Caveat

If you fall in love with the road,
you will forget the destination

Zen Saying

Outline

- **Preliminaries**
- **IST for the Korteweg – de Vries equation**
- **References**

but also

- **Zakharov-Shabat spectral problem and AKNS method (NLS and NLS-related systems)**
- **Semi-classical asymptotics in the IST method**

Introduction

Preamble: linear PDEs and solution via Fourier transform

Consider an IVP for a linear PDE (the linearised KdV equation)

$$u_t + u_{xxx} = 0\,, \quad u(x,0) = u_0(x)\,. \tag{1}$$

Assume that $u(x,t)$ can be expressed as a Fourier integral,

$$u(x,t) = \int_{-\infty}^{\infty} a(k,t)e^{ikx}\,dk\,, \tag{2}$$

Then for (1) one finds that $a_t = ik^3 a$ hence $a(k,t) = a(k,0)e^{ik^3 t}$ where $a(k,0)$ is found as the Fourier transform of the initial condition $u(x,0)$. Thus the solution of the IVP (1) becomes

$$u(x,t) = \int_{-\infty}^{\infty} a(k,0)e^{i(kx+k^3 t)}\,dk\,. \tag{3}$$

So the integration procedure is as follows:

$$u(x,0) \to a(k,0) \to a(k,0)e^{ik^3 t} \to u(x,t)$$

Nonlinear example: Burgers equation

The Burgers equation (nonlinearity + dissipation)

$$u_t + uu_x = \nu u_{xx}, \quad \nu > 0$$

Direct application of the Fourier transform method is not possible due to nonlinearity. However, the **Cole-Hopf transformation**

$$u = -2\nu \frac{\phi_x}{\phi}$$

reduces the Burgers equation to the linear heat equation

$$\phi_t = \nu \phi_{xx} .$$

so that we can use the Fourier transform to solve it

Now, what about the KdV equation

$$u_t + uu_x + u_{xxx} = 0$$

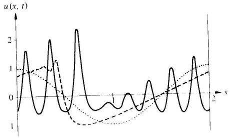
and other nonlinear dispersive PDEs (NLS, sin-Gordon, ...)?

Nonlinear PDEs and solitons

- **John Scott Russell (1834); Boussinesq (1871 - 1877); Korteweg & de Vries (1895)**: solitary waves on shallow water, $u = a\,\mathrm{sech}^2[\beta(x - ct)]$, $c \sim a$.
- **Zabusky** & **Kruskal (1965)**: numerical simulation of the continuum limit of the **Fermi-Pasta-Ulam (1955)** problem. The KdV equation

$$u_t + uu_x + \delta^2 u_{xxx} = 0\,, \qquad \delta = 0.022 \qquad (4)$$

with initial conditions $u(x, 0) = \cos \pi x$, $0 \le x \le 2$ and u, u_x, u_{xx} periodic on $[0, 2]$ for all t.



Generation of solitary waves elastically interacting with each other; wave-particle duality: solitons

- IST: **Gardner, Green, Kruskal & Miura (1967)** – KdV; **Zakharov & Shabat (1972)** – NLS; **AKNS (1974)** – many other equations.

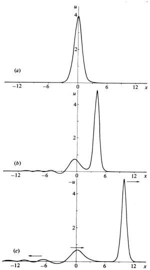
IVP for the KdV equation

We consider the KdV equation in its canonical dimensionless form

$$u_t + 6uu_x + u_{xxx} = 0\,. \qquad (5)$$

We shall be interested in solving the KdV equation (5) in the class of functions decaying sufficiently fast together with their first derivatives far from the origin. With this aim in view, we consider initial data

$$u(x,0) = u_0(x)\,, \qquad u_0(x) \to 0\,, \;\; u_0'(x) \to 0 \;\; \text{as} \;\; |x| \to \infty\,. \qquad (6)$$



Decay of a localised initial profile into solitons and some radiation

Digression 1 for NLS lovers: NLS to KdV

The defocusing nonlinear Schrödinger (NLS) equation

$$i\psi_t + \frac{1}{2}\psi_{xx} - |\psi^2|\psi = 0\,, \quad \psi \in \mathbb{C}$$

can be asymptotically reduced to the KdV equation in the uni-directional, weakly nonlinear long-wave limit. Multiple-scale perturbation procedure: one introduces $\psi \mapsto (\rho, v)$, where $\rho, v \in \mathbb{R}$ by

$$\psi = \sqrt{\rho}e^{i\phi}\,, \qquad v = \phi_x$$

and considers weakly nonlinear expansions for ρ and v

$$v = \epsilon v_1(x,t) + \epsilon^2 v_2(x,t) + \dots\,, \quad \rho = \rho_0 + \epsilon\rho_1(x,t) + \epsilon^2\rho_2(x,t) + \dots\,,$$

$\epsilon \ll 1$, together with the coordinate transformation

$$\xi = \epsilon^{1/2}(x \pm \sqrt{\rho_0}t)\,, \quad \tau = \epsilon^{3/2}t\,.$$

As a result one obtains the KdV equation for $v_1 \sim \rho_1$ (modulo coefficients):

$$v_{1,\tau} \mp v_1 v_{1,\xi} + v_{1,\xi\xi\xi} = 0$$

for the left- (upper sign) and right-(lower sign) propagating wave.

Digression 2 for NLS lovers: KdV to NLS

The defocusing NLS equation can be derived from the KdV equation

$$u_t + 6\varepsilon u u_x + u_{xxx} = 0\,, \qquad 0 < \varepsilon \ll 1\,,$$

by considering an *envelope* of nearly linear narrow-band wave packet KdV solution. The derivation technique is often called the *singular multiple-scale perturbation theory*.

One introduces an asymptotic expansion $u = \varepsilon u_1 + \varepsilon u_2 + \varepsilon^2 u_3 + \dots,$, where

$$u_1 = A(\tau, \chi, T, X)e^{i\theta} + c.c., \tag{7}$$

with $\theta = kx - \omega t, \tau = \varepsilon t, \chi = \varepsilon x, T = \varepsilon^2 t, X = \varepsilon^2 x$.
Here $\omega = -k^3$ is the linear dispersion relation of the KdV equation.

As a result of the substitution in the KdV equation, collecting the terms for the like powers of ϵ and eliminating secular terms, the defocusing NLS equation arises in the order $O(\varepsilon^3)$ as

$$iA_T + \beta A_{\chi\chi} + \gamma |A|^2 A = 0\,,$$

where $\beta = \frac{1}{2}\frac{d^2\omega}{dk^2} = -3k < 0,\ \gamma = \frac{1}{6k} > 0$.

Lax pairs, Zakharov-Shabat spectral problem and AKNS method

KdV equation as a compatibility condition

GGKM (1967): the KdV equation $u_t - 6uu_x + u_{xxx} = 0$ can be viewed as a compatibility condition for two linear differential equations for the **same** auxiliary function $\phi(x,t;\lambda)$:

$$\mathcal{L}\phi = -\phi_{xx} + u(x,t)\phi = \lambda\phi\,, \tag{8}$$

$$\phi_t = \mathcal{A}\phi = (-u_x + \gamma)\phi + (4\lambda + 2u)\phi_x\,. \tag{9}$$

Here λ is a complex parameter (can generally depend on time) and γ is a constant (which is determined by normalisation of $\phi(x,t;\lambda)$).

Important:

- In (8) t is a parameter so (8) is an ODE (a stationary spectral problem);
- System (8), (9) is overdetermined (i.e. compatible only for certain potentials $u(x,t)$)

Direct calculation shows that the compatibility condition $(\phi_{xx})_t = (\phi_t)_{xx}$ for all λ yields the KdV equation for $u(x,t)$ provided

$$\lambda_t = 0\,, \tag{10}$$

that is the KdV evolution of the potential $u(x,t)$ is isospectral i.e. preserves the spectrum of $\mathcal{L}$.

Lax Pair

Lax (1968) put the GGKM formulation into a general operator form. Consider two compatible linear equations:

$$\mathcal{L}\phi = \lambda\phi\,, \quad (11)$$
$$\phi_t = \mathcal{A}\phi\,. \quad (12)$$

Taking $\frac{\partial}{\partial t}$ of (11) and using (12) we obtain

$$[\mathcal{L}_t + (\mathcal{L}\mathcal{A} - \mathcal{A}\mathcal{L})]\phi = \lambda_t\phi$$

Then introducing $[\mathcal{L}, \mathcal{A}] := \mathcal{L}\mathcal{A} - \mathcal{A}\mathcal{L}$ we obtain the Lax equation

$$\mathcal{L}_t + [\mathcal{L}, \mathcal{A}] = 0\,, \quad (13)$$

for nontrivial solutions $\phi(x,t)$ *if and only if* $\lambda_t = 0$.
Take

$$\mathcal{L} = -\partial_{xx}^2 + u(x,t)\,,$$
$$\mathcal{A} = \gamma - 4\partial_{xxx}^3 + 6u\partial_x + 3u_x\,.$$

Then we see that (13) is satisfied iff $u(x,t)$ satisfies the KdV equation.

Zakharov-Shabat spectral problem

The stationary Schrödinger equation

$$\phi_{xx} - u\phi = -k^2\phi$$

can be represented as a system of *two equations of the first order*:

$$\begin{aligned} \phi_x &= ik\phi + \psi, \\ \psi_x &= -ik\psi + u\phi \end{aligned} \qquad (14)$$

Zakharov and Shabat (1972) proposed a generalisation of (14):

$$\begin{aligned} \psi_{1,x} &= -ik\psi_1 + q\psi_2, \\ \psi_{2,x} &= ik\psi_2 + r\psi_1 \end{aligned} \qquad (15)$$

i.e. we arrive at a matrix spectral problem

$$\psi_x = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix} \psi \quad \text{or} \quad \begin{pmatrix} \partial_x & -q \\ r & -\partial_x \end{pmatrix} \psi = -ik\psi\,,$$

where $r(x,t)$ and $q(x,t)$ are *potentials* and

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

The spectral problem (15) with $r = \pm q^*$ is associated with NLS eqn.

AKNS method

Motivated by the Lax and ZS constructions Ablowitz, Kaup, Newell and Segur (1974) proposed a method for generating integrable equations.
Consider two linear equations

$$\mathbf{v}_x = \mathrm{X}\mathbf{v}\,, \quad \mathbf{v}_t = \mathrm{T}\mathbf{v}$$

where $\mathbf{v}$ is an n-dimensional vector and X and T are $n \times n$ matrices. The compatibility condition $\mathbf{v}_{xt} = \mathbf{v}_{tx}$ yields (cf. (13))

$$\mathrm{X}_t - \mathrm{T}_x + [\mathrm{X}, \mathrm{T}] = 0\,. \tag{16}$$

Consider the X-T pair (often called a Lax pair as well)

$$\mathrm{X} = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix}, \qquad \mathrm{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $q(x,t)$ and $r(x,t)$ are (generally) complex functions, k is the spectral parameter and A, B, C and D are scalar functions of r, q and their derivatives, and k. (Note that the choice $r = -1$ leads to the Schrödinger scattering problem $v_{2,xx} + (k^2 + q)v_2 = 0$, i.e. the KdV case).

AKNS method and NLS equations

The compatibility condition

$$\mathrm{X}_t - \mathrm{T}_x + [\mathrm{X}, \mathrm{T}] = 0 \qquad (17)$$

and the isospectrality assumption $k_t = 0$, impose a set of conditions for A, B, C, D. These conditions are solvable only if another condition (on r and q) is satisfied, this condition being an evolution equation.

The AKNS method: Representing A, B, C, D as polynomials in the spectral parameter k one can obtain various evolution equations associated with the same spectral problem $\mathbf{v}_x = \mathrm{X}\mathbf{v}$.

Example (*Zakharov and Shabat (1972)*)
Choosing $r = -\sigma q^*$, where $\sigma = \pm 1$, and

$$\begin{aligned} D = -A, \quad A &= -2ik^2 + i\sigma q q^*\,, \\ B = 2qk + iq_x\,, \quad C &= -2\sigma q^* k + i\sigma q_x^* \end{aligned} \qquad (18)$$

we obtain from (17) the NLS equation

$$iq_t + q_{xx} + 2\sigma q^2 q^* = 0\,, \qquad (19)$$

$\sigma = \pm 1$ corresponding to the focusing and defocusing cases respectively.

Consequences of the existence of the Lax pair

- Infinite number of conserved quantities;
- Combined with the Hamiltonian structure, this leads to the notion of infinitely-dimensional completely integrable system (Zakharov & Faddeev (1971))
- IST – the method of solution by finding the appropriate angle-action variables (spectral data), determining their evolution in time and performing the inverse map.

IST method

IST: The Big Picture

Consider an IVP for a nonlinear evolution equation

$$u_t = F(u, u_x, u_{xx}, \dots), \quad u(x,0) = u_0(x). \tag{20}$$

Assume that (20) can be represented as a compatibility condition for two linear equations

$$\mathcal{L}\phi = \lambda\phi, \tag{21}$$
$$\phi_t = \mathcal{A}\phi. \tag{22}$$

Let $\{S(\lambda, t)\}$ be spectral (scattering) data for $u(x,t)$ in (21):
the discrete eigenvalues, the norming coefficients of eigenfunctions, and the reflection and transmission coefficients.

Then the IST steps are (cf. solution via the Fourier Transform):

$$u_0(x) \mapsto \{S(\lambda, 0)\} \mapsto \{S(\lambda, t)\} \mapsto u(x,t)$$

At each step we have to solve a linear problem!

KdV: Forward Scattering Problem

We consider the KdV equation in the class of functions sufficiently rapidly decaying as $|x| \to \infty$ To be more precise, we require boundedness of the integral (Faddeev's condition)

$$\int_{-\infty}^{+\infty} (1 + |x|)|u(x)|dx < \infty \,, \tag{23}$$

which ensures applicability of the scattering analysis in the sequel. Now we turn to the linear Schrödinger equation

$$-\phi_{xx} + u(x,t)\phi = \lambda\phi \tag{24}$$

associated with the KdV evolution.

The **spectrum** of the Schrödinger operator is the set of values λ for which there is a bounded solution $\phi(x)$ for all x.

The spectrum is divided into two parts: discrete and continuous.

Properties of the Schrödinger operator

$$\mathcal{L}\phi = (-\frac{d^2}{dx^2} + u(x))\phi = \lambda\phi . \qquad (25)$$

- **Self-adjoint** (Hermitian) $\Rightarrow$ real eigenvalues λ
- **Discrete spectrum**: $\lambda_n = -\kappa_n^2 < 0$, $n = 1, 2, \ldots N$
  - Non-degenerate: $\lambda_n \leftrightarrow \phi_n$;
  - $\min\{u(x)\} < \lambda_1 < \lambda_2 < \cdots < 0$;
  - The eigenfunctions (bound states) ϕ_n are square integrable; can use the normalisation $\|\phi_n\|^2 = \int_{-\infty}^{\infty} \phi_n^2 dx = 1$;
- **Continuous spectrum**: $\lambda > 0$
  - Doubly degenerate;
  - Eigenfunctions are not square integrable;
  - Abel's theorem: If ϕ and ψ are two independent solutions of (25) corresponding to the same eigenvalue, then the Wronskian

$$W = \begin{vmatrix} \phi & \psi \\ \phi' & \psi' \end{vmatrix} = \text{constant}$$

(to prove just consider $\frac{dW}{dx}$).

Jost solutions

Consider an asymptotic behaviour as $x \to \pm\infty$: $u \to 0$ so $\phi'' \sim -\lambda\phi$. Hence ϕ is asymptotically a linear combination of $e^{\pm i\sqrt{\lambda}}$ implying that ϕ decays exponentially at infinity if $\lambda < 0$ and oscillates at infinity if $\lambda > 0$.
One can introduce a basis of solutions defined by the asymptotic behaviour at infinity (continuous spectrum):

$$\begin{aligned} \phi_1(x,k) &= e^{-ikx} + o(1) \\ \phi_2(x,k) &= e^{ikx} + o(1)\,, \qquad x \to +\infty\,. \end{aligned}$$

The functions $\phi_{1,2}$ defined above are called the Jost solutions (another set of Jost solutions can be specified by the analogous conditions at $-\infty$).
Any solution corresponding to the continuous spectrum can be obtained by linear combination of the Jost functions.

A similar set of basis solutions can be introduced for the discrete spectrum as well (several different normalisations are possible).

Forward Scattering Problem: Continuous Spectrum

$\lambda > 0$: **scattering solutions.**

We introduce $k^2 \equiv \lambda\,,\ k \in \mathbb{R}$ and, assuming that $u \to 0$ as $x \to \pm\infty$ fix an asymptotic behaviour of the function $\phi(x; k^2)$ far from the origin :

$$\phi \sim e^{-ikx} + b(k)e^{ikx} \ \text{ as } \ x \to +\infty\,, \tag{26}$$

$$\phi \sim a(k)e^{-ikx} \ \text{ as } \ x \to -\infty\,. \tag{27}$$

This solution of the Schrödinger Eq. (24) describes scattering *from the right* of the incident wave $\exp(-ikx)$ on the potential $u(x)$. Then $b(k)$ represents a reflection coefficient and $a(k)$ a transmission coefficient.

Note:

- $|a|^2 + |b|^2 = 1$ (follows from constancy of the Wronskian $W(\phi, \phi^*$ + asymptotic behaviours (26), (27));
- $a(k) \equiv a(k, t)$, $b(k) \equiv b(k, t)$.

Direct Scattering Transform: $u(x, 0) \mapsto \{b(k, 0),\ a(k, 0)\}$.

Time evolution of scattering data

Let the potential $u(x,t)$ evolve according to the KdV equation $u_t = 6uu_x + u_{xxx} = 0$. Then the corresponding evolution of the scattering data $a(x,t), b(x,t)$ is found by substituting asymptotics (26), (27) into the second Lax equation

$$\phi_t = \mathcal{A}\phi = (-u_x + \gamma)\phi + (4\lambda + 2u)\phi_x \,.$$

(note that $u, u_x \to 0$ as $|x| \to \infty$ and $k_t = 0$).

As a result we obtain $\gamma = 4ik^3$ and

$$b_t = 8ik^3 b\,, \quad a_t = 0\,. \tag{28}$$

Hence

$$b(k,t) = b(k,0)e^{8ik^3 t}\,, \qquad a(k,t) = a(k,0)\,. \tag{29}$$

Forward Scattering Problem: Discrete Spectrum

(i) $\lambda = \lambda_n < 0$: **bound states**

If the potential $u(x)$ is sufficiently negative near the origin of the x-axis, the spectral problem (8) implies existence of finite number of bound states $\phi = \phi_n(x;\lambda)$, $n = 1, \dots, N$ corresponding to the discrete admissible values of the spectral parameter $\lambda = \lambda_n = -\kappa_n^2$, $\kappa_n \in \mathbb{R}$, $\kappa_1 > \kappa_2 > \cdots > \kappa_n$. We require the following asymptotic behaviour consistent with $\phi'' \sim -\lambda\phi$ at $|x| \to \infty$:

$$\phi_n \sim c_n e^{-\kappa_n x} \quad \text{as} \quad x \to +\infty\,, \tag{30}$$

where c_n are called the *norming constants*. We need to complete the specification of ψ_n by either fixing the behaviour $\psi \sim e^{\kappa_n x}$ at $x \to -\infty$ or by the normalisation

$$\int_{-\infty}^{\infty} \phi_n^2 dx = 1. \tag{31}$$

Thus, for the case of discrete spectrum we have an analog of the scattering transform: $u \mapsto \{\kappa_n, c_n\}$.

Discrete Spectrum: evolution of the spectral data

We are interested in the evolution of the spectral data $\{\kappa_n, c_n\}$ as the potential $u(x,t)$ evolves according to the KdV equation $u_t - 6uu_x + u_{xxx} = 0$.

First of all, we already proved that $d\kappa_n/dt = 0$ so $\kappa_n = \text{const}$. Next, we use the second Lax equation

$$\phi_t = (-u_x + \gamma)\phi + (4\lambda + 2u)\phi_x \, . \tag{32}$$

and normalisation (31) of the bound states $\phi = \phi_n$ to obtain $\gamma = 0$. Then, setting (30) into (32) we obtain

$$\frac{dc_n}{dt} = 4\kappa_n^3 c_n \ \text{ so that } \ c_n(t) = c_n(0)e^{4\kappa_n^3 t} \, . \tag{33}$$

Remark: We note that the bound state problem can be viewed as an analytic continuation of the scattering problem defined on the real k-axis, to the upper half of the complex k-plane. Then the discrete points of the spectrum are found as *simple* poles $k = i\kappa_n$ of the reflection coefficient $b(k)$ and $b \to 0$ as $|k| \to \infty$.

Forward Scattering Problem: Summary

Step I

Find scattering data S for the KdV initial condition $u(x,0) = u_0(x)$ by solving

$$\phi_{xx} + (\lambda - u_0(x))\phi = 0 \, .$$

As a result we get

$$S(0) = \{\kappa_n, c_n(0); a(k;0), b(k;0)\}$$

Step II

If the potential $u(x,t)$ evolves according to the KdV equation then the scattering data S evolve according to simple equations

$$\kappa_n = \text{const}\,, \quad c_n(t) = c_n(0)e^{4\kappa_n^3 t}\,, \quad a(k,t) = a(k,0), \quad b(k,t) = b(k,0)e^{8ik^3 t}\,.$$

Next step: Inverse Scattering transform $S(T) \mapsto u(x,t)$.

Inverse Scatering Problem: GLM equation

It was established in the 1950s that the potential $u(x)$ of the Schrödinger equation can be completely reconstructed from the scattering data S. The corresponding IST mapping $S \mapsto u$ is accomplished through the Gelfand - Levitan - Marchenko (GLM) linear integral equation.

We define the function $F(x)$ as

$$F(x,t) = \sum_{n=1}^{N} c_n^2(t) e^{-\kappa_n x} + \frac{1}{2\pi} \int_{-\infty}^{+\infty} b(k,t) e^{ikx} dk \, . \tag{34}$$

Then the potential $u(x,t)$ is restored from the formula

$$u(x,t) = -2 \frac{\partial}{\partial x} K(x,x,t) \, , \tag{35}$$

where $K(x,y,t)$ is found from the linear integral (GLM) equation

$$K(x,y) + F(x+y) + \int_{x}^{+\infty} K(x,z) F(y+z) dz = 0 \tag{36}$$

defined for any moment t.

Reflectionless potentials and N-soliton solutions

There exists a remarkable class of potentials characterised by zero reflection coefficient, $b(k) = 0$. Such potentials are called reflectionless and can be expressed in terms of elementary functions.

Example: $N = 1$

Assuming $b(k) = 0$, $N = 1$ in (34) we obtain:
$F(x,t) = c(0)^2 \exp(-\kappa x + 8\kappa^3 t)$, where $c \equiv c_1$, $\kappa \equiv \kappa_1$. Then the solution of the GLM equation(36) can be sought in the form $K(x,y,t) = M(x,t)\exp(-\kappa y)$. After simple algebra we get

$$M(x,t) = \frac{-2\kappa c(0)^2 \exp(-\kappa x + 8\kappa^2 t)}{2\kappa + c(0)^2 \exp(-2\kappa x + 8\kappa^2 t)} . \qquad (37)$$

As a result, we obtain from (35)

$$u = -2\kappa^2 \mathrm{sech}^2(\kappa(x - 4\kappa^2 t - x_0)) , \qquad (38)$$

which is just the soliton with the amplitude $a_s = 2\kappa^2$ propagating to the right with the velocity $c_s = 4\kappa^2 = 2a_s$ and having the initial phase

$$x_0 = \frac{1}{2\kappa} \ln \frac{c(0)^2}{2\kappa} . \qquad (39)$$

Reflectionless potentials and N-soliton solutions

For arbitrary $N \in \mathbb{N}$ and $b(k) \equiv 0$ we have from (34)

$$F(x,t) = \sum_{n=1}^{N} c_n(t)^2 \exp(-\kappa_n x)\,, \tag{40}$$

and therefore, seek the solution of the GLM equation (36) in the form

$$K(x,y,t) = \sum_{n=1}^{N} M_n(x,t) \exp(-\kappa_n y)\,. \tag{41}$$

Now, on using (35), (36) one arrives, after some algebra, at the general (Kay-Moses) representation for the reflectionless potential $u_N(x,t)$

$$u_N(x,t) = -2\frac{\partial^2}{\partial x^2} \ln \det \mathrm{A}(x,t)\,. \tag{42}$$

Here A is the $N \times N$ matrix given by

$$A_{mn} = \delta_{mn} + \frac{c_n(0)^2}{\kappa_n + \kappa_m} e^{-(\kappa_n+\kappa_m)x + 8\kappa_n^3 t}\,, \tag{43}$$

δ_{km} being the Kronecker delta.

Reflectionless potentials and N-soliton solutions

Analysis of formulae (42), (43) shows that for $t \to \pm\infty$ the solution of the KdV equation corresponding to the reflectionless potential can be asymptotically represented as a superposition of N single-soliton solutions propagating to the right and ordered in space by their speeds (amplitudes):

$$u_N(x,t) \sim -\sum_{n=1}^{N} 2\kappa_n^2 \mathrm{sech}^2[\kappa_n(x - 4\kappa_n^2 t - x_n^{\pm})] \quad \text{as} \ \ t \to \pm\infty\,, \qquad (44)$$

where the amplitudes of individual solitons are given by $a_n = 2\kappa_n^2$ and the positions $\mp x_n$ of the n-th soliton as $t \to \mp\infty$ are given by the relationship (cf. (39) for a single soliton)

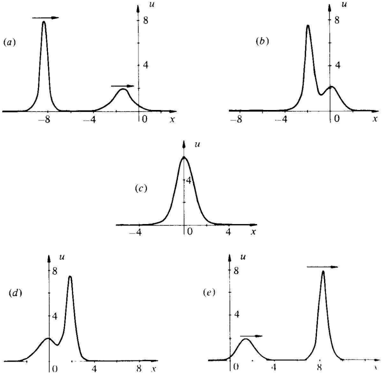
$$x_n^{\pm} = \frac{1}{2\kappa_n} \ln \frac{c_n(0)^2}{2\kappa_n} \pm \frac{1}{2\kappa_n} \left\{ \sum_{m=1}^{n-1} \ln \left| \frac{\kappa_n - \kappa_m}{\kappa_n + \kappa_m} \right| - \sum_{m=n+1}^{N} \ln \left| \frac{\kappa_n - \kappa_m}{\kappa_n + \kappa_m} \right| \right\}.$$

Reflectionless potentials and N-soliton solutions

One can infer from Eq. (44) that at $t \gg 1$, the tallest soliton with $n = N$ is at the front followed by the progressively shorter solitons behind, forming thus the triangle amplitude (velocity) distribution characteristic for noninteracting particles. At $t \to -\infty$ we get the reversed picture. The full solution (42), (43) thus describes the interaction (collision) of N solitons at finite times. For this reason it is called N-soliton solution.

The N-soliton solution is characterised by $2N$ parameters $\kappa_1, \dots, \kappa_N$, $c_1(0), \dots, c_N(0)$. Owing to isospectrality ($\kappa_n = \text{constant}$), the solitons preserve their amplitudes (and velocities) in the interactions; the only change they undergo is an additional phase shift $\delta_n = x_n^+ - x_n^-$ due to collisions.

2-soliton solution: interaction of two solitons



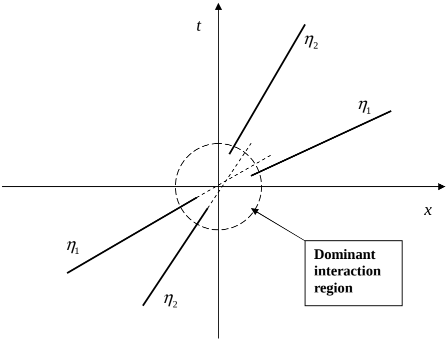
$$-u_2(x,t) \sim 2\kappa_1^2 \text{sech}^2[\kappa_1(x - 4\kappa_1^2 t \mp x_1)] + 2\kappa_2^2 \text{sech}^2[\kappa_2(x - 4\kappa_2^2 t \mp x_2)]$$

as $t \to \pm\infty$

Reflectionless potentials. N=2

For a two-soliton collision with $\kappa_1 > \kappa_2$ the phase shifts as $t \to +\infty$ are

$$\delta_1 = 2x_1 = \frac{1}{\kappa_1} \ln \left(\frac{\kappa_1 + \kappa_2}{\kappa_1 - \kappa_2} \right) , \quad \delta_2 = 2x_2 = -\frac{1}{\kappa_2} \ln \left(\frac{\kappa_1 + \kappa_2}{\kappa_1 - \kappa_2} \right) . \quad (45)$$



It follows from formula above that, as a result of the interaction, the taller soliton gets an additional shift forward by the distance δ_1 while the shorter soliton is shifted backwards by the distance $-\delta_2$.

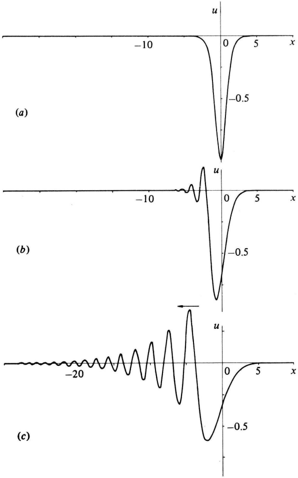
Solitonsless potentials: nonlinear radiation

In contrast to the reflectionless potentials, characterised by discrete spectrum, there are potentials characterised by pure continuous spectrum. In particular, this is the case for all positive potentials $u_0(x) \geq 0$. Now one has to deal with the second term alone in formula (34). In this case, the general expression for the solution similar to the N-soliton solution is not available.

An asymptotic analysis shows that, under the long-time evolution, such a potential transforms into the linear dispersive wave (the radiation) described by the linearised KdV equation but the detailed structure of this wave and its dependence on the initial data are very complicated. However, the qualitative behaviour is physically transparent: the linear radiation propagates to the left with the velocity close to the group velocity $c_g = -3k^2$ of a linear wavepacket and the lowest rate of the amplitude decay is $\sim t^{-1/2}$ which is consistent with the wave energy conservation in linear theory.

Evolution of a “solitonless” profile

The plot shows evolution of $-u(x,t)$



Evolution of an arbitrary decaying profile

The long-time asymptotic outcome of the general KdV initial-value problem for decaying initial data can be represented in the form

$$u(x,t) \sim -\sum_{n=1}^{N} 2\kappa_n^2 \mathrm{sech}^2(\kappa_n(x - 4\kappa_n^2 t - x_n)) + \text{linear radiation}\,, \qquad (46)$$

where the soliton amplitudes $a_n = 2\kappa_n^2$ and the initial phases x_n, as well as the parameters of the radiation component, are determined from the scattering data for the initial potential.

Unfortunately, even the direct scattering problem can be solved explicitly only for very few potential forms. In most cases, one has to use numerical simulations or asymptotic estimates.

Arbitrary initial potential: concluding remarks

For a given initial potential $u_0(x)$:

- A simple **sufficient** condition for the appearance of at least one bound state in the spectrum (i.e. of a soliton in the solution of the KdV equation $u_t - 6uu_x + u_{xxx} = 0$) is

$$\int_{-\infty}^{+\infty} u_0(x)dx < 0 . \tag{47}$$

- The upper bound for the number N of solitons in the solution can be estimated by the formula

$$N \leq 1 + \int_{-\infty}^{+\infty} |x||u_0(x)|dx . \tag{48}$$

Example: KdV evolution of a delta-function profile

$$u_t - 6uu_x + u_{xxx} = 0\,, \qquad u(x,0) = -u_0\delta(x)\,. \tag{49}$$

Scattering Problem

Consider $-\phi_{xx} + V(x)\phi = \lambda\phi$, where $V(x) = -u_0\delta(x)$.

Simple analysis shows that

- $\phi(x)$ is continuous for all $x \in \mathbb{R}$.
- $\phi'(x)$ has a jump at $x = 0$: $\phi'(+0) - \phi'(-0) = -u_0\phi(0)$

Then for a discrete spectrum $\lambda_n = -\kappa_n^2 < 0$ we have

$$\phi_n = \begin{cases} \alpha_n e^{-\kappa_n x}, & x > 0, \\ \beta_n e^{\kappa_n x}, & x < 0. \end{cases}$$

Using continuity of $\phi(x)$ at $x = 0$ and normalisation $\int_{-\infty}^{\infty} \phi_n^2 dx = 1$ we obtain $\alpha_n = \beta_n = \sqrt{\kappa_n}$. Then the jump condition for ϕ' at $x = 0$ implies that there is only one discrete eigenvalue $\lambda_1 = -\kappa_1^2$, where $\kappa_1 = u_0/2$ (note that there is no discrete spectrum if $u_0 < 0$).

Example: KdV evolution of a delta-function profile

Continuous spectrum

Let $\lambda = k^2$, $k \in \mathbb{R}$. Then the scattering solution is

$$\phi_n = \begin{cases} e^{-ikx} + b(k)e^{ikx}, & x > 0, \\ a(k)e^{-ikx}, & x < 0. \end{cases}$$

Continuity at $x = 0$ yields $1 + b(k) = a(k)$. Then from the jump condition for ϕ' at $x = 0$ we obtain the reflection coefficient.

$$b(k) = -\frac{u_0}{2ik + u_0}\,.$$

One can see that

- continuous spetrum exists for both signs of u_0;
- $b(k)$ has a simple pole at $k = iu_0/2$ in the upper half-plane, the discrete spectrum value $\lambda = k^2 = -u_0^2/4$.

Example: KdV evolution of a delta-function profile

The single discrete eigenvalue $\lambda_1 = -\kappa_1 = -u_0/2$ and non-zero reflection coefficient $b(k)$ imply the KdV solution in the form of a single soliton with the amplitude $a = 2\kappa_1^2 = u_0^2/2$ plus some radiation due to continuous spectrum, so that for $t \gg 1$ we have

$$u(x,t) \simeq -\frac{u_0^2}{2}\mathrm{sech}^2\left(\frac{u_0}{2}(x - u_0^2 t - x_0)\right) + \text{radiation}\,, \qquad (50)$$

where $x_0 = \frac{1}{u_0}\ln\frac{1}{2}$.

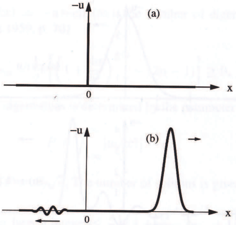


Fig. 9.13 (a) Initial profile and (b) solution at a later time.

Semi-classical asymptotics

Semi-classical asymptotics in the IST method

One of the important cases where some explicit analytic results of rather general form become available, occurs when the initial potential is a 'large-scale' function.
We consider the KdV equation in the form

$$u_t + 6uu_x + u_{xxx} = 0 \qquad (51)$$

with large-scale positive initial data

$$u(x,0) = u_0(x/L) > 0\,, \qquad L \gg 1\,. \qquad (52)$$

For simplicity, we assume that initial function (52) has a form of a single positive hump satisfying an additional condition

$$\int_{-\infty}^{\infty} {u_0}^{1/2} dx \gg 1\,, \qquad (53)$$

An estimate following from Eq. (53) is $A^{1/2}L \gg 1$, where $A = \max(u_0)$.

Semi-classical asymptotics in the IST method

The Schrödinger operator associated with the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{54}$$

is

$$-\phi_{xx} - u\phi = \lambda\phi\,. \tag{55}$$

Then, for large positive initial condition $u_0(x) > 0$ the operator (55) has a large number of bound states located close to each other so that the discrete spectrum can be characterized by a single continuous distribution function. In this case, an effective asymptotic description of the spectrum can be obtained with the use of the semi-classical Wentzel-Kramers-Brillouin (WKB) method.

Semi-classical asymptotics in the IST method

Assuming $A = \mathcal{O}(1)$ we introduce 'slow' variables $X = \epsilon x$, $T = \epsilon t$, where $\epsilon = 1/L \ll 1$ is a small parameter, into Eq. (54) to get the small-dispersion KdV equation:

$$u_T + 6uu_X + \epsilon^2 u_{XXX} = 0\,, \qquad \epsilon \ll 1 \tag{56}$$

with the initial condition

$$u(X, 0) = u_0(X) \geq 0\,, \tag{57}$$

where

$$u_0(X) \;\text{ is } \mathrm{C}^1\,, \quad \text{and} \quad \int_{-\infty}^{\infty} u_0^{1/2} dX = \mathcal{O}(1)\,. \tag{58}$$

The associated Schrödinger equation (55) in the Lax pair assumes the form

$$-\epsilon^2 \phi_{XX} - u\phi = \lambda\phi\,, \qquad \epsilon \ll 1\,. \tag{59}$$

Semi-classical asymptotics in the IST method

The WKB analysis of the Schrödinger equation (59) yields that, for the potential $-u_0(X) \leq 0$ satisfying condition (58) the reflection coefficient is asymptotically zero,

$$\lim_{\epsilon \to 0} b(k) = 0\,, \tag{60}$$

while the eigenvalues $\lambda_n = -\eta_n^2$ ($n = 1, \dots, N$ and $\eta_1 > \eta_2 > \cdots > \kappa_n \geq 0$) are distributed in the range $-A \leq -\eta^2 \leq 0$ so that the density of the distribution of η_k's is given by the formula (Weyl's law)

$$\phi(\eta) = \frac{1}{\pi\epsilon} \int_{X^-(\eta)}^{X^+(\eta)} \frac{\eta}{\sqrt{u_0(X) - \eta^2}} dX\,, \tag{61}$$

so that $\phi(\eta) d\eta$ is the number of η_k's in the interval $(\eta; \eta + d\eta)$. Here the limits of integration $X^-(\eta) < X^+(\eta)$ are defined by $u_0(X^{\pm}) = \eta^2$.

Semi-classical asymptotics in the IST method

The total number of bound states N can be estimated as

$$N \sim \int_0^{A^{1/2}} \phi(\eta) d\eta = \frac{1}{\pi\epsilon} \int_{-\infty}^{+\infty} u_0^{1/2}(X) dX \gg 1\,. \qquad (62)$$

The inequality in (62) is equivalent to the condition (53) and clarifies its physical meaning. The norming constants $c_n(0)$ of the scattering data in the semi-classical limit are given by formulae

$$c_n = \exp\{\chi(\kappa_n)/\epsilon\}\,, \qquad (63)$$

where

$$\chi(\eta) = \eta X^+(\eta) + \int_{X^+(\eta)}^{\infty} \left(\eta - \sqrt{\eta^2 - u_0(X)}\right) dX\,. \qquad (64)$$

Semi-classical asymptotics in the IST method

Now we interpret the semi-classical scattering data (60) – (64) in terms of the solution $u(X, T)$ of the small-dispersion KdV equation (56). First of all, the relation (60) implies that the potential $-u_0(X)$ is *asymptotically reflectionless* and, hence, the initial data $u_0(X)$ can be approximated by the N-soliton solution (42), (43),

$$u_0(X) \approx u_N(X/\epsilon) \qquad \text{for} \quad \epsilon \ll 1 \,, \tag{65}$$

where $N[u_0] \sim \epsilon^{-1}$ is given by (62) and the discrete spectrum is defined by (61), (63), (64). Now one can use the known N-soliton dynamics for the description of the evolution of an arbitrary initial potential satisfying the condition (53). This observation served as a starting point in the series of papers by Lax, Levermore and Venakides (see their review (1994) and references therein), where the singular zero-dispersion limit of the KdV equation has been introduced and thoroughly studied.

Semi-classical asymptotics in the IST method

While the description of multisolitons at finite T turns out to be quite complicated in the zero-dispersion limit, the asymptotic behaviour as $T \to \infty$ can be readily predicted using formula (44) which implies that the asymptotic as $T \to \infty$ outcome of the evolution will be a 'soliton train' consisting of N free solitons ordered by their amplitudes $a_n = 2\kappa_n^2\,,\; n = 1, \dots, N$ and propagating on a zero background. The number of solitons in the train having the amplitude within the interval $(a, a + da)$ is $f(a)da$ where the soliton amplitude distribution function $f(a)$ follows from Weyl's law (61):

$$f(a) = \frac{1}{8\pi\epsilon} \oint \frac{dX}{\sqrt{u_0(X) - a/2}}\,. \tag{66}$$

The formula (66) was obtained for the first time by Karpman (1967).

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