Toric forms of elliptic curves and their arithmetic

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Abstract

We scan a large class of one-parameter families of elliptic curves for efficient arithmetic. The construction of the class is inspired by toric geometry, which provides a natural framework for the study of various forms of elliptic curves. The class both encompasses many prominent known forms and includes thousands of new forms. A powerful algorithm is described that automatically computes the most compact group operation formulas for any parameterized family of elliptic curves. The generality of this algorithm is further illustrated by computing uniform addition formulas and formulas for generalized Montgomery arithmetic.

Key words: elliptic curve, cryptography, arithmetic, newton polytope, toric geometry

1 Introduction

Since the discovery of the elliptic curve factorization method (Lenstra, 1987) and the introduction of elliptic curve cryptography by Miller (1986) and Koblitz (1987), there has been a continuous interest in speeding up addition/doubling and (multi-)scalar multiplication on elliptic curves. Whereas

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Lenstra and Koblitz suggested to simply use the short Weierstrass equation and normal affine coordinates, Miller already proposed the use of Jacobian coordinates.

In a plethora of papers, many new coordinate systems and different forms were proposed. The most notable proposals are the following: Chudnovsky Jacobian coordinates (Chudnovsky and Chudnovsky, 1986) and modified Jacobian coordinates (Cohen et al., 1998), both using the short Weierstrass equation, Jacobi intersections (Chudnovsky and Chudnovsky, 1986; Liardet and Smart, 2001), the Hessian form (Joye and Quisquater, 2001; Smart, 2001), the Jacobi quartic form (Billet and Joye, 2003; Chudnovsky and Chudnovsky, 1986), the Montgomery form (Montgomery, 1987), the Doche/Icart/Kohel forms (Doche et al., 2006) and finally, the Edwards and twisted Edwards forms (Bernstein and Lange, 2007; Bernstein et al., 2008a). All these forms and coordinate systems have been gathered in the Explicit Formulas Database by Bernstein and Lange (EFD), which also includes numerous speed-ups, mainly due to Bernstein and Lange themselves, and to Hisil et al. (2007, 2008).

The discovery of these different forms raises the question whether there are more unknown forms of interest that lead to efficient arithmetic. The goal of this paper is to provide an answer within a certain large class. In this, we will always assume that we work over a field of sufficiently large characteristic. The class is inspired by classical results from toric geometry that give a natural classification of elliptic curves based on the Newton polytope of the defining polynomial, provided the latter satisfies a certain generic condition. This idea was presented at (Castryck, 2008) – it independently proved useful (Lange, 2008) in the construction of a characteristic 2 variant of Edwards arithmetic (Bernstein et al., 2008b). On the highest level, there are 16 non-equivalent base forms, only 6 of which seem to have appeared in the literature so far. On a somewhat lower level, i.e. by using $\mathbb{Z}$-affine transformations and specializing coefficients, one obtains an infinite number of forms, out of which we selected our class. It consists of over 50000 one-parameter families of elliptic curves, all of which we scanned for efficient arithmetic.

Of course, computing group operation formulas, let alone efficient formulas, in a large number of parameterized families soon becomes impossible by hand. To solve this problem, we propose a very general algorithm to compute efficient group operation formulas based on a combination of interpolation and lattice reduction. Alternatively, we could have used a rational simplication algorithm due to Monagan and Pearce (2006), but our method is more robust and avoids capricious Gröbner basis computations. The robustness of our algorithm is illustrated by its capability of computing efficient affine or projective addition/doubling/negation formulas, efficient uniform addition formulas (i.e. formulas that can also be used for doubling), and efficient formulas for generalized Montgomery arithmetic. In each case we provide a non-trivial example obtained by our algorithm.

At no point in this article, we claim immediate cryptographic applicability, neither are we blind for the current limitations of our scan: we restrict to prime fields of large enough characteristic, we restrict to affine doubling formulas, some well-known forms are not covered by our class, and we tightly link efficiency with compactness. However, we emphasize that these limitations are not intrinsic to our method.

Despite these limitations, there are some interesting conclusions to be drawn. First, we prove that within our class, the Edwards form is essentially the only form admitting quadratic doubling and addition formulas having comparatively small coefficients. Although our class is finite, it seems big enough to detect patterns, and in Proposition 11 we will give some theoretical evidence suggesting that the above conclusion is not a coincidence. Another conclusion is that relatively good formulas are very common, so that designers for which other curve features are more important should not feel limited to the list of well-known forms. Our interpolation algorithm can then serve in finding efficient formulas for arithmetic.

Along the way, we obtain a number of theoretical results and side-way observations, regardless of our class. For instance, we obtain a better understanding of what can be expected from Montgomery arithmetic, in its most general setting: we prove that one can never improve upon Montgomery's doubling formula in the rough sense explained in Proposition 12. Next, we illustrate that by studying the toric resolution of the curve, one can often guess good candidates for projective coordinate systems suited for efficient arithmetic. It also provides inspiration for the design of elliptic curve forms admitting complete addition and doubling formulas – this was used in (Bernstein et al., 2008b; Lange, 2008). Two off-topic contributions are an explanation, within a random matrix model, of various statistics concerning the number of isomorphism classes of elliptic curves in certain families, and a proof of the fact that elliptic curves defined by trinomials always have j-invariant 0 or 1728 (over fields of suffiently large characteristic).

The remainder of this paper is organized as follows. Section 2 recalls the notions of elliptic curve, addition formulas, doubling formulas, uniformity, and completeness. Five well-known and well working examples are discussed, and corresponding statistics are proven within the random matrix model. Generalized Montgomery arithmetic is introduced. Section 3 presents our framework based on toric geometry, along with the construction of our class of over 50000 forms. It is shown that many prominent known forms are contained in this class, including our five selected examples. Section 4 describes our interpolation algorithm to compute efficient formulas for arithmetic in families of elliptic curves and provides examples illustrating the robustness of this algorithm. Section 5 discusses the results of our scan, while proving some prudent optimality results on Edwards and Montgomery doubling. Finally, Section 6 concludes the paper.

All computations were carried out using the Magma computer algebra system (Bosma et al., 1997).

2 Elliptic curves, addition, and doubling

2.1 Theoretical framework

Throughout this article, k denotes a perfect field (typically a finite field or a field containing $\mathbb{Q}$) and $\overline{k}$ denotes an algebraic closure.

An *elliptic curve* over k is a pair $(E, \mathcal{O})$. Here E is a curve of geometric genus one in $\mathbb{P}^2$, defined by the homogenization of an absolutely irreducible polynomial $C(x, y) \in k[x, y]$. We do not impose E to be non-singular: this is not standard, but it allows us to consider e.g. Edwards curves and Jacobi quartics as elliptic curves in a more natural way. In any case, there always exists a non-singular curve $\widetilde{E}/k$ along with a k-rational birational morphism $\lambda : \widetilde{E} \to E$ under which the non-singular part E_{ns} of E can be identified with a Zariski open subset of $\widetilde{E}$, i.e. $\lambda|_{\lambda^{-1}(E_{ns})}$ is an isomorphism. The points of $\widetilde{E}$ are called *places* of E, and a place $\mathcal{P}$ is said to *dominate* $\lambda(\mathcal{P})$. The singular points of E may be dominated by several places. The second parameter $\mathcal{O}$ is a k-rational place of E. By the above identification this is typically just a non-singular k-rational point.

The curve $\widetilde{E}$ is endowed with unique k-rational morphisms $\psi : \widetilde{E} \times \widetilde{E} \to \widetilde{E}$ and $\chi : \widetilde{E} \to \widetilde{E}$ that can be interpreted as addition and negation, turning $\widetilde{E} = \widetilde{E}(\overline{k})$ into an abelian group in which $\mathcal{O}$ serves as neutral element. Note that, for each intermediate field $k \subset k' \subset \overline{k}$, the k'-rational points $\widetilde{E}(k')$ form a subgroup of $\widetilde{E}$. We will write $\mathcal{P} + \mathcal{Q}$ for $\psi(\mathcal{P}, \mathcal{Q})$ and $-\mathcal{P}$ for $\chi(\mathcal{P})$. Changing the base place boils down to translating the group law: let $\mathcal{O}'$ be a new base place inducing new operations $+'$ and $-'$, then $\mathcal{P} +' \mathcal{Q} = ((\mathcal{P} - \mathcal{O}') + (\mathcal{Q} - \mathcal{O}')) + \mathcal{O}'$ and $-'\mathcal{P} = -(\mathcal{P} - \mathcal{O}') + \mathcal{O}'$. For each $n \in \mathbb{Z}$ the notation $[n]\mathcal{P}$ abbreviates

$$\operatorname{sgn}(n) \cdot \Bigl(\underbrace{\mathcal{P} + \cdots + \mathcal{P}}_{|n| \text{ times}}\Bigr).$$

The map $\varphi_n : \widetilde{E} \to \widetilde{E} : \mathcal{P} \to [n]\mathcal{P}$ is a degree n^2 morphism. We will be particularly interested in the *doubling map* φ_2.

The function fields of $\widetilde{E}$ and $\widetilde{E} \times \widetilde{E}$ are identified with the fraction fields of

$$\frac{k[x,y]}{(C(x,y))} \quad \text{and} \quad \frac{k[x_1,y_1,x_2,y_2]}{(C(x_1,y_1),C(x_2,y_2))}$$

respectively. We define a set of *addition formulas* on an elliptic curve $(E, \mathcal{O})$ to be a quartet of non-zero polynomials $f_1, g_1, f_2, g_2 \in k[x_1, y_1, x_2, y_2]$ such that

$$x \circ \psi = \frac{f_1}{g_1}, \qquad y \circ \psi = \frac{f_2}{g_2}$$

inside the function field $k(\widetilde{E} \times \widetilde{E})$. A set of *doubling formulas* is a quartet of non-zero polynomials $f_1, g_1, f_2, g_2 \in k[x, y]$ such that

$$x \circ \varphi_2 = \frac{f_1}{g_1}, \qquad y \circ \varphi_2 = \frac{f_2}{g_2}$$

inside the function field $k(\widetilde{E})$.

Let $U_{ns} = E_{ns} \cap \mathbb{A}^2$. Then addition formulas and doubling formulas can be used to perform arithmetic on generically chosen points of U_{ns}. For instance, let f_1, g_1, f_2, g_2 be a set of addition formulas and take points $\mathcal{P} = (p_1, p_2)$, $\mathcal{Q} = (q_1, q_2) \in U_{ns}$. Then unless the denominators are zero, it makes sense to compute

$$\left(\frac{f_1(p_1,p_2,q_1,q_2)}{g_1(p_1,p_2,q_1,q_2)}, \frac{f_2(p_1,p_2,q_1,q_2)}{g_2(p_1,p_2,q_1,q_2)} \right).$$

If the result is in U_{ns} again, this exactly matches with $\mathcal{P} + \mathcal{Q}$. Point pairs of $(\widetilde{E} \times \widetilde{E}) \setminus (U_{ns} \times U_{ns})$, as well as point pairs of $U_{ns} \times U_{ns}$ where the above method fails, are called *exceptional point pairs* with respect to the given addition formulas. With respect to doubling formulas, it is straightforward to define the similar notion of *exceptional points*. Exceptional point (pair) sets are always of codimension ≥ 1.

2.2 Some well-known examples

Here are five famous shapes of elliptic curves. Evidently, the existing literature contains a lot more forms that have proven useful (see the references in the introduction), but the examples below are both very classical – even the Edwards form, which in fact dates back to Gauss – and illustrative for the remainder of this paper.

(1) Assume $\operatorname{char} k \neq 2, 3$. A *Weierstrass curve* is an elliptic curve E defined by

$$C(x,y) = y^2 - x^3 - Ax - B \in k[x,y], \quad 4A^3 + 27B^2 \neq 0,$$

along with the unique point $\mathcal{O} = (0, 1, 0)$ at infinity. Such a curve is non-singular (thus $\widetilde{E} = E$ and $\lambda = \mathrm{id}$) and the group operations can be described using the well-known tangent-chord method. A naive calculation then gives the following addition formulas:

$$x \circ \psi = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2 - x_1 - x_2, \qquad y \circ \psi = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x_1 - x_3) - y_1,$$

where x_3 abbreviates $x \circ \psi$. Note that all point pairs for which $x_1 = x_2$ are exceptional. In particular, the above expressions are unsuitable for doubling, for which instead

$$x_3 = x \circ \varphi_2 = \left(\frac{3x^2 + A}{2y}\right)^2 - 2x, \qquad y \circ \varphi_2 = \left(\frac{3x^2 + A}{2y}\right)(x - x_3) - y$$

can be used.

(2) Assume $\mathrm{char}\, k \neq 3$. A *Hessian curve* is an elliptic curve E defined by

$$C(x, y) = x^3 + y^3 + 1 - 3dxy \in k[x, y], \quad d^3 \neq 1,$$

along with $\mathcal{O} = (-1, 1, 0)$. Hessian curves are non-singular, and again tangent-chord arithmetic applies. We refer to (Joye and Quisquater, 2001) for details on how to obtain the addition formulas

$$x \circ \psi = \frac{y_1^2 x_2 - y_2^2 x_1}{x_2 y_2 - x_1 y_1}, \qquad y \circ \psi = \frac{x_1^2 y_2 - x_2^2 y_1}{x_2 y_2 - x_1 y_1}$$

and for explicit doubling formulas. An interesting property of Hessian curves is that, although the diagonal of $U_{ns} \times U_{ns}$ belongs to the exceptional locus of the addition formulas, these can nevertheless be used to perform doubling, using the relation $[2](\alpha, \beta, \gamma) = (\gamma, \alpha, \beta) + (\beta, \gamma, \alpha)$. This feature is interesting against side-channel attacks. See also (Smart, 2001).

(3) Assume $\mathrm{char}\, k \neq 2$. An *Edwards curve* is an elliptic curve E which is defined by a polynomial

$$C(x, y) = x^2 + y^2 - 1 - dx^2y^2 \in k[x, y], \quad d \neq 0, 1,$$

along with the non-singular affine point $\mathcal{O} = (0, 1)$. Edwards curves allow the following elegant addition formulas:

$$x \circ \psi = \frac{x_1 y_2 + y_1 x_2}{1 + dx_1 x_2 y_1 y_2}, \qquad y \circ \psi = \frac{y_1 y_2 - x_1 x_2}{1 - dx_1 x_2 y_1 y_2}.$$

These are *uniform*, in the sense that under $x_1, x_2 \mapsto x$ and $y_1, y_2 \mapsto y$ they specialize to doubling formulas (see also Section 2.3). The curve E has two singular points, namely $(1,0,0)$ and $(0,1,0)$. It desingularizes to an intersection of quadrics

$$\widetilde{E} : \begin{cases} xy - zw = 0 \\ x^2 + y^2 - z^2 - dw^2 = 0 \end{cases}$$

in $\mathbb{P}^3$, which naturally projects onto $E \subset \mathbb{P}^2$ (the projection $\lambda : \widetilde{E} \to E$ corresponds to substituting $x \leftarrow xz$, $y \leftarrow yz$, $z \leftarrow z^2$, $w \leftarrow xy$). The place dominating $\mathcal{O}$ is $(0,1,1,0)$. The places dominating $(1,0,0)$ are $(\sqrt{d},0,0,1)$ and $(-\sqrt{d},0,0,1)$, and the places dominating $(0,1,0)$ are $(0,\sqrt{d},0,1)$ and $(0,-\sqrt{d},0,1)$. Note that if d is a non-square, then $\widetilde{E}(k) \subset \mathbb{A}^2$, which is related to the *completeness* of Edwards addition in that case (see Section 2.3). The main references on Edwards arithmetic are (Bernstein et al., 2008a; Bernstein and Lange, 2007).

(4) Assume $\operatorname{char} k \neq 2$. A *Jacobi quartic* is an elliptic curve E defined by

$$C(x,y) = y^2 - x^4 + 2Ax^2 - 1 \in k[x,y], \quad A \neq \pm 2,$$

along with the affine point $\mathcal{O} = (0,1)$. In (Billet and Joye, 2003), the following formulas were computed:

$$x \circ \psi = \frac{x_1y_2 + x_2y_1}{1 - x_1^2x_2^2}, \quad y \circ \psi = \frac{(1 + x_1^2x_2^2)(y_1y_2 - 2Ax_1x_2) + 2x_1x_2(x_1^2 + x_2^2)}{1 - x_1^2x_2^2},$$

which are again uniform. The Jacobi quartic has a singular point $(0,1,0)$ at infinity. The desingularization map λ is the projection from the intersection in $\mathbb{P}^3$ of the quadrics $x^2 - zw$ and $y^2 - w^2 + 2Ax^2 - z^2$ to $\mathbb{P}^2$ (corresponding to substituting $x \leftarrow xz$, $y \leftarrow yz$, $z \leftarrow z^2$, $w \leftarrow x^2$). The place dominating $\mathcal{O}$ is $(0,1,1,0)$. The places dominating $(0,1,0)$ are $(0,1,0,1)$ and $(0,-1,0,1)$.

(5) Assume $\operatorname{char} k \neq 2$. A *Montgomery curve* is an elliptic curve E defined by a polynomial

$$C(x,y) = By^2 - x^3 - Ax^2 - x \in k[x,y], \quad B \neq 0,\ A \neq \pm 2$$

along with the unique point $\mathcal{O} = (0,1,0)$ at infinity. Montgomery curves are non-singular and tangent-chord arithmetic applies. Montgomery (1987) proved the following efficient *x-coordinate only formulas*:

$$x \circ \varphi_2 = \frac{(x+1)^2(x-1)^2}{4x((x-1)^2 + \frac{A+2}{4}((x+1)^2 - (x-1)^2))} \tag{1}$$

$$x_{m+n}x_{m-n} = \frac{((x_m-1)(x_n+1) + (x_m+1)(x_n-1))^2}{((x_m-1)(x_n+1) - (x_m+1)(x_n-1))^2},$$

where $x_i = x \circ \varphi_i$. We will say more on formulas of this type in Section 2.4.

Intermezzo: classification of the above forms.

In the following discussion, we always assume that k is a finite field having an appropriate characteristic ($\operatorname{char} k \geq 5$ will work everywhere). It is well-known that every elliptic curve is in k-rational birational equivalence with a Weierstrass curve, but the same is no longer true for the other forms **(2-5)**. In this intermezzo, we will give a brief classification, both up to k-birational equivalence and up to k-isogeny, and explain corresponding statistics within the random matrix model; see (Achter, 2008) and (Castryck and Hubrechts, 2010). It is essentially a summary of existing (yet fragmentary) material. Similar statistics have been observed in (Bernstein et al., 2008a, Section 4). Classification up to $\overline{k}$-birational equivalence can be done through a j-invariant computation, which was carried out in (Rezaeian Farashahi and Shparlinski, 2009) in a number of cases.

(2) If $(E, \mathcal{O})$ is Hessian, then $\widetilde{E}(k)$ has a subgroup $\{\mathcal{O}, (-1, 0, 1), (0, -1, 1)\}$ of order 3. In particular, $3 \mid \#\widetilde{E}(k)$ is a necessary condition for an elliptic curve $(E, \mathcal{O})$ to be k-birationally equivalent to a Hessian curve. If $\#k \equiv 2 \bmod 3$ then this condition is also sufficient (Cohen et al., 2006, 13.1.5.b). If $\#k \equiv 1 \bmod 3$, then $(E, \mathcal{O})$ is Hessian if and only if $\widetilde{E}(k)$ contains all nine 3-torsion points of $\widetilde{E}(\overline{k})$. For the if-part, the proof goes as follows. Take a model in which $\mathcal{O}$ is a flex (e.g. a Weierstrass model). Then by the tangent-chord rule, its other flexes are exactly its other 3-torsion points, hence k-rational. But then an additional projective transformation puts our curve into Hessian form (Hirschfeld, 1998, Lemma 11.36). For the only-if-part it suffices to observe that the flexes of a Hessian curve are precisely the intersection points with the three coordinate axes, all of them being rational if $\#k \equiv 1 \bmod 3$.

The probability that a randomly chosen Weierstrass curve can be shaped into a Hessian form can then be estimated as

$$P(3 \,|\, \#\widetilde{E}(k)) \approx \frac{1}{2} \qquad \text{if } \#k \equiv 2 \bmod 3$$

$$P(\widetilde{E}[3](k) \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}) \approx \frac{1}{\#\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})} = \frac{1}{24} \qquad \text{if } \#k \equiv 1 \bmod 3,$$

following the random matrix model. The error term is $O(\#k^{-1/2})$.

Two elliptic curves $(E, \mathcal{O})$ and $(E', \mathcal{O}')$ are k-isogenous if and only if $\#\widetilde{E}(k) = \#\widetilde{E}'(k)$ by Tate's theorem. So if $\#k \equiv 2 \bmod 3$, a necessary and sufficient condition for an elliptic curve $(E, \mathcal{O})$ to be k-isogenous to a Hessian curve is $3 \mid \#\widetilde{E}(k)$. If $\#k \equiv 1 \bmod 3$, then the condition $9 \mid \#\widetilde{E}(k)$ is necessary and almost always sufficient. This follows from Tsfasman et al. (2007, Theorem 3.3.15). All exceptions are supersingular.

(5) Montgomery curves can be intrinsically characterized by the existence of a point $\mathcal{P} \in \widetilde{E}(k)$ for which $\left(\widetilde{E}/\langle\mathcal{P}\rangle\right)(k)$ contains all four 2-torsion points: combine (Okeya et al., 2000, Proposition 5) with Vélu's formulas to see this. Equivalently, an elliptic curve $(E, \mathcal{O})$ is Montgomery if the curve or its quadratic twist have a k-rational point of order 4; see also (Bernstein et al., 2008a, Theorems 3.2 and 3.3). In case $\#k \equiv 3 \bmod 4$, it suffices to check whether $(E, \mathcal{O})$ itself has a point of order 4 (Bernstein et al., 2008a, Theorem 3.4). In case $\#k \equiv 1 \bmod 4$, the curve $(E, \mathcal{O})$ is Montgomery if and only if $4 \mid \#\widetilde{E}(k)$: one can verify that every 2×2 matrix over $\mathbb{Z}/4\mathbb{Z}$ having trace 2 and determinant 1 is conjugated to a matrix of the form

$$\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \quad \text{or of the form} \quad \begin{pmatrix} -1 & w \\ 0 & -1 \end{pmatrix}$$

implying that $(E, \mathcal{O})$ resp. its quadratic twist have a k-rational point of order 4.

The probability that a randomly chosen Weierstrass curve can be shaped into a Montgomery form can then be estimated as

$$P(\widetilde{E}(k) \text{ contains point of order 4}) \approx \frac{3}{8} \qquad \text{if } \#k \equiv 3 \bmod 4$$

$$P(4 \,|\, \#\widetilde{E}(k)) \approx \frac{5}{12} \qquad \text{if } \#k \equiv 1 \bmod 4,$$

following the random matrix model. The error term is again $O(\#k^{-1/2})$.

Clearly, a necessary condition for an elliptic curve $(E, \mathcal{O})$ to be k-isogenous to a Montgomery curve is $4 \mid \#\widetilde{E}(k)$. This is also sufficient: an explicit isogeny is given in (Bernstein et al., 2008a, Theorem 5.1).

(3) Up to k-rational birational equivalence, Edwards curves are precisely those elliptic curves having a k-rational point of order 4, see (Bernstein et al., 2008a, Theorem 3.3). Note that in particular, every Edwards curve is a Montgomery curve and, conversely, every Montgomery curve is a twist of an Edwards curve.

The probability that a randomly chosen Weierstrass curve can be shaped into an Edwards form can then be estimated as

$$P(\widetilde{E}(k) \text{ contains point of order 4}) \approx \frac{3}{8} \qquad \text{if } \#k \equiv 3 \bmod 4$$

$$P(\widetilde{E}(k) \text{ contains point of order 4}) \approx \frac{1}{3} \qquad \text{if } \#k \equiv 1 \bmod 4,$$

following the random matrix model. The error term is $O(\#k^{-1/2})$.

In the case where k is finite, an application of (Tsfasman et al., 2007, Theorem 3.3.15) classifies Edwards curves up to isogeny: if $4 \mid \#\widetilde{E}(k)$, then $(E, \mathcal{O})$ only fails to be k-isogenous to an Edwards curve if $\#k$ is a square and $\widetilde{E}$ is a supersingular curve having $(\sqrt{\#k} \pm 1)^2$ rational points, the sign to be chosen such that $4 \nmid \sqrt{\#k} \pm 1$. In particular, if $\#k \equiv 3 \bmod 4$, then $\#k$ is never a square and an explicit k-isogeny can be constructed following (Bernstein et al., 2008a, Theorems 3.2, 3.4 and 5.1).

(4) A Weierstrass curve can be k-birationally transformed to a Jacobi quartic if and only if all 2-torsion points are k-rational. This can be read along the lines in (Billet and Joye, 2003, Section 3). Using the random matrix model, the proportion of Weierstrass curves shapable into a Jacobi quartic form is then given by

$$P(\widetilde{E}[2](k) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \approx \frac{1}{\#\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})} = \frac{1}{6}.$$

The error term is $O(\#k^{-1/2})$.

Again by (Tsfasman et al., 2007, Theorem 3.3.15), apart from some explicitly known supersingular exceptions, if $4 \mid \#\widetilde{E}(k)$ then $(E, \mathcal{O})$ is k-isogenous to a Jacobi quartic.

2.3 Uniformity and completeness

We call a set of addition formulas *uniform* if they specialize under $x_i \mapsto x$, $y_i \mapsto y$ $(i = 1, 2)$ to a set of doubling formulas. This will be the case whenever g_1 and g_2 do not identically vanish (over $\overline{k}$) on the diagonal of $U_{ns} \times U_{ns}$. Uniform addition formulas always exist (whatever E and $\mathcal{O}$ are), simply because ψ is a morphism. Indeed, it suffices to take a point $\mathcal{P} \in U_{ns}(\overline{k})$ such that $x \circ \psi$ and $y \circ \psi$ are defined on $(\mathcal{P}, \mathcal{P})$. Then there exists an open neighborhood $W \ni (\mathcal{P}, \mathcal{P})$ in $U_{ns} \times U_{ns}$ and polynomials $f_1, g_1, f_2, g_2 \in k[x_1, y_1, x_2, y_2]$ for which $x \circ \psi = f_1/g_1$ and $y \circ \psi = f_2/g_2$, such that g_1, g_2 nowhere vanish on W. Uniformity is an interesting feature against side-channel attacks. We already encountered uniform addition formulas for Edwards and Jacobi quartic curves. In Section 4.3.3 we provide an example for the short Weierstrass form computed by the algorithm described in Section 4.1.

A set of addition formulas (resp. doubling formulas) is said to be *complete* if $U_{ns}(k) \times U_{ns}(k)$ (resp. $U_{ns}(k)$) contains no exceptional point pairs (resp. exceptional points). As soon as $U_{ns}(k) \neq \emptyset$, complete addition formulas are automatically uniform. However, whereas uniformity is a property that is invariant under base field extension, completeness is not.

Lemma 1. *If $k = \overline{k}$, then complete addition or doubling formulas do not exist.*

Proof. $U_{ns} = U_{ns}(\overline{k}) \neq \emptyset$, thus it suffices to prove that complete doubling formulas cannot exist. Let $\mathcal{P}$ be a place above a point at infinity. Since $\widetilde{E} \setminus U_{ns}$ is finite, there is a minimal r such that $\varphi_{2^r}^{-1}\{\mathcal{P}\}$ contains a point $\mathcal{Q} \in U_{ns}$. This will be an exceptional point. ■

The best-known example of complete addition formulas (and hence of complete doubling formulas) is provided by Edwards addition, as described above. Indeed, if d is taken to be non-square, then $dx_1x_2y_1y_2$ can never be ± 1. See (Bernstein and Lange, 2007, Theorem 3.3) for more details.

2.4 Generalized Montgomery arithmetic

Let $(E, \mathcal{O})$ be an elliptic curve. The subfield of $k(\widetilde{E})$ consisting of functions f that satisfy $f = f \circ \chi$ is of the form $k(t)$ for a non-constant function $t \in k(\widetilde{E})$. Equivalently, $k(t)$ consists of all functions f that satisfy $f(\mathcal{P}) = f(-\mathcal{P})$ for all pairs $\pm\mathcal{P} \in \widetilde{E}(\overline{k})$ at which f is defined. It is a subfield of index 2, corresponding to a k-rational degree 2 morphism $\widetilde{E} \to \mathbb{P}^1$. For example, in **(1)** the Weierstrass setting, the map $f \mapsto f \circ \chi$ is determined by $x \mapsto x, y \mapsto -y$. So the subfield is just $k(x)$, and one can take $t = x$. In **(2)** the Hessian setting, the map $f \mapsto f \circ \chi$ is determined by $x \mapsto y, y \mapsto x$ and one can take $t = x + y$. In **(3)** the Edwards setting, we have $x \mapsto -x, y \mapsto y$ so we can take $t = y$. In **(4)** the Jacobi quartic setting, the map is $x \mapsto -x, y \mapsto y$ and one can take $t = (y+1)/x^2$. Note that $k(y) \subsetneq k(t)$. Finally, **(5)** for Montgomery curves one can take $t = x$ as in the Weierstrass setting. The following lemma is easy to prove by noting that $\varphi_n \circ \chi = \chi \circ \varphi_n$.

Lemma 2. *For all $n \in \mathbb{Z}$, we have $t \circ \varphi_n \in k(t)$.*

A *t-only doubling formula* is a couple of non-zero polynomials $f, g \in k[t]$ such that $t \circ \varphi_2 = f/g$ inside $k(t)$. Concerning addition, it is in general impossible to derive $t(\mathcal{P} + \mathcal{Q})$ from $t(\mathcal{P})$ and $t(\mathcal{Q})$. Instead, one makes use of the next statement, which is easy to verify using the classical Weierstrass addition formulas (it holds in any characteristic).

Lemma 3. *There exists a bivariate rational function F over k such that for all $m, n \in \mathbb{Z}$*

$$(t \circ \varphi_{m+n})(t \circ \varphi_{m-n}) = F(t \circ \varphi_n, t \circ \varphi_m) \quad \text{in } k(t). \tag{2}$$

A *t-only addition formula* is a couple of non-zero polynomials $f, g \in k[t_n, t_m]$ such that $F = f/g$ satisfies (2). Here t_n and t_m are formal variables.

Then t-only arithmetic can be used to compute $[n]\mathcal{P}$ by subsequently obtaining $([n]\mathcal{P}, [n+1]\mathcal{P})$ from $([\lfloor n/2 \rfloor]\mathcal{P}, [\lfloor n/2 \rfloor + 1]\mathcal{P})$, using one t-only doubling and one t-only addition. This is the so-called *Montgomery ladder*; for more details we refer to (Cohen et al., 2006, 13.2.3). Montgomery (1987) proposed this method in the context of speeding up the elliptic curve factorization algorithm (Lenstra, 1987), although soon after it found its way to cryptography, e.g. in `curve25519` (Bernstein, 2006).

2.5 Projective coordinates

To avoid time-costly field inversions, addition and doubling are commonly done using projective coordinates, see for instance (Cohen et al., 2006, 13.2.1.b). The same principle is used for Montgomery arithmetic: one then works on the projective t-line $\mathbb{P}^1$. Now instead of projective coordinates, one can often gain a speed-up using alternative coordinate systems, the most famous being weighted projective coordinates. E.g., in the Weierstrass setting, it is more natural to work in $\mathbb{P}(2;3;1)$, these are called *Jacobian coordinates*; see (Cohen et al., 2006, 13.2.1.c) for some details. Similarly, one preferably works in a $\mathbb{P}(1;2;1)$-related coordinate system for Jacobi quartics. It can also be useful to work with hyperboloidal coordinates, i.e. to work on $\mathbb{P}^1 \times \mathbb{P}^1$ (which is the quadric $xy = zw$ in $\mathbb{P}^3$). For that, one embeds a point (x, y) as $(x, y, 1, xy)$. This setting gave some of the best operation counts so far for Edwards arithmetic (Hisil et al., 2008). We refer to the Explicit Formulas Database (Bernstein and Lange, EFD) for an overview of the various other inversion-free coordinate systems that have been proposed.

3 Toric forms of elliptic curves

3.1 Non-degenerate polynomials, lattice polytopes, and equivalence

For the general background on toric varieties and non-degenerate polynomials, we refer to (Fulton, 1993; Batyrev, 1993).

Let $C(x, y) \in k[x, y]$ be an absolutely irreducible polynomial. Let $S \in \mathbb{Z}^2$ be the set of exponent vectors appearing in C, and denote by $\Delta = \Delta(C)$ its convex hull in $\mathbb{R}^2$. It is called the *Newton polytope* of C, which is an example of a *lattice polytope*, i.e. a convex polytope whose vertices lie in $\mathbb{Z}^2$. A *face* of Δ is either Δ itself, either a non-empty intersection of Δ with a line $aX + bY = c$ for which $\Delta \subset \{(X, Y) \in \mathbb{R}^2 \,|\, aX + bY \leq c\}$. The 1-dimensional faces are called *edges*, the 0-dimensional faces are referred to as *vertices*. The union over the edges of Δ is called the *boundary* and is denoted by $\partial\Delta$. For each subset $\tau \subset \mathbb{R}^2$, let $C_\tau(x, y)$ be obtained from $C(x, y)$ by erasing all terms whose exponent vectors lie outside of τ.

Definition/Theorem 4. Suppose that for each face $\tau \subset \Delta$, the system of equations

$$C_\tau(x, y) = \frac{\partial}{\partial x} C_\tau(x, y) = \frac{\partial}{\partial y} C_\tau(x, y) = 0$$

has no solutions in the *torus* $\mathbb{T}^2 = (\overline{k} \setminus 0)^2 \subset \mathbb{A}^2$, then $C(x, y)$ is called *non-degenerate with respect to its Newton polytope.* In that case, the geometric genus of the curve defined by $C(x, y) = 0$ equals $\#((\Delta \setminus \partial\Delta) \cap \mathbb{Z}^2)$.

This result is due to Hovanskiĭ (1978). See e.g. (Castryck et al., 2006, Corollary 2.8) for an elementary proof. Regardless of the condition of non-degeneracy, $\#((\Delta \setminus \partial\Delta) \cap \mathbb{Z}^2)$ is an upper bound for the geometric genus of the curve defined by $C(x, y) = 0$. This is called *Baker's inequality*, for a recent proof see (Beelen, 2009).

It is worth noting that non-degeneracy can be interpreted in terms of the non-vanishing of a certain polynomial expression in the coefficients of $C(x, y)$. I.e., for any lattice polytope $\Delta \subset \mathbb{R}^2$ there exists a polynomial $r \in \mathbb{Z}[c_{ij}]_{(i,j)\in\Delta\cap\mathbb{Z}^2}$ such that for any field k and for any polynomial $C(x, y) \in k[x, y]$ that is supported in Δ, one has that $r(C) \neq 0$ if and only if $\Delta(C) = \Delta$ and C is non-degenerate with respect to its Newton polytope. The evaluation $r(C)$ is given by

$$\operatorname{res}_\Delta \left(C, x\frac{\partial C}{\partial x}, y\frac{\partial C}{\partial y} \right)$$

where $\operatorname{res}_\Delta$ is the *sparse resultant* or *principal* $(\Delta \cap \mathbb{Z}^2)$*-determinant* in the sense of (Gel'fand et al., 1994, Chapter 10).

We now consider $\mathbb{Z}$*-affine maps*

$$f : \mathbb{R}^2 \to \mathbb{R}^2 : \begin{bmatrix} X \\ Y \end{bmatrix} \mapsto A \cdot \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

for $a, b \in \mathbb{Z}$ and $A \in \mathrm{GL}_2(\mathbb{Z})$. Two lattice polytopes $\Delta, \Delta' \subset \mathbb{R}^2$ are called *equivalent* if there exists a $\mathbb{Z}$-affine map f such that $f(\Delta) = \Delta'$. Two absolutely irreducible polynomials $C(x, y), C'(x, y) \in k[x, y]$ are called *equivalent* if C' can be obtained from C by applying a $\mathbb{Z}$-affine map to its exponent vectors. This procedure actually induces an isomorphism between their respective loci in $\mathbb{T}^2$. Equivalent polynomials have equivalent Newton polytopes, and share their being non-degenerate or not.

In the spirit of Theorem 4, we define the *genus* of a lattice polytope Δ to be $\#((\Delta \setminus \partial\Delta) \cap \mathbb{Z}^2)$. For a fixed $g \geq 1$, there is a finite number of equivalence classes of lattice polytopes of genus g. If $g = 1$, there are 16 equivalence classes. Lattice polytopes representing these are shown in Figure 1 below, which was taken from (Poonen and Rodriguez-Villegas, 2000).

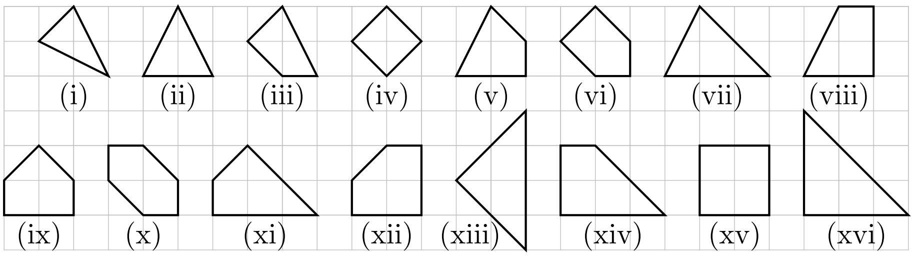


Figure 1: The 16 equivalence classes of lattice polytopes of genus 1

Using the above theory, we can prove the following simple observation, which seems new.

Lemma 5. *Let k be a field of characteristic 0. Any geometric genus one curve over k which is defined by a bivariate trinomial has j-invariant 0 or 1728.*

PROOF. After rescaling and applying a suitable $\mathbb{Z}$-affine map to the exponent vectors, one sees that the locus in $\mathbb{T}^2$ is isomorphic to a curve defined by

$$1 + \alpha y^j + \beta x^k y^\ell \quad \in k[x, y],$$

with $\alpha, \beta \neq 0$ and $j, k > 0$. The $\overline{k}$-isomorphism

$$x \leftarrow \alpha^{\frac{\ell}{jk}} \beta^{-\frac{1}{k}} x, \qquad y \leftarrow \alpha^{-\frac{1}{j}} y$$

transforms the defining polynomial into

$$1 + y^j + x^k y^\ell,$$

which is non-degenerate with respect to its Newton polytope. This Newton polytope therefore contains exactly one interior lattice point. According to Figure 1, up to a $\mathbb{Z}$-affine map we remain with one of the following forms:

$$y^2 + y + x^3, \quad y^2 + x^3 + x, \quad y^2 + x^3 + 1, \quad y^3 + x^3 + 1, \quad y^2 + x^4 + 1.$$

Their j-invariants are 0, 1728, 0, 0 and 1728 respectively. ■

It follows from the proof that the lemma is still true if k is of sufficiently large finite characteristic (when compared to the degree of the trinomial).

From now on, we fix an irreducible polynomial $C(x,y) \in k[x,y]$, and we assume that it is non-degenerate with respect to its Newton polytope Δ. We also assume that $C(x,y)$ defines a curve $E \subset \mathbb{P}^2$ of geometric genus one, although most of the statements below are true for arbitrary genus. By Theorem 4, Δ has exactly one $\mathbb{Z}^2$-point in its interior. Hence up to equivalence, it is one of the polytopes listed in Figure 1.

We maintain the notation introduced in Section 2.1. The desingularization map

$$\lambda : \widetilde{E} \to E$$

can be described very explicitly. To each point $(i,j) \in \Delta \cap \mathbb{Z}^2$, associate a variable z_{ij}. These will be considered as homogeneous coordinate functions on $\mathbb{P}^N$, where $N = \#(\Delta \cap \mathbb{Z}^2) - 1$. The combinatorics of Δ gives rise to a set of *binomial relations* in $k[z_{ij}]$: for $\sum_{s=1}^n (i_s, j_s) = \sum_{s=1}^n (k_s, \ell_s)$, where $(i_s, j_s), (k_s, \ell_s) \in \Delta \cap \mathbb{Z}^2$, we have the degree n relation

$$\prod_{s=1}^n z_{i_s j_s} - \prod_{s=1}^n z_{k_s \ell_s} = 0. \tag{3}$$

These relations can be shown to define a projective surface $X(\Delta) \subset \mathbb{P}^N$, which is called the *toric surface associated to* Δ. The torus $\mathbb{T}^2$ can be canonically embedded in $X(\Delta)$ by

$$\mathbb{T}^2 \hookrightarrow X(\Delta) : (x,y) \mapsto (x^i y^j)_{(i,j) \in \Delta \cap \mathbb{Z}^2}. \tag{4}$$

One can prove that it suffices to restrict to $n \leq 3$ and even to $n \leq 2$ whenever $\#(\partial\Delta \cap \mathbb{Z}^2) > 3$ (Koelman, 1993).

The faces $\tau \subset \Delta$ naturally partition $X(\Delta)$ into sets of the form

$$O(\tau) = \left\{ (\alpha_{ij})_{(i,j) \in \Delta \cap \mathbb{Z}^2} \in X(\Delta) \,\middle|\, \alpha_{ij} \neq 0 \iff (i,j) \in \tau \right\},$$

which are called the *toric orbits* of $X(\Delta)$. Note that $O(\Delta)$ is precisely the image of the above map (4), hence it has the structure of a torus $\mathbb{T}^2$. More generally, each orbit $O(\tau)$ is canonically isomorphic to a torus $\mathbb{T}^{\dim \tau}$. Points in $X(\Delta) \setminus O(\Delta)$ are said to lie at *toric infinity.*

Now $C(x,y)$ itself defines one additional, linear relation in $\mathbb{P}^N$: if

$$C(x,y) = \sum_{(i,j) \in \Delta \cap \mathbb{Z}^2} c_{ij} x^i y^j, \quad \text{then it is} \quad \sum_{(i,j) \in \Delta \cap \mathbb{Z}^2} c_{ij} z_{ij} = 0.$$

This cuts out a curve $\widetilde{E}$ in $X(\Delta)$ which is birationally equivalent to E: this is easily seen using (4). More generally, for a positive integer n, we define the

Minkowski multiple $n\Delta$ of a lattice polytope Δ as the lattice polytope obtained by 'dilating Δ with a factor n', i.e. by taking the convex hull in $\mathbb{R}^2$ of all points (na, nb) for which $(a, b) \in \Delta$. A straightforward calculation shows that there is a natural isomorphism $X(\Delta) \to X(n\Delta)$ such that the torus embedding $\mathbb{T}^2 \hookrightarrow X(n\Delta)$ is in fact the composition of the torus embedding $\mathbb{T}^2 \hookrightarrow X(\Delta)$ with this isomorphism. Now suppose the Newton polytope of $C(x, y)$ is $n\Delta$. Then its image in $X(\Delta)$ is cut out by a hypersurface of degree n. We still denote this image by $\widetilde{E}$.

The following theorem is the main statement on non-degenerate polynomials.

Theorem 6. *The curve $\widetilde{E}$ is non-singular and intersects the 1-dimensional orbits $O(\tau)$ (corresponding to the edges τ of Δ) transversally in $\#(\tau \cap \mathbb{Z}^2) - 1$ points. It does not contain the 0-dimensional orbits (corresponding to the vertices of Δ). In particular, the number of points at toric infinity equals $\#(\partial\Delta \cap \mathbb{Z}^2)$. Moreover, these properties fully characterize the non-degeneracy of $C(x, y)$.*

We can now describe the desingularization map $\lambda : \widetilde{E} \to E$. The restriction map $\lambda \mid_{O(\Delta)}$ is an isomorphism onto $E \cap \mathbb{T}^2 \subset U_{ns}$ whose inverse is given by the embedding (4). Now suppose $\mathcal{P} \in O(\tau) \cap \widetilde{E}$ for an edge $\tau \subset \Delta$. Let $\theta \in [0, 2\pi[$ be such that $(\cos\theta, \sin\theta)$ is a normal vector on τ that points towards the interior of Δ. Write $\mathcal{P} = (\alpha_{ij})_{(i,j)\in\Delta\cap\mathbb{Z}^2}$. Then

(1) if $\theta = 0$, then $\lambda(P) = (0, \alpha_{0,k}, \alpha_{0,k-1})$ where $(0, k), (0, k-1) \in \tau \cap \mathbb{Z}^2$;
(2) if $\theta \in]0, \pi/2[$, then $\lambda(P) = (0, 0, 1)$;
(3) if $\theta = \pi/2$, then $\lambda(P) = (\alpha_{k,0}, 0, \alpha_{k-1,0})$ where $(k, 0), (k-1, 0) \in \tau \cap \mathbb{Z}^2$;
(4) if $\theta \in]\pi/2, 5\pi/4[$, then $\lambda(P) = (1, 0, 0)$;
(5) if $\theta = 5\pi/4$, then $\lambda(P) = (\alpha_{k+1,\ell}, \alpha_{k,\ell+1}, 0)$ where $(k+1, \ell), (k, \ell+1) \in \tau \cap \mathbb{Z}^2$;
(6) if $\theta \in]5\pi/4, 2\pi[$, then $\lambda(P) = (0, 1, 0)$.

In cases (1), (3) and (5), the restriction map $\lambda \mid_{O(\tau)}$ is one-to-one.

3.3 The toric framework of well-known forms

The reader can verify that the basic forms **(1-5)** of Section 2.2 all fit in the above setting, i.e. they are all defined by a non-degenerate polynomial whose Newton polytope is therefore contained in Figure 1 (up to $\mathbb{Z}$-affine equivalence): Weierstrass curves are represented by (vii), Hessian curves by (xvi), Edwards curves by (xv), Jacobi quartics by (xiii) and Montgomery curves by (v). These five polytope classes seem to be the only cases that have been addressed in the literature so far, with the recent exception of binary Edwards curves (Bernstein et al., 2008b), represented by (xii), that were designed for usage over fields of characteristic two only. Note that in all cases, the base point $\mathcal{O}$ lies at toric infinity. Let us have a look at the toric picture of these forms in closer detail.

(1) The Newton polytope Δ_W of a Weierstrass curve E defines the toric surface

$$X(\Delta_W): \quad z_{00}z_{20} = z_{10}^2, \quad z_{10}z_{20} = z_{00}z_{30}, \quad z_{11}z_{00} = z_{10}z_{0,1}, \quad z_{02}z_{00} = z_{01}^2$$

in $\mathbb{P}^6$. The hyperplane $z_{02} = z_{30} + Az_{10} + Bz_{00}$ cuts out a non-singular model $\widetilde{E}$ of E, which of course was itself already non-singular. The unique place dominating $\mathcal{O} = (0,1,0)$ is $(0,0,0,1,0,0,1)$, where the 1's correspond to the variables z_{30} and z_{02}. Now for practical applications, we do not suggest to work with coordinates in $\mathbb{P}^6$. But we remark that $X(\Delta_W)$ is exactly how the weighted projective space $\mathbb{P}(2;3;1)$ is canonically realized as a projective surface: it is the image of

$$\varphi : \mathbb{P}(2;3;1) \hookrightarrow \mathbb{P}^6 : (x,y,z) \mapsto (z^6, xz^4, x^2z^2, x^3, yz^3, xyz, y^2)$$

and under this map, the natural embedding of E in $\mathbb{P}(2;3;1)$ is precisely sent to $\widetilde{E}$. Thus, the toric picture of a Weierstrass curve is its natural embedding in $\mathbb{P}(2;3;1)$.

(2) The Newton polytope of a Hessian curve E is

$$\Delta_H = \text{Conv}\,\{(0,0),(3,0),(0,3)\},$$

which is 3Σ, where Σ is the standard 2-simplex in $\mathbb{R}^2$. The toric surface $X(\Sigma)$ is simply $\mathbb{P}^2$, and the toric model $\widetilde{E}$ is cut out by the cubic relation

$$z_{00}^3 + z_{10}^3 + z_{01}^3 = 3dz_{10}z_{01}z_{00},$$

hence the toric picture of a Hessian curve is the curve itself. One can verify that $X(\Delta_H)$ is the 3-uple embedding of $\mathbb{P}^2$ in $\mathbb{P}^9$, where the Hessian curve becomes a hyperplane section.

(3) The Newton polytope of an Edwards curve E is

$$\Delta_E = \text{Conv}\,\{(0,0),(2,0),(0,2),(2,2)\},$$

which is the Minkowski double of $\square = \text{Conv}\,\{(0,0),(1,0),(0,1),(1,1)\}$. The toric surface $X(\square)$ is the surface in $\mathbb{P}^3$ defined by $z_{01}z_{10} = z_{00}z_{11}$, that is: it is $\mathbb{P}^1 \times \mathbb{P}^1$. The toric model $\widetilde{E}$ of E is cut out by an additional quadratic relation

$$z_{10}^2 + z_{01}^2 = z_{00}^2 + dz_{11}^2.$$

Renaming $z_{00} \leftarrow z$, $z_{10} \leftarrow x$, $z_{10} \leftarrow y$, $z_{11} \leftarrow w$ reveals a complete match with the description of Edwards curves given in Section 2.2, to which we refer for further details.

(4) Similarly, one can verify that the toric surface associated to the Newton Polytope Δ_J of a Jacobi quartic E is the conic $z_{10}^2 = z_{00}z_{10}$ (which is in fact $\mathbb{P}(1;2;1)$) and that $\widetilde{E}$ is cut out by $z_{01}^2 = z_{20}^2 + Az_{10}^2 + z_{00}^2$. Again compare this with the description given in Section 2.2.

(5) The toric surface associated to the Newton polytope of a Montgomery curve is the blow-up of $\mathbb{P}(2;3;1)$ in $(0,0,1)$. The blow-up is only necessary to ensure that the Montgomery curve does not contain the 0-dimensional toric orbit $(0,0,0)$; cf. Theorem 6.

3.4 Some toric design criteria

3.4.1 Projective coordinate systems

Toric surfaces are generalizations of projective space, and can serve as an inspiration for the choice of a coordinate system in which to perform efficient arithmetic (see also Section 2.5). Weighted projective coordinates for Weierstrass curves ($X(\Delta_W) \cong \mathbb{P}(2;3;1)$) and Jacobi quartics ($X(\Delta_J) \cong \mathbb{P}(1;2;1)$) have proven useful (Chudnovsky and Chudnovsky, 1986). It is probably not a coincidence that Hisil et al. (2008) established their speed-records for Edwards curve arithmetic using hyperboloidal coordinates ($X(\Delta_E) \cong \mathbb{P}^1 \times \mathbb{P}^1$) and that ordinary projective coordinates remain in many aspects the better system for Hessian curves ($X(\Delta_H) \cong \mathbb{P}^2$).

3.4.2 Completeness

Lattice polytopes of genus one can also be helpful in the design of elliptic curve forms allowing complete addition and doubling formulas. If $C(x,y)$ and Δ are such that, for each *edge* $\tau \subset \Delta$, the (essentially univariate) polynomials $C_\tau(x,y)$ have no k-rational root, then one will have that $\widetilde{E}(k) \subset \mathbb{T}^2$, which is necessary if one wants to avoid ending up at toric infinity during arithmetic. For example, if $\operatorname{char} k \neq 2$ and $d \in k$ is nonsquare, then this condition is satisfied for

$$C(x,y) = x^2 + y^2 - d - dx^2y^2, \quad d \neq 0, \pm 1,$$

where all C_τ are essentially of the form $t^2 - d$. In this example, Δ is of type (xv), which has the feature that all edges contain at least three lattice points.

As such one avoids linear polynomials C_τ, which certainly have a k-rational root. Other classes sharing this feature are (xiii) and (xvi).

In many cases, $X(\Delta)$ appears as a completion of $\mathbb{A}^2$ instead of $\mathbb{T}^2$. In these cases, there is the weaker threat of ending up at infinity instead of *toric* infinity. Here one can restrict to the edges whose inwards-pointing normal vector has negative X- or Y-coordinate. E.g., our above example can then be simplified to the Edwards form

$$C(x,y) = x^2 + y^2 - 1 - dx^2y^2, \quad d \neq 0, 1$$

(which moreover allows for a natural choice of the base point $\mathcal{O}$). Note that the condition is now no longer an invariant of the equivalence class of the Newton polytope.

We remark that avoiding infinity (toric or non-toric) does not guarantee completeness: it remains a property of the concrete formulas.

The above line of thought was followed in (Bernstein et al., 2008b; Lange, 2008) in the construction of a complete addition law that works in characteristic 2.

3.5 A vast class of toric forms

We will now algorithmically describe a large class of families of elliptic curves, that will be scanned for efficient arithmetic in Section 4.1. For sake of simplicity, all families depend on one parameter, which appears as the coefficient of a certain fixed monomial. Fix an integer $d \geq 3$.

(i) First enumerate all lattice polytopes that
  (1) have exactly one interior lattice point (genus one);
  (2) have at least one vertex on the X-axis and one vertex on the Y-axis, not necessarily distinct (irreducibility);
  (3) are contained in $d\Sigma = \mathrm{Conv}\{(0,0),(d,0),(0,d)\}$ (degree at most d).
  This can be done fairly naively, by iteratively adjoining a vertex (note that all forms have at most six vertices due to Figure 1). The numbers of such lattice polytopes for $d = 3, \ldots, 8$ are 79, 208, 433, 650, 884, 1244.

(ii) For each such polytope Δ and each edge $\tau_b \subset \Delta$ (called the *base edge*), we label the interior lattice points of τ_b with 0, the most clockwise oriented vertex of τ_b with 1, and the most counter-clockwise oriented vertex of τ_b with -1.

(iii) For each such partially labeled pair (Δ, τ_b), complete the labeling in all possible ways in accordance with the following rules.
  (1) One lattice point v_C of $\Delta \setminus \tau_b$ gets the label 'A';

(2) The vertices of Δ that were not labeled so far, become equipped with a '1'.

(3) The lattice points of Δ that were not labeled so far, get a '0' or a '1'.

(iv) Finally, to each completely labeled pair (Δ, τ_b), associate a polynomial

$$C_A(x,y) = \sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} (\text{label of } (i,j)) \cdot x^i y^j \quad \in \mathbb{Q}(A)[x,y].$$

In the spirit of Lemma 5, erase all trinomials from this list: such 'families' will define the same elliptic curve for each specialized choice of A.

All pairs (C_A, τ_b) are collected in the output set S_d. The numbers of elements of S_d for $d = 3, \ldots, 8$ are 5292, 14553, 32643, 55758, 73332, 103908. In practice we will take $d = 6$.

Lemma 7. *Every $(C_A, \tau_b) \in S_d$ defines a smooth genus one curve in $X(\Delta)$ over $\mathbb{Q}(A)$, where $\Delta = \Delta(C_A)$. It contains the point $\mathcal{O} = (\alpha_{ij})_{(i,j)\in\Delta\cap\mathbb{Z}^2}$, where $\alpha_{ij} = 1$ if $(i,j) \in \tau_b$ and $\alpha_{ij} = 0$ if $(i,j) \notin \tau_b$.*

Proof. It suffices to verify the statement up to $\mathbb{Z}$-affine equivalence. Since all polytopes of Figure 1 have a representative in 4Σ, it therefore suffices to prove the statement for $d = 4$. A finite computation then shows that all $(C_A, \tau_b) \in S_4$ define a curve of geometric genus one.

Now, since we may even assume that the Newton polytope of C_A is contained in 3Σ, in $\mathrm{Conv}\{(0,0),(2,0),(0,2),(2,2)\}$ or in $\mathrm{Conv}\{(0,0),(4,0),(0,2)\}$, it is easy to see that this curve can have no singular points in $\mathbb{T}^2$. Indeed, suppose $(x_0, y_0) \in \mathbb{T}^2$ were a singular point. Then the Newton polytope of $C(x-x_0, y-y_0)$ has no lattice points in its interior and, by Baker's inequality, we run into a contradiction. This proves the non-degeneracy of C_A with respect to Δ itself.

The non-degeneracy conditions with respect to the vertices are immediate. Verifying the non-degeneracy conditions with respect to the edges boils down to verifying the being square-free of polynomials of the form

$$x^k - 1 \ (k = 1, \ldots, 4) \quad \text{and} \quad \alpha_4 x^4 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + 1$$

with the $\alpha_i \in \{0, 1, A\}$ (at most one α_i equalling A). In the former case, this is immediate. In the latter case, if $\alpha_4 = 0$ then a finite computation proves their being square-free. This proves full non-degeneracy as soon as Δ is not of type (xiii) in Figure 1, in which case the lemma follows.

It remains to deal with the subtle case where Δ is of type (xiii). Then C_A may accidentally *fail* to be non-degenerate: $C_A(x,y) = -y^2 + Ay + x^4 + x^3 + x + 1$ is an example. However, again following a reasoning using Baker's inequality one can prove that the non-degeneracy failure can only be due to tangency with toric infinity. In particular, the curve defined by $C_A(x,y) = 0$ still embeds smoothly in $X(\Delta)$. (In fact, $X(\Delta)$ will contain a copy of $\mathbb{A}^2$ and a horizontal or vertical translation will put the curve in non-degenerate position.) $\blacksquare$

The point $\mathcal{O}$ will be called the *base point* of (C_A, τ_b): when speaking about arithmetic on (C_A, τ_b) it will always be with respect to this base point. In Section 5, we will report on an exhaustive scan of all $(C_A, \tau_b) \in S_6$ for efficient arithmetic over $\mathbb{Q}(A)$. Note that doubling and addition formulas over $\mathbb{Q}(A)$ are suited for arithmetic over any finite field k of sufficiently large characteristic, for almost all specializations of A in k. E.g., in case $C_A(x, y)$ is non-degenerate, it suffices that the sparse resultant does not reduce to 0.

We conclude this section with a discussion on both the vastness and the limitations of the family S_d (and hence of our scan in Section 5). We tried to make a choice that is both practical and natural, and remark that:

- The methods for finding efficient arithmetic, explained in Section 4.1, do not depend on the particular construction of S_d.
- In the search for efficient arithmetic, it is a priori sufficient to consider one-parameter families only. If a family depending on two parameters has some remarkable arithmetical properties, then specializing one of the parameters will result in a one-parameter family having the same remarkable arithmetical properties.
- The fact that all constants are '0', '1' or '-1' is less restrictive than it seems at first sight. The efficiency of doubling and/or addition formulas is hardly affected by substitutions of the type $x \leftarrow \alpha x$, $y \leftarrow \beta y$, for small $\alpha, \beta \in k$.

  Conversely, as in the proof of Lemma 5, up to three non-zero coefficients (whose corresponding exponent vectors are not collinear) can always be transformed to '1' for some suitable choice of $\alpha, \beta \in \overline{k}$. In general however, this might involve the introduction of large constants that are defined over an extension field only.

With these remarks in mind, S_d essentially contains all our working examples (if $d \geq 4$).

Form	$C_A(x,y)$	τ_b	Fig. 1
(1) Weierstrass (with $B = 1$)	$-y^2 + x^3 + Ax + 1$	$[< 3,0 >, < 0,2 >]$	(vii)
(2) Hessian (modulo $y \leftarrow -y$)	$x^3 - y^3 + 1 + Axy$	$[< 3,0 >, < 0,3 >]$	(xvi)
(3) Edwards	$x^2 + y^2 - 1 + Ax^2y^2$	$[< 0,2 >, < 0,0 >]$	(xv)
(4) Jacobi quartic	$-y^2 + x^4 + Ax^2 + 1$	$[< 4,0 >, < 0,2 >]$	(xiii)
(5) Montgomery (with $B = 1$)	$-y^2 + x^3 + Ax^2 + x$	$[< 3,0 >, < 0,2 >]$	(v)

Of course, we also indirectly cover the doubly parameterized twisted Edwards curves (Bernstein et al., 2008a) and twisted Hessian curves (Bernstein and Lange, EFD). Thus, despite its apparent narrowness, S_d contains most of the prominent known forms whose arithmetical properties have been studied in the literature so far (over fields of large characteristic). The Doche/Icart/Kohel forms (Doche et al., 2006) and (if one refuses to proceed to $\overline{k}$) the popular Weierstrass form $y^2 - x^3 - 3x + A$ are the most important absentees.

3.6 Efficient preliminary arithmetic on toric forms

The main prerequisite of the algorithm described in Section 4.1 is a relatively efficient method to perform arithmetic on the above toric forms. Very general algorithms based on Riemann-Roch computations are currently too slow for this purpose. Let (C_A, τ_b) be one of the above forms. Then our method consists of, using an appropriate $\mathbb{Z}$-affine map, transforming the curve to either an intersection of quadrics in $\mathbb{P}^3$ (corresponding to the cases (xiii) and (xv) of Figure 1), or a plane cubic in $\mathbb{P}^2$ (corresponding to the other cases), and perform arithmetic there. This can be done very quickly: in the plane cubic case tangent-chord applies, whereas in the quadric intersection case a projection from the base point takes us to the plane cubic case (Harris, 1992, Example 18.16). We go into more detail for two exemplary situations:

Example. $C_A(x, y) = A + y + x^2y - x^2y^2, \tau_b = [< 2, 1 >, < 2, 2 >]$

Let $\mathcal{P}$ be a $\mathbb{T}^2$-point of the curve defined by $C(x, y)$, and let n be a positive integer. Suppose we wish to compute the sequence $\mathcal{P}$, $[2]\mathcal{P}$, $[4]\mathcal{P}$, $\ldots$, $[2^n]\mathcal{P}$, in order to interpolate doubling formulas (see Section 4.1). For our purposes it suffices to suppose that all these points are in $\mathbb{T}^2$ again. Note that $\Delta(C_A)$ has 4 lattice points $v_1, \ldots, v_4$ (enumerated counterclockwise) on the boundary, all of which are vertices. Together with the relation $v_1 + v_3 = v_2 + v_4$, this implies that $\Delta(C_A)$ is of type (iv). The transformation

$$f : \mathbb{R}^2 \to \mathbb{R}^2 : \begin{bmatrix} X \\ Y \end{bmatrix} \mapsto \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is a corresponding $\mathbb{Z}$-affine map, taking $C_A(x, y)$ to $C'_A(x, y) = Ax + y + x^2y - xy^2$. The point $\mathcal{P} = (a, b)$ is sent to

$$\mathcal{P}' = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} a \\ b \end{bmatrix}.$$

The base edge becomes $[< 2, 1 >, < 1, 2 >]$ and, following the explicit description of λ given in Section 3.2, this corresponds to taking $\mathcal{O}' = (1, 1, 0)$ as neutral element. Now using tangent-chord arithmetic, one can compute $\mathcal{P}'$, $[2]\mathcal{P}'$, $[4]\mathcal{P}'$, …, $[2^n]\mathcal{P}'$. Transforming back gives the requested answer.

Example. $C_A(x, y) = x^4 - x^4y^2 + x^3y^2 + y^2 + Ax^3y, \tau_b = [< 4, 0 >, < 4, 2 >]$

Let $\mathcal{P}$ be a $\mathbb{T}^2$-point of the curve defined by $C_A(x, y)$, and let n be a positive integer. Suppose we wish to compute the sequence $\mathcal{P}$, $[2]\mathcal{P}$, $[3]\mathcal{P}$, …, $[n]\mathcal{P}$, in order to interpolate addition formulas (see Section 4.1). Again we suppose that all these points are in $\mathbb{T}^2$. Note that $\Delta(C_A)$ has an edge containing 5 lattice points, hence it is of type (xiii). For this type, we use the representant $\text{Conv}\{(0, 0), (4, 0), (0, 2)\}$, the Newton polytope of a Jacobi quartic form. Then under an appropriate $\mathbb{Z}$-affine transformation, our polynomial is sent to $C'_A(x, y) = y^2 - 1 + Axy + x + x^4$. Note that $\Delta(C'_A) = 2\Gamma$ for the smaller triangle $\Gamma = \text{Conv}\{(0, 0), (2, 0), (1, 0)\}$: then the non-singular model of $C'_A(x, y)$ can be realized by a quadratic relation

$$z_{10}^2 - z_{00}^2 + Az_{10}z_{01} + z_{00}z_{10} + z_{20}^2$$

in $X(\Gamma) : z_{10}^2 - z_{00}z_{20}$ in $\mathbb{P}^3$. The desingularization map λ can again be described explicitly, which allows one to trace back the neutral element $\mathcal{O}'$ and the place of interest $\mathcal{P}'$. Now projecting from $\mathcal{O}'$, one obtains a plane cubic in which it is possible to use tangent-chord arithmetic. As such, one can quickly compute $\mathcal{P}'$, $[2]\mathcal{P}'$, $[3]\mathcal{P}'$, …, $[n]\mathcal{P}'$. Transforming back gives the requested answer.

4 Efficient formulas via lattice reduction

4.1 An interpolation algorithm

Each elliptic curve allows an infinite number of doubling and addition formulas among which one would like to find the most 'efficient' ones. In this section, we describe a simple but powerful algorithm that scans for addition and doubling formulas of a prescribed compact form. The algorithm is very robust: it can be used to find efficient formulas in affine or (various) projective coordinate systems, or for generalized Montgomery arithmetic, or for uniform addition, … We will use it in Section 5 to scan all curves of S_6 for efficient doubling formulas. But in fact our method applies to *any* family of elliptic curves on which arithmetic is a priori feasible.

We will look for formulas that are valid over $\mathbb{Q}(A)$, such that they can be used for all finite fields of large enough characteristic and for almost all specializations of A. To simplify the exposition, we will only describe the algorithm in the case of computing efficient doubling formulas in affine coordinates. The method is easily adapted to each of the above-mentioned alternative settings: the necessary adaptations will be briefly discussed in Section 4.3.

Given a curve $\widetilde{E}_A$ over $\mathbb{Q}(A)$ corresponding to some $(C_A, \tau_b) \in S_d$, we need to compute a quartet of non-zero polynomials $f_1, g_1, f_2, g_2 \in \mathbb{Q}[A][x, y]$ such that

$$x \circ \varphi_2 = \frac{f_1}{g_1}, \qquad y \circ \varphi_2 = \frac{f_2}{g_2}$$

inside the function field $\mathbb{Q}(A)(\widetilde{E}_A)$. To do this, we first select a support set S of monomials in A, x, y that are allowed to appear in the f_i, g_i. Note that the parameter A could appear non-linearly in the support set S. The polynomials f_i, g_i can then be written as $\mathbb{Q}$-linear combinations of the monomials in S, i.e.

$$x \circ \varphi_2 = \frac{f_1}{g_1} = \frac{\sum_{m_i \in S} f_{1,i} \cdot m_i}{\sum_{m_i \in S} g_{1,i} \cdot m_i}, \qquad y \circ \varphi_2 = \frac{f_2}{g_2} = \frac{\sum_{m_i \in S} f_{2,i} \cdot m_i}{\sum_{m_i \in S} g_{2,i} \cdot m_i}.$$

To obtain a description of all possible doubling formulas, we would like to use an evaluation strategy to compute a linear system of equations in the unknown coefficients. For this we would like to find a non-trivial point $\mathcal{P}$ on $\widetilde{E}_A$ and compute $\mathcal{P}, [2]\mathcal{P}, \cdots, [2^n]\mathcal{P}$. However, if we specialize the family in a value $\bar{A} \in \mathbb{Q}$, this would result in a non-trivial rational point on the elliptic curve $\widetilde{E}_{\bar{A}}$ over $\mathbb{Q}$, which is known to be hard. To solve this and other related problems coming from working over $\mathbb{Q}$, we reduce the whole setup modulo a large prime p and work over the finite field $\mathbb{F}_p$.

Therefore, choose a large prime p and choose a large random $\bar{A} \in \mathbb{F}_p$ to obtain the polynomial $\bar{C}_{\bar{A}} \in \mathbb{F}_p[x, y]$. Now it becomes trivial to pick a generic point $\bar{\mathcal{P}}$ on the corresponding curve and using the method described in Section 3.6, we obtain the sequence $\bar{\mathcal{P}}, [2]\bar{\mathcal{P}}, \cdots, [2^n]\bar{\mathcal{P}}$. By 'generic', we mean that all these points should be in $\mathbb{T}^2$. Let $\bar{m}_{i,j}$ denote the evaluation of the monomial m_i in $\bar{A}$ and the coordinates of $[2^j]\bar{\mathcal{P}}$. Each tuple $([2^j]\bar{\mathcal{P}}, [2^{j+1}]\bar{\mathcal{P}})$ results in two linear equations

$$\begin{aligned} x([2^{j+1}]\bar{\mathcal{P}}) \sum_{m_i \in S} g_{1,i} \cdot \bar{m}_{i,j} - \sum_{m_i \in S} f_{1,i} \cdot \bar{m}_{i,j} = 0 \\ y([2^{j+1}]\bar{\mathcal{P}}) \sum_{m_i \in S} g_{2,i} \cdot \bar{m}_{i,j} - \sum_{m_i \in S} f_{2,i} \cdot \bar{m}_{i,j} = 0\,. \end{aligned}$$

Therefore if $n \gg 2\#S$ we obtain an overdetermined system M_x (resp. M_y) of linear equations over $\mathbb{F}_p$ such that all possible formulas for the x-coordinate (resp. y-coordinate) of the doubling for the curve defined by $\bar{C}_{\bar{A}}$ are contained in $\mathrm{Ker}(M_x)$ (resp. $\mathrm{Ker}(M_y)$).

Two problems remain: how to find the most efficient parameterized doubling formulas in the kernel and how to lift the situation from $\mathbb{F}_p$ back to $\mathbb{Q}$. Both problems can be solved simultaneously by finding shortest vectors in the following lattice over $\mathbb{Z}$ spanned by the columns of

$$\left[b_1,\, b_2,\, \cdots,\, b_n,\, pI_{2|S|}\right] \,,$$

where $\{b_1, \cdots, b_n\}$ is a basis of $\mathrm{Ker}(M_x)$ (resp. $\mathrm{Ker}(M_y)$) with $b_i \in \mathbb{F}_p^{2|S|}$ and I_n denotes the $n \times n$ identity matrix. Finding shortest vectors indeed solves both problems: firstly, a formula with only a few monomials will lead to a shorter vector than a formula consisting of many monomials. Secondly, since $\bar{A}$ was chosen randomly and large, the lattice reduction will automatically make the correct choice between using a large coefficient in front of a monomial not involving A and using a small coefficient in front of the corresponding monomial with A included.

Example. To illustrate this behavior, assume that in the final formula, there is a monomial of the form $(A + 2)xy$. Let $m_u = xy$ and $m_v = Axy$, then over the finite field $\mathbb{F}_p$ the monomial $(\bar{A} + 2)xy$ can be written as any of the following linear combinations of $\bar{m}_u$ and $\bar{m}_v$, namely

$$(\bar{A} + 2)xy = (\alpha + 2)\bar{m}_u + (1 - \alpha\bar{A}^{-1})\bar{m}_v \,, \quad \alpha \in \mathbb{F}_p \,.$$

However, it is easy to see that the shortest linear combination corresponds precisely to the choice $\alpha = 0$, since for $\alpha \neq 0$, either α or $\alpha\bar{A}^{-1}$ will be large.

If the set S contains all monomials appearing in the equation of the curve, the kernel of M_x (resp. M_y) will also contain short vectors that correspond to polynomials f_i and g_i that are zero in the function field, i.e. are multiples of the equation of the curve. Therefore, when f_1/g_1 (resp. f_2/g_2) is computed, a final verification is necessary to ensure that the formula is not one of these trivial cases.

As a result, we obtain efficient parameterized formulas for the family (C_A, τ_b). Whereas they are a priori valid mod p and mod $(A - \overline{A})$ only, they will most likely be valid over $\mathbb{Q}(A)$. If so, they will be valid over any finite field of large enough characteristic, and for any sufficiently generic evaluation of A. We will comment on the phrase ‘most likely’ in Section 4.2 below.

Remark. Note that the length of the vectors b_i appearing in the lattice is $2|S|$, so when S contains many monomials, finding a short vector in the lattice becomes a major bottleneck of the algorithm. One solution to overcome this problem is to assign several different *small* values to $\bar{A}$ (since we want the corresponding vectors to be short), run the lattice reduction to obtain efficient formulas for the different curves $C_{\overline{A}}$ and then use interpolation to find expressions for the coefficients that depend on A.

4.2 The size of p

The primary aim of the above algorithm is to serve as a supporting tool for finding efficient formulas. A priori, it merely outputs *candidate* group operation formulas. Once an interesting candidate set of formulas has been found, an additional analysis allows us to decide whether it is indeed the straightforward reduction mod p and mod $(A - \overline{A})$ of a set of group operation formulas over $\mathbb{Q}(A)$.

However, as already indicated, this additional analysis turns out to be superfluous in practice. That is, if p is big enough, and 'efficient' means that the corresponding vector is short enough, an efficient set of group operation formulas that is valid mod p and mod $(A - \overline{A})$, will automatically be valid over $\mathbb{Q}(A)$.

This can be made rigorous, but there is a big discrepancy between the bounds following from the theory below, and the bounds that we observe in practice. Again, for simplicity, we will only sketch this in the case of computing doubling formulas.

Lemma 8. *Let d and e be positive integers, and let $S \subset \mathbb{Z}[A, x, y]$ be a set of monomials. Then there exists an explicitly computable integer $N(d, e, S)$ such that, if $p > N(d, e, S)$ and $\overline{A} \in \mathbb{F}_p$ is chosen uniformly at random, the following holds with probability at least $(p - N(d, e, S))/p$. Let $(C_A, \tau_b) \in S_d$ and let $\widetilde{E}_A$ be the corresponding elliptic curve over $\mathbb{Q}(A)$. Then it reduces mod p and mod $(A - \overline{A})$ to an elliptic curve over $\mathbb{F}_p$. Moreover, if $f_1, g_1, f_2, g_2 \in \mathbb{Z}[A, x, y]$ are polynomials that are supported on S, that have coefficients whose absolute value is bounded by e, and that reduce to a set of doubling formulas mod p and mod $(A - \overline{A})$, then it is a set of doubling formulas for $\widetilde{E}_A/\mathbb{Q}_A$.*

Proof. A first ingredient is that the elliptic curves $\widetilde{E}_A/\mathbb{Q}(A)$ allow for *relatively efficient* doubling formulas. That is, there exist explicitly computable positive integers $\delta(d)$ and $M(d)$, for which any $(C_A, \tau_b) \in S_d$ admits a set of doubling formulas $F_1, G_1, F_2, G_2 \in \mathbb{Z}[A, x, y]$ of degree at most $\delta(d)$ and having coefficients that are bounded in absolute value by $M(d)$. This can be seen by following the machinery of Section 3.6.

Secondly, and more easily, similar bounds $\delta_r(d)$ and $M_r(d)$ can be computed for the sparse resultant

$$r(A) = \mathrm{Res}_{\Delta(C_A)}\left(C_A, x\frac{\partial C}{\partial x}, y\frac{\partial C}{\partial y}\right) \quad \in \mathbb{Z}[A],$$

the non-vanishing of which is equivalent to the non-degeneracy of C_A (over any base field) – see e.g. (Khetan, 2003). Recall from the proof of Lemma 7 that there were a few C_A's that are *not* non-degenerate with respect to their Newton polytope. These can be covered using the sparse resultant of a translate of the curve.

Third, let $F, G \in \mathbb{Z}[A, x, y]$ be polynomials that are bounded by $\delta(d)$ and $M(d)$ in the above sense, and let $f, g \in \mathbb{Z}[A, x, y]$ be polynomials that are supported on S and whose coefficients are bounded in absolute value by e. Let r be the remainder of $fG - gF$ under division by C_A, carried out in $\mathbb{Q}(A)[x, y]$ with respect to the lexicographical monomial ordering. It will be of the form h/A for $h \in \mathbb{Z}[A, x, y]$. Analyzing the division algorithm, it is easy to see that there exist integers $\delta'(d, e, S) \geq \delta(d)$ and $M'(d, e, S) \geq M(d)$, independent of f, g, F, G, and C_A, such that h is bounded by $\delta'(d, e, S)$ and $M'(d, e, S)$ in the foregoing sense.

Then let $N(d, e, S)$ be

$$\delta'(d, e, S)(2M'(d, e, S) + 1)^{\delta'(d,e,S)+1} + M_r(d) + \delta_r(d).$$

Let $p > N(d, e, S)$, and let $\overline{A} \in \mathbb{F}_p$ be chosen uniformly at random. Note that, since $p > M_r(d)$, the sparse resultant $r(A)$ will not vanish identically mod p. Then with probability at least $(p - \delta_r(d))/p$, we will have that

(i) $\overline{A}$ is a non-root of $r(A)$ mod p.

Second, there exist $(2M'(d, e, S)+1)^{\delta'(d,e,S)+1}-1$ non-zero polynomials in $\mathbb{Z}[A]$ of degree at most $\delta'(d, e, S)$ and whose coefficients are bounded in absolute value by $M'(d, e, S)$. Every such polynomial will not vanish mod p, and over $\mathbb{F}_p$ it will have at most $\delta'(d, e, S)$ roots. Hence, with probability at least

$$\left(p - \delta'(d, e, S)(2M'(d, e, S) + 1)^{\delta'(d,e,S)+1}\right)/p$$

we will have that

(ii) $\overline{A}$ is not the root of such a polynomial mod p.

With probability at least $(p - N(d, e, S))/p$, both (i) and (ii) will be satisfied, which from now on we assume.

As mentioned, since $p > M_r(d)$, it will be a prime of good reduction. Moreover, since $p > M(d)$, the polynomials F_1, G_1, F_2, G_2 will be a valid set of doubling formulas for $\widetilde{E}_A/\mathbb{F}_p(A)$. Finally, by hypothesis (i) above, one obtains valid doubling formulas for $\widetilde{E}_{\overline{A}}/\mathbb{F}_p$. Now let $f_1, g_1, f_2, g_2 \in \mathbb{Z}[A, x, y]$ be a set of efficient doubling formulas that are supported on S, whose coefficients are bounded in absolute value by e, and that are valid mod p and mod $(A - \overline{A})$. Then mod p and mod $(A - \overline{A})$, it must be true that

$$f_1 G_1 - g_1 F_1 \equiv 0 \bmod C_{\overline{A}}.$$

Carrying out multivariate division of $f_1 G_1 - g_1 F_1$ by C_A in $\mathbb{Q}(A)[x, y]$, one must therefore end up with h/A for some $h \in \mathbb{Z}[A][x, y]$, all of whose $\mathbb{Z}[A]$-coefficients have $\overline{A}$ as a root (when considered mod p). By hypothesis (ii) above, these coefficients must therefore be identically zero. Since, by analogy, the same conclusion holds for $f_2 G_2 - g_2 F_2$, we deduce that the doubling formulas f_1, g_1, f_2, g_2 are valid over $\mathbb{Q}(A)$. ■

Note that the above proof is indeed constructive: for given d, e, S, a valid instance for $N(d, e, S)$ can in principle be devised by hand. But the outcome is huge. E.g., for $d = 6$, a naive computation of $M(d)$ already gives approximately 2^{500}, in great contrast with our practical observations (where $M(d)$ tends to be no larger than 6).

Whereas Lemma 8, at least in theory, deals with the question whether all formulas that we obtain are actually valid over $\mathbb{Q}(A)$, we conclude this section with the dual question whether we do not miss any efficient formulas. The following lemma somehow provides a negative answer. At the same time, the lemma essentially allows us to get rid of the dependency on e in Lemma 8. But again, there is a big discrepancy between the bounds we observe in practice, and the bounds that can be theoretically proven.

Lemma 9. *Let d be a positive integer and let $S \subset \mathbb{Z}[A, x, y]$ be a set of monomials. There exists an effectively computable integer $e(d, S)$ such that the following holds. Let $(C_A, \tau_b) \in S_d$ and let $\widetilde{E}_A$ be the corresponding elliptic curve over $\mathbb{Q}(A)$. Suppose that it allows a set of doubling formulas $f_1, g_1, f_2, g_2 \in \mathbb{Z}[A, x, y]$ that are supported on S. Then it also allows a set of doubling formulas $f_1', g_1', f_2', g_2' \in \mathbb{Z}[A, x, y]$ that are supported on S and whose coefficients are bounded in absolute value by $e(d, S)$.*

PROOF. Take $M(d), \delta(d), F_1, G_1, F_2, G_2$ as in the proof of Lemma 8. Suppose that there exists a set of doubling formulas $f_1, g_1, f_2, g_2 \in \mathbb{Z}[A, x, y]$ whose monomials are contained in S. Although the coefficients of f_1 and g_1 may be very large, this expresses that the equation

$$f_1 G_1 - g_1 F_1 = D_1 C_A$$

is solvable for f_1, g_1, D_1 in $\mathbb{Q}(A)[x, y]$. In fact, by the construction of C_A, one easily sees that $D_1 \in \mathbb{Q}[A, x, y]$. Next, one notices that D_1 is supported on a bounded set of monomials, depending only on S and d (using $\delta(d)$). Thus if we replace the coefficients of f_1, g_1, D_1 by indeterminates, we end up with a system of linear equations over $\mathbb{Q}$ having at least one non-trivial solution. The equations have coefficients that can be bounded in terms of S and d (using $\delta(d)$ and $M(d)$). Therefore, there must also exist a solution f_1', g_1', D_1' whose coefficients have small height (e.g. by Cramer's rule). Clearing denominators in f_1'/g_1' concludes the argument. Similarly, one constructs f_2' and g_2'. ■

4.3 Applications

4.3.1 Results for a new family

In this paragraph we provide a fully worked example for one family out of the many we have tested. The family corresponds to type (ii) in Figure 1 and is defined by the equation

$$C_A = Ax + x^2 - xy^2 + 1 \qquad \sigma_b = [< 2, 0 >, < 1, 2 >] .$$

Negation is simply given by $-(x, y) = (x, -y)$ and affine doubling is

$$[2](x, y) = \left(\frac{(x^2 - 1)^2}{(2xy)^2}, \frac{-(x^2 - 1)^2 + 2xy^2(x^2 + 1)}{2xy(x^2 - 1)} \right) .$$

Affine addition formulas are as follows:

$$(x_1, y_1) + (x_2, y_2) = \left(\frac{(x_1x_2 - 1)^2}{x_1x_2(y_1 + y_2)^2}, \frac{x_1x_2(x_1y_2 - x_2y_1) + (x_1y_1 - x_2y_2)}{x_1x_2(y_1^2 - y_2^2)} \right) .$$

Note that the formula for the x-coordinate of addition is uniform, i.e. by setting $x_1 = x_2$ and $y_1 = y_2$ we obtain the x-coordinate of the double. The negation formula implies that $k(x)$ is the index 2 invariant subfield, so x-only arithmetic is possible. The resulting formulas are:

$$x_{2n} = \frac{(x_n^2 - 1)^2}{4x_n(x_n^2 + Ax_n + 1)} \qquad x_{m-n}x_{m+n} = \frac{(x_mx_n - 1)^2}{(x_m - x_n)^2}$$

4.3.2 Generalized Montgomery arithmetic

To provide a non-trivial example of generalized Montgomery arithmetic, we revisit the Jacobi quartic. In this case, the invariant index 2 subfield is generated by $t = (y + 1)/x^2$, and t-only doubling and addition formulas are:

$$t_{2n} = \frac{t_n^4 - (2t_n - A)^2 + 2t_n^2 + 1}{(2t_n^2 + 2)(2t_n - A)}$$

$$t_{m+n}t_{m-n} = \frac{(t_m t_n - 1)^2 + 2A(t_m + t_n) - A^2}{(t_m - t_n)^2}.$$

4.3.3 Uniform addition formulas

Recall that addition formulas are called uniform if they can also be used for doubling. The algorithm in Section 4.1 can be easily adapted to return uniform addition formulas by generating half of the total number of linear equations using $\psi(P, Q)$ with $P \neq Q$ and half of them using $\psi(P, P)$. As such, the resulting addition formulas will automatically be uniform.

To illustrate this approach, we give uniform addition formulas for the Weierstrass curve $y^2 - x^3 - Ax - B$ over a field k of characteristic > 3, where as usual $\mathcal{O}$ is the point at infinity:

$$x \circ \psi = \frac{(x_1x_2 - 2A)x_1x_2 - 4B(x_1 + x_2) + A^2}{(x_1x_2 + A)(x_1 + x_2) + 2y_1y_2 + 2B}$$

$$y \circ \psi = \frac{x_1x_2(x_1 + x_2) - x_3\left((x_1 + x_2)^2 - x_1x_2 + A\right) - y_1y_2 - B}{y_1 + y_2}$$

here x_3 abbreviates $x \circ \psi$. Note that here we used the technique described at the end of Section 4.1, i.e. we first derived the above formulas for different small values of A and B using lattice reduction, and then used interpolation to recover the coefficients that depend on A and B. We remark that similar uniform Weierstrass addition formulas were already devised by hand (Brier and Joye, 2002, Corollary 1).

4.3.4 Projective coordinates

Our algorithm can also be adapted to return compact projective addition/doubling/negation formulas. This is done by forcing a common denominator and working with a unified system M_{xy} instead of M_x and M_y. Also other inversion-free coordinate systems ($\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}(2; 3; 1)$, ...) can be addressed by using an appropriately chosen set of monomials S. We leave the details to the reader.

5 Quasi-optimality of Edwards and Montgomery doubling

5.1 Quasi-optimality of Edwards doubling

Using the equation of the curve, the doubling law on an Edwards curve $C(x,y) = x^2 + y^2 - 1 - dx^2y^2$ can be rewritten as

$$(x,y) \mapsto \left(\frac{2xy}{x^2+y^2}, \frac{(y^2-x^2)}{2-(x^2+y^2)} \right).$$

Thus Edwards curves allow for affine doubling formulas consisting of *quadratic polynomials*, which is an attractive property putting Edwards curves among the most efficient known-to-date models for point doubling in characteristic $\neq 2$. One can immediately deduce more families of curves having this property: translates $C(x - x_0, y - y_0)$ for $x_0, y_0 \in k$, flips of the type $y^2C(x, y^{-1})$ or $x^2C(x^{-1}, y)$, and combinations of these. However, using our interpolation algorithm, applied to the support set

$$S = \{1, x, y, x^2, xy, y^2, A, Ax, Ay, Ax^2, Axy, Ay^2\},$$

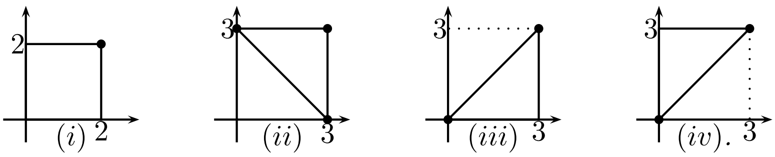
we computationally proved:

Proposition 10. *S_6 does not contain any non-Edwards-related families having doubling formulas $f_1, g_1, f_2, g_2 \in \mathbb{Z}[A, x, y]$ that are supported on S, such that $||(f_i, g_i)|| \leq 2^{100}$ (for $i = 1, 2$). Here, $|| \cdot ||$ is the Euclidean norm, and (f_i, g_i) should be thought of as a $\mathbb{Z}$-valued vector of length $2|S| = 24$.*

Note that the theoretical considerations made in Section 4.2 do not affect this statement due to its negative outcome: in the non-Edwards-related case, we did not find *any* formula of norm less than 2^{100} that is valid modulo our prime $p \approx 2^{200}$ and modulo our random choice of $\overline{A}$ (hence we cannot have found any false formula). The only concern is that the elliptic curves behave well under reduction mod p and mod $(A - \overline{A})$. But this is checked separately for each curve during the preliminary arithmetic stage described in Section 3.6. To illustrate the functioning of our program, we have included the result of this search over S_6 in the appendix.

The following prudently suggests that Proposition 10 is not a coincidence.

Proposition 11. *Let k be a perfect field and let $C(x,y) \in k[x,y]$ be a non-degenerate polynomial defining a curve E of geometric genus one. Let $\mathcal{O}$ be a k-rational place of E, and suppose there are polynomials $f_1, g_1, f_2, g_2 \in k[x,y]$ of degree at most two defining a set of doubling formulas on $(E, \mathcal{O})$. Then the Newton polytope of $C(x,y)$ is contained in one of the following:*



Moreover, in each case all of the bold-marked lattice points appear as vertices.

Proof. Write Δ for the Newton polytope of $C(x, y)$. By the irreducibility of $C(x, y)$, Δ has at least one vertex on the X-axis and at least one vertex on the Y-axis. This proof only makes use of the fact that φ_2 is a morphism of degree 4. In particular, $\mathcal{O}$ plays no role. We will write E' for $E \cap \mathbb{A}^2$.

We first prove that there is an $m \in \mathbb{Z}_{\geq 2}$ for which (m, m) appears as a vertex of Δ, such that Δ is in its turn contained in

$$\operatorname{Conv}\{(0,0),(m,0),(0,m),(m,m)\}.$$

This property, as well as the property of having quadratic doubling formulas, is invariant under replacing C by $C(x - x_0, y - y_0)$ for any $x_0, y_0 \in \overline{k}$. The replacement might spoil the property of non-degeneracy, but we will not use this. We may therefore assume that E' satisfies the following generic conditions:

(1) the x-axis intersects E' in $\deg_y C$ points (counting multiplicities),
(2) the y-axis intersects E' in $\deg_x C$ points (counting multiplicities),
(3) none of the coordinate axes contains singular points of E', or regular points $\mathcal{P} \in E'$ for which there is a place $\mathcal{Q}$ above a singularity or at infinity such that $\varphi_2(\mathcal{Q}) = \mathcal{P}$.

The first assumption implies that $\operatorname{div}(x) = D_0 - D_\infty$ for effective degree $\deg_y C$ divisors D_0 and D_∞ that are supported on E' and $E \setminus E'$ respectively. Then

$$\operatorname{div}(x \circ \varphi_2) = \varphi_2^* D_0 \ - \ \varphi_2^* D_\infty$$

is the difference of two effective degree $4 \deg_y C$ divisors with disjoint support, so in particular $x \circ \varphi_2$ has $4 \deg_y C$ zeroes. By our third assumption these are all in the affine part E' of E, hence they must be realized as zeroes of f_1. By Bezout's theorem, this number is bounded by $2 \deg C$ and we conclude $2 \deg_y C \leq \deg C$. Similarly, $2 \deg_x C \leq \deg C$. When combined, these inequalities are seen to become equalities, and the statement follows with $m = \deg_x C = \deg_y C$. Since E is of geometric genus one, it is immediate that $m \geq 2$.

Now if $m = 2$ then we are in situation *(i)*. Therefore suppose $m \geq 3$. As mentioned above, Δ contains at least one point of the form $(a, 0)$ and one point of the form $(0, b)$ (not necessarily distinct). Suppose $a, b \neq 0$, then a non-empty part of the line segment connecting $(0, 0)$ and (m, m) is contained in the interior of $\Delta(f)$. This part contains two or more lattice points unless

$m = a = b = 3$, hence we are in situation (ii). Suppose on the other hand that Δ contains $(0, 0)$. Since the line segment connecting $(0, 0)$ and (m, m) contains $m - 1 \geq 2$ interior lattice points, it must be an edge. From Figure 1, we see that the maximal number of lattice points in the interior of an edge is 2. Therefore, $m = 3$ and we are in situation (iii) or (iv). ■

We were not able to eliminate the cases $(ii) - (iv)$, neither could we construct quadratic doubling laws corresponding to such a Newton polytope. Apart from that, it *is* possible to obtain alternative quadratic formulas if one allows the y-coordinate to depend on $x_3 = x \circ \varphi_2$. For example let $C_A(x, y) = A + x^3 y - x^2 y^2 + x^2$ and $\tau_b = [< 3, 1 >, < 2, 2 >]$, then

$$x_3 = x \circ \varphi_2 = \frac{-x^2 - 4xy + 4y^2 - 4}{2x - 4y}, \quad y \circ \varphi_2 = \frac{x^2 + xx_3 - 2yx_3}{2x_3}.$$

These and many similar formulas were again computed using the methods described in Section 4.1.

5.2 Quasi-optimality of Montgomery doubling

A crucial observation in t-only arithmetic is that the specific form of the elliptic curve is actually of little importance. Indeed, let $(E', \mathcal{O}')$ be isomorphic to $(E, \mathcal{O})$. Then there is an isomorphism of function fields $\psi : k(\widetilde{E}) \to k(\widetilde{E}')$ such that $k(\psi(t))$ is exactly the set of functions $f \in k(\widetilde{E}')$ that satisfy $f = f \circ \chi'$ (where χ' is the negation morphism on $\widetilde{E}'$). Therefore, the $\psi(t)$-only doubling and addition formulas on $(E', \mathcal{O}')$ are exact copies of the t-only doubling and addition formulas on $(E, \mathcal{O})$.

Thus, the only thing that matters is the choice of the transcendental generator t. Every function generating $k(t)$ corresponds to an automorphism of $\mathbb{P}^1$ and is of the form

$$t' = \frac{at + b}{ct + d}, \quad a, b, c, d \in k, \quad ad - bc \neq 0, \quad \text{or conversely,} \quad t = \frac{dt' - b}{-ct' + a}.$$

Suppose $\operatorname{char} k \neq 2$. Using Montgomery's doubling formula (1) one can verify that every t-only doubling formula must be of the form

$$t \circ \varphi_2 = \frac{\alpha_4 t^4 + \alpha_3 t^3 + \alpha_2 t^2 + \alpha_1 t + \alpha_0}{\beta_4 t^4 + \beta_3 t^3 + \beta_2 t^2 + \beta_1 t + \beta_0}, \tag{5}$$

where the α_i, β_i are long (but manageable) polynomial expressions in a, b, c, d and Montgomery's curve parameter A. Note that a, b, c, d, A might live over an extension field only: an isomorphism with a Montgomery curve might not exist over k.

A Gröbner basis computation then shows that the ideal generated by α_4 and β_4 contains $(c-d)^2(c+d)^2(ad-bc)$. If $c = d \neq 0$ then β_4 can only vanish if $A = 1$, and if $c = -d \neq 0$ then β_4 can only vanish if $A = 0$. Hence for generic A, it is impossible to let α_4 and β_4 vanish at the same time. Similarly, it is impossible that both α_0 and β_0 vanish and to get rid of the curve parameter A. An interesting corollary is the following (somewhat loosely stated) proposition:

Proposition 12. *Let* $\operatorname{char} k \neq 2$. *On a randomly chosen elliptic curve in any nontrivial family over* k*, it is impossible to do* t*-only doubling using projective coordinates* (t, z) *in less than five field multiplications (which include squarings and multiplications with curve constants).*

Proof. By the above discussion, $t \circ \varphi_2$ must be of the form (5). Now $\alpha_4 \neq 0$ or $\beta_4 \neq 0$ and $\alpha_0 \neq 0$ or $\beta_0 \neq 0$, so one at least has to compute a term in t^4 and in z^4, already accounting for 4 multiplications. Since the curve is randomly chosen, the constant A will account for at least one additional multiplication.■

Montgomery's doubling formula attains this bound, so in this sense it is optimal. But note that the above statement does not make a distinction between ordinary field multiplications, field squarings (considerably faster), and multiplications with curve constants (often chosen small, often to be multiplied with multiple times). And indeed, from a practical point of view, there is room for improvement over Montgomery arithmetic. For small curve parameters, the current speed record is due to Gaudry and Lubicz (2009, Section 6.2).

6 Conclusion and future research paths

In this paper we used toric geometry to study different forms of elliptic curves and their arithmetical properties. Within this framework, we scanned a large class of over 50000 elliptic curve forms for efficient affine doubling formulas. Some prudent optimality results on Edwards and Montgomery doubling were presented. We also illustrated how toric geometry might serve as a source of inspiration in finding good projective coordinate systems and in finding elliptic curve shapes allowing for complete group operation formulas.

To compute compact group operation formulas for each of these forms, we described a simple but powerful algorithm based on interpolation and lattice reduction. We illustrated its generality by not only computing affine addition and doubling formulas, but also generalized Montgomery formulas and uniform addition formulas.

Given the lack of speed-wise improvements upon the existing literature, a possible future research path is to adapt our search to genus 2, where such improvements are more likely. There seem to be no theoretical obstructions for doing so. One point of concern is however that, when implemented naively, our interpolation algorithm might reach the limits of what is computationally feasible. Note for instance that, when searching for efficient addition formulas, we are to interpolate a non-linear polynomial expression in 8 variables, given that we represent a point on the Jacobian by using 4 coordinates.

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Quasi-optimality of Edwards doubling

In this appendix we give the results generated by running our program on the list S_6 using the support set: $S = \{1, x, y, x^2, xy, y^2, A, Ax, Ay, Ax^2, Axy, Ay^2\}$. For each elliptic curve, the first line of output contains its index (in our list S_6) and the curve equation. The second line is the base face τ_b. The third and fourth line contain the doubling formulas that were generated accordingly.

```
9056 A*y^2 + x^2*y^2 - x^2 + 1
[ <0, 0>, <1, 0>, <2, 0> ]
X-double [(-x^2 - y^2*A)/(x^2 - y^2*A - 2)]
Y-double [2*x*y/(x^2 - y^2*A)]

9088 A*x^2*y^2 - x^2 + y^2 + 1
[ <0, 0>, <1, 0>, <2, 0> ]
X-double [(-x^2 - y^2)/(x^2 - y^2 - 2)]
Y-double [2*x*y/(x^2 - y^2)]

9120 A*y^2 - x^2*y^2 + x^2 + 1
[ <2, 0>, <2, 1>, <2, 2> ]
X-double [(1/2*x^2 + 1/2*y^2*A)/(x*y)]
Y-double [(x^2 + y^2*A + 2)/(x^2 - y^2*A)]

9160 A - x^2*y^2 + x^2 + y^2
[ <2, 0>, <2, 1>, <2, 2> ]
X-double [(1/2*x^2 + 1/2*y^2)/(x*y)]
```

```
Y-double [(x^2 + y^2 + 2*A)/(x^2 - y^2)]

9184 A*x^2 + x^2*y^2 - y^2 + 1
[ <2, 2>, <1, 2>, <0, 2> ]
X-double [(-x^2*A + y^2 - 2)/(x^2*A + y^2)]
Y-double [(-1/2*x^2*A + 1/2*y^2)/(x*y)]

9216 A + x^2*y^2 + x^2 - y^2
[ <2, 2>, <1, 2>, <0, 2> ]
X-double [(-x^2 + y^2 - 2*A)/(x^2 + y^2)]
Y-double [(-1/2*x^2 + 1/2*y^2)/(x*y)]

9241 A*x^2 + x^2*y^2 + y^2 - 1
[ <0, 2>, <0, 1>, <0, 0> ]
X-double [2*x*y/(x^2*A + y^2)]
Y-double [(x^2*A - y^2)/(x^2*A + y^2 - 2)]

9281 A*x^2*y^2 + x^2 + y^2 - 1
[ <0, 2>, <0, 1>, <0, 0> ]
X-double [2*x*y/(x^2 + y^2)]
Y-double [(x^2 - y^2)/(x^2 + y^2 - 2)]
```