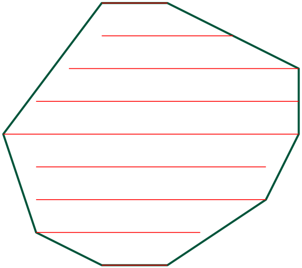
A $K_{p,1}$ conjecture for toric surfaces

(joint work in progress with Filip Cools and Alexander Lemmens)



Wouter Castryck (Universiteit Gent)

Minimal free resolutions, Betti numbers, and combinatorics
Edinburgh, June 2, 2015

I. Introduction

Set-up

$$k \quad : \quad \text{algebraically closed field of characteristic 0}$$
$$\Delta \quad : \quad \text{two-dimensional lattice polygon}$$
$$\mathbb{T}^2 = (k \setminus \{0\})^2 = \operatorname{Spec} k[x^{\pm 1}, y^{\pm 1}]$$
$$N_\Delta = \sharp(\Delta \cap \mathbb{Z}^2)$$
$$S_\Delta = k[X_{i,j} \,|\, (i,j) \in \Delta \cap \mathbb{Z}^2]$$

We consider the morphism

$$\varphi_\Delta : \mathbb{T}^2 \hookrightarrow \underbrace{\mathbb{P}^{N_\Delta - 1}}_{\text{Proj } S_\Delta} : (x,y) \mapsto (x^i y^j)_{(i,j)\in(\Delta\cap\mathbb{Z}^2)}$$

and define $X(\Delta)$ as the Zariski closure of the image.

What does the graded Betti table of $X(\Delta)$ look like?
How does it depend on the combinatorics of Δ?

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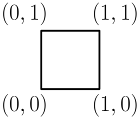
Example

$\Delta =$ 

Here

$$\varphi_\Delta : \mathbb{T}^2 \hookrightarrow \mathbb{P}^3 : (x, y) \mapsto (1, x, y, xy),$$

so

$$X(\Delta) : X_{0,0}X_{1,1} - X_{1,0}X_{0,1} =: f.$$

The Betti diagram reads

	0	1
0	1	0
1	0	1

because of

$$0 \to S_\Delta(-2) \xrightarrow{g \mapsto gf} S_\Delta \to S_\Delta/(f) \to 0.$$

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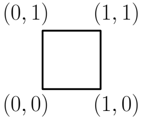

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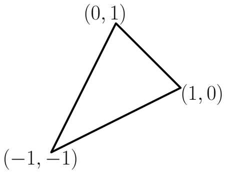

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$$X(\Delta) : X_{-1,-1}X_{1,0}X_{0,1} - X_{0,0}^3 =: f.$$

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	0	1
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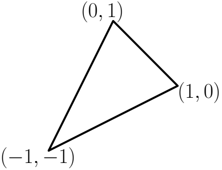

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Example: rational normal surface scrolls

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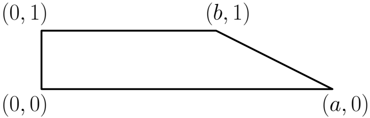

$$\varphi_\Delta : \mathbb{T}^2 \hookrightarrow \mathbb{P}^{a+b+1} : (x, y) \mapsto (1, x, \ldots, x^a, y, xy, \ldots, x^b y)$$

Here $X(\Delta)$ is cut out by

$$\operatorname{rank}\begin{pmatrix} X_{0,0} & X_{1,0} & \ldots & X_{a-1,0} & X_{0,1} & X_{1,1} & \ldots & X_{b-1,1} \\ X_{1,0} & X_{2,0} & \ldots & X_{a,0} & X_{1,1} & X_{2,1} & \ldots & X_{b,1} \end{pmatrix} \leq 1,$$

i.e. it is a rational normal surface scroll of type (a, b).

Betti diagram:

	0	1	2	3	...	$N_\Delta - 4$	$N_\Delta - 3$
0	1	0	0	0	...	0	0
1	0	$\binom{N_\Delta - 2}{2}$	$2\binom{N_\Delta - 2}{3}$	$3\binom{N_\Delta - 2}{4}$	...	$(N_\Delta - 4)\binom{N_\Delta - 2}{N_\Delta - 3}$	$N_\Delta - 3$

because the Eagon-Northcott complex is exact.

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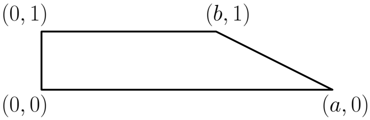
$\Delta =$ 

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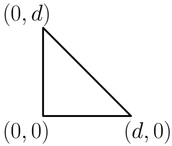

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Here $X(\Delta)$ is the image of the d-uple embedding

$$\rho_d : \mathbb{P}^2 \hookrightarrow \mathbb{P}^{\binom{d+2}{2}-1} \qquad \text{(Veronese surface).}$$

For $d = 2$: classical Veronese surface, with Betti diagram

	0	1	2	3
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1	0	6	8	3

(= Eagon-Northcott with $N_\Delta = 6$).

In general: unknown (computed for $d \le 5$ by Greco–Martino).

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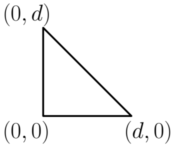

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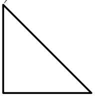

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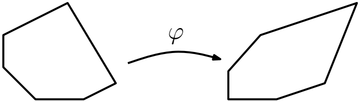
Remark

Preliminary remark: call Δ and Δ' (unimodularly) equivalent if there exists

$$\psi : \mathbb{R}^2 \to \mathbb{R}^2 : \binom{i}{j} \mapsto A \binom{i}{j} + B, \qquad A \in \mathrm{GL}_2(\mathbb{Z}), B \in \mathbb{Z}^2$$

such that $\Delta' = \psi(\Delta)$.



Notation: $\Delta \cong \Delta'$.

Fact: $\Delta \cong \Delta' \;\Rightarrow\; X(\Delta) \cong_{\text{proj. eq.}} X(\Delta') \;\Rightarrow\;$ same Betti table.

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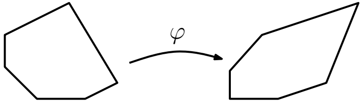

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Known facts on the Betti table

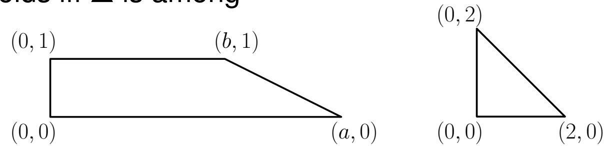
Hering: shape of the Betti table of $X(\Delta)$ is

	0	1	2	3	...	$N_\Delta - 4$	$N_\Delta - 3$
0	1	0	0	0	...	0	0
1	0	b_1	b_2	b_3	...	$b_{N_\Delta - 4}$	$b_{N_\Delta - 3}$
2	0	$c_{N_\Delta - 3}$	$c_{N_\Delta - 4}$	$c_{N_\Delta - 5}$	...	c_2	c_1

where c-row (cubic strand) is zero iff

$$\Delta^{(1)} := \text{conv}\left(\Delta^{\text{int}} \cap \mathbb{Z}^2\right) = \emptyset$$

which holds iff Δ is among



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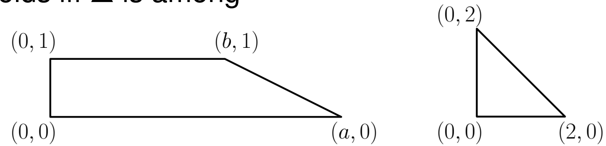

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What else is known?

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By Pick's theorem the Hilbert function of $X(\Delta)$ equals

$$d \mapsto \mathrm{Vol}(\Delta)d^2 + \frac{\sharp(\partial\Delta \cap \mathbb{Z}^2)}{2}d + 1,$$

which implies that

$$b_i - c_{N-1-i} = i\binom{N-1}{i+1} - 2\binom{N-3}{i-1}\mathrm{Vol}(\Delta)$$

for all i.

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1	0	$\binom{N_\Delta - 1}{2} - 2\mathrm{Vol}(\Delta)$	b_2	b_3	...	$b_{N_\Delta - 4}$	$b_{N_\Delta - 3}$
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Koelman: ideal of $X(\Delta)$ is generated by quadrics, i.e.

$$c_{N_\Delta - 3} = 0,$$

if and only if $\sharp(\partial\Delta \cap \mathbb{Z}^2) > 3$. If not then in addition one needs

$$c_{N_\Delta - 3} = \begin{cases} 1 & \text{if } \Delta^{(1)} \text{ is two-dimensional,} \\ N_\Delta - 3 & \text{if not} \end{cases}$$

cubics (C.–Cools).

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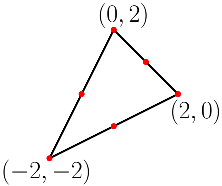
Vast generalization of Koelman's result:

	0	1	2	3	...	$N_\Delta - 5$	$N_\Delta - 4$	$N_\Delta - 3$
0	1	0	0	0	...	0	0	0
1	0	b_1	b_2	b_3	...	$b_{N_\Delta - 5}$	$b_{N_\Delta - 4}$	$b_{N_\Delta - 3}$
2	0	$c_{N_\Delta - 3}$	$c_{N_\Delta - 4}$	$c_{N_\Delta - 5}$	...	c_3	c_2	c_1

The number of leading c_i's that are zero equals

$$\sharp(\partial\Delta \cap \mathbb{Z}^2) - 3$$

by Schenck, Gallego–Purnaprajna, Hering.



Example:

	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	0	24	84	126	84	20	0	0
2	0	0	0	0	20	36	21	4

Known facts on the Betti table

At the other end: Green's $K_{p,1}$ theorem.

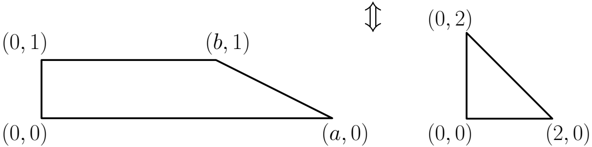
	0	1	2	3	...	$N_\Delta - 5$	$N_\Delta - 4$	$N_\Delta - 3$
0	1	0	0	0	...	0	0	0
1	0	b_1	b_2	b_3	...	$b_{N_\Delta - 5}$	$b_{N_\Delta - 4}$	$b_{N_\Delta - 3}$
2	0	$c_{N_\Delta - 3}$	$c_{N_\Delta - 4}$	$c_{N_\Delta - 5}$	...	c_3	c_2	c_1

One has $b_{N_\Delta - 3} \neq 0$ iff $X(\Delta)$ is a variety of minimal degree

$\Updownarrow$ (recall)

$$\Delta^{(1)} = \emptyset$$

$\Updownarrow$



If not then $b_{N_\Delta - 3} = N_\Delta - 3$ by Eagon-Northcott.

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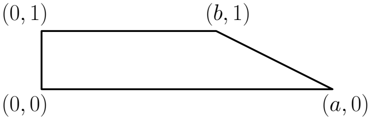
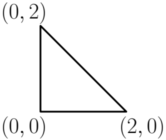
$\Updownarrow$





If not then $b_{N_\Delta-3} = N_\Delta - 3$ by Eagon-Northcott.

Known facts on the Betti table

At the other end: Green's $K_{p,1}$ theorem.

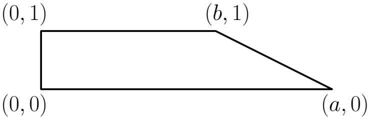
	0	1	2	3	...	$N_\Delta - 5$	$N_\Delta - 4$	$N_\Delta - 3$
0	1	0	0	0	...	0	0	0
1	0	b_1	b_2	b_3	...	$b_{N_\Delta-5}$	$b_{N_\Delta-4}$	$b_{N_\Delta-3}$
2	0	$c_{N_\Delta-3}$	$c_{N_\Delta-4}$	$c_{N_\Delta-5}$	...	c_3	c_2	c_1

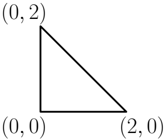
One has $b_{N_\Delta-3} \neq 0$ iff $X(\Delta)$ is a variety of minimal degree

$\Updownarrow$ (recall)

$$\Delta^{(1)} = \emptyset$$

$\Updownarrow$





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Our goal

Similar generalization of $K_{p,1}$ theorem (for toric surfaces)?

	0	1	2	3	...	$N_\Delta - 5$	$N_\Delta - 4$	$N_\Delta - 3$
0	1	0	0	0	...	0	0	0
1	0	b_1	b_2	b_3	...	$b_{N_\Delta - 5}$	$b_{N_\Delta - 4}$	$b_{N_\Delta - 3}$
2	0	$c_{N_\Delta - 3}$	$c_{N_\Delta - 4}$	$c_{N_\Delta - 5}$	...	c_3	c_2	c_1

Goal: find a combinatorial interpretation for the number of concluding b_i's that are zero.

Remainder of this talk:

- state a concrete guess/conjecture,
- give a partial proof,
- digress on Green's canonical syzygy conjecture for curves.

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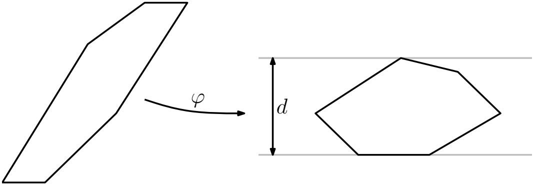
II. A $K_{p,1}$ conjecture for toric surfaces

Lattice width

The lattice width of a lattice polygon Δ is

$$\mathrm{lw}(\Delta) = \min\{\, d \mid \exists \text{ unimodular transf. } \varphi : \varphi(\Delta) \subset \mathbb{R} \times [0, d] \,\},$$

i.e. it is the minimal possible height of an equivalent polygon.



Has been studied since the 70s (in a.o. complexity theory!).

Easy to compute in practice:

$$\mathrm{lw}(\Delta) = \begin{cases} \mathrm{lw}(\Delta^{(1)}) + 3 & \text{if } \Delta \cong d\Sigma \text{ for some } d \geq 2, \\ \mathrm{lw}(\Delta^{(1)}) + 2 & \text{if not,} \end{cases}$$

where $\mathrm{lw}(\emptyset) = -1$ (Lubbes–Schicho, C.–Cools).

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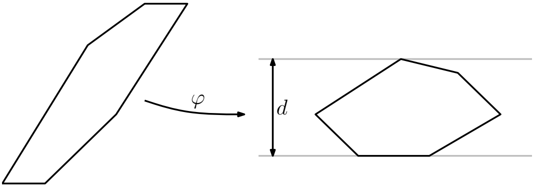

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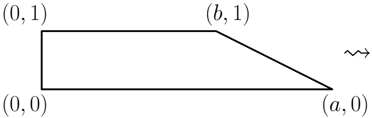
Lattice width

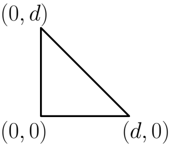
Examples:

$\Delta =$



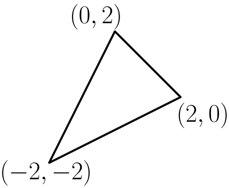
$\rightsquigarrow \mathrm{lw}(\Delta) = 1$

$\Delta = d\Sigma =$



$\rightsquigarrow \mathrm{lw}(\Delta) = d$

$\Delta = 2\Upsilon =$



$\rightsquigarrow \mathrm{lw}(\Delta) = 4$

$K_{p,1}$ conjecture for toric surfaces

Betti table of $X(\Delta)$:

	0	1	2	3	...	$N_\Delta - 5$	$N_\Delta - 4$	$N_\Delta - 3$
0	1	0	0	0	...	0	0	0
1	0	b_1	b_2	b_3	...	$b_{N_\Delta-5}$	$b_{N_\Delta-4}$	$b_{N_\Delta-3}$
2	0	$c_{N_\Delta-3}$	$c_{N_\Delta-4}$	$c_{N_\Delta-5}$	...	c_3	c_2	c_1

Guess/conjecture: the number of concluding b_i's that are zero equals

$$\mathrm{lw}(\Delta) - 1$$

unless

- $\Delta \cong d\Sigma$ **for some** $d \geq 2$**, then it is** $\mathrm{lw}(\Delta) - 2 = d - 2$**,**
- $\Delta \cong 2\Upsilon$**, then it is** $\mathrm{lw}(\Delta) - 2 = 4 - 2 = 2$**.**

Note: this value is 0 iff $\Delta^{(1)} = \emptyset$ iff $X(\Delta)$ has minimal degree.

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	0	1	2	3	...	$N_\Delta - 5$	$N_\Delta - 4$	$N_\Delta - 3$
0	1	0	0	0	...	0	0	0
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2	0	$c_{N_\Delta-3}$	$c_{N_\Delta-4}$	$c_{N_\Delta-5}$	...	c_3	c_2	c_1

Guess/conjecture: the number of concluding b_i's that are zero equals

$$\mathbf{lw}(\Delta) - \mathbf{1}$$

unless

- $\Delta \cong d\Sigma$ **for some** $d \geq$ **2, then it is** $\mathbf{lw}(\Delta) - \mathbf{2} = d - \mathbf{2}$**,**
- $\Delta \cong 2\Upsilon$**, then it is** $\mathrm{lw}(\Delta) - \mathbf{2} = \mathbf{4} - \mathbf{2} = \mathbf{2}$**.**

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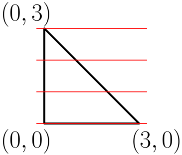
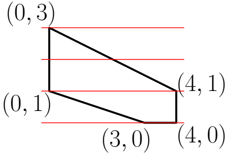
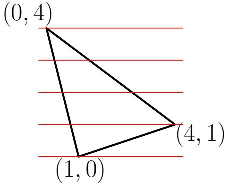
$K_{p,1}$ conjecture for toric surfaces: examples



	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	0	27	105	189	189	105	27	0
2	0	0	0	0	0	0	0	1

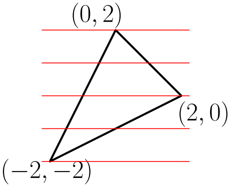


	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	0	32	136	266	280	140	12	0	0
2	0	0	0	0	0	20	49	24	4

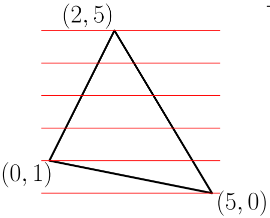


	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	0	15	35	21	0	0	0
2	0	1	6	36	55	30	6

$K_{p,1}$ conjecture for toric surfaces: examples



	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	0	24	84	126	84	20	0	0
2	0	0	0	0	20	36	21	4



	0	1	2	3	4	5	6	7	8	9	10	11
0	1	0	0	0	0	0	0	0	0	0	0	0
1	0	56	330	936	1528	1405	741	175	0	0	0	0
2	0	0	1	10	85	609	1330	1540	1056	430	99	10

Motivation 1: computational data

Using Magma intrinsic, we have verified the conjecture for several hundreds of polygons.

Problem: N_Δ grows at least quadratically with $\mathrm{lw}(\Delta)$, so hard to gather data for large lattice widths.

Current goal: write an algorithm for computing Betti tables of toric surfaces that exploits

- known facts about Betti table,
- the action of $\mathbb{T}^2$ on Koszul cohomology.

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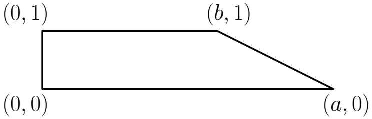
Motivation 2: rational normal scrolls

Recall: if Δ is



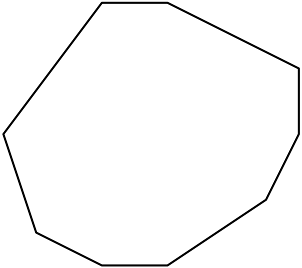
then $X(\Delta)$ is cut out by

$$\operatorname{rank}\begin{pmatrix} X_{0,0} & X_{1,0} & \dots & X_{a-1,0} & X_{0,1} & X_{1,1} & \dots & X_{b-1,1} \\ X_{1,0} & X_{2,0} & \dots & X_{a,0} & X_{1,1} & X_{2,1} & \dots & X_{b,1} \end{pmatrix} \leq 1,$$

i.e. it is a rational normal surface scroll of type (a, b).

Motivation 2: rational normal scrolls

If Δ has a more general shape



then this is no longer true, but . . . the ideal of $X(\Delta)$ still contains the rank ≤ 1 locus of

$$\begin{pmatrix} x_{i_0,0} & \cdots & x_{i_0+a_0-1,0} & x_{i_1,1} & \cdots & x_{i_1+a_1-1,1} & & x_{i_d,d} & \cdots & x_{i_d+a_d-1,d} \\ x_{i_0+1,0} & \cdots & x_{i_0+a_0,0} & x_{i_1+1,1} & \cdots & x_{i_1+a_1,1} & \cdots & x_{i_d+1,d} & \cdots & x_{i_d+a_d,d} \end{pmatrix},$$

$\Rightarrow X(\Delta)$ contained in rational normal scroll of type $(a_0, \ldots, a_d)$.

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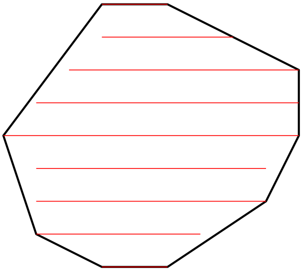

$(i_d, d), \ldots, (i_d + a_d, d)$

$\vdots$

$(i_1, 1), \ldots, (i_1 + a_1, 1)$
$(i_0, 0), \ldots, (i_0 + a_0, 0)$

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Motivation 2: rational normal scrolls

Betti table of rational normal scroll is

	0	1	2	3	...	$f-2$	$f-1$
0	1	0	0	0	...	0	0
1	0	$\binom{f}{2}$	$2\binom{f}{3}$	$3\binom{f}{4}$	...	$(f-2)\binom{f}{f-1}$	$(f-1)\binom{f}{f}$

with $f = N_\Delta - d - 1$, because of Eagon-Northcott.

Scroll contains $X(\Delta) \Rightarrow$ the above is a subtable of the Betti table of $X(\Delta)$:

	0	1	2	3	...	$N_\Delta - 5$	$N_\Delta - 4$	$N_\Delta - 3$
0	1	0	0	0	...	0	0	0
1	0	b_1	b_2	b_3	...	$b_{N_\Delta-5}$	$b_{N_\Delta-4}$	$b_{N_\Delta-3}$
2	0	$c_{N_\Delta-3}$	$c_{N_\Delta-4}$	$c_{N_\Delta-5}$	...	c_3	c_2	c_1

We find: number of zero b_i's is at most

$$N_\Delta - 3 - (f-1) = N_\Delta - 3 - (N_\Delta - d - 1) = d - 1.$$

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2	0	$c_{N_\Delta - 3}$	$c_{N_\Delta - 4}$	$c_{N_\Delta - 5}$	...	c_3	c_2	c_1

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2	0	$c_{N_\Delta - 3}$	$c_{N_\Delta - 4}$	$c_{N_\Delta - 5}$	...	c_3	c_2	c_1

For the optimal choice $d = \mathrm{lw}(\Delta)$ we find:

The number of concluding zero b_i's is at most $\mathrm{lw}(\Delta) - 1$.

Alternatively: possible to construct explicit non-zero classes in Koszul cohomology.

Finer analysis then moreover gives:

- $\Delta \cong d\Sigma$ **for some** $d \geq 2$**, then at most** $\mathrm{lw}(\Delta) - 2 = d - 2$**,**
- $\Delta \cong 2\Upsilon$**, then at most** $\mathrm{lw}(\Delta) - 2 = 4 - 2 = 2$**.**

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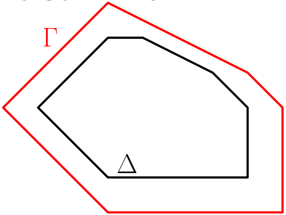
Motivation 3: Green's canonical syzygy conjecture

Suppose that $\Delta = \Gamma^{(1)}$ for a larger lattice polygon Γ.



Let

$$f = \sum_{(i,j)\in\Gamma\cap\mathbb{Z}^2} c_{i,j} x^i y^j \in k[x^{\pm 1}, y^{\pm 1}]$$

be sufficiently generic. Let $C_f \subset \mathbb{T}^2$ be the curve it defines.

Khovanskii: $\exists$ canonical divisor K_Γ on C_f such that

$$H^0(C_f, K_\Gamma) = \langle x^i y^j \rangle_{(i,j)\in\Gamma^{(1)}\cap\mathbb{Z}^2} = \langle x^i y^j \rangle_{(i,j)\in\Delta\cap\mathbb{Z}^2}.$$

Corollary:

- genus g of C_f equals N_Δ,
- C_f canonically embeds in $X(\Delta)$.

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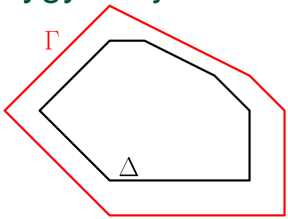

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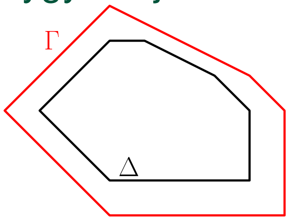

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Motivation 3: Green's canonical syzygy conjecture

Betti table of $X(\Delta)$; recall $g = N_\Delta$:

	0	1	2	3	...	$g-5$	$g-4$	$g-3$
0	1	0	0	0	...	0	0	0
1	0	b_1	b_2	b_3	...	b_{g-5}	b_{g-4}	b_{g-3}
2	0	c_{g-3}	c_{g-4}	c_{g-5}	...	c_3	c_2	c_1

Betti table of C_f^{can}:

	0	1	2	3	...	$g-5$	$g-4$	$g-3$	$g-2$
0	1	0	0	0	...	0	0	0	0
1	0	a_1	a_2	a_3	...	a_{g-5}	a_{g-4}	a_{g-3}	0
2	0	a_{g-3}	a_{g-4}	a_{g-5}	...	a_3	a_2	a_1	0
3	0	0	0	0	...	0	0	0	1

$X(\Delta)$ contains $C_f^{\text{can}} \Rightarrow$ for all i we have $b_i \leq a_i$.

The number of concluding zero b_i's is at least the number of concluding zero a_i's.

Motivation 3: Green's canonical syzygy conjecture

Betti table of C_f^{can}:

	0	1	2	3	...	$g-5$	$g-4$	$g-3$	$g-2$
0	1	0	0	0	...	0	0	0	0
1	0	a_1	a_2	a_3	...	a_{g-5}	a_{g-4}	a_{g-3}	0
2	0	a_{g-3}	a_{g-4}	a_{g-5}	...	a_3	a_2	a_1	0
3	0	0	0	0	...	0	0	0	1

Conjecturally (Green): the number of concluding zero a_i's equals $\mathrm{ci}(C_f) - 1$ (ci = Clifford index).

Kawaguchi, C.–Cools: the Clifford index of C_f equals

- $\mathrm{lw}(\Delta) - 1$ if $\Delta \cong d\Sigma$ for some $d \geq 5$ or $\Delta \cong 2\Upsilon$,
- $\mathrm{lw}(\Delta)$ if not.

(assuming $g \geq 5$ to exclude pathological cases).

Green's conjecture $\Rightarrow$ **$K_{p,1}$ conjecture for toric surfaces associated to *interior* polygons.**

Motivation 3: Green's canonical syzygy conjecture

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1	0	a_1	a_2	a_3	...	a_{g-5}	a_{g-4}	a_{g-3}	0
2	0	a_{g-3}	a_{g-4}	a_{g-5}	...	a_3	a_2	a_1	0
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Green's conjecture $\Rightarrow$ $K_{p,1}$ **conjecture for toric surfaces associated to *interior* polygons.**

Motivation 3: Green's canonical syzygy conjecture

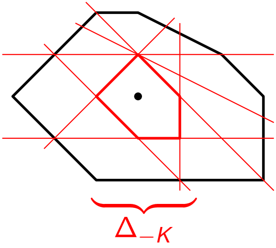
Green's conjecture unfortunately open, but:

Lelli-Chiesa: Green's conjecture is true if

$$H^0(X(\Delta), -K_{X(\Delta)}) \geq 3.$$

Latter condition can be formulated in combinatorial terms:

- move edges to distance 1 from origin: should cut out a polygon with ≥ 3 lattice points,
- true for all 'nice' polygons
- suffices that $X(\Delta)$ is Gorenstein weak Fano.



$K_{p,1}$ conjecture for toric surfaces: conclusion

Betti table of $X(\Delta)$:

	0	1	2	3	...	$N_\Delta - 5$	$N_\Delta - 4$	$N_\Delta - 3$
0	1	0	0	0	...	0	0	0
1	0	b_1	b_2	b_3	...	$b_{N_\Delta -5}$	$b_{N_\Delta -4}$	$b_{N_\Delta -3}$
2	0	$c_{N_\Delta -3}$	$c_{N_\Delta -4}$	$c_{N_\Delta -5}$	...	c_3	c_2	c_1

Let us restate:

Guess/conjecture: the number of concluding b_i's that are zero equals $\mathrm{lw}(\Delta) - 1$ unless

- **$\Delta \cong d\Sigma$ for some $d \geq 2$, then it is $\mathrm{lw}(\Delta) - 2 = d - 2$,**
- **$\Delta \cong 2\Upsilon$, then it is $\mathrm{lw}(\Delta) - 2 = 4 - 2 = 2$.**

Known:

- always an upper bound (scrolls / explicit classes),
- true for small polygons (computation),
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1	0	b_1	b_2	b_3	...	$b_{N_\Delta - 5}$	$b_{N_\Delta - 4}$	$b_{N_\Delta - 3}$
2	0	$c_{N_\Delta - 3}$	$c_{N_\Delta - 4}$	$c_{N_\Delta - 5}$	...	c_3	c_2	c_1

Let us restate:

Guess/conjecture: the number of concluding b_i's that are zero equals $\mathrm{lw}(\Delta) - 1$ unless

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0	1	0	0	0	...	0	0	0
1	0	b_1	b_2	b_3	...	$b_{N_\Delta - 5}$	$b_{N_\Delta - 4}$	$b_{N_\Delta - 3}$
2	0	$c_{N_\Delta - 3}$	$c_{N_\Delta - 4}$	$c_{N_\Delta - 5}$	...	c_3	c_2	c_1

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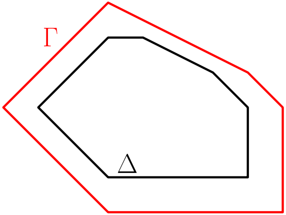
III. Digression on Green's conjecture

Revisiting...

- $f = \sum_{(i,j)\in\Gamma\cap\mathbb{Z}^2} c_{i,j}x^iy^j \in k[x^{\pm1}, y^{\pm1}]$
- $C_f \subset \mathbb{T}^2$ curve of genus $g = N_\Delta$
- $C_f^{\mathrm{can}} \subset X(\Delta) \subset \mathbb{P}^{g-1}$



Betti table of $X(\Delta)$:

	0	1	2	3	...	$g-5$	$g-4$	$g-3$
0	1	0	0	0	...	0	0	0
1	0	b_1	b_2	b_3	...	b_{g-5}	b_{g-4}	b_{g-3}
2	0	c_{g-3}	c_{g-4}	c_{g-5}	...	c_3	c_2	c_1

Betti table of C_f^{can}:

	0	1	2	3	...	$g-5$	$g-4$	$g-3$	$g-2$
0	1	0	0	0	...	0	0	0	0
1	0	a_1	a_2	a_3	...	a_{g-5}	a_{g-4}	a_{g-3}	0
2	0	a_{g-3}	a_{g-4}	a_{g-5}	...	a_3	a_2	a_1	0
3	0	0	0	0	...	0	0	0	1

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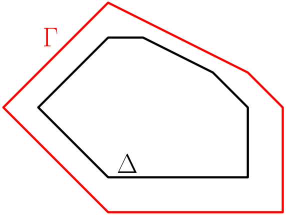

Betti table of $X(\Delta)$:

	0	1	2	3	...	$g-5$	$g-4$	$g-3$
0	1	0	0	0	...	0	0	0
1	0	b_1	b_2	b_3	...	b_{g-5}	b_{g-4}	b_{g-3}
2	0	c_{g-3}	c_{g-4}	c_{g-5}	...	c_3	c_2	c_1

Betti table of C_f^{can}:

We noted: $b_i \leq a_i$.

	0	1	2	3	...	$g-5$	$g-4$	$g-3$	$g-2$
0	1	0	0	0	...	0	0	0	0
1	0	a_1	a_2	a_3	...	a_{g-5}	a_{g-4}	a_{g-3}	0
2	0	a_{g-3}	a_{g-4}	a_{g-5}	...	a_3	a_2	a_1	0
3	0	0	0	0	...	0	0	0	1

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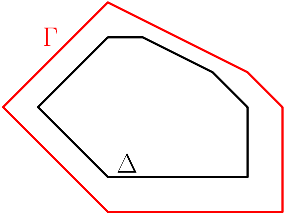

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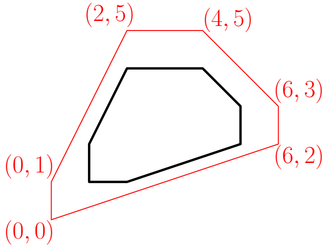
	0	1	2	3	...	$g-5$	$g-4$	$g-3$
0	1	0	0	0	...	0	0	0
1	0	b_1	b_2	b_3	...	b_{g-5}	b_{g-4}	b_{g-3}
2	0	c_{g-3}	c_{g-4}	c_{g-5}	...	c_3	c_2	c_1

Betti table of C_f^{can}: Connection seems to be much tighter!

	0	1	2	3	...	$g-5$	$g-4$	$g-3$	$g-2$
0	1	0	0	0	...	0	0	0	0
1	0	a_1	a_2	a_3	...	a_{g-5}	a_{g-4}	a_{g-3}	0
2	0	a_{g-3}	a_{g-4}	a_{g-5}	...	a_3	a_2	a_1	0
3	0	0	0	0	...	0	0	0	1

Example

$\Gamma =$
$\Delta =$



$(g = 14)$

	0	1	2	3	4	5	6	7	8	9	10	11
0	1	0	0	0	0	0	0	0	0	0	0	0
1	0	60	374	1155	2178	2640	1980	714	88	9	0	0
2	0	0	0	0	0	0	21	308	405	210	55	6

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	0	0	0	0	0	0	0	0	0	0	0	0
1	0	66	429	1365	2583	2948	2001	714	88	9	0	0	0
2	0	0	0	9	88	714	2001	2948	2583	1365	429	66	0
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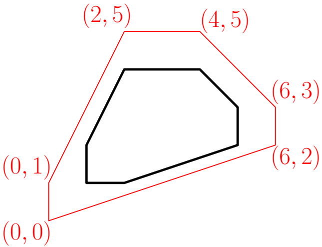

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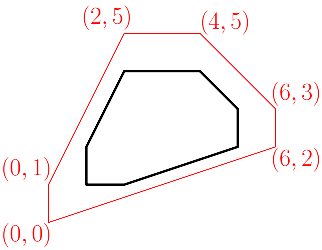

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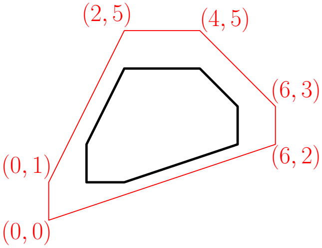

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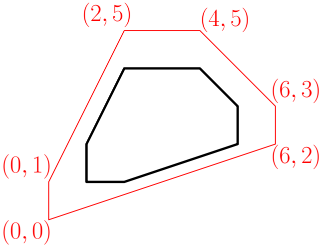

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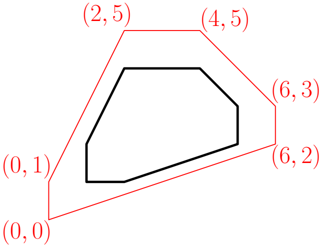

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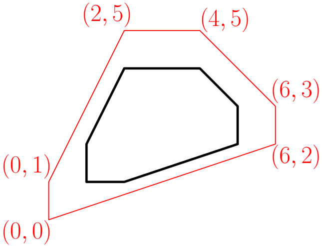

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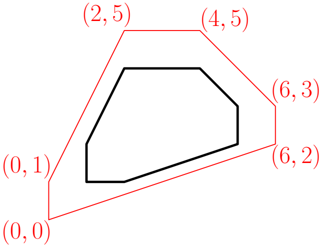

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3	0	0	0	0	0	0	0	0	0	0	0	0	1

Example

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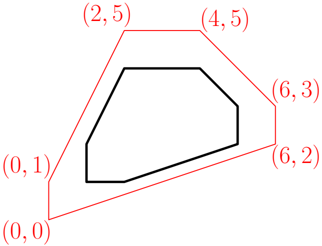
$\Delta =$



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3	0	0	0	0	0	0	0	0	0	0	0	0	1

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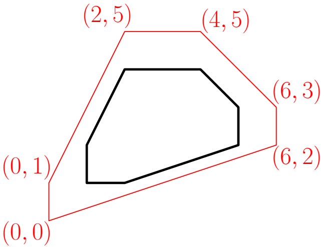


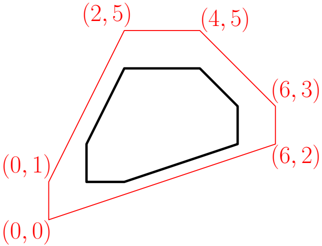
$(g = 14)$

	0	1	2	3	4	5	6	7	8	9	10	11
0	1	0	0	0	0	0	0	0	0	0	0	0
1	0	60	374	1155	2178	2640	1980	714	88	9	0	0
2	0	0	0	0	0	0	21	308	405	210	55	6

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	0	0	0	0	0	0	0	0	0	0	0	0
1	0	66	429	1365	2583	2948	2001	714	88	9	0	0	0
2	0	0	0	9	88	714	2001	2948	2583	1365	429	66	0
3	0	0	0	0	0	0	0	0	0	0	0	0	1

Example

$\Gamma =$
$\Delta =$



$(g = 14)$

	0	1	2	3	4	5	6	7	8	9	10	11
0	1	0	0	0	0	0	0	0	0	0	0	0
1	0	60	374	1155	2178	2640	1980	714	88	9	0	0
2	0	0	0	0	0	0	21	308	405	210	55	6

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	0	0	0	0	0	0	0	0	0	0	0	0
1	0	66	429	1365	2583	2948	2001	714	88	9	0	0	0
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3	0	0	0	0	0	0	0	0	0	0	0	0	1

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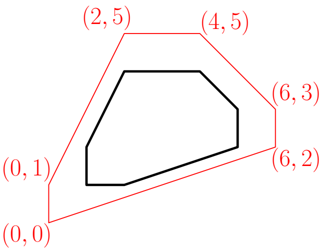

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	0	1	2	3	4	5	6	7	8	9	10	11
0	1	0	0	0	0	0	0	0	0	0	0	0
1	0	60	374	1155	2178	2640	1980	714	88	9	0	0
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2	0	0	0	9	88	714	2001	2948	2583	1365	429	66	0
3	0	0	0	0	0	0	0	0	0	0	0	0	1

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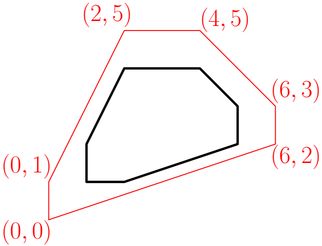

$(g = 14)$

So: $a_i = b_i + c_i$?

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0	1	0	0	0	0	0	0	0	0	0	0	0
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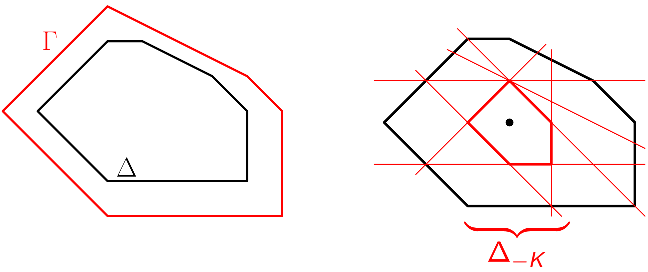
3	0	0	0	0	0	0	0	0	0	0	0	0	1

Do we always have $a_i = b_i + c_i$ for all i?

Very often: yes.

Sufficient condition:
Δ_{-K} is a lattice polygon ($\Leftrightarrow X(\Delta)$ is Gorenstein weak Fano).



True for $d\Sigma$, $d\Upsilon$, ...

Main property (for us): if $p \in \Gamma \cap \mathbb{Z}^2$ then $p = p_1 + p_2$, with

$$p_1 \in \Delta \cap \mathbb{Z}^2, \qquad p_2 \in \Delta_{-K} \cap \mathbb{Z}^2.$$

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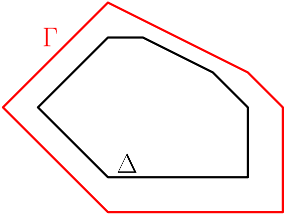

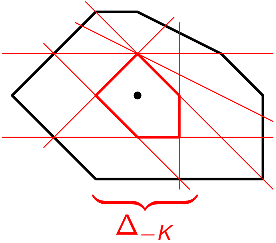

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Proof (very brief)

Using

- adjunction formula,
- vanishing theorems in toric geometry,
- long exact sequence in Koszul cohomology,
- Serre duality

one finds an exact sequence

$$0 \longrightarrow \underbrace{K_{i,1}(X(\Delta),\Delta)}_{b_i} \longrightarrow \underbrace{K_{i,1}(C_f^{\mathrm{can}},K_C)}_{a_i} \longrightarrow \underbrace{K_{i-1,1}(X(\Delta);K,\Delta)}_{c_i}$$

$$\xrightarrow{\cdot f} \underbrace{K_{i-1,2}(X(\Delta),\Delta)}_{c_{g-1-i}} \longrightarrow \underbrace{K_{i-1,2}(C_f^{\mathrm{can}},K_C)}_{a_{g-1-i}} \longrightarrow \underbrace{K_{i-2,2}(X(\Delta);K,\Delta)}_{b_{g-1-i}}$$

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$$\longrightarrow 0$$

Proof (very brief)

Closer look at $K_{i-1,1}(X(\Delta);K,\Delta) \xrightarrow{\cdot f} K_{i-1,2}(X(\Delta),\Delta)$:

$$\begin{array}{ccccc}
\cdots & \xrightarrow{\delta} & \bigwedge^{i-1} H^0(\Delta)\otimes H^0(\Delta^{(1)}) \; (\downarrow +K) & \xrightarrow{\delta} & \bigwedge^{i-2} H^0(\Delta)\otimes H^0((2\Delta)^{(1)}) \\
 & & \Big\downarrow \cdot f \quad \begin{array}{c}\sum_j \alpha_{j,1}\wedge\cdots\wedge\alpha_{j,i-1}\otimes\omega_j \\ \mapsto \\ \sum_j \alpha_{j,1}\wedge\cdots\wedge\alpha_{j,i-1}\otimes f\omega_j\end{array} & & \\
\bigwedge^{i} H^0(\Delta)\otimes H^0(\Delta) & \xrightarrow{\delta} & \bigwedge^{i-1} H^0(\Delta)\otimes H^0(2\Delta) & \xrightarrow{\delta} & \cdots
\end{array}$$

$$2\Delta = \Delta^{(1)}+\Gamma$$

Now:

- $f = \sum_{(r,s)\in\Gamma\cap\mathbb{Z}^2} c_{r,s}x^r y^s \;\Rightarrow\;$ suffices to show that $\cdot x^r y^s = 0$,
- Δ_{-K} lattice polygon $\Rightarrow$ can write $(r,s) = \underbrace{(r_1,s_1)}_{\in\Delta} + \underbrace{(r_2,s_2)}_{\in\Delta_{-K}}$.

Then

$$\sum_j \alpha_{j,1}\wedge\cdots\wedge\alpha_{j,i-1}\otimes x^r y^s\omega_j = \delta\left(\sum_j x^{r_1}y^{s_1}\wedge\alpha_{j,1}\wedge\cdots\wedge\alpha_{j,i-1}\otimes x^{r_2}y^{s_2}\omega_j\right).$$

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\bigwedge^{i} H^0(\Delta) \otimes H^0(\Delta) & \xrightarrow{\delta} & \bigwedge^{i-1} H^0(\Delta) \otimes H^0(\underset{\substack{|| \\ \Delta^{(1)}+\Gamma}}{2\Delta}) & \xrightarrow{\delta} & \cdots
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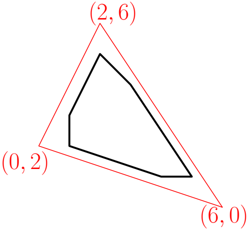
Open question

Formula $a_i = b_i + c_i$ seems to hold much more generally!
How to characterize / prove?

So far: found only one counterexample (in $g = 12$).



	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0	0
1	0	39	186	414	504	295	69	0	0	0
2	0	0	0	0	1	105	189	136	45	6

	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	0	45	231	550	693	399	69	0	0	0	0
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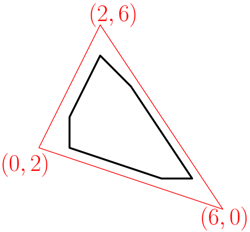

	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0	0
1	0	39	186	414	504	295	69	0	0	0
2	0	0	0	0	1	105	189	136	45	6

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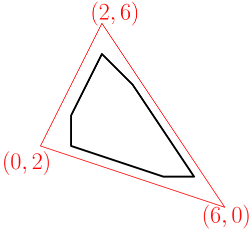

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$$0 \to 295 \to 399 \to 105$$
$$\xrightarrow{\cdot f} 1 \to 69 \to 69 \to 0$$

	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0	0
1	0	39	186	414	504	295	69	0	0	0
2	0	0	0	0	1	105	189	136	45	6

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Other questions

If $a_i = b_i + c_i$ does not hold, then is it at least true that

- a_i depends on b_i and c_i only, and not on f? Guess: yes.
- $c_i \leq a_i$ for all i?
  Remark: this would imply that $\mathrm{ci}(C_f^{\mathrm{can}}) - 1 \leq \sharp(\partial\Delta \cap \mathbb{Z}^2) - 3$

$$\Updownarrow \text{ (usually)}$$

$$\mathrm{lw}(\Delta) - 1 \leq \sharp(\partial\Delta \cap \mathbb{Z}^2) - 3.$$

But there are (interior) polygons Δ violating this.

$N_\Delta = 36$
$\sharp(\partial\Delta \cap \mathbb{Z}^2) = 9$
$\mathrm{lw}(\Delta) = 8$

This would contradict Green's conjecture in genus 36.

Guess: no.

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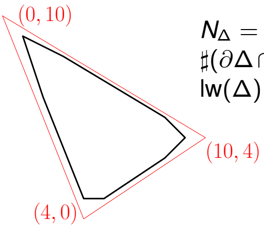

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