EXTENDED PREFACE TO:

Smooth curves in toric surfaces

(Les courbes lisses dans les surfaces toriques)

by Wouter CASTRYCK

Let k be an algebraically closed field and let

$$f = \sum_{(i,j)\in\mathbf{Z}^2} c_{ij}x^iy^j \in k[x^{\pm 1}, y^{\pm 1}]$$

be an irreducible bivariate Laurent polynomial, defining a curve U_f inside the two-dimensional torus $\mathbf{T}^2 := (k^*)^2 = \mathbf{A}^2 \setminus$ coordinate axes. This manuscript is devoted to connections between the birational geometry of U_f and the combinatorics of the Newton polygon

$$\Delta(f) = \mathrm{conv}\{\, (i,j) \in \mathbf{Z}^2 \,|\, c_{ij} \neq 0 \,\} \subseteq \mathbf{R}^2$$

(assumed to be two-dimensional) of f. The earliest such connection is surprisingly old, dating back to 1893, when Baker observed [Bak93] that the geometric genus of U_f is bounded by the number of lattice points (= $\mathbf{Z}^2$-valued points) in the interior of $\Delta(f)$. In the 1970s, after toric geometry had made its appearance, a more satisfactory proof was given by Khovanskii [Kho77], who moreover showed that Baker's bound is generically met. Recently developed tools such as tropical geometry and Berkovich theory conceptualized this remarkable result further, although these topics will not be addressed here.

A well-known generically satisfied condition which is sufficient for meeting Baker's bound [CDV06, Prop. 1] is that f is **nondegenerate with respect to its Newton polygon**, meaning that for all faces $\tau \subseteq \Delta(f)$ of any dimension (i.e. vertices, edges and $\Delta(f)$ itself), the system of equations

$$f_\tau = x\frac{\partial f_\tau}{\partial x} = y\frac{\partial f_\tau}{\partial y} = 0 \qquad \text{with} \qquad f_\tau = \sum_{(i,j)\in\tau\cap\mathbf{Z}^2} c_{ij}x^iy^j$$

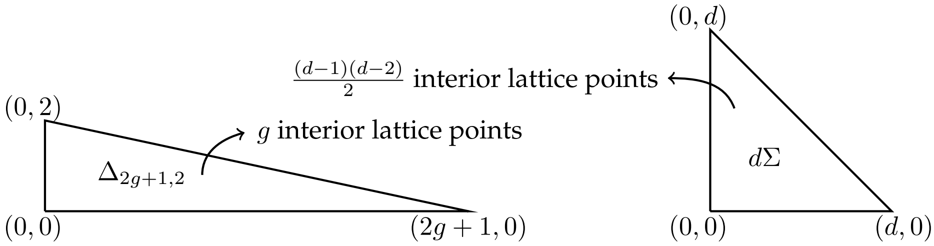
has no solutions in $\mathbf{T}^2$. For Δ a lattice polygon (= the convex hull in $\mathbf{R}^2$ of finitely many points in $\mathbf{Z}^2$) we say that f is **Δ-nondegenerate** if it is nondegenerate with respect to its Newton polygon and $\Delta(f) = \Delta$. In general, the condition of nondegeneracy is not strictly needed for meeting Baker's bound, which leads to the following slight and seemingly bland relaxation:

Definition 1. *Let Δ be a two-dimensional lattice polygon and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be irreducible. We say that f is* **weakly Δ-nondegenerate** *if*

- *$\Delta(f) \subseteq \Delta$ and for each edge $\tau \subseteq \Delta$ one has $\Delta(f) \not\subseteq \Delta \setminus \tau$,*
- *the genus of U_f equals the number of lattice points in the interior of Δ.*

Weak nondegeneracy is the assumption underlying most of the results presented in this manuscript. Besides being (slightly) weaker than nondegeneracy and thereby leading to stronger statements, the notion allows for more combinatorial freedom, in the sense that a weakly Δ-nondegenerate Laurent polynomial might also be weakly Δ'-nondegenerate for some other (potentially easier) lattice polygon Δ', which has important proof-technical advantages. This freedom does not apply to $\Delta^{(1)}$, the convex hull of the lattice points in the interior of Δ, which is fixed and in fact turns out to play a more important role than Δ itself.

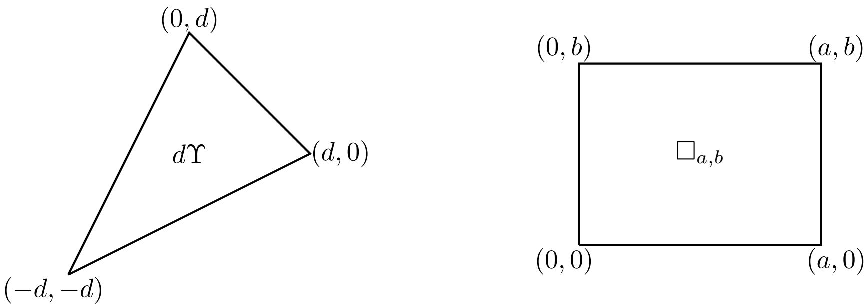
Well-known examples. Familiar examples include the Weierstrass polynomials $f = y^2 - h(x)$, where $\operatorname{char} k \neq 2$ and $h(x) \in k[x]$ is squarefree of degree $2g+1$ for some integer $g \geq 1$: these are weakly $\Delta_{2g+1,2}$-nondegenerate. Other examples are the dehomogeniza-



tions $f \in k[x, y]$ with respect to z of the homogeneous degree $d \geq 1$ forms $F \in k[x, y, z]$ that define a smooth curve in $\mathbf{P}^2$: such polynomials are weakly $d\Sigma$-nondegenerate. In both cases the reader sees that Baker's bound confirms the well-known formula for the genus.

Remark. More generally for $a, b \in \mathbf{Z}_{\geq 1}$ we use $\Delta_{a,b}$ to denote $\operatorname{conv}\{(0,0), (a,0), (0,b)\}$. If $\gcd(a, b) = 1$ then the corresponding curves are said to be $C_{a,b}$; this notion was introduced by Miura in the context of coding theory [Miu93].

More examples. Other recurring examples are weakly $d\Upsilon$-nondegenerate Laurent polynomials and weakly $\square_{a,b}$-nondegenerate Laurent polynomials, where $d, a, b \geq 1$, which



define curves of genus $\frac{3}{2}d^2 - \frac{3}{2}d + 1$ and $(a-1)(b-1)$, respectively.

For an irreducible (not necessarily smooth or complete) algebraic curve C/k and a two-dimensional lattice polygon Δ, we say that C is **weakly Δ-nondegenerate** if it is birationally equivalent to U_f for some weakly Δ-nondegenerate Laurent polynomial $f \in k[x^{\pm 1}, y^{\pm 1}]$ — similarly we say that C is **Δ-nondegenerate** if f can moreover be taken Δ-nondegenerate.

The presented work groups together a number of research papers that are devoted to connections between the birational geometry of such a weakly Δ-nondegenerate curve

C and the combinatorics of Δ. Their joint goal is to extend the geometry-combinatorics dictionary that started with Baker's formula for the genus, although we stress that several entries remain to be added and/or enhanced by future researchers. For reasons of coauthorship and efficiency I have left the papers in their original shape, even though when put together the treatment is not entirely uniform: some statements assume nondegeneracy rather than weak nondegeneracy, while others are presented subject to the condition that the base field k is of characteristic 0. One source for this non-uniformity is that the material has matured over time, with some insights postdating the publication of the earliest papers. Another cause is that several important references assume that $\operatorname{char} k = 0$ or even $k = \mathbf{C}$, and unfortunately I was not able to sift each of these to the bottom to verify the need for this (possibly often unneeded) assumption.

In view of these considerations, the goal of this preface is not only to give an overview of the results obtained, but also to update the exposition: in the text below, all main results are stated under the weak nondegeneracy assumption, which is always sufficient, and certain characteristic zero statements have been reformulated in arbitrary characteristic, along with some lines of explanation why this is allowed.

Remark on terminology. Unfortunately the non-uniformity also affects the terminology of being *weakly Δ-nondegenerate*, for which a.o. in Chapters 5 and 11 the phrasing *Δ-toric* is used.

Contents. Concretely, the following papers are included in this HDR thesis:

- **Chapter 1**: *On nondegeneracy of curves*, Algebra & Number Theory **3**(3), pp. 255-281 (2009), written jointly with John Voight
- **Chapter 2**: *Moving out the edges of a lattice polygon*, Discrete and Computational Geometry **47**(3), pp. 496-518 (2012)
- **Chapter 3**: *The lattice size of a lattice polygon*, Journal of Combinatorial Theory, Series A **136**, pp. 64-95 (2015), written jointly with Filip Cools
- **Chapter 4**: *A minimal set of generators for the canonical ideal of a non-degenerate curve*, Journal of the Australian Mathematical Society **98**(3), pp. 311-323 (2015), written jointly with Filip Cools
- **Chapter 5**: *Linear pencils encoded in the Newton polygon*, to appear in International Mathematics Research Notices (2017), written jointly with Filip Cools
- **Chapter 6**: *Computing graded Betti tables of toric surfaces*, preprint, written jointly with Filip Cools, Jeroen Demeyer and Alexander Lemmens
- **Chapter 7**: *A lower bound for the gonality conjecture*, preprint
- **Chapter 8**: *On graded Betti tables of curves in toric surfaces*, preprint, written jointly with Filip Cools, Jeroen Demeyer and Alexander Lemmens
- **Chapter 9**: *A combinatorial interpretation for Schreyer's tetragonal invariants*, Documenta Mathematica **20**, pp. 903-918 (2015), written jointly with Filip Cools
- **Chapter 10**: *Intrinsicness of the Newton polygon for smooth curves on $\mathbf{P}^1 \times \mathbf{P}^1$*, to appear in Revista Matemática Complutense, written jointly with Filip Cools
- **Chapter 11**: *Curves in characteristic 2 with non-trivial 2-torsion*, Advances in Mathematics of Communications **8**(4), pp. 479-495 (2014), written jointly with Marco Streng and Damiano Testa

I have also made these chapters, as well as the current preface, available in electronic form on http://math.univ-lille1.fr/~castryck/HDR/.

Acknowledgements. My Ph.D. thesis was on the development of a Kedlaya-style algorithm for computing Hasse-Weil zeta functions of nondegenerate curves over finite fields of small characteristic [CDV06], which is how I got acquainted with the world of smooth curves in toric surfaces. I wish to thank my former supervisor Jan Denef for his enthusiastic introduction to this beautiful topic, and him and my collaborator Frederik Vercauteren for their guidance over the first hurdles. The direct provocation for the currently presented work was to gain a better understanding of to which curves exactly our algorithm applies, a problem which I attacked together with John Voight. Later the research diverged in the direction of linear systems on smooth curves in toric surfaces, sparked by connections with tropical geometry [Bak08; CC12] and by recent work of Kawaguchi [Kaw16]; here, most of the results were obtained in collaboration with Filip Cools. I would like to thank John and Filip, and also my other coauthors Jeroen Demeyer, Alexander Lemmens, Marco Streng and Damiano Testa for the fruitful collaboration. My hope is that our work turns out useful for future algebraic geometers in verifying hypotheses and proving existence results, and as such contributes to Fulton's qualification of toric geometry as a remarkably fertile testing ground for general theories [Ful93, Pref.]. Finally I would like to express my gratitude to my garant Raf Cluckers, for his stimulating and genuinely positive attitude, to Pierre Dèbes, Anne Moreau, Sam Payne, Josef Schicho and Frank-Olaf Schreyer for willing to be part of the jury, and to my parents, sister, brother in law, niece, and other family and friends, for their continuous support and for the moments of much-welcomed relaxation.

1. Weakly nondegenerate curves as smooth curves in toric surfaces (Chapters 4 and 5)

To every two-dimensional lattice polygon Δ one can associate a projectively embedded toric surface X_Δ over k, obtained by taking the Zariski closure of the image of

$$\varphi_\Delta : \mathbf{T}^2 \hookrightarrow \mathbf{P}^{\#(\Delta\cap\mathbf{Z}^2)-1} : (x,y) \mapsto (x^i y^j)_{(i,j)\in\Delta\cap\mathbf{Z}^2}.$$

If $f \in k[x^{\pm 1}, y^{\pm 1}]$ is weakly Δ-nondegenerate then $\varphi_\Delta(U_f)$ closes along with $\varphi_\Delta(\mathbf{T}^2)$ to the smooth hyperplane section

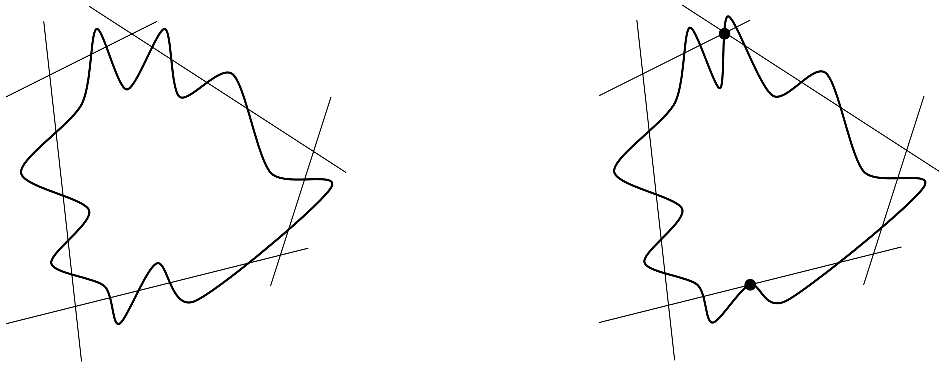
$$\sum_{(i,j)\in\Delta\cap\mathbf{Z}^2} c_{ij} X_{i,j} = 0$$

of X_Δ, where $X_{i,j}$ denotes the projective coordinate corresponding to the lattice point (i,j). Thus, weakly Δ-nondegenerate Laurent polynomials f allow for an explicit smooth complete model of U_f, which we denote by C_f.

Remark. Informally one can think of a weakly Δ-nondegenerate Laurent polynomial f as defining a smooth curve in $\mathbf{T}^2$, the singularities of whose planar completion are 'no worse' than what Δ prescribes and that therefore can be resolved using toric geometry.

When viewed as a divisor on X_Δ the curve C_f is Cartier and very ample. From the theory of toric varieties [CLS11; Ful93] it follows that C_f is linearly equivalent to a torus-invariant divisor D, to which one can naturally associate a polygon $P_D \subseteq \mathbf{R}^2$. It turns out that this polygon is precisely Δ, modulo translation over an element of $\mathbf{Z}^2$ that depends on the specific choice of D; this issue will be ignored from now on. Conversely consider a smooth complete Cartier curve C on a toric surface $X \supseteq \mathbf{T}^2$, where to avoid certain pathologies we assume that C is non-rational. Consider a torus-invariant divisor $D \sim C$ and let $P_D \subseteq \mathbf{R}^2$ be the associated polygon. Then this is automatically a two-dimensional lattice polygon and $C \cap \mathbf{T}^2$ is defined by a weakly P_D-nondegenerate Laurent polynomial $f \in k[x^{\pm 1}, y^{\pm 1}]$.

In this sense weak nondegeneracy is a geometrically more pleasing notion than nondegeneracy, which on top of smoothness requires that the curve intersects toric infinity $X \setminus \mathbf{T}^2$ transversally. For instance, while $d\Sigma$-nondegenerate Laurent polynomials merely



nondegenerate

weakly nondegenerate but not nondegenerate (allowed to pass through non-singular $\mathbf{T}^0$'s, tangency allowed to $\mathbf{T}^1$'s)

correspond to smooth degree d curves in $\mathbf{P}^2$, $d\Sigma$-nondegeneracy moreover forces the curve not to pass through the coordinate points and to be non-tangent to the coordinate axes. On the other hand every weakly $d\Sigma$-nondegenerate *curve* (i.e. when considered modulo birational equivalence) is also $d\Sigma$-nondegenerate because using an automorphism of $\mathbf{P}^2$ one can enforce appropriate intersection behaviour with the coordinate axes. This trick does not always work: there exist two-dimensional lattice polygons Δ along with weakly Δ-nondegenerate curves that are genuinely *non*-Δ-nondegenerate. An example is given in Chapter 5.

Remark on the non-Cartier case. Let C be a smooth complete non-rational curve on a toric surface $X \supseteq \mathbf{T}^2$ which is not necessarily Cartier. Let D be a linearly equivalent torus-invariant divisor. Then P_D need not be a lattice polygon, which complicates matters slightly. Nevertheless C is weakly nondegenerate, as one can show that $C \cap \mathbf{T}^2$ is defined by a weakly $\mathrm{conv}(P_D \cap \mathbf{Z}^2)$-nondegenerate Laurent polynomial.

Khovanskii's proof of Baker's formula for the genus $g(U_f)$ essentially amounts to an application of the adjunction formula to the inclusion $C_f \subseteq X_\Delta$, in combination with a well-known combinatorial interpretation for the Riemann-Roch space associated to a torus-invariant divisor D (the statement involves the polygon P_D). In fact this yields much finer information than merely $g(U_f) = g(C_f) = \#(\Delta^{(1)} \cap \mathbf{Z}^2)$: it entails an explicit canonical divisor K_Δ on C_f that satisfies

$$H^0(C_f, K_\Delta) = \langle x^i y^j \rangle_{(i,j) \in \Delta^{(1)} \cap \mathbf{Z}^2},$$

where x, y are viewed as functions on C_f through φ_Δ. This leads to the following classification based on $\dim \Delta^{(1)}$:

(i) C_f is rational if and only if $\Delta^{(1)} = \emptyset$.

(ii) C_f is elliptic if and only if $\dim \Delta^{(1)} = 0$.

(iii) C_f is hyperelliptic if and only if $\dim \Delta^{(1)} = 1$.

(iv) If $\dim \Delta^{(1)} = 2$ then the canonical embedding 'factors' through $\varphi_{\Delta^{(1)}}$ and therefore the canonical image C_f^{can} is contained in the toric surface

$$X_{\Delta^{(1)}} \subseteq \mathbf{P}^{\#(\Delta^{(1)} \cap \mathbf{Z}^2)-1}.$$

Even though these four claims are easy consequences of Khovanskii's proof method, as far as we know, prior to our articles the only explicit mention of the last two statements can be found in Koelman's (unpublished) Ph.D. thesis see [Koe91, Lem. 3.1.3 and Lem. 3.2.9] and are therefore not well-known. We hope that our work helps to publicize these interesting facts.

Chapters 4 and 5 contain a number of new accompanying facts, one of which is the following geometric interpretation in case (iv) of $\Delta^{\max}$, the **maximal polygon** with respect to inclusion whose interior polygon equals $\Delta^{(1)}$ — from a combinatorial perspective the existence of such a maximum was observed by Koelman [Koe91, §2.2] and rediscovered by Haase and Schicho [HS09] (see also the next section).

Lemma 2. *If in case* (iv) *one considers a torus-invariant divisor on $X_{\Delta^{(1)}}$ that is linearly equivalent to C_f^{can}, then its associated polygon equals $\Delta^{\max}$.*

Another contribution is an explicit minimal set of generators for the ideal $\mathcal{I}(C_f^{\text{can}})$ of C_f^{can}, again in case (iv). These are obtained by starting from a minimal set of generators for the ideal of $X_{\Delta^{(1)}}$, consisting of

$$\binom{g-1}{2} - 2\,\text{vol}(\Delta^{(1)}) \text{ quadrics} \quad \text{and} \quad \begin{cases} 0 & \text{if } \Delta^{(1)} \not\cong \Upsilon \\ 1 & \text{if } \Delta^{(1)} \cong \Upsilon \end{cases} \text{ cubics}$$

(here $\cong$ denotes unimodular equivalence). Extending this to a minimal set of generators for the canonical ideal of C_f can be done following a so-called *rolling factors recipe*. This amounts to adding

$$\begin{cases} 1 \text{ quartic} & \text{if } \Delta^{(1)} \cong \Sigma, \\ g-3 \text{ cubics} & \text{if } \Delta^{(2)} = \emptyset \text{ but } \Delta^{(1)} \not\cong \Sigma, \\ \#(\Delta^{(2)} \cap \mathbf{Z}^2) \text{ quadrics} & \text{if } \Delta^{(2)} \neq \emptyset \end{cases}$$

(where $\Delta^{(2)}$ abbreviates $\Delta^{(1)(1)}$). For instance in the last case the quadrics are

$$Q_w = \sum_{(i,j) \in \Delta \cap \mathbf{Z}^2} c_{ij} X_{u_{ij}} X_{v_{ij}} \in k[\, X_{i,j} \,|\, (i,j) \in \Delta^{(1)} \cap \mathbf{Z}^2 \,]$$

where w runs through $\Delta^{(2)} \cap \mathbf{Z}^2$ and u_{ij}, v_{ij} are chosen such that $(i,j) - w = (u_{ij} - w) + (v_{ij} - w)$. For more details we refer to Chapter 4. We have implemented the resulting algorithm in Magma [BCP97], allowing for a quick computation of a minimal set of generators for the canonical ideal of any concretely given weakly nondegenerate curve of genus up to about 100. For general curves within this range this is currently an infeasible task.

2. The combinatorics of lattice polygons (Chapters 2, 3 and 5)

Even though there is always some toric geometric motivation in the background, several parts of the presented chapters are purely combinatorial. Mostly these parts are concerned with the question of how the operation $\Delta \mapsto \Delta^{(1)}$ affects certain *combinatorial invariants*, i.e. quantities that do not change when applying a unimodular transformation.

One example of such a combinatorial invariant is the number of lattice points on the boundary, in which case the question amounts to relating $\#(\partial\Delta \cap \mathbf{Z}^2)$ to $\#(\partial\Delta^{(1)} \cap \mathbf{Z}^2)$. An answer was obtained through a beautiful application of Poonen and Rodrigues-Villegas' 12 theorem, by Haase and Schicho [HS09], from whom we have copied the superscript notation $^{(1)}$. We omit a detailed statement. Indirectly their work also treats the number of lattice points in the interior $\#(\Delta^\circ \cap \mathbf{Z}^2) = \#(\Delta^{(1)} \cap \mathbf{Z}^2)$, which in view of Baker's theorem is called the **genus**. An important property of the genus is given by the following result:

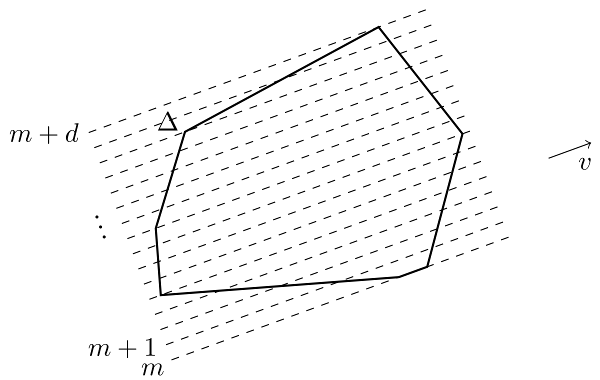
Lemma 3. *Up to unimodular equivalence the number of lattice polygons having a given genus $g \geq 1$ is finite.*

See e.g. [LZ91]. An alternative proof can be found in Chapter 2.

For our needs, besides the genus the most important combinatorial invariant is the **lattice width**, which is defined as follows. For each primitive vector $v = (a, b) \in \mathbf{Z}^2$ define the width $w(\Delta, v)$ to be the smallest integer d for which there exists an $m \in \mathbf{Z}$ such that

$$m \leq aj - bi \leq m + d \qquad \text{for all } (i, j) \in \Delta, \tag{1}$$

as illustrated below. This definition assumes that Δ is a non-empty lattice polygon; one



lets $w(\emptyset, v) = -1$. The lattice width $\mathrm{lw}(\Delta)$ is then defined as $\min_v w(\Delta, v)$. Alternatively, if $\Delta \neq \emptyset$ then $\mathrm{lw}(\Delta)$ is the minimal height $d \in \mathbf{Z}_{\geq 0}$ of a horizontal strip $\mathbf{R} \times [0, d]$ in which Δ can be mapped using a unimodular transformation. The question of relating $\mathrm{lw}(\Delta)$ to $\mathrm{lw}(\Delta^{(1)})$ has the following surprisingly simple answer: if Δ is two-dimensional then

$$\mathrm{lw}(\Delta) = \begin{cases} \mathrm{lw}(\Delta^{(1)}) + 3 & \text{if } \Delta \cong d\Sigma \text{ for some } d \geq 2, \\ \mathrm{lw}(\Delta^{(1)}) + 2 & \text{if not.} \end{cases} \tag{2}$$

Note that this allows one to compute $\mathrm{lw}(\Delta)$ recursively; we have implemented this in Magma.

Remark. This implies that $\mathrm{lw}(\Delta) = \mathrm{lw}(\Delta^{\max})$ whenever $\Delta^{(1)}$ is two-dimensional, except possibly if $\Delta^{\max} \cong d\Sigma$ for some $d \geq 4$.

A proof of the recursive formula (2) can be found in the paper [CC12], which is not considered part of this thesis because independently a more complete result, discussing the concrete primitive vectors v for which $\mathrm{lw}(\Delta) = w(\Delta, v)$, was obtained by Lubbes and Schicho [LS11, Thm. 13]. Note that such vectors always arise in pairs $\pm v$. One can prove [DMN12] that the number of pairs realizing the lattice width is at most 4 as soon as Δ is two-dimensional. These and some accompanying properties are reported upon in more detail in Chapter 5, which also includes a number of new facts and introduces the following refined quantity.

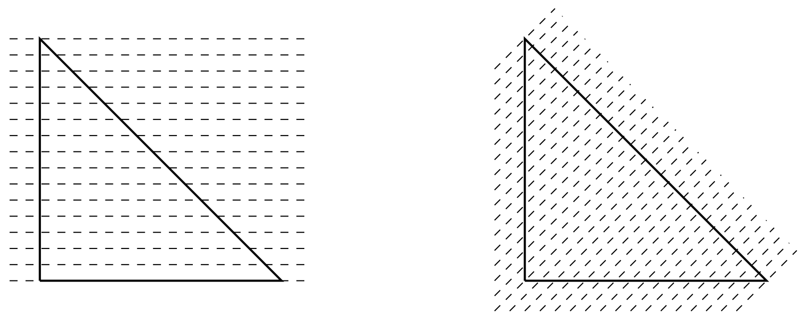
Definition 4. *The multi-set of* **width invariants** *of a lattice polygon Δ with non-empty interior associated to a primitive $v \in \mathbf{Z}^2$ is defined as*

$$E(\Delta, v) = \left\{ -1 + \#\{ (i,j) \in \Delta^{(1)} \cap \mathbf{Z}^2 \,|\, aj - bi = m + \ell \} \,\middle|\, \ell = 1, \ldots, d-1 \right\},$$

where $m, d \in \mathbf{Z}$ are the values from (1). (Its cardinality is $w(\Delta, v) - 1$, counting multiplicities.)

Using a unimodular transformation if needed one can always assume that $v = (1,0)$ and $\Delta \subseteq \mathbf{R} \times [0, w(\Delta, v)]$. In this setting the width invariants are given by the multi-set $\{E_j\}_{j=1,\ldots,w(\Delta,v)-1}$, with E_j the number of lattice points minus one that are contained in $\Delta^{(1)}$ at height j.

Example. Consider $d\Sigma$ for some $d \geq 3$. Then $\mathrm{lw}(d\Sigma) = d$ and there are three pairs of primitive vectors realizing the lattice width, namely $\pm(1,0), \pm(0,1), \pm(1,-1)$. For each of these vectors v the multi-set $E(d\Sigma, v)$ of width invariants equals $\{-1, 0, 1, 2, \ldots, d-4, d-3\}$. On the other hand one verifies that $w(d\Sigma, (1,1)) = 2d$ and $E(d\Sigma, (1,1)) =$



$w(d\Sigma, (1,0)) = d$ $\qquad$ $w(d\Sigma, (1,1)) = 2d$

$\{-1^4, 0^4, 1^4, 2^4, \ldots, \lfloor (d-6)/2 \rfloor^4, \lfloor (d-4)/2 \rfloor^\epsilon\}$, where the superscripts denote the multiplicities and $\epsilon = 1$ or 3, depending on whether d is odd or even, respectively.

Note that the width invariants are elements of $\mathbf{Z}_{\geq -1}$. In Chapter 5 it is shown that if v realizes the lattice width and $\Delta \not\cong d\Sigma$ for any $d \geq 3$, then the width invariants associated to v are all non-negative.

Chapter 3 introduces the following generalization of the lattice width:

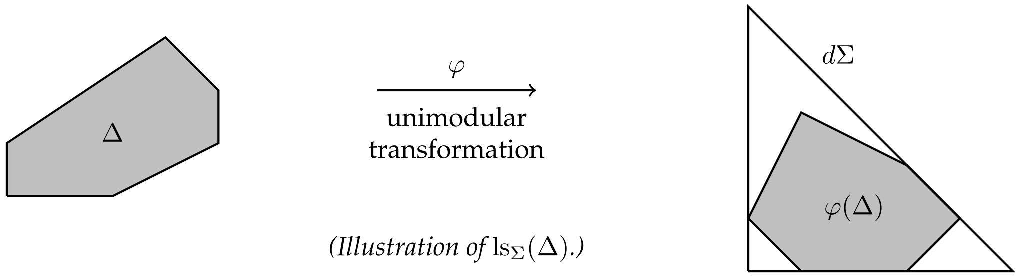
Definition 5. *The* **lattice size** $\mathrm{ls}_X(\Delta)$ *of a non-empty lattice polygon Δ with respect to a given set $X \subseteq \mathbf{R}^2$ having positive Jordan measure is defined as the minimal $d \in \mathbf{Z}_{\geq 0}$ such that Δ can be mapped inside dX by means of a unimodular transformation.*

For $X = \mathbf{R} \times [0,1]$ one recovers the lattice width. The chapter focuses entirely on $X = \Sigma$ and $X = \square := \square_{1,1}$. In order to state our main results relating $\mathrm{ls}_X(\Delta)$ to $\mathrm{ls}_X(\Delta^{(1)})$ it is convenient to define $\mathrm{ls}_\Sigma(\emptyset) = -2$ and $\mathrm{ls}_\square(\emptyset) = -1$.

Theorem 6. *Let Δ be a two-dimensional lattice polygon. Then $\mathrm{ls}_\Sigma(\Delta) = \mathrm{ls}_\Sigma(\Delta^{(1)}) + 3$, except in the following situations:*

- $\Delta \cong \mathrm{conv}\{(0,0),(a,0),(b,1),(0,1)\}$ *where* $a = b = 1$ *or* $2 \leq a \geq b \geq 0$*, in which case* $\mathrm{ls}_\Box(\Delta^{(1)}) = -2$ *while*
$$\mathrm{ls}_\Sigma(\Delta) = \begin{cases} a+1 & \text{if } a = b \\ a & \text{if } a > b. \end{cases}$$
- $\Delta \cong 2\Sigma$ *in which case* $\mathrm{ls}_\Sigma(\Delta^{(1)}) = -2$ *while* $\mathrm{ls}_\Sigma(\Delta) = 2$.
- $\Delta \cong \Delta_{4,2}$ *in which case* $\mathrm{ls}_\Sigma(\Delta^{(1)}) = 0$ *while* $\mathrm{ls}_\Sigma(\Delta) = 4$.
- $\Delta \cong \Box_{a,b}$ *for* $a, b \geq 2$*, in which case* $\mathrm{ls}_\Sigma(\Delta^{(1)}) = a + b - 4$ *while* $\mathrm{ls}_\Sigma(\Delta) = a + b$.
- *There exist parallel edges* $\tau \subseteq \Delta$ *and* $\tau' \subseteq \Delta^{(1)}$ *whose supporting lines are at integral distance* 1 *of each other, such that*
$$\#(\tau \cap \mathbf{Z}^2) - \#(\tau' \cap \mathbf{Z}^2) \geq 4,$$
in which case $\mathrm{ls}_\Sigma(\Delta^{(1)}) = \#(\tau' \cap \mathbf{Z}^2)$ *and* $\mathrm{ls}_\Sigma(\Delta) = \#(\tau \cap \mathbf{Z}^2)$.

As in the case of the lattice width, this result can be converted into a recursive algorithm for computing $\mathrm{ls}_\Sigma(\Delta)$ in practice, which we have again implemented in Magma. From the proof of the foregoing theorem one sees that Schicho's algorithm for simplifying rational surface parametrizations [Sch03a] works optimally.



(Illustration of $\mathrm{ls}_\Sigma(\Delta)$*.)*

Theorem 7. *Let* Δ *be a two-dimensional lattice polygon. Then* $\mathrm{ls}_\Box(\Delta) = \mathrm{ls}_\Box(\Delta^{(1)}) + 2$*, except in the following situations:*

- $\Delta \cong \mathrm{conv}\{(0,0),(a,0),(b,1),(0,1)\}$ *where* $2 \leq a \geq b \geq 0$*, in which case* $\mathrm{ls}_\Box(\Delta^{(1)}) = -1$ *while* $\mathrm{ls}_\Box(\Delta) = a$.
- $\Delta \cong 2\Sigma$ *in which case* $\mathrm{ls}_\Box(\Delta^{(1)}) = -1$ *while* $\mathrm{ls}_\Box(\Delta) = 2$.
- $\Delta \cong 3\Sigma, \Delta_{3,2}, \mathrm{conv}\{(0,0),(3,0),(2,1)(0,2)\}$ *or* $\mathrm{conv}\{(0,0),(3,0),(1,2),(0,2)\}$ *in which case* $\mathrm{ls}_\Box(\Delta^{(1)}) = 0$ *while* $\mathrm{ls}_\Box(\Delta) = 3$.
- $\Delta \cong \Delta_{4,2}$ *in which case* $\mathrm{ls}_\Box(\Delta^{(1)}) = 0$ *while* $\mathrm{ls}_\Box(\Delta) = 4$.
- *There exist parallel edges* $\tau \subseteq \Delta$ *and* $\tau' \subseteq \Delta^{(1)}$ *whose supporting lines are at integral distance* 1 *of each other, such that*
$$\#(\tau \cap \mathbf{Z}^2) - \#(\tau' \cap \mathbf{Z}^2) \geq 3,$$
in which case $\mathrm{ls}_\Box(\Delta^{(1)}) = \#(\tau' \cap \mathbf{Z}^2)$ *and* $\mathrm{ls}_\Box(\Delta) = \#(\tau \cap \mathbf{Z}^2)$.

Again the resulting recursive method has been implemented in Magma. The proof of the foregoing theorem has a remarkable byproduct: it turns out that the unimodular transformation mapping Δ inside $\mathrm{ls}_\Box(\Delta) \cdot \Box$ can be chosen such that it also maps inside $\mathbf{R} \times [0, \mathrm{lw}(\Delta)]$. As a consequence:

Corollary 8. *For each non-empty lattice polygon* Δ *the set*

$$\left\{ \, (a,b) \in \mathbf{Z}^2_{\geq 0} \, \middle| \, a \leq b \text{ and } \exists \Delta' \cong \Delta \text{ with } \Delta' \subseteq [0,a] \times [0,b] \, \right\}$$

has a minimum with respect to the product order on $\mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0}$*, namely* $(\mathrm{lw}(\Delta), \mathrm{ls}_\square(\Delta))$.

We conclude by stressing that the map $\Delta \mapsto \Delta^{(1)}$ is not surjective. In fact for a two-dimensional lattice polygon Γ to be of the form $\Delta^{(1)}$ for some larger lattice polygon Δ is a rather restrictive property. The following criterion was proved by Haase and Schicho [HS09]. Each edge $\tau \subseteq \Gamma$ lies on the boundary of a unique half-plane $a_\tau X + b_\tau Y \leq c_\tau$ containing Γ, where $a_\tau, b_\tau, c_\tau \in \mathbf{Z}$ are chosen to satisfy $\gcd(a_\tau, b_\tau) = 1$. Consider the polygon

$$\Gamma^{(-1)} = \bigcap_\tau (\text{half-plane } a_\tau X + b_\tau Y \leq c_\tau + 1)\,,$$

said to be obtained from Γ by moving out the edges. Then $\Gamma = \Delta^{(1)}$ for some lattice polygon Δ if and only if $\Gamma^{(-1)}$ is a lattice polygon. If this is the case then one can simply let $\Delta = \Gamma^{(-1)}$, and this is the maximal possible choice with respect to inclusion. In other words if Δ is a lattice polygon having a two-dimensional interior then $\Delta^{\max} = \Delta^{(1)(-1)}$.

For any lattice polygon Δ, a repeated application of $\Delta \mapsto \Delta^{(1)}$ eventually leads to a lattice polygon whose interior is at most one-dimensional. Such polygons have been classified explicitly by Koelman [Koe91, §4]. Conversely, starting from these basic cases one can algorithmically produce all lattice polygons up to a given genus by repeatedly applying $\Delta \mapsto \Delta^{(-1)}$, verifying Haase and Schicho's criterion and making local tweaks (clipping off vertices). The details can be found in Chapter 2, which comes along with a Magma implementation by means of which we have produced a list containing exactly one representative within each unimodular equivalence class of lattice polygons of genus $1 \leq g \leq 30$. This list is useful for testing hypotheses and detecting patterns; we have mainly applied this to the study of syzygies of toric surfaces (and of smooth curves therein) in Chapters 6 and 8. But there are also some purely combinatorial consequences which seem interesting in their own right. For instance prior to our work the concrete number of equivalence classes of lattice polygons of genus $g \geq 1$ was unknown for g as small as 3, even though asymptotically for $g \to \infty$ it was shown to be $O(\exp(g^{1/3}))$ by Bárány [BT04]. Another consequence (albeit slightly indirect; see Chapter 2 for the details) is:

Lemma 9. *The minimal genus of a lattice* 15*-gon is* 45.

This fills in the smallest open entry of a list whose study began with Arkinstall [Ark80]. Recently our data set was used to give tight bounds on the generalized Helly numbers of $\mathbf{Z}^2$; see [Ave+15].

3. The number of moduli (Chapter 1)

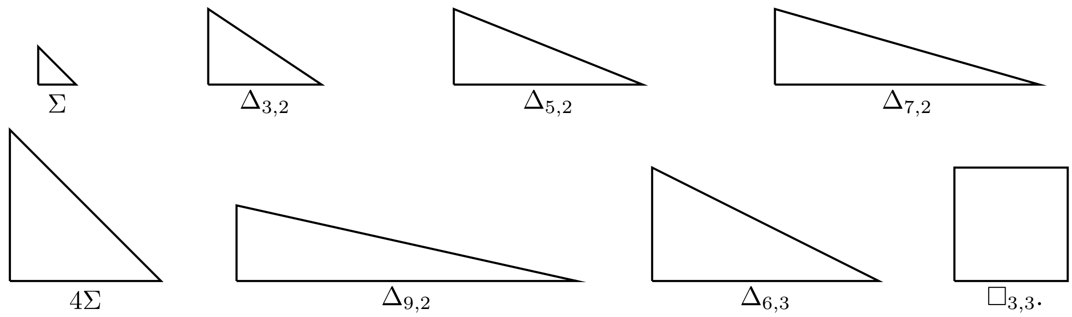
The generic Laurent polynomial that is supported on a given two-dimensional lattice polygon $\Delta \subseteq \mathbf{R}^2$ is Δ-nondegenerate, and weakly Δ-nondegenerate in particular. As a consequence, if the man in the street would be asked to scribble down a random curve, the outcome is likely to be weakly nondegenerate, and most curves that can be found *in the wild* are indeed of this kind, including all hyperelliptic curves and smooth curves in $\mathbf{P}^2$ as we have seen above, but also all trigonal curves, $C_{a,b}$ curves, and several more well-studied families. For a moment this might tempt one to conclude that the generic curve, in the proper moduli-theoretic sense, is weakly non-degenerate. But a second thought quickly reveals that this is far from true. Some obstructions are:

- The moduli space $\mathcal{M}_g$ of curves of genus g is not unirational for $g \geq 22$ [Far09].
- The gonality of a weakly Δ-nondegenerate genus g curve is bounded by $\operatorname{lw}(\Delta)$, which is $O(\sqrt{g})$ by [FTM74], while the general curve has gonality $\lfloor (g+3)/2 \rfloor$ by Brill-Noether theory. This was recently elaborated in detail by Smith [Smi15] who proved that weakly nondegenerate curves cannot be Brill-Noether general from $g \geq 7$ onwards.
- The canonical ideal of a weakly nondegenerate curve contains (many) quadratic binomials, i.e. quadrics of rank 3 or 4.

The latter obstruction seems best-suited for proving that a certain concretely given curve C/k is *not* weakly nondegenerate.

In Chapter 1 we try to obtain a more precise understanding of which curves are weakly nondegenerate. For curves of genus at most four, we prove:

Theorem 10. *Every curve C/k of genus $g \leq 4$ is weakly Δ-nondegenerate for exactly one choice of Δ among the lattice polygons listed below.*



(The polygons referred to in the statement of Theorem 10.)

The theorem remains true upon replacement of 'weakly Δ-nondegenerate' by 'Δ-nondegenerate'. Also, slightly modified versions hold over fields that are not necessarily algebraically closed. For instance over finite fields the theorem is true except when C is of genus 4 and canonically embeds into an elliptic quadric in $\mathbf{P}^3$; see also [CV10].

The main result of Chapter 1 is a determination of the number of moduli of the family of weakly nondegenerate curves, through a parameter count that builds on Haase and Schicho's aforementioned work [HS09] and a combinatorial description of the automorphism group of X_Δ due to Bruns and Gubeladze [BG09]. For a two-dimensional lattice polygon Δ, we denote by $\mathcal{M}_\Delta$ the Zariski closure of the locus inside $\mathcal{M}_g$ of all weakly Δ-nondegenerate curves; these spaces had already been introduced and studied by Koelman [Koe91, §2]. For each $g \geq 1$ let

$$\mathcal{M}_g^{\text{wnd}} = \bigcup_{\substack{\Delta \text{ for which} \\ \sharp(\Delta^{(1)} \cap \mathbf{Z}^2) = g}} \mathcal{M}_\Delta,$$

which in view of Lemma 3 is a finite union because unimodularly equivalent lattice polygons give rise to the same curves. We show:

Theorem 11. *One has*

$$\begin{cases} \dim \mathcal{M}_1^{\text{wnd}} = 1, \\ \dim \mathcal{M}_2^{\text{wnd}} = 3, \\ \dim \mathcal{M}_3^{\text{wnd}} = 6, \\ \dim \mathcal{M}_7^{\text{wnd}} = 16, \\ \dim \mathcal{M}_g^{\text{wnd}} = 2g+1 \quad \textit{if } g \geq 4 \textit{ and } g \neq 7. \end{cases}$$

In particular from genus five on the generic curve is not weakly nondegenerate (let alone nondegenerate). For $g \geq 4$ a top-dimensional subvariety of $\mathcal{M}_g^{\text{wnd}}$ is given by the trigonal locus $\mathcal{M}_g^{\text{tri}}$, except when $g = 7$, where the trigonal curves are beaten by the trinodal plane sextics. Recently Brodsky, Joswig, Morrison and Sturmfels used a similar approach to obtain the same moduli count for tropical plane curves [Bro+15].

4. Gonality, Clifford index, and related invariants (Chapters 3 and 5)

This section adds a number of entries to the geometry-combinatorics dictionary for weakly nondegenerate curves, related to linear systems. The main reference for the results presented below is Chapter 5. The most important new entry is the **gonality**, which is defined as the minimal possible degree of a non-constant rational map to $\mathbf{P}^1$:

Theorem 12. *Let Δ be a two-dimensional lattice polygon and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly Δ-nondegenerate Laurent polynomial. Then the gonality of U_f equals $\operatorname{lw}(\Delta^{(1)})+2$, unless $\Delta^{(1)} \cong \Upsilon$ in which case it is $\operatorname{lw}(\Delta^{(1)}) + 1$.*

This theorem arises as a consequence of a stronger result. Let f be a weakly Δ-nondegenerate Laurent polynomial and let $v = (a,b) \in \mathbf{Z}^2$ be a primitive vector. We define the **combinatorial pencil** g_v on C_f associated to v as the trace of the linear system on X_Δ swept out by $\mathbf{T}^2 \to \mathbf{T}^1 : (x,y) \mapsto x^a y^b$. Notice that $g_v = g_{-v}$ is of degree $w(\Delta, v)$, in other words it concerns a $g^1_{w(\Delta,v)}$.

Remark. In almost all cases g_v equals the basepoint free pencil associated to the map $U_f \to \mathbf{T}^1 : (x,y) \mapsto x^a y^b$. However when $\Delta(f) \subsetneq \Delta$, in certain cases this needs to be extended by basepoints.

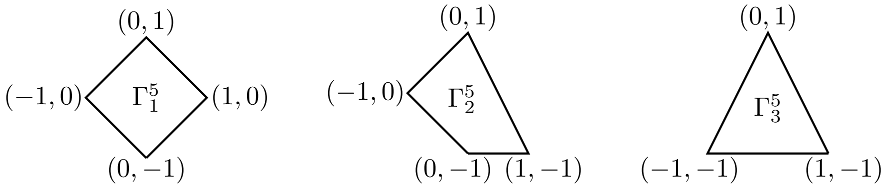
Our strengthening of Theorem 12 reads as follows:

Theorem 13. *Let Δ be a two-dimensional lattice polygon such that $\Delta^{(1)}$ is not unimodularly equivalent to any of the following:*

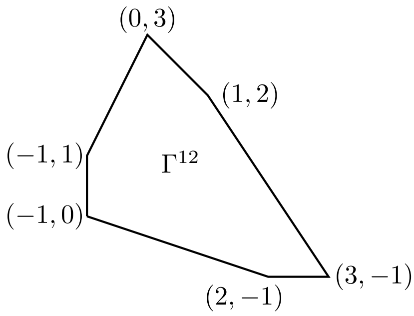
$$\emptyset, \quad (d-3)\Sigma \text{ (for some } d \geq 3), \quad \Upsilon, \quad 2\Upsilon, \quad \Gamma_1^5, \quad \Gamma_2^5, \quad \Gamma_3^5.$$

If $\operatorname{char} k > 0$ then we also exclude Γ^{12}. Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly Δ-nondegenerate Laurent polynomial. Then every linear pencil on C_f which realizes the gonality is combinatorial.

One byproduct of the above theorem is that besides the gonality itself, one also knows the **number of gonality pencils** by merely looking at the Newton polygon, except possibly when $\Delta^{(1)} \cong \Upsilon$ in which case U_f is always tetragonal but the number of gonality pencils depends on the concrete choice of f (it is either 1 or 2), and except possibly when $\operatorname{char} k > 0$ and $\Delta^{(1)} \cong \Gamma^{12}$ where the situation is not fully understood. If $\Delta^{(1)} = \emptyset$ then there is a unique gonality pencil. In the other exceptional cases $\Delta^{(1)} \cong (d-3)\Sigma, 2\Upsilon, \Gamma_i^5$ the number of gonality pencils can be shown to be infinite.



(Polygons excluded in the statement of Theorem 13, corresponding to curves of genus 5.*)*



(Polygon excluded by Theorem 13 in positive characteristic, corresponding to curves of genus 12.*)*

Our main proof ingredient is a result due to Serrano [Ser87] which given a curve C inside some surface X, provides sufficient conditions under which a morphism $C \to \mathbf{P}^1$ can be extended to a morphism $X \to \mathbf{P}^1$. We stress that this approach, and as a matter of fact the entire statement of Theorem 13, is due to Kawaguchi [Kaw16], modulo two relaxations:

- Kawaguchi made the technical assumption that U_f is not birationally equivalent to a smooth plane curve of degree $d \geq 5$. We got rid of this condition, essentially by invoking the formula (2) at the proof step where this assumption was used.

- Both Kawaguchi and we proved these statements subject to $\operatorname{char} k = 0$. However one can obtain the same results in positive characteristic by using [Ser87, Rmk. 3.12], which discusses sufficient conditions for the extension of separable morphisms from curves to rational surfaces, in combination with the fact that every morphism $C_f \to \mathbf{P}^1$ decomposes into a purely inseparable part and a separable part

$$C_f \to C_f^{\text{Frob}} \to \mathbf{P}^1,$$

  along with the observation that Frobenius preserves weak nondegeneracy. This approach does not work when $\Delta^{(1)} \cong \Gamma^{12}$, in which case one of the extra conditions mentioned in [Ser87, Rmk. 3.12] is violated. Therefore Γ^{12} pops up as a new exception, although this may well be just a proof artefact (unlike the other exclusions, which are really needed). More details will be included in the forthcoming version of [CT].

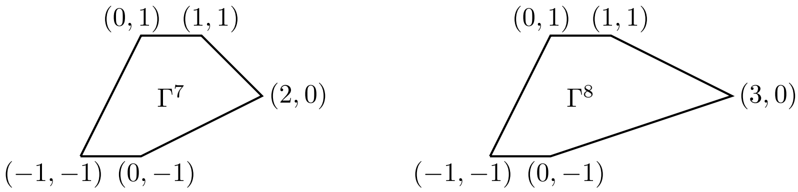
In addition, our database of polygons having small genus allowed us to skip a large and combinatorially tedious part of Kawaguchi's proof.

Using the same techniques we can deduce an analogous result for **near-gonal pencils**, by which we mean base-point free linear pencils of degree $\gamma + 1$, where γ is the gonality of U_f:

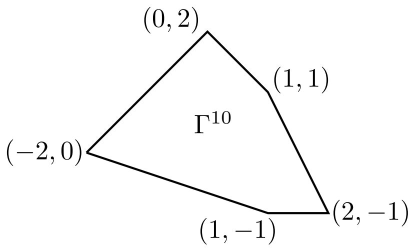
Theorem 14. *Let Δ be a two-dimensional lattice polygon such that*

$$\operatorname{ls}_\Sigma(\Delta^{(1)}) \geq \operatorname{lw}(\Delta^{(1)}) + 2$$

and such that $\Delta^{(1)} \not\cong 2\Upsilon, 3\Upsilon, \Gamma^7, \Gamma^8$*. If* $\operatorname{char} k > 0$ *then we also exclude* Γ^{10}*. Let* $f \in k[x^{\pm 1}, y^{\pm 1}]$ *be a weakly* Δ*-nondegenerate Laurent polynomial. Then every near-gonal pencil on* C_f *is combinatorial.*



(Polygons excluded in the statement of Theorem 14, corresponding to curves of genus 7 *and* 8*.)*



(Polygon excluded by Theorem 14 in positive characteristic, corresponding to curves of genus 10*.)*

Again the exclusion of Γ^{10} might be a proof artefact; as explained in Chapter 5 the other exclusions are necessary. Also, one can verify that the list of excluded polygons is a strict extension of its counterpart from Theorem 13.

In principle it should be possible to obtain similar statements for basepoint free $g^1_{\gamma+n}$'s with $n = 2, 3, \ldots$, but we expect the proof to become increasingly case-distinctive and the number of excluded polygons to grow. Nevertheless for small n it might be worth the try, in order to gain some feeling on how $\dim W^1_{\gamma+n}$ can grow with n, a question which has sparked much interest in view of connections with Green's canonical syzygy conjecture, through Aprodu's linear growth condition [Apr05].

Another entry to the dictionary is given by the **scrollar invariants** associated to a combinatorial pencil (e.g. any gonality pencil in the case of a polygon that is non-exceptional for Theorem 13). The scrollar invariants associated to a linear pencil g^1_d on a non-hyperelliptic genus g curve C/k are defined as follows. View the g^1_d as a 1-dimensional family of effective divisors D on the canonical model $C_f^{\text{can}} \subseteq \mathbf{P}^{g-1}$ and let $S \subseteq \mathbf{P}^{g-1}$ be the ruled variety obtained by taking the union of all linear spans $\langle D \rangle$. A theorem by Eisenbud and Harris [EH87, Thm. 2] states that S is a rational normal scroll. The scrollar invariants associated to g^1_d are defined as the multi-set of invariants (= the degrees of the spanning rational normal curves) of this scroll. If our g^1_d is complete and basepoint free then the $\langle D \rangle$'s are planes of dimension $d-2$, and S is of dimension $d-1$. In this case the scrollar invariants $0 \leq e_0 \leq e_1 \leq \ldots \leq e_{d-1}$ satisfy $e_{d-1} \leq (2g-2)/d$.

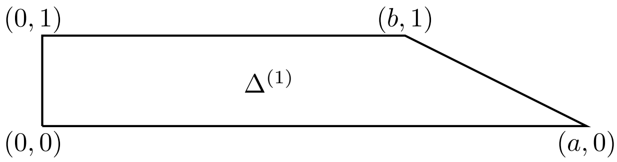
Theorem 15. *Let* Δ *be a lattice polygon such that* $\Delta^{(1)}$ *is two-dimensional. Let* $v \in \mathbf{Z}^2$ *be a primitive vector and let* $f \in k[x^{\pm 1}, y^{\pm 1}]$ *be a weakly* Δ*-nondegenerate Laurent polynomial. Then the multi-set of scrollar invariants of* C_f *with respect to* g_v *equals the multi-set of non-negative width invariants of* Δ *with respect to* v*.*

The proof can be found in Chapter 5 and has the following corollary:

Corollary 16. *The rank of the complete linear system spanned by* g_v *equals the number of negative width invariants (counting multiplicities) plus one. In particular* g_v *is complete if and only if all width invariants are non-negative.*

Example. Consider $d\Sigma$ for some $d \geq 4$ along with the primitive vector $v = (1,0)$. Recall from Section 2 that $E(d\Sigma, v) = \{-1, 0, 1, 2, \ldots, d-3\}$. By Theorem 15 the scrollar invariants associated to $g_{(1,0)}$ are $\{0, 1, 2, \ldots, d-3\}$. By Corollary 16 our $g_{(1,0)}$ is a subsystem of a g^2_d, hence it is not complete. But this just confirms a well-known fact, because C_f is a smooth projective degree d curve in $\mathbf{P}^2$ and $g_{(1,0)}$ is cut out by the pencil of lines through a fixed point of the plane. By varying the point one obtains the g^2_d.

Example (Maroni invariants). Consider a lattice polygon Δ with $\mathrm{lw}(\Delta) = 3$ and $\Delta \not\cong 3\Sigma$. Then up to unimodular equivalence $\Delta^{(1)}$ is of the form below, for certain integers $1 \leq a \geq b \geq 0$. Notice that weakly Δ-nondegenerate Laurent polynomials $f \in k[x^{\pm 1}, y^{\pm 1}]$



give rise to trigonal curves C_f of genus $g = \#(\Delta^{(1)} \cap \mathbf{Z}^2) = a + b + 2$. Then $g_{(1,0)}$ is a gonality pencil and the corresponding scrollar invariants are seen to be $\{a, b\}$; if $a = 1$ then there may exist other combinatorial gonality pencils but the associated scrollar invariants are the same. The numbers a and b are classical invariants called the **Maroni invariants**[1] of C_f. Every trigonal curve arises as a weakly Δ-nondegenerate curve with Δ a lattice polygon of the above form. A fun fact is that the well-known bound $a \leq (2g-2)/3$ which is usually proven through the Riemann-Roch theorem, can also be obtained in a purely combinatorial way, using Haase and Schicho's criterion for $\Delta^{(1)}$ to be an interior polygon.

Another observation is that Theorem 13 can be combined with results of Coppens and Martens [CM91] to obtain combinatorial interpretations for the **Clifford index** and the **Clifford dimension**, which are defined for curves of genus $g \geq 4$ only: the Clifford index is

$$\min\{\, d - 2r \mid C_f \text{ carries a divisor } D \text{ with } |D| = g^r_d \text{ and } h^0(C_f, D), h^0(C_f, K_\Delta - D) \geq 2 \,\}$$

which is a non-negative integer due to Clifford's theorem. The Clifford dimension is the smallest r for which the minimum is realized; this concept was introduced in [Eis+89]. Coppens and Martens assume $\mathrm{char}\, k = 0$; we inherit this condition since it is not clear to us how to circumvent it.

Theorem 17. *Let Δ be a two-dimensional lattice polygon and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly Δ-nondegenerate Laurent polynomial. Assume that $\#(\Delta^{(1)} \cap \mathbf{Z}^2) \geq 4$ and $\mathrm{char}\, k = 0$. Then:*

- *The Clifford index of U_f equals $\mathrm{lw}(\Delta^{(1)})$, unless $\Delta^{(1)} \cong (d-3)\Sigma$ for some $d \geq 5$, $\Delta^{(1)} \cong \Upsilon$, or $\Delta^{(1)} \cong 2\Upsilon$, in which cases it is $\mathrm{lw}(\Delta^{(1)}) - 1$.*
- *The Clifford dimension of U_f equals 2 if $\Delta^{(1)} \cong (d-3)\Sigma$ for some $d \geq 5$, it equals 3 if $\Delta^{(1)} \cong 2\Upsilon$, and it is 1 in all other cases.*

The possibility of combining Theorem 13 with [CM91] was already mentioned by Kawaguchi [Kaw16]. However we recall Kawaguchi's assumption that U_f is not birationally equivalent to a smooth plane curve of degree $d \geq 5$, or in other words that the Clifford dimension is different from 2. So by getting rid of this condition we end up with a more

[1] The existing literature is ambiguous at this point: sometimes one talks about a single Maroni invariant, in which case one means either b or $a - b$.

complete and pleasing statement. The main ingredient taken from [CM91] is that there always exist infinitely many gonality pencils as soon as the Clifford dimension is at least 2. Because the number of combinatorial pencils is necessarily finite, through Theorem 13 this reduces one's task to analyzing the exceptions $\Delta^{(1)} \cong \emptyset, (d-3)\Sigma, \Upsilon, 2\Upsilon, \Gamma_1^5, \Gamma_2^5, \Gamma_3^5$. For smooth plane curves, Coppens and Martens' result is classical and holds in any characteristic [Har86], allowing one to obtain the following corollary:

Corollary 18. *Let Δ be a lattice polygon with non-empty interior. Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be weakly Δ-nondegenerate and assume that U_f is birationally equivalent to a smooth projective curve in $\mathbf{P}^2$, say of degree $d \geq 3$. Then $\Delta^{(1)} \cong (d-3)\Sigma$.*

Therefore, in the case of smooth plane curves, one can view $\Delta^{(1)}$ in some sense as a geometric invariant. We refer to this property as **intrinsicness** of the interior polygon and will state a few more results of this kind in Section 8.

Remark. Stated more geometrically, this means that if a toric surface X contains a smooth projective curve C that is abstractly isomorphic to a smooth plane projective curve, then there exists a *toric* blow-down $\pi : X \to \mathbf{P}^2$ such that $\pi|_C : C \to \mathbf{P}^2$ is an embedding.

$\Delta^{(1)}$	gonality	Clifford ind.	Clifford dim.
$(d-3)\Sigma$ (for some $d \geq 5$)	$d-1$	$d-4$	2
Υ	3	1	1
2Υ	6	3	3
everything else	$\mathrm{lw}(\Delta)$	$\mathrm{lw}(\Delta^{(1)})$	1

(Overview in char. 0 of combinatorial interpretations for gonality, Clifford index, Clifford dimension.)

A consequence of Theorems 12 and 17 is that the gonality and (if $\operatorname{char} k = 0$) the Clifford index and dimension of U_f depend on Δ only, rather than on the specific choice of our weakly Δ-nondegenerate Laurent polynomial $f \in k[x^{\pm 1}, y^{\pm 1}]$. This is an a priori non-trivial fact that can be rephrased as constancy among the smooth curves in linear systems of curves on toric surfaces. The existing literature contains other results of this type, which are usually stated in characteristic zero only. For instance recent work of Lelli-Chiesa proves constancy of the gonality and the Clifford index for curves in certain linear systems on other types of rational surfaces [LC13]. An important theorem by Green and Lazarsfeld states that constancy of the Clifford index holds in linear systems on K3 surfaces [GL87], although here constancy of the gonality is not necessarily true. In the next section we will state a constancy result for the entire canonical graded Betti table (which subject to Green's canonical syzygy conjecture is a vast generalization of the Clifford index).

We end this section with a brief discussion (more details to be found in Chapter 3) of two other invariants that we have put to a combinatorial analysis, albeit with less conclusive results:

- The minimal degree $s_2(U_f)$ of a possibly singular curve in $\mathbf{P}^2$ that is birationally equivalent to U_f; equivalently this asks for the minimal degree of a simple linear system of rank 2.
- The minimum $s_{1,1}(U_f)$ of

$$\left\{ \, (a,b) \in \mathbf{Z}_{\geq 0}^2 \,\middle|\, a \leq b \text{ and } \exists\, C \subseteq \mathbf{P}^1 \times \mathbf{P}^1 \text{ of bidegree } (a,b) \text{ with } C \simeq U_f \, \right\} \qquad (3)$$

where $\simeq$ denotes birational equivalence. The minimum is taken with respect to the lexicographic order on $\mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0}$ (but see the 'open question' at the end of this section).

Unfortunately we can only provide upper bounds, and leave it as an unsolved problem whether these statements are sharp. In particular we do not know whether the quantities $s_2(U_f)$ and $s_{1,1}(U_f)$ are independent of the concrete choice of $f \in k[x^{\pm 1}, y^{\pm 1}]$.

Theorem 19. *Let Δ be a two-dimensional lattice polygon and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly Δ-nondegenerate Laurent polynomial. Then $s_2(U_f) \leq \mathrm{ls}_\Sigma(\Delta^{(1)}) + 3$. If $\Delta^{(1)} \cong d\Upsilon$ for some $d \geq 1$ then the sharper bound $s_2(U_f) \leq \mathrm{ls}_\Sigma(\Delta^{(1)}) + 2$ applies.*

Theorem 20. *Let Δ be a two-dimensional lattice polygon and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly Δ-nondegenerate Laurent polynomial. Then $s_{1,1}(U_f) \leq (\mathrm{lw}(\Delta^{(1)}) + 2, \mathrm{ls}_\square(\Delta^{(1)}) + 2)$. If $\Delta^{(1)} \cong \Upsilon$ then the sharper bound $s_{1,1}(U_f) \leq (3, 4)$ applies.*

Remark that by Theorem 12 the first components of the upper bounds stated in Theorem 20 are equal to the gonality. Therefore this part of the statement is optimal and one sees that the bounds necessarily hold with respect to the product order on $\mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0}$. In particular, if it is indeed true that Theorem 20 is always sharp, then the set (3) always admits a minimum with respect to the product order. Please compare this to Corollary 8, which can be viewed as a combinatorial analogue of this statement.

Open question. As an even wilder shot in the dark, we wonder whether it is true for *every* algebraic curve C/k (i.e. not necessarily weakly nondegenerate) that the set of bidegrees (a, b) with $a \leq b$ of curves in $\mathbf{P}^1 \times \mathbf{P}^1$ that are birationally equivalent to C admits a minimum with respect to the product order on $\mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0}$.

5. Canonical graded Betti numbers (Chapters 6, 7 and 8)

Let Δ be a lattice polygon with two-dimensional interior and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly Δ-nondegenerate Laurent polynomial. Recall from Section 1 that $C_f \subseteq X_\Delta$ is non-hyperelliptic and that its canonical model satisfies

$$C_f^{\mathrm{can}} \subseteq X_{\Delta^{(1)}} \subseteq \mathbf{P}^{g-1} = \operatorname{Proj} S_{\Delta^{(1)}} \tag{4}$$

where $S_{\Delta^{(1)}} = k[\, X_{i,j} \,|\, (i,j) \in \Delta^{(1)} \cap \mathbf{Z}^2 \,]$ and $g = \#(\Delta^{(1)} \cap \mathbf{Z}^2)$ denotes the genus of C_f. In this section we report on a combinatorial analysis of the Betti numbers β_{ij} appearing in a minimal free resolution of the homogeneous coordinate ring of C_f^{can} as a graded $S_{\Delta^{(1)}}$-module:

$$\cdots \to \bigoplus_{q \geq 2} S_{\Delta^{(1)}}(-q)^{\beta_{2,q}} \to \bigoplus_{q \geq 1} S_{\Delta^{(1)}}(-q)^{\beta_{1,q}} \to \bigoplus_{q \geq 0} S_{\Delta^{(1)}}(-q)^{\beta_{0,q}} \to S_{\Delta^{(1)}} / \mathcal{I}(C_f^{\mathrm{can}}) \to 0.$$

These numbers are usually gathered in what is called the **canonical graded Betti table** of C_f, by writing $\beta_{p,p+q}$ in the pth column and the qth row. Alternatively, and often more conveniently, this entry equals the dimension of the Koszul cohomology space $K_{p,q}(C_f, K_\Delta)$. The canonical graded Betti table is known to be of the form

	0	1	2	3	$\dots$	$g-4$	$g-3$	$g-2$
0	1	0	0	0	$\dots$	0	0	0
1	0	a_1	a_2	a_3	$\dots$	a_{g-4}	a_{g-3}	0
2	0	a_{g-3}	a_{g-4}	a_{g-5}	$\dots$	a_2	a_1	0
3	0	0	0	0	$\dots$	0	0	1,

(5)

where omitted entries are understood to be zero. If one assumes Green's canonical syzygy conjecture [Gre84], the settlement of which is arguably the most important unsolved problem concerning linear series on algebraic curves, then the canonical graded Betti table is a vast generalization of the Clifford index. Indeed, Green's conjecture predicts that the latter is equal to $\min\{\,\ell \,|\, a_{g-\ell} \neq 0\,\} - 2$.

Remark. Assume that $\operatorname{char} k = 0$. If $X_{\Delta^{(1)}}$ carries an anticanonical pencil, or equivalently if the polygon P_{-K} associated to an anticanonical torus-invariant divisor $-K$ on $X_{\Delta^{(1)}}$ contains at least two lattice points, then one can invoke a result of Lelli-Chiesa [LC13] to settle Green's conjecture for all weakly Δ-nondegenerate curves. This includes the cases where $X_{\Delta^{(1)}}$ is Gorenstein and weak Fano, which are discussed further down. The details of these claims are explained in Chapter 8.

Remark. It is known that Green's conjecture may fail over fields of very small positive characteristic [Sch03b], but we do not know of any weakly nondegenerate counterexamples.

Having a combinatorial description of the Clifford index at hand (at least if $\operatorname{char} k = 0$, see Theorem 17), it is a natural step to look for a similar description of the entire canonical Betti table. At this moment this seems to be an infeasible task, both from a combinatorial and a geometric perspective. In view of (4) we hope for an explicit relationship with the graded Betti table of $X_{\Delta^{(1)}}$, which is of the form

$$\begin{array}{c|ccccccc} & 0 & 1 & 2 & 3 & \dots & g-4 & g-3 \\ \hline 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & b_1 & b_2 & b_3 & \dots & b_{g-4} & c_{g-3} \\ 2 & 0 & c_{g-3} & c_{g-4} & c_{g-5} & \dots & c_2 & c_1 . \end{array} \qquad (6)$$

Concretely, we distill the following three research questions, each of which we leave unanswered in their general form, although we can offer several partial results:

(i) **What would such an explicit relationship look like?**

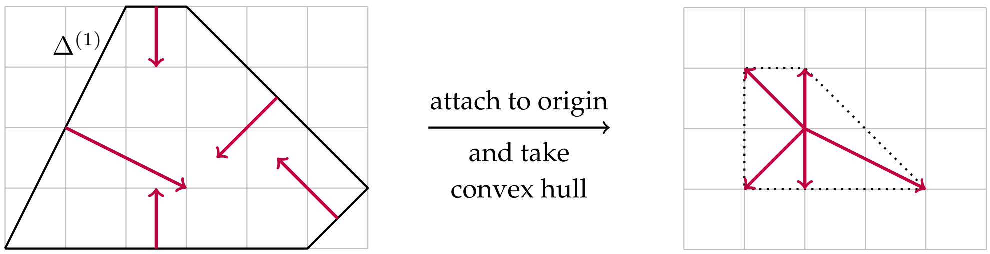
The inclusion (4) gives rise to an exact sequence

$$0 \longrightarrow b_\ell \longrightarrow a_\ell \longrightarrow c_\ell \xrightarrow{\mu_{\ell,f}} c_{g-1-\ell} \longrightarrow a_{g-1-\ell} \longrightarrow b_{g-1-\ell} \longrightarrow 0 \qquad (7)$$

for each value of $\ell = 1, 2, \dots, g-2$. Here we abusingly write the dimensions of the Koszul cohomology spaces, rather than the spaces themselves, and it is understood that $a_{g-2} = b_{g-2} = c_{g-2} = 0$. The map $\mu_{\ell,f}$ is a morphism between two cohomology spaces associated to $X_{\Delta^{(1)}}$ that is induced by multiplication by f; we refer to Chapter 8 for the precise construction. This shows that $a_\ell = b_\ell + c_\ell - \dim \operatorname{im} \mu_{\ell,f}$, and the question reduces to a determination of the last term. Our main theorem is that $\mu_{\ell,f} = 0$ in the cases where $X_{\Delta^{(1)}}$ is Gorenstein and weak Fano.

Theorem 21. *Let Δ be a lattice polygon with two-dimensional interior. Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly Δ-nondegenerate Laurent polynomial and let $g = \#(\Delta^{(1)} \cap \mathbf{Z}^2)$. Denote by $a_1, a_2, \dots, a_{g-3}$ the canonical graded Betti numbers of C_f as in (5), and similarly let $b_1, c_1, b_2, c_2, \dots, b_{g-3}, c_{g-3}$ be the graded Betti numbers of $X_{\Delta^{(1)}}$ as in (6). If $X_{\Delta^{(1)}}$ is Gorenstein and weak Fano then for all $\ell = 1, 2, \dots, g-3$ we have $a_\ell = b_\ell + c_\ell$.*

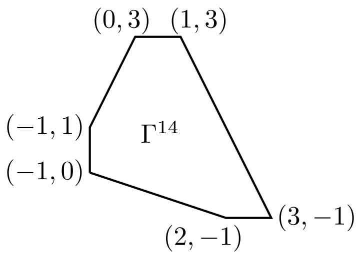
Being Gorenstein and weak Fano has an easy combinatorial interpretation: it means that the convex hull of the primitive inward pointing normal vectors to the edges is a reflexive polygon (= a lattice polygon of genus one). An example is depicted below. This is a rather strong condition, but we note that Theorem 21 applies to



(Illustration of the Gorenstein weak Fano property from the combinatorial viewpoint.)

most of our introductory examples, including the cases where $\Delta \cong d\Sigma$ for some $d \geq 4$, where $\Delta \cong d\Upsilon$ for some $d \geq 2$, where $\Delta \cong \square_{a,b}$ for some $a, b \geq 3$, and so on. Moreover, experimentally we observe that $\mu_{\ell,f} = 0$ much more frequently than under the Gorenstein weak Fano assumption. Of course, an obvious reason could be that $c_\ell = 0$ or $c_{g-1-\ell} = 0$: by Theorem 24 below we perfectly understand when this happens. But often $\mu_{\ell,f} = 0$ for reasons we do not know.

Example. In this example we let k be (the algebraic closure of) the finite field $\mathbf{F}_{10007}$; this is mainly for computational efficiency, we expect the same analysis to apply over $\mathbf{C}$. The toric surface $X_{\Delta^{(1)}}$ over k corresponding to the interior polygon $\Delta^{(1)}$ shown below is Gorenstein but not weak Fano. One computationally verifies that



the corresponding graded Betti table is

	0	1	2	3	4	5	6	7	8	9	10	11
0	1	0	0	0	0	0	0	0	0	0	0	0
1	0	59	363	1100	2013	2310	1525	343	24	0	0	0
2	0	0	0	0	0	7	112	574	561	265	66	7,

while the canonical graded Betti table of C_f for an aimlessly chosen[2] Laurent polynomial $f \in \mathbf{F}_{10007}[x^{\pm 1}, y^{\pm 1}]$ that is weakly $\Delta^{\max}$-nondegenerate when considered over k was found to be

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	0	0	0	0	0	0	0	0	0	0	0	0
1	0	66	429	1365	2574	2884	1637	350	24	0	0	0	0
2	0	0	0	0	24	350	1637	2884	2574	1365	429	66	0
3	0	0	0	0	0	0	0	0	0	0	0	0	1.

[2] We equipped the lattice points on the boundary of $\Delta^{\max}$ with the prime coefficients $2, 3, 5, 7, 11, 13, 17, 19$, starting at the left-most vertex and proceeding counterclockwise. The coefficients corresponding to the interior lattice points were chosen 0 in view of Lemma 23 below.

(The computation started from our explicit minimal set of generators for the canonical ideal; see Section 2.) Thus for $\ell = 6$ the exact sequence (7) reads

$$0 \longrightarrow 1525 \longrightarrow 1637 \longrightarrow 112 \stackrel{\mu_{6,f}}{\longrightarrow} 7 \longrightarrow 350 \longrightarrow 343 \longrightarrow 0,$$

implying that $\mu_{6,f} = 0$, and similarly one sees that $\mu_{7,f} = 0$. We do not understand why, as this is not explained by Theorem 21. In all other cases $\mu_{\ell,f} = 0$ because either $c_\ell = 0$ or $c_{g-1-\ell} = 0$.

Example. The same computer experiment, when applied to the polygon Γ^{12} from Section 4, respectively resulted in the graded Betti tables

	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	0	45	231	550	693	399	69	0	0	0	0
2	0	0	0	0	69	399	693	550	231	45	0
3	0	0	0	0	0	0	0	0	0	0	1

and

	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0	0
1	0	39	186	414	504	295	69	0	0	0
2	0	0	0	0	1	105	189	136	45	6.

Here one sees that the exact sequence (7) for $\ell = 5$ reads:

$$0 \longrightarrow 295 \longrightarrow 399 \longrightarrow 105 \stackrel{\mu_{5,f}}{\longrightarrow} 1 \longrightarrow 69 \longrightarrow 69 \longrightarrow 0.$$

So $\mu_{5,f}$ is *not* trivial in this case, but rather surjective onto its one-dimensional codomain. In fact, $\Delta^{(1)}$ is the only interior polygon for which we have observed deviating behavior with respect to the formula $a_\ell = b_\ell + c_\ell$, although we expect more exceptions to pop up beyond the range of polygons that we have computed (if not then Green's conjecture would be violated, as explained at the end of this section).

(ii) **Is it true at all that the canonical graded Betti table of C_f only depends on the graded Betti table of $X_{\Delta^{(1)}}$, rather than on the specific choice of f?**

In other words, do we have constancy in the sense discussed in Section 4? It is clear from Theorem 21 that the answer is yes if $X_{\Delta^{(1)}}$ is Gorenstein and weak Fano:

Corollary 22. *Let Δ be a lattice polygon with two-dimensional interior and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly Δ-nondegenerate Laurent polynomial. If $X_{\Delta^{(1)}}$ is Gorenstein and weak Fano then the canonical graded Betti numbers of C_f do not depend on the specific choice of f.*

For example, this implies that the canonical graded Betti table of a smooth plane projective degree $d \geq 4$ curve depends on d only. For general lattice polygons Δ we can show that only the coefficients that are supported on the boundary potentially matter:

Lemma 23. *Let Δ be a two-dimensional lattice polygon and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly Δ-nondegenerate Laurent polynomial. Then the canonical graded Betti table of C_f at most depends on the coefficients of f that are supported on $\partial\Delta \cap \mathbf{Z}^2$.*

See again Chapter 8 for a proof. As a modest new application of this, we obtain constancy of the canonical graded Betti table for triangles whose only lattice points on the boundary are its vertices. Indeed, using the action of $\mathbf{T}^2$ the three corresponding coefficients can always be set to 1.

Remark. If the answer to (ii) is no, then question (i) still makes sense by restricting to *sufficiently generic* weakly Δ-nondegenerate Laurent polynomials $f \in k[x^{\pm 1}, y^{\pm 1}]$.

(iii) **What does the graded Betti table of $X_{\Delta^{(1)}}$ look like?**

In order to have a combinatorial description of the canonical graded Betti table of C_f it does not suffice to merely relate it to the graded Betti numbers of $X_{\Delta^{(1)}}$: we also need to describe these numbers in a combinatorial way. This is a difficult question in its own right, with several partial results available in the existing literature, most notably in the Ph.D. thesis [Her06] of Hering (who in fact studied syzygies of toric varieties of arbitrary dimension). Much of our recent research time was devoted to complementing the existing statements, but the overall picture remains far from complete. Because of the independent interest we studied graded Betti numbers of *arbitrary* projectively embedded toric surfaces, i.e. not necessarily of the form $X_{\Delta^{(1)}}$. An overview of our findings can be found in the chart on the next page. For an accompanying discussion and proofs we refer to Chapter 6, but let us highlight two statements that can be viewed as analogues of Green's canonical syzygy conjecture. At the lower-left end of the graded Betti table we have:

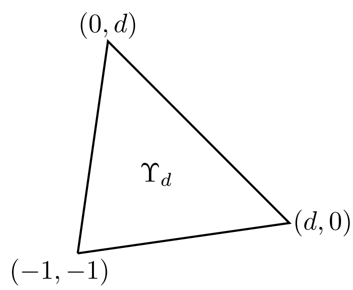
Theorem 24 (Hering, Schenck, Lemmens)**.** *Let Δ be a lattice polygon such that $\Delta^{(1)} \neq \emptyset$. The number of leading zeroes on row $q = 2$ of the graded Betti table of the toric surface X_Δ over k, counting from column index $p = 1$, is given by $\#(\partial\Delta \cap \mathbf{Z}^2) - 3$.*

(If $\Delta^{(1)} = \emptyset$ then the entire row $q = 2$ is trivial.) In characteristic zero the above theorem was proven by Hering [Her06], following an observation of Schenck [Sch04] and building on work of Gallego–Purnaprajna [GP01]. Recently Lemmens [Lem] gave a proof that works in arbitrary characteristic; he also provided an explicit formula for the first non-zero entry. At the upper-right end of the graded Betti table we conjecture:

Conjecture 25. *Let Δ be a two-dimensional lattice polygon such that $\Delta \not\cong \Sigma, \Upsilon$. The number of concluding zeroes on row $q = 1$ of the graded Betti table of the toric surface X_Δ over k, counting backwards from column index $p = \#(\Delta \cap \mathbf{Z}^2) - 3$, is given by $\mathrm{lw}(\Delta) - 1$, except if*

$$\Delta \cong d\Sigma \text{ for some } d \geq 2 \quad \text{or} \quad \Delta \cong \Upsilon_d \text{ for some } d \geq 2 \quad \text{or} \quad \Delta \cong 2\Upsilon \qquad (8)$$

in which case it is given by $\mathrm{lw}(\Delta) - 2$.



Graded Betti table of the toric surface X_Δ associated to a two-dimensional lattice polygon $\Delta \ncong \Sigma, \Upsilon$: known and conjectural facts.

Notation: $N_\Delta = \#(\Delta \cap \mathbf{Z}^2)$.

	0	1	2	3	4	5	…	$N_\Delta - 6$	$N_\Delta - 5$	$N_\Delta - 4$	$N_\Delta - 3$
0	1	0	0	0	0	0	…	0	0	0	0
1	0	b_1	b_2	b_3	b_4	b_5	…	$b_{N_\Delta-6}$	$b_{N_\Delta-5}$	$b_{N_\Delta-4}$	$b_{N_\Delta-3}$
2	0	$c_{N_\Delta-3}$	$c_{N_\Delta-4}$	$c_{N_\Delta-5}$	$c_{N_\Delta-6}$	$c_{N_\Delta-7}$	…	c_4	c_3	c_2	c_1

$b_1 = \binom{N_\Delta - 1}{2} - 2\mathrm{vol}(\Delta)$

$b_2 = 2\binom{N_\Delta - 1}{3} - 2(N_\Delta - 3)\mathrm{vol}(\Delta) + c_{N_\Delta-3}$

Conjecture: number of concluding zeroes on first row is $\mathrm{lw}(\Delta) - 1$, unless $\Delta \cong 2\Upsilon$, $\Delta \cong d\Sigma$ for some $d \geq 2$, or $\Delta \cong \Upsilon_d$ for some $d \geq 2$, in which case it is $\mathrm{lw}(\Delta) - 2$

Conjecture: if $\Delta \cong d\Sigma$ for some $d \geq 2$ then last non-zero b-value equals $d^3(d^2-1)/8$ *(no general conjecture)*

$b_{N_\Delta-4} = (N_\Delta - 4)B_\Delta$ where B_Δ is given by

$$\begin{cases} 0 & \text{if } \dim \Delta^{(1)} = 2 \text{ and } \Delta \ncong \Upsilon_2 \\ 1 & \text{if } \dim \Delta^{(1)} = 1 \text{ or } \Delta \cong \Upsilon_2 \\ (N_\Delta - 1)/2 & \text{if } \dim \Delta^{(1)} = 0 \\ N_\Delta - 2 & \text{if } \Delta^{(1)} = \emptyset \end{cases}$$

$b_{N_\Delta-3} = \begin{cases} 0 \text{ if } \Delta^{(1)} \neq \emptyset \\ N_\Delta - 3 \text{ else} \end{cases}$

c-row is zero iff $\Delta^{(1)} = \emptyset$

$c_1 = N_{\Delta^{(1)}}$

$c_2 = \max\left(0, (N_\Delta - 3)(N_{\Delta^{(1)}} - 1)\right)$

$c_3 = (N_\Delta - 4)\left((N_\Delta - 3)\mathrm{vol}(\Delta) - \binom{N_\Delta - 1}{2} + B_\Delta\right)$ with B_Δ as defined above

$c_{N_\Delta-3} = \begin{cases} 0 & \text{if } \#(\partial\Delta \cap \mathbf{Z}^2) > 3 \\ 1 & \text{if } \#(\partial\Delta \cap \mathbf{Z}^2) = 3 \text{ and } \dim \Delta^{(1)} = 2 \\ N_\Delta - 3 & \text{if } \#(\partial\Delta \cap \mathbf{Z}^2) = 3 \text{ and } \dim \Delta^{(1)} \leq 1 \end{cases}$

number of leading zeroes on second row is $\#(\Delta \cap \mathbf{Z}^2) - 3$ (Hering – Schenck in char 0, Lemmens in gen'l)

if $\Delta^{(1)} \neq \emptyset$ then first non-zero c-value equals $\binom{N_{\Delta^{(1)}} + t - 2}{N_{\Delta^{(1)}} - 1}$ with t the number of integral translates of $\Delta^{(1)}$ that fit inside Δ (Lemmens)

antidiagonal differences $b_i - c_{N_\Delta - 1 - i} = i\binom{N_\Delta - 1}{i+1} - 2\binom{N_\Delta - 3}{i-1}\mathrm{vol}(\Delta)$

(If $\Delta \cong \Sigma, \Upsilon$ then the entire row $q = 1$ is trivial.) We have the following evidence in favour of this conjecture:

Theorem 26. *Let Δ be a two-dimensional lattice polygon such that $\Delta \not\cong \Sigma, \Upsilon$. The number of concluding zeroes on row $q = 1$ of the graded Betti table of the toric surface X_Δ over k is at most the quantity predicted by Conjecture 25. Moreover:*

- *If $\operatorname{char} k = 0$ and $\#(\Delta \cap \mathbf{Z}^2) \leq 32$ then equality holds.*
- *If $\operatorname{char} k = 0$, $\Delta = \Gamma^{(1)}$ for a larger lattice polygon Γ, and Green's canonical syzygy conjecture holds for some weakly Γ-nondegenerate curve (e.g. if X_Δ carries an anti-canonical pencil [LC13]), then equality holds.*

  In particular, if $\operatorname{char} k = 0$ and X_Δ is Gorenstein and weak Fano then equality holds.
- *If equality holds for a certain instance of Δ not among (8), then it also holds for every lattice polygon containing Δ and having the same lattice width.*

  Using [CL] and (a) it follows that if $\operatorname{char} k = 0$ and $\operatorname{lw}(\Delta) \leq 6$ then equality holds.

For proofs we refer to Chapter 6, although let us note that the first claim was obtained using an explicit determination of the relevant entries in the graded Betti table of X_Δ, for all two-dimesional lattice polygons Δ containing at most 32 lattice points, using our database from Chapter 2. For reasons of efficiency the computation was carried out in finite characteristic, leading to the stated result through a semi-continuity argument.

Remark. The underlying algorithm can be used to gather all sorts of related data. It explicitly computes the Koszul cohomology of X_Δ, using duality, the action of $\mathbf{T}^2$ and some of the features stated in the chart on the previous page to reduce the time and memory requirements. It was implemented in SageMath and for instance allowed us to explicitly determine the graded Betti numbers of the Veronese surface $X_{6\Sigma} = \nu_6(\mathbf{P}^2)$ in characteristic $40\,009$; we expect this to match with characteristic zero. Up to $\nu_5(\mathbf{P}^2)$ these data were recently gathered by Greco and Martino [GM16].

We end this section with two applications. Recall that Green's conjecture helped us in settling special instances of Conjecture 25. But there is also an implication in the opposite direction: the instances of Conjecture 25 that were established through explicit computation in turn imply new cases of Green's conjecture.

Theorem 27. *Let $\operatorname{char} k = 0$, let X/k be a toric surface, and let $C \subseteq X$ be a non-hyperelliptic smooth projective curve of genus $4 \leq g \leq 32$. Then Green's canonical syzygy conjecture is true for C.*

Omitting exceptional cases, the proof uses that

$$a_{g-\operatorname{lw}(\Delta^{(1)})-1} = b_{g-\operatorname{lw}(\Delta^{(1)})-1}$$

as soon as

$$c_{g-\operatorname{lw}(\Delta^{(1)})-1} = 0, \tag{9}$$

which is immediate from the exact sequence (4). From Theorem 24 we know that (9) holds if and only if $\#(\partial\Delta^{(1)} \cap \mathbf{Z}^2) \geq \operatorname{lw}(\Delta^{(1)}) + 2$. Using our database of lattice polygons we computationally verified that this last inequality is true whenever $\#(\Delta^{(1)} \cap \mathbf{Z}^2) \leq 32$, except if $\Delta^{(1)} \cong \Upsilon$. The result then follows from Theorem 26, which says that $b_{g-\operatorname{lw}(\Delta^{(1)})-1} = 0$, and our combinatorial interpretation for the Clifford index stated in Theorem 17, which says that Green's conjecture amounts to $a_{g-\text{lw}(\Delta^{(1)})-1} = 0$. We refer to Chapter 8 for additional details.

Remark. In higher genus there exist more counterexamples to $\#(\partial\Delta^{(1)} \cap \mathbf{Z}^2) \geq \text{lw}(\Delta^{(1)}) + 2$, with $\Delta = \text{conv}\{(4,0),(10,4),(0,10)\}$ being the smallest instance that we have found, corresponding to $g = 36$. Here $\#(\partial\Delta^{(1)} \cap \mathbf{Z}^2) = 9$ and $\text{lw}(\Delta^{(1)}) = 8$, and as a consequence $c_{27} \neq 0$. In this case Green's canonical syzygy conjecture amounts to $a_{27} = 0$, but unlike the foregoing cases it is insufficient to verify that $b_{27} = 0$. In fact, if the sum formula $a_{27} = b_{27} + c_{27}$ from Theorem 21 would be true here (which we do not think it is), then this would show that $a_{27} \neq 0$ and hence that weakly Δ-nondegenerate curves are counterexamples to Green's conjecture!

A second application is concerned with the gonality conjecture due to Green and Lazarsfeld [GL86], which was recently proven by Ein and Lazarsfeld [EL15].

Theorem 28 (Gonality conjecture, proven by Ein–Lazarsfeld). *Let* $\text{char}\, k = 0$ *and let* C/k *be a smooth projective curve of gonality* $\gamma \geq 2$. *Let* L *be a globally generated divisor on* C *of sufficiently large degree, and assume that* $C \subseteq \mathbf{P}^{\text{rk}|L|}$ *is embedded using the linear system* $|L|$. *Then the number of non-zero entries on row* $q = 1$ *of the graded Betti table of the homogeneous coordinate ring of* C *equals* $h^0(C, L) - \gamma - 1$.

Concretely Ein and Lazarsfeld showed that $\deg L \geq g^3$ is sufficient, a bound which was recently improved to $\deg L \geq 4g - 3$ by Rathmann [Rat]. It is expected that this is not yet optimal, although Green and Lazarsfeld already noted that one needs at least $\deg L \geq 2g + \gamma - 1$. In a first draft of [FK] Farkas and Kemeny speculated that the latter bound might always be sufficient. However we show:

Theorem 29. *For each* $\gamma \geq 3$ *there exists a curve* C/k *of genus* $g = \gamma(\gamma-1)/2$ *along with a very ample divisor* L *of degree* $2g + \gamma - 1$ *such that the number of non-zero entries on row* $q = 1$ *of the graded Betti table of the homogeneous coordinate ring of the correspondingly embedded curve is at least* $h^0(C, L) - \gamma$.

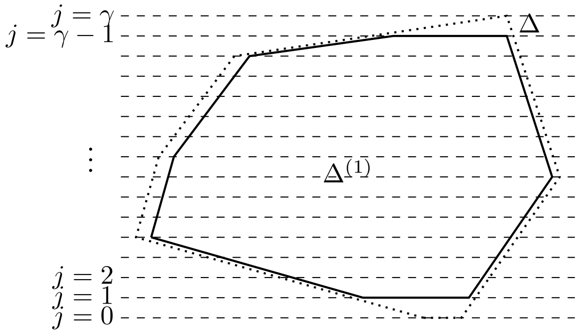
The curve C we construct is weakly $\Upsilon_{\gamma-1}$-nondegenerate, and its exceptional behaviour is tightly connected with the fact that $\Upsilon_{\gamma-1}$ is exceptional for Conjecture 25. We refer to Chapter 7 for the details.

6. Scrollar ruling degrees (Chapters 9 and 10)

As in the previous section we consider a lattice polygon Δ with two-dimensional interior along with a weakly Δ-nondegenerate Laurent polynomial $f \in k[x^{\pm 1}, y^{\pm 1}]$. Write $g = \#(\Delta^{(1)} \cap \mathbf{Z}^2)$ and let $\gamma \geq 3$ denote the lattice width $\text{lw}(\Delta)$. Assume that the latter equals the gonality of C_f, or in other words that there exists at least one combinatorial gonality pencil g_v.

Remark. In view of the material from Chapter 5 this means that we exclude $\Delta \cong 2\Upsilon$ and $\Delta \cong d\Sigma$ for any $d \geq 4$. However, in the last case one can circumvent this by noting that a weakly $d\Sigma$-nondegenerate curve is also weakly $\text{conv}\{(1,0),(d,0),(0,d),(0,1)\}$-nondegenerate: simply replace f by $f(x + x_0, y + y_0)$ for some $(x_0, y_0) \in U_f$. This does not affect the interior polygon, in terms of which all results below are stated.

For convenience we assume that $v = (1,0)$ and that $\Delta \subseteq \mathbf{R} \times [0, \gamma]$; in particular $g_v = |\text{fiber of } (x,y) \mapsto x|$. This can always be achieved by means of a unimodular transformation. It is not hard to see that the rational normal scroll $S \subseteq \mathbf{P}^{g-1}$ swept out by g_v equals



the $(\gamma - 1)$-dimensional toric variety associated to the polytope that one obtains from $\Delta^{(1)}$ by 'forgetting' that horizontal lines are coplanar, which amounts to omitting certain defining equations; we refer to Chapter 5 for more details. In fact this observation is the central ingredient in the proof of Theorem 15, which in our case states that the scrollar invariants are given by the width invariants

$$E_j := i^{(+)}(j) - i^{(-)}(j)$$

for $j = 1, \ldots, \gamma - 1$, where

$$i^{(-)}(j) = \min\{i \in \mathbf{Z} \,|\, (i, j) \in \Delta^{(1)}\} \quad \text{and} \quad i^{(+)}(j) = \max\{i \in \mathbf{Z} \,|\, (i, j) \in \Delta^{(1)}\}.$$

For our current purposes an important conclusion is that the series of inclusions (4) extends to $C_f^{\text{can}} \subseteq X_{\Delta^{(1)}} \subseteq S \subseteq \mathbf{P}^{g-1}$.

Recall that in Chapter 4 we provided a recipe for obtaining a minimal set of generators of the canonical ideal of C_f. In this section we report on a similar method for determining a minimal set of defining equations for our canonical curve *relative to the scroll* S. This concept was introduced and made precise by Schreyer [Sch86]. Informally spoken, the goal is to realize C_f^{can} as the scheme-theoretic intersection of as few divisors on S as possible. Modulo linear equivalence these divisors can be expressed as linear combinations of the hyperplane section class H and the ruling class R, which generate the Picard group. The coefficient of H matches with our intuitive notion of degree and therefore has little added value. But the coefficient of R can give interesting new discrete information.

Example. In the trigonal case C_f^{can} is a divisor itself. It concerns a 'cubic', as it is contained in the class $3H - (g - 4)R$.

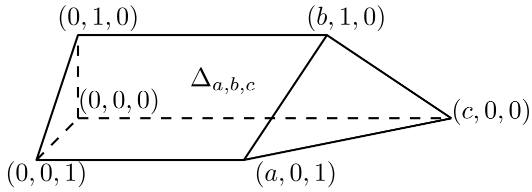
Subtlety. If one of the rational normal curves spanning S is of degree 0 (i.e. is a point) then the Picard group may not be *freely* generated by H and R [Fer01]. To give a rather degenerated example, consider $\Delta^{(1)} = \Sigma$: here the scroll is just $\mathbf{P}^2$ where $R = H$, and we recover that $C_f^{\text{can}} \in 3H + R = 4R$ is a plane quartic. To avoid non-unique expansions and various other theoretical issues one should actually work with the strict transforms of C_f^{can} and $X_{\Delta^{(1)}}$ under the natural birational morphism $j : S' \to S$ induced by increasing the degrees of the spanning rational normal curves by some fixed positive amount, but we will ignore this technicality here.

From $\gamma \geq 4$ on it turns out that our curve is minimally cut out by $(\gamma^2 - 3\gamma)/2$ 'quadrics', i.e. divisors whose classes are of the form

$$2H - b_1 R, \quad 2H - b_2 R, \quad 2H - b_3 R, \quad \ldots, \quad 2H - b_{(\gamma^2 - 3\gamma)/2} R$$

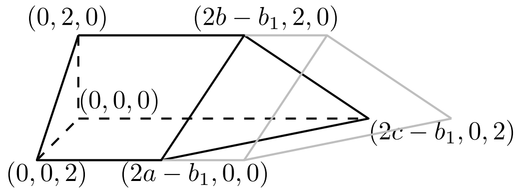
for integers b_i that sum up to $(\gamma-3)(g-\gamma-1)$ and which turn out to be independent of the chosen divisors, up to order. See [BH15; Sch86] for more details. We call these numbers the **scrollar ruling degrees**[3] with respect to g_v.

We first concentrate on the case $\gamma = 4$, which is particularly interesting. Assume that g_v has scrollar invariants $0 \le a \le b \le c$, so that $S \subseteq \mathbf{P}^{g-1}$ is the toric threefold associated to the polytope $\Delta_{a,b,c}$ depicted below. Note that we can view it as a completion of affine



space $\mathbf{A}^3$ rather than just the torus $\mathbf{T}^3$. Inside S our curve C_f^{can} arises as a complete intersection of two divisors Y, Z whose respective classes are $2H - b_1R$ and $2H - b_2R$ with $b_1 + b_2 = g - 5$, where we can assume that $b_1 \ge b_2$. It can be shown that $b_2 \ge -1$, where equality occurs if and only if C_f is isomorphic to a smooth plane quintic [Sch86, §6], i.e. if and only if $\Delta^{(1)} \cong 2\Sigma$.

In terms of defining equations this means that $Y \cap \mathbf{A}^3$ is defined by a polynomial $f_Y \in k[x,y,z]$ which is supported on the horizontally shrunk version of $2\Delta_{a,b,c}$ shown below (and that this is no longer true if we shrink it further). The analogous claim applies



to the polynomial $f_Z \in k[x,y,z]$ associated to Z. Since Z corresponds to the bigger polytope it moves in a family: one is free to replace f_Z by $f_Z + gf_Y$ for some $g \in k[x]$ of degree at most $b_1 - b_2$. On the other hand if $b_1 > b_2$ then Y is immovable. In fact Schreyer proved that the invariants b_1, b_2 are independent of the chosen g^1_4 and that the same is true for the surface Y as soon as $b_1 > b_2$. The main result of Chapter 9 is:

Theorem 30. *Let Δ be a two-dimensional lattice polygon, let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be weakly Δ-nondegenerate, and assume that C_f is tetragonal. Then the scrollar ruling degrees $\{b_1, b_2\}$ of C_f are given by*

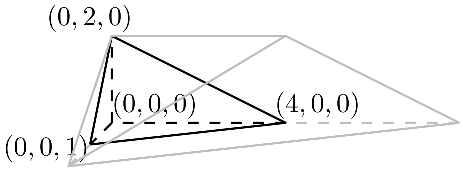
$$\left\{ \#(\partial\Delta^{(1)} \cap \mathbf{Z}^2) - 4 \, , \; \#(\Delta^{(2)} \cap \mathbf{Z}^2) - 1 \right\}.$$

Moreover if $\#(\partial\Delta^{(1)} \cap \mathbf{Z}^2) - 4 > \#(\Delta^{(2)} \cap \mathbf{Z}^2) - 1$ then the surface Y is given by $X_{\Delta^{(1)}}$.

The last equality is almost always satisfied, with all counterexamples to be found in Chapter 9.

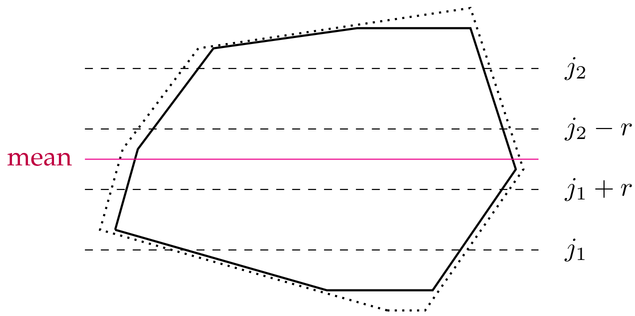
Example. Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly $\Delta_{4,7}$-nondegenerate Laurent polynomial. This is a $C_{4,7}$ curve, which by Baker's bound is of genus 9, and by Theorem 13 carries a unique g^1_4. According to Theorem 15 the corresponding scrollar invariants are $0, 2, 4$ and the above theorem says that $b_1 = 4$ and $b_2 = 0$. The surface Y, whose corresponding polytope is depicted above, is the toric surface $X_{\Delta^{(1)}_{4,7}}$, which on $\mathbf{A}^3$ is cut out by $f_Y = y^2 - z$.

[3] In Chapter 10 we have called these invariants the 'first scrollar Betti numbers', but since these numbers do not appear as dimensions of cohomology spaces, this terminology was up for improvement.



Next in Chapter 10 we work towards a combinatorial interpretation for the scrollar ruling degrees of a weakly nondegenerate curve of gonality $\gamma \geq 5$. For this we assume that Δ and v satisfy two technical conditions $\mathcal{P}_1(v)$ and $\mathcal{P}_2(v)$, which are explained in more detail below. Essentially these conditions impose on $\Delta^{(1)}$ a certain combinatorial compatibility between its 'left-hand side' and its 'right-hand side', i.e. between the numbers $i^{(-)}(j)$ and the numbers $i^{(+)}(j)$.

In a first phase we look for divisors that cut out our toric surface $X_{\Delta^{(1)}}$. Recall that S was obtained from $X_{\Delta^{(1)}}$ by forgetting that horizontal lines are coplanar, so the idea is to rivet these lines gradually back together. Concretely for each pair $j_1, j_2 \in \{1, 2, \ldots, \gamma - 1\}$ such that $j_2 - j_1 \geq 2$ we define a toric $(\gamma - 2)$-fold $D_{j_1,j_2} \subseteq S$ which reminds the scroll of the fact that the pair of lines at heights j_1, j_2 and the pair of lines at heights $j_1 + r, j_2 - r$ have the same 'mean':



Here $r \in \{1, 2, \ldots, (j_2 - j_1)/2\}$ should be chosen carefully, which is where the condition $\mathcal{P}_1(v)$ shows up. Concretely, for each r we define

$$\begin{aligned} \epsilon^{(-)}_{j_1,j_2,r} &= \begin{cases} 0 & \text{if} \quad i^{(-)}(j_1 + r) + i^{(-)}(j_2 - r) \leq i^{(-)}(j_1) + i^{(-)}(j_2) \\ 1 & \text{if} \quad i^{(-)}(j_1 + r) + i^{(-)}(j_2 - r) > i^{(-)}(j_1) + i^{(-)}(j_2) \end{cases} \\ &= \max\{0, (i^{(-)}(j_1 + r) + i^{(-)}(j_2 - r)) - (i^{(-)}(j_1) + i^{(-)}(j_2))\} \end{aligned}$$

and

$$\begin{aligned} \epsilon^{(+)}_{j_1,j_2,r} &= \begin{cases} 0 & \text{if} \quad i^{(+)}(j_1 + r) + i^{(+)}(j_2 - r) \geq i^{(+)}(j_1) + i^{(+)}(j_2) \\ 1 & \text{if} \quad i^{(+)}(j_1 + r) + i^{(+)}(j_2 - r) < i^{(+)}(j_1) + i^{(+)}(j_2) \end{cases} \\ &= \max\{0, (i^{(+)}(j_1) + i^{(+)}(j_2)) - (i^{(+)}(j_1 + r) + i^{(+)}(j_2 - r))\}. \end{aligned}$$

Condition $\mathcal{P}_1(v)$ imposes that

- either we can find an r for which $\epsilon^{(-)}_{j_1,j_2,r} = \epsilon^{(+)}_{j_1,j_2,r} = 0$, in which case we define $\epsilon_{j_1,j_2} = 0$,
- or there is no r for which $\epsilon^{(-)}_{j_1,j_2,r} = 0$ but there is an r for which $\epsilon^{(+)}_{j_1,j_2,r} = 0$, in which case we define $\epsilon_{j_1,j_2} = 1$,
- or, similarly, there is no r for which $\epsilon^{(+)}_{j_1,j_2,r} = 0$ but there is an r for which $\epsilon^{(-)}_{j_1,j_2,r} = 0$, in which case we define $\epsilon_{j_1,j_2} = 1$,

- or there is no r for which $\epsilon^{(-)}_{j_1,j_2,r} = 0$ and neither there is an r for which $\epsilon^{(+)}_{j_1,j_2,r} = 0$, in which case we define $\epsilon_{j_1,j_2} = 2$.

This should be true for all $(\gamma-2)(\gamma-3)/2$ pairs j_1, j_2. We refer to Chapter 10 for further details on the construction of D_{j_1,j_2}, for some clarifying examples, and eventually for a proof of the following statement:

Theorem 31. *Inheriting the notation from above and assuming that condition $\mathcal{P}_1(v)$ is satisfied, one has that the $(\gamma-2)(\gamma-3)/2$ divisors $D_{j_1,j_2} \subseteq S$ together cut out $X_{\Delta^{(1)}}$. Moreover*

$$D_{j_1,j_2} \in 2H - B_{j_1,j_2}R \text{ for all } j_1, j_2, \text{ where } B_{j_1,j_2} = E_{j_1} + E_{j_2} - \epsilon_{j_1,j_2},$$

and the B_{j_1,j_2}'s sum up to $(\gamma-4)g - (\gamma^2 - 3\gamma) + \#(\partial\Delta^{(1)} \cap \mathbf{Z}^2)$.

The next step is to add divisors that slice this further down to C_f^{can}. Recall from Section 2 that the canonical ideal of $C_f \subseteq \mathbf{P}^{g-1}$ is spanned by the ideal of $X_{\Delta^{(1)}}$ and the quadrics

$$Q_w = \sum_{(i,j)\in\Delta\cap\mathbf{Z}^2} c_{ij} X_{u_{ij}} X_{v_{ij}}, \qquad w \in \Delta^{(2)} \cap \mathbf{Z}^2,$$

where u_{ij}, v_{ij} should be chosen such that $(i,j) - w = (u_{ij} - w) + (v_{ij} - w)$. Typically the choice of u_{ij} and v_{ij} is not unique. Condition $\mathcal{P}_2(v)$ amounts to the existence for each horizontal line L of horizontal lines M_1 and M_2 such that it is possible to choose $u_{ij} \in M_1$ and $v_{ij} \in M_2$ for all $w \in L$. If this is indeed possible then we obtain our requested divisors by grouping together all Q_w's that correspond to lattice points on the same horizontal line L. More precisely we define for each $j = 2, 3, \ldots, \gamma - 2$ a divisor

$$D_j := S \cap \{\, Q_w \mid w \in \Delta^{(2)} \cap \mathbf{Z}^2 \text{ lies at height } j \,\}.$$

We refer to Chapter 10 for explanation why this indeed results in a subscheme of codimension one (in contrast with what happens if one does not choose the u_{ij}'s and v_{ij}'s in a consistent way), and for a proof of the following statement:

Theorem 32. *Inheriting the notation from above and assuming that condition $\mathcal{P}_2(v)$ is satisfied, one has that the $\gamma - 3$ divisors $D_j \subseteq S$ together with $X_{\Delta^{(1)}} \subseteq S$ cut out C_f^{can}. Moreover*

$$D_j \in 2H - B_j R \text{ for all } j, \text{ where } B_j = -1 + \#\{\, i \in \mathbf{Z} \mid (i,j) \in \Delta^{(2)} \cap \mathbf{Z}^2 \,\},$$

and the B_j's sum up to $\#(\Delta^{(2)} \cap \mathbf{Z}^2) - (\gamma - 3)$.

Notice that the multi-set of B_j's equals the multi-set of width invariants $E(\Delta^{(1)}, v)$. Finally, by combining both results and noticing that

$$(\gamma-2)(\gamma-3)/2 + (\gamma-3) = (\gamma^2 - 3\gamma)/2,$$

we arrive at our desired interpretation of the scrollar ruling degrees:

Corollary 33. *Inheriting the notation from above and assuming that conditions $\mathcal{P}_1(v)$ and $\mathcal{P}_2(v)$ are satisfied, one has that the scrollar ruling degrees of C_f with respect to g_v are given by*

$$\{B_j\}_{j\in\{2,\ldots,\gamma-2\}} \cup \{B_{j_1,j_2}\}_{\substack{j_1,j_2\in\{1,\ldots,\gamma-1\} \\ j_2-j_1\geq 2}}.$$

These scrollar Betti numbers indeed add up to $(\gamma-3)(g-\gamma-1)$, as announced.

Remark. The conditions $\mathcal{P}_1(v)$ and $\mathcal{P}_2(v)$ are milder than one might fear at a first glance. In fact condition $\mathcal{P}_1(v)$ is void for $\gamma = 5$ and $\gamma = 6$ and we believe that the same is true for $\mathcal{P}_2(v)$, although we could not prove this. The smallest pair Δ, v violating $\mathcal{P}_2(v)$ that we managed to find corresponds to curves of genus 46 and gonality 10; see Chapter 10.

7. Arithmetic features (Chapter 11)

Consider a field k that is not necessarily algebraically closed. Let Δ be a two-dimensional lattice polygon and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a Laurent polynomial that is weakly Δ-nondegenerate when considered over $k^{\text{alg.cl.}}$. Then $C_f \subseteq X_\Delta$ is a smooth projective curve that is defined over k. In this section we wish to illustrate that besides geometric information, our polygon Δ potentially also contains some arithmetic data, even though we do not expect the existence of a large one-to-one arithmetic-combinatorics dictionary as in the geometric case.

A first arithmetic feature is that if an edge $\tau \subseteq \Delta$ has integral length one then the corresponding torus-invariant divisor $D_\tau \subseteq X_\Delta$ contains a k-rational point of C_f. The reason is that the intersection locus of C_f with D_τ is locally given by a linear equation with coefficients in k. Thus if there are many such edges then this yields meaningful lower bounds on $\#C_f(k)$. This was used by Kresch, Wetherell and Zieve to prove the following fact:

Theorem 34 (Kresch, Wetherell, Zieve [KWZ02]). *For every integer $g \geq 0$ and every prime power q define $N_q(g) := \max_C \#C(\mathbf{F}_q)$, where C ranges over all smooth projective curves of genus g over $\mathbf{F}_q$. Then $\lim_{g\to\infty} N_q(g) = \infty$. More precisely $\liminf_{g\to\infty} N_q(g)/g^{1/3} > 0$.*

This statement is no longer the best available: in 2004 the same authors, in cooperation with Elkies, Howe and Poonen [Elk+04], managed to replace the denominator $g^{1/3}$ by g, which is optimal in view of the Hasse-Weil bound. Unfortunately this relies on other techniques, but nevertheless Theorem 34 remains a beautiful application of smooth curves in toric surfaces.

Remark. More generally the presence of an edge $\tau \subseteq \Delta$ of integeral length r ensures the existence of a k-rational divisor of degree r. Using a classification due to Fisher [Fis08], we used this in Chapter 1 to prove that a genus one curve C/k is k-birationally equivalent to a weakly nondegenerate curve if and only if it has a k-rational divisor of degree at most 3.

A second arithmetic feature is that the k-rational gonality of C_f, by which we mean the minimal degree of a k-rational map $C_f \to \mathbf{P}^1$, equals its geometric gonality, except possibly if $\Delta \cong 2\Upsilon$ or if $\Delta \cong d\Sigma$ for some $d \geq 2$. This is a trivial consequence of Theorem 13, because combinatorial pencils are clearly k-rational. In particular C_f is hyperelliptic if and only if it is geometrically hyperelliptic.

Remark. By letting $k = \mathbf{C}((t))$, through specialization of divisors this gives (very) prudent support in the case of planar graphs in favour of a conjecture by Baker [Bak08, Conj. 3.14], saying that the gonality of a graph equals the gonality of its metrization.

If $\Delta \cong 2\Upsilon$ then the k-rational gonality equals the geometric gonality (namely 3) except if C_f canonically embeds into an elliptic quadric in $\mathbf{P}^3$ in which case it equals 4; the occurrence of this event may depend on the specific choice of f. If $\Delta \cong d\Sigma$ for some $d \geq 2$ then the k-rational gonality equals the geometric gonality (namely $d-1$) except if $\#C_f(k) = \emptyset$ in which case it equals d; again this may depend on the specific choice of f.

Chapter 11 is devoted to yet another arithmetic phenomenon, which has a geometric intake. One can verify that the canonical divisor K_Δ on C_f that one obtains from adjunction theory (see Section 1) equals

$$\sum_\tau (-\langle \nu_\tau, p_\tau \rangle - 1)(D_\tau \cap C_f), \tag{10}$$

where:

- the sum runs through all edges $\tau \subseteq \Delta$,
- D_τ denotes the torus-invariant divisor on X_Δ associated to τ,
- $\nu_\tau \in \mathbf{Z}^2$ is the primitive inward pointing normal vector to τ,
- p_τ is any point on $\tau \cap \mathbf{Z}^2$.

Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbf{R}^2$. It is the divisor of the differential

$$\frac{dx}{xy\frac{\partial f}{\partial y}},$$

unless $\partial f / \partial y = 0$, which happens if x is not a separating variable, in which case[4] one should exchange the role of x and y. Similarly one verifies that the set

$$\left\{ x^i y^j \frac{dx}{xy\frac{\partial f}{\partial g}} \right\}_{(i,j) \in \Delta^{(1)} \cap \mathbf{Z}^2} \tag{11}$$

is a basis of holomorphic differentials. We refer to [CDV06] for more details.

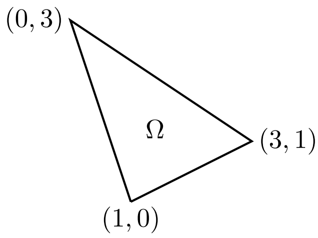
One observes that if all $\langle \nu_\tau, p_\tau \rangle$'s are odd then all coefficients in (10) are even, so that we obtain a theta characteristic Θ_{geom} by halving them. Note that Θ_{geom} is k-rational. If not all inner products are odd then it might be possible to achieve this by translating Δ over some $(i,j) \in \mathbf{Z}^2$, amounting to multiplying f by the monomial $x^i y^j$. If this is indeed possible then Δ is called **canonically even**. If it is moreover possible to do this in such a way that $(0,0)$ becomes contained in Δ, then Δ is called **effectively canonically even** because the resulting theta characteristic is effective. (Note that then $(0,0)$ is automatically contained in $\Delta^{(1)}$, otherwise one of the inner products would be zero, hence even.)

Remark. If translation over a vector $v \in \mathbf{Z}^2$ makes all $\langle \nu_\tau, p_\tau \rangle$'s odd, then so does translating over $v + w$ for any $w \in (2\mathbf{Z})^2$.

Examples. All triangles $\Delta_{a,b}$ where $\gcd(a,b,2) = 1$ are canonically even. If moreover $a, b \geq 2$ then they are effectively canonically even. This covers all smooth plane curves of odd degree and all $C_{a,b}$ curves. Also all triangles $\Delta_{2,2g+2}$ where $g \geq 3$ is odd are effectively canonically even; this corresponds to hyperelliptic curves of odd genus. The polygon Ω depicted below is an example of a lattice polygon which is canonically even but not effectively canonically even.

Now assume $\operatorname{char} k = 2$. Then C_f automatically carries another k-rational theta characteristic Θ_{arith}, which was introduced by Mumford [Mum71]. It is simply defined as $\Theta_{\text{arith}} := (\operatorname{div} dx)/2$, where we note that dx indeed has even orders of vanishing because

[4] This event is extremely unlikely but it can happen. Example: $f = y^2 + x^2 + x + 1$ in characteristic 2.



only even terms remain when differentiating a Laurent series in characteristic two.[5] A theorem by Stöhr and Voloch [SV87] says that $h^0(C_f, \Theta_{\text{arith}}) = g - r$, where g is the genus of C_f and r is the rank of the Cartier operator acting on the space of holomorphic differentials. It is well-known that $r = g$ if and only if C_f is an ordinary curve. This implies:

Lemma 35. *Let Δ be a two-dimensional effectively canonically even lattice polygon and let k be a field of characteristic 2. Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly Δ-nondegenerate Laurent polynomial. If C_f is ordinary then $\text{Jac}(C_f)$ carries a non-trivial k-rational 2-torsion point.*

Indeed, if C_f is ordinary then $h^0(C_f, \Theta_{\text{arith}}) = 0$ and therefore Θ_{arith} is not linearly equivalent to an effective divisor, in particular not to Θ_{geom}. Then the divisor $\Theta_{\text{arith}} - \Theta_{\text{geom}}$ maps to a non-trivial k-rational 2-torsion point on the Jacobian.

Alternatively, we obtain Lemma 35 as a corollary to the following stronger result, which is proven in Chapter 11 by explicit computation, using the basis (11).

Theorem 36. *Let Δ be a two-dimensional canonically even lattice polygon and let k be a field of characteristic 2. Let*

$$f = \sum_{(i,j) \in \Delta \cap \mathbf{Z}^2} c_{ij} x^i y^j \in k[x^{\pm 1}, y^{\pm 1}]$$

be a weakly Δ-nondegenerate Laurent polynomial. Let P be the set of vectors $(i, j) \in \Delta \cap \mathbf{Z}^2$ such that translating over $(-i, -j)$ makes all $\langle \nu_\tau, p_\tau \rangle$'s odd. Let $\rho = \#P$.

- *If $c_{i,j} \neq 0$ for at least one $(i, j) \in P$ then Θ_{arith} and Θ_{geom} are not linearly equivalent, and therefore $\text{Jac}(C_f)$ carries a non-trivial k-rational 2-torsion point.*
- *If $c_{i,j} = 0$ for all $(i, j) \in P$ then the rank of the Cartier operator is at most $g - \rho$.*

In particular if Δ is effectively canonically even, then $\rho > 0$ and

$$C_f \text{ ordinary} \quad \Rightarrow \quad c_{i,j} \neq 0 \ \text{ for some } (i,j) \in P \quad \Rightarrow \quad \text{Jac}(C_f)(k)[2] \neq 0.$$

As a consequence, in characteristic two, a sufficiently generic Laurent polynomial that is supported on an effectively canonically even lattice polygon defines a curve with a non-trivial k-rational two-torsion point on its Jacobian. This observation was first made by Cais, Ellenberg and Zureick-Brown in the case of smooth plane curves of odd degree [CEZB13]. In the case of $\Delta_{a,b}$ with $\gcd(a, b) = 1$ this explains why Denef and Vercauteren [DV06] had to tolerate a factor 2 in $\#\text{Jac}(C)(k)$ when trying to generate cryptographically secure $C_{a,b}$ curves C over finite fields of characteristic two.

We actually conjecture that under the assumptions of the theorem the rank of the Cartier operator is *at least* $g - \rho$, where equality holds if and only if $c_{i,j} = 0$ for all $(i, j) \in P$. Chapter 11 contains proofs of this conjecture for $\Delta \cong \Delta_{2g+2,2}$ with g odd (hyperelliptic curves of odd genus), for $\Delta \cong d\Sigma$ with d odd (smooth plane curves of odd degree), and also for $\Delta \cong \Omega$. In the latter case $\rho = 0$, so this converts into the following fact:

[5] If x is not a separating variable then $dx = 0$, in which case we again exchange the role of x and y.

Lemma 37. *Let k be a field of characteristic 2 and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be weakly Ω-nondegenerate. Then C_f is ordinary.*

8. Intrinsicness (Chapters 5, 9 and 10)

In this section we reinstall the assumption that k is an algebraically closed field. Let Δ be a two-dimensional lattice polygon and let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly Δ-nondegenerate Laurent polynomial. Given the long list of geometric invariants that can be told from the combinatorics of Δ, one can wonder to what extent it is possible to recover the polygon *itself* from the abstract birational geometry of U_f (or of C_f). The best one can hope for is to find back Δ up to unimodular equivalence, because unimodular transformations correspond to automorphisms of $\mathbf{T}^2$. Another relaxation is that (usually) one can only expect to recover $\Delta^{(1)}$, rather than all of Δ. For example, recall from the first remark in Section 6 that every weakly $d\Sigma$-nondegenerate Laurent polynomial is also weakly Δ-nondegenerate, where Δ is obtained from $d\Sigma$ by clipping off the point $(0,0)$. More generally, pruning a vertex off a two-dimensional lattice polygon Δ without affecting its interior boils down to forcing the curve through a certain non-singular point of X_Δ, which is usually not an intrinsic property. One is naturally led to the following definition.

Definition 38. *Let Δ be a two-dimensional lattice polygon and let C/k be a weakly Δ-nondegenerate curve. We say that $\Delta^{(1)}$ is* ***intrinsic to C*** *if for all two-dimensional lattice polygons Δ' for which C is weakly Δ'-nondegenerate it holds that $\Delta^{(1)} \cong \Delta'^{(1)}$. We say that $\Delta^{(1)}$ is* ***intrinsic*** *if it is intrinsic to every weakly Δ-nondegenerate curve.*

A few first cases in which $\Delta^{(1)}$ is intrinsic are:

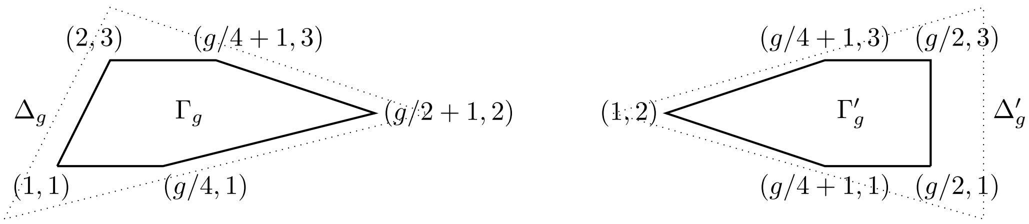
- $\Delta^{(1)} = \emptyset$, which occurs if and only if C_f is rational,
- $\dim \Delta^{(1)} = 0$, which holds if and only if C_f is elliptic,
- $\dim \Delta^{(1)} = 1$, which holds if and only if C_f is hyperelliptic of genus $\#(\Delta^{(1)} \cap \mathbf{Z}^2)$,
- $\Delta^{(1)} \cong (d-3)\Sigma$ for some $d \geq 3$, which by Corollary 18 occurs if and only if C_f is birationally equivalent to a smooth projective plane curve of degree d.

From Theorem 17 we see that if $\operatorname{char} k = 0$ then also $\Delta^{(1)} \cong 2\Upsilon$ is intrinsic, because this occurs if and only if C_f is of Clifford index 3. Most likely this result is also true in positive characteristic.

As with many statements in this manuscript, the case where $\Delta^{(1)} \cong \Upsilon$ turns out to be an exception. Indeed, recall from Theorem 10 that every genus 4 curve is either weakly $\square_{3,3}$-nondegenerate or weakly $\Delta_{6,3}$-nondegenerate. The respective interiors of these polygons are $\square_{1,1}$ and $\Delta_{2,1}$, while Υ is equivalent to neither of both. Since both cases occur, this turns all three interior genus 4 polygons $\square_{1,1}, \Delta_{2,1}, \Upsilon$ into exceptions.

More counterexamples. Our polygon Υ belongs to a larger family of counterexamples. Let $g \geq 4$ satisfy $g \equiv 0 \bmod 4$, and consider the lattice polygons Γ_g and Γ'_g depicted below, which are non-equivalent. We note that $\Gamma_4 \cong \Upsilon$. If $\operatorname{char} k = 0$ or $\operatorname{char} k > g/2 + 1$ then the polynomials $f = 1 - x^2y^4 - x^{\frac{g}{2}+2}y^2$ and $f' = (y^4 - 1)x^{\frac{g}{2}+1} + 4y^2$ are weakly Δ_g-nondegenerate and weakly Δ'_g-nondegenerate, respectively. Here Δ_g and Δ'_g are as depicted above and satisfy $\Delta_g^{(1)} = \Gamma_g$ and $\Delta_g'^{(1)} = \Gamma'_g$. Since the rational maps

$$U_f \to U_{f'} : (x,y) \mapsto \left(x, \frac{1 - xy^2}{x^{\frac{g}{4}+1}y} \right)$$



$$U_{f'} \to U_f : (x,y) \mapsto \left(x, \frac{2y}{x^{\frac{g}{4}+1}(1+y^2)} \right)$$

are inverses of each other, we conclude that C_f and $C_{f'}$ are isomorphic. Therefore Γ_g is not intrinsic to C_f, and neither is Γ'_g.

In spite of these exceptions we believe that 'most' interior lattice polygons are intrinsic, but making this statement precise (let along proving this) seems to be a hard task. Using Theorem 15 and Theorem 30 we can settle some additional cases, though:

- $\#(\Delta^{(1)} \cap \mathbf{Z}^2) \geq 5$ and $\Delta^{(2)} = \emptyset$, which holds if and only if C_f is trigonal of genus $g \geq 5$, or isomorphic to a smooth plane quintic,
- $\mathrm{lw}(\Delta^{(1)}) = 2$ and $\#(\partial\Delta^{(1)} \cap \mathbf{Z}^2) \geq \#(\Delta^{(2)} \cap \mathbf{Z}^2) + 5$, which holds if and only if C_f is tetragonal and $b_1 \geq b_2 + 2$.

In both cases, the bare line of accompanying text does not suffice to conclude intrinsicness: more details and refined statements can be found in Chapter 9. Let us remark that in both situations $X_{\Delta^{(1)}}$ can be easily recovered from the canonical model $C_f^{\mathrm{can}} \subseteq \mathbf{P}^{g-1}$. Indeed, in the former case it arises as the intersection of all quadrics containing C_f^{can}. In the latter case it is the unique surface containing C_f^{can} that is linearly equivalent to $2H - b_1 R$, when viewed as a divisor inside the scroll spanned by a g^1_4. Our most subtle intrinsicness result, which strongly relies on Corollary 33, is:

Theorem 39. *Let $a, b \geq 1$ be integers that are not both equal to 1. Then the interior polygon $\Box_{a,b}$ is intrinsic. More precisely let Δ be a two-dimensional lattice polygon, let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be a weakly Δ-nondegenerate Laurent polynomial, and assume that U_f is birationally equivalent to a smooth projective curve in $\mathbf{P}^1 \times \mathbf{P}^1$ of bidegree $(a+2, b+2)$. Then $\Delta^{(1)} \cong \Box_{a,b}$.*

A proof can be found in Chapter 10.

Remark. One can also target weaker intrinsicness questions, by only distinguishing between polygons that belong to some given family:

- A weakly $\Delta_{a,b}$-nondegenerate curve cannot be weakly $\Delta_{a',b'}$-nondegenerate for distinct pairs of coprime positive integers $\{a, b\}$ and $\{a', b'\}$. This is immediate from our combinatorial interpretations for the genus

  $$\#(\Delta^{(1)}_{a,b} \cap \mathbf{Z}^2) = (a-1)(b-1)/2$$

  and the gonality

  $$\mathrm{lw}(\Delta^{(1)}_{a,b}) + 2 = \mathrm{lw}(\Delta_{a,b}) = \min\{a, b\}.$$

  In other words a $C_{a,b}$ curve cannot be $C_{a',b'}$.
- A similar reasoning involving the scrollar invariants shows that if a smooth non-hyperelliptic curve C/k of genus $g \geq 2$ can be embedded in the n^{th} Hirzebruch surface $\mathcal{H}_n$ for some $n \geq 0$, then this value of n is unique and can therefore be considered an invariant of C. We refer to Chapter 5 for an elaboration of the details.

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