Low frequency estimates and local energy decay for asymptotically Euclidean Laplacians

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Abstract

For Riemannian metrics G on $\mathbb{R}^d$ which are long range perturbations of the flat one, we prove estimates for $(-\Delta_G - \lambda - i\epsilon)^{-n}$ as $\lambda \to 0$, which are uniform with respect to ϵ, for all $n \leq [d/2]+1$ in odd dimension and $n \leq d/2$ in even dimension. We also give applications to the time decay of Schrödinger and Wave (or Klein-Gordon) equations.

1 Introduction and results

Let $G = (G^{jk})$ be a Riemannian metric on $\mathbb{R}^d$ which is asymptotically Euclidean in the sense that, for some $\rho > 0$,

$$|\partial^\alpha (G^{jk}(x) - \delta_{jk})| \leq C_\alpha \langle x \rangle^{-\rho - |\alpha|}, \tag{1.1}$$

δ_{jk} being the Kronecker symbol. In other words, (the coefficients of) $G - I$ belongs to the symbol class $S^{-\rho}$ of functions such that $|\partial^\alpha a(x)| \leq C_\alpha \langle x \rangle^{-\rho-|\alpha|}$. In the sequel we shall also refer to G as a long range metric. The Laplacian Δ_G reads

$$\Delta_G = \det\, G(x)^{-1/2} \frac{\partial}{\partial x_j} \left(\det\, G(x)^{1/2} G_{jk}(x) \frac{\partial}{\partial x_k} \right), \tag{1.2}$$

using the summation convention as well as the standard notation $(G_{jk}) := (G^{jk})^{-1}$, and is (formally) self-adjoint with respect to the measure

$$d_G x = \det\, G(x)^{1/2} dx.$$

Since $\det\, G(x)^{1/2}$ is bounded from above and below, the spaces $L^2(\mathbb{R}^d, dx)$ and $L^2(\mathbb{R}^d, d_G x)$ coincide and have equivalent norms. We will thus use the unambiguous notation $L^2(\mathbb{R}^d)$ (or L^2) in the sequel. By Δ_G we will also denote the self-adjoint realization of (1.2), whose domain is H^2.

We are interested in the low frequency estimates for powers of the resolvent of $-\Delta_G$, namely the behaviour of

$$(-\Delta_G - z)^{-n} \quad \text{as } z \text{ approaches } 0,$$

in suitably weighted L^2 or L^p spaces, and their applications to time dependent equations.

The purpose of this paper is twofold. The first one is to prove in detail the resolvent estimates announced in [3] (note however that the sketches of proofs therein concern operators in divergence form) and the second one is to derive applications to the local energy decay for wave equations (which were not considered in [3]).

We first consider resolvent estimates. The study of the limiting absorption principle, namely the behaviour of (powers of) the resolvent of self-adjoint operators as the spectral parameter approaches the absolutely continuous spectrum is a basic problem in scattering theory and there is a huge literature on this topic which we can not review here. For the operators considered in this paper (and more general Schrödinger operators), the analysis of $(-\Delta_G - z)^{-n}$ is rather well known as long as $\mathrm{Re}(z)$ remains away from 0; by the results of [22, 21] (and those of [23] to ensure that $-\Delta_G$ has no embedded eigenvalues in its (absolutely continuous) spectrum $[0,\infty)$), we know that, for any $I \Subset (0,\infty)$ and $n \geq 1$, the limits $\lim_{\epsilon \to 0^{\pm}} (-\Delta_G - \lambda - i\epsilon)^{-n}$ exist as bounded operators between dual weighted L^2 spaces, provided that $\lambda \in I$. The asymptotics as $\lambda \to +\infty$ have also been widely studied in various contexts, perhaps more for the resolvent itself than for its powers, but this is not a serious restriction since, in the high energy or semiclassical regime, one can get estimates for powers in terms of estimates of the resolvent (see [20, 21] and Subsection 6.2 below): basically $||(-\Delta_G - \lambda - i0)^{-n}||'$ grows as $||(-\Delta_G - \lambda - i0)^{-1}||^n$, if $|| \cdot ||$ and $|| \cdot ||'$ are operator norms between suitable weighted L^2 spaces. In this regime, the asymptotics depend crucially on whether the geodesic flow is non trapping, namely if all geodesics escape to infinity as time goes to infinity, or trapping: see [32, 40, 16, 31, 38] for the non trapping case, [26, 28] for weak trapping, and [6, 7, 8] in the general case, ie without condition on the geodesic flow.

The situation is definitely different as $\mathrm{Re}(z) \to 0$. At first, we note that the geodesic flow plays no role in this non semiclassical regime. More importantly, there is no hope to deduce bounds on powers of the resolvent from bounds on the resolvent $(-\Delta_G - z)^{-1}$ as above. We know indeed that $(-\Delta_G - z)^{-1}$ remains bounded for z close to 0 (see [2, 5, 17] for the long range metric case) but, as we shall see below, its powers start to blow up as $z \to 0$ if n is large enough (essentially $n > d/2$). This can be seen on the example of the flat Laplacian on $\mathbb{R}^3$ whose kernel of the resolvent reads

$$G_z(x,y) = \frac{1}{4\pi} \frac{e^{iz^{1/2}|x-y|}}{|x-y|}, \qquad \mathrm{Im}(z^{1/2}) > 0.$$

Indeed, since $(-\Delta - z)^{-2} = \frac{d}{dz}(-\Delta - z)^{-1}$, we see that for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$||(-\Delta - z)^{-1}||_{L^{6/5} \to L^6} \leq C, \qquad ||(-\Delta - z)^{-2}||_{L^1 \to L^\infty} \approx |z|^{-1/2}, \tag{1.3}$$

where the first estimate follows from the Hardy-Littlewood-Sobolev inequality and the second one means that we have upper and lower bounds by constants times $|z|^{-1/2}$. Of course, such $L^p \to L^{p'}$ estimates imply weighted L^2 estimates using, in the present case, the boundedness of

$$\langle x \rangle^{-1-\varepsilon} : L^2(\mathbb{R}^3) \to L^{6/5}(\mathbb{R}^3), \qquad \langle x \rangle^{-\frac{3}{2}-\varepsilon} : L^2(\mathbb{R}^3) \to L^1(\mathbb{R}^3),$$

and their adjoints (for any $\varepsilon > 0$).

The literature on powers of the resolvent near the 0 energy is rather lacunary in the long range case. Actually, this topic seems to have been studied for Schrödinger operators $-\Delta + V$ only, in [27] for V of definite sign and in [15, 12] for V sufficiently negative at infinity (see also [42] in the radial case). We note that, for such potentials, the resolvent behaves differently from the free resolvent in that its powers are uniformly bounded as $\mathrm{Re}(z) \to 0^+$, unlike (1.3). Our first purpose is to show that, for variable coefficients metrics, we get the same kind of estimates as in the free case.

To state our results, we introduce the notation

$$\bar{r}(d) = \text{the largest integer strictly smaller than } d/2. \tag{1.4}$$

In other words $\bar{r}(d)$ is the integer part $[d/2]$ of $d/2$ if d is odd, and $\frac{d}{2} - 1$ if d is even. The notation r refers to the fact that it will be interpreted as some regularity index further on. Let us remark that, in all cases, $\bar{r}(d) \geq 1$. We also introduce the conjugate Lebesgue exponents

$$p(n) = \frac{2d}{d+2n}, \qquad q(n) = \frac{2d}{d-2n}, \qquad \text{for} \;\; 1 \leq n \leq \bar{r}(d), \tag{1.5}$$

which belong to $(1,\infty)$ since $n < d/2$ by definition of $\bar{r}(d)$. Finally, we denote by A the (self-adjoint realization of the) generator of L^2 dilations, namely

$$A = \frac{x \cdot \nabla}{i} + \frac{d}{2}. \tag{1.6}$$

Our main result is the following.

Theorem 1.1. *Fix $d \geq 3$. There exists $\kappa > 0$ and $C > 0$ such that, for $1 \leq n \leq \bar{r}(d)$,*

$$\big|\big|(\kappa A + i)^{-n}(-\Delta_G - z)^{-n}(\kappa A - i)^{-n}\big|\big|_{L^{p(n)} \to L^{q(n)}} \leq C, \qquad |\mathrm{Re}(z)| < 1,$$

and, for $n = N := \bar{r}(d) + 1$,

1. *if d is odd,*

$$\big|\big|(\kappa A + i)^{-N}(-\Delta_G - z)^{-N}(\kappa A - i)^{-N}\big|\big|_{L^1 \to L^\infty} \leq C|\mathrm{Re}(z)|^{-1/2},$$

2. *if d is even, then for all $q > 2d$ there exists $C_q > 0$ such that,*

$$\big|\big|(\kappa A + i)^{-N}(-\Delta_G - z)^{-N}(\kappa A + i)^{-N}\big|\big|_{L^{q/(q-1)} \to L^q} \leq C_q|\mathrm{Re}(z)|^{-\frac{2d}{q}},$$

both for $0 < |\mathrm{Re}(z)| < 1$.

Up to the weights $(\kappa A \pm i)^{-n}$, this theorem generalizes the estimates (1.3) to long range metrics in all dimensions greater than 2. In odd dimensions, our result is sharp from the point of view of the singularity at $z = 0$, as shown by (1.3). We also point out that for small long range perturbations of the flat Laplacian (when $G - I$ is small everywhere on $\mathbb{R}^d$, not only at infinity as imposed by (1.1)), such estimates actually hold for all z, ie also for large ones, and are scale invariant (see Subsection 5.1). We emphasize that, up to a scaling and a convenient choice of scaling covariant norms for the coefficients of the operators, the most important ingredient to prove this result is the Jensen-Mourre-Perry theory on multiple commutators estimates [22]. This theory gives general conditions under which one can prove weighted estimates for powers of the resolvent, under a positive commutator assumption. The issue we address is that we don't have such a (uniform) positive commutator estimate close to the bottom of the spectrum but, by a suitable scaling, our main observation is that we can reduce the problem to estimates close to energy 1 where one has a positive commutator.

The $L^p \to L^{p'}$ estimates of Theorem 1.1 can be turned into the following weighted $L^2 \to L^2$ estimates. In fact, as one can see from Subsection 5.2 below, Theorem 1.1 and the following one are equivalent.

Theorem 1.2. *Let $N = \bar{r}(d) + 1$. For all $1 \leq n \leq N$ and $\nu > 2n$, we have: for $n \leq N - 1$,*

$$\big|\big|\langle x \rangle^{-\nu}(-\Delta_G - z)^{-n}\langle x \rangle^{-\nu}\big|\big|_{L^2 \to L^2} \leq C, \qquad |\mathrm{Re}(z)| < 1, \tag{1.7}$$

and, for $n = N$,

1. if d is odd,

$$\big|\big|\langle x\rangle^{-\nu}(-\Delta_G - z)^{-N}\langle x\rangle^{-\nu}\big|\big|_{L^2\to L^2} \leq C|\mathrm{Re}(z)|^{-1/2},$$

2. if d is even, then for all $\epsilon > 0$,

$$\big|\big|\langle x\rangle^{-\nu}(-\Delta_G - z)^{-N}\langle x\rangle^{-\nu}\big|\big|_{L^2\to L^2} \leq C_\epsilon|\mathrm{Re}(z)|^{-\epsilon},$$

both for $0 < |\mathrm{Re}(z)| < 1$.

We implicitly assume that $\mathrm{Im}(z) \neq 0$ in these estimates but, since one knows that pointwise limits exist as $\pm\mathrm{Im}(z) \to 0^+$ (if $\mathrm{Re}(z) \neq 0$), this is not a restriction.

We now study the applications to the local energy decay. Let us consider

$$\begin{aligned} i\partial_t u + \Delta_G u &= 0, \qquad u_{|t=0} = u_0, & (1.8)\\ \partial_t^2 w - \Delta_G w + m^2 w &= 0, \qquad w_{|t=0} = f, \;\; \partial_t w_{|t=0} = g, & (1.9) \end{aligned}$$

which are respectively the Schrödinger equation and Klein-Gordon equation ($m > 0$) or wave equation ($m = 0$). We are interested in the decay as $t \to \infty$ of

$$||\chi u(t)||_{L^2}, \qquad ||\chi\partial_t w(t)||_{L^2} + ||\chi w(t)||_{H^1}, \tag{1.10}$$

for some spatial localization χ, typically $\chi \in C_0^\infty$ or more generally $\chi(x) = \langle x\rangle^{-\nu}$ for some $\nu > 0$. Using estimates on the resolvent alone (ie those for $n = 1$ in (1.7)), it is well known that one can recover $L^2(\mathbb{R}, dt)$ estimates for (1.10) which is a weak form of time decay (see for instance [33, 34, 25, 4, 37] in contexts close to ours). Proving quantitative decay rates requires more information, for instance estimates on powers of the resolvent as we recall now.

The flows of the equations (1.8) and (1.9) are functions of Δ_G namely,

$$u(t) = e^{it\Delta_G}u_0, \qquad w(t) = \cos\left(t\sqrt{m^2 - \Delta_G}\right)f + \frac{\sin\left(t\sqrt{m^2-\Delta_G}\right)}{\sqrt{m^2-\Delta_G}}g.$$

If we denote by $(E_\lambda)_{\lambda\in\mathbb{R}}$ the family of spectral projections associated to $-\Delta_G$, we have, for instance for the Schrödinger equation,

$$e^{it\Delta_G} = \int e^{-it\lambda}dE_\lambda, \tag{1.11}$$

where the spectral measure dE_λ can be recovered from the resolvent by the following Stone's formula (see [30] for a proof and Lemma 6.9 below for a precise statement)

$$dE_\lambda = \lim_{\epsilon\downarrow 0}\frac{1}{2i\pi}\left((-\Delta_G - \lambda - i\epsilon)^{-1} - (-\Delta_G - \lambda + i\epsilon)^{-1}\right)d\lambda.$$

Thus, by using the Stone formula in (1.11) and integration by parts, one expects to recover time decay for (1.10) from the smoothness of the resolvent with respect to λ, that is from the integrability in λ of $\lim_{\epsilon\to 0^+}(-\Delta_G - \lambda \pm i\epsilon)^{-n}$.

This is of course a rough formal description but this approach is well known and can be made rigorous (see Section 6), provided that the corresponding integrals are convergent with respect to λ. Its justification requires two types of estimates: high frequency estimates ($\lambda \to \infty$) and low frequency estimates ($\lambda \to 0$). As we recalled above, the high frequency estimates are often a delicate question, especially when there are trapped geodesics. But independently of this question, whatever the classical dynamic looks like and whatever methods are used to study the resolvent (*e.g.* resonances theory or Mourre theory), one also needs to deal with the low frequencies. For the local energy decay, this topic has been treated in the literature for fast decaying perturbations (from [24, 36, 6] for compactly supported perturbations to [41, 29] in the short range case, with radial assumptions in [29]). For long range metrics, there are either conditional results ([10] which assume that the resolvent can be continued accross the absolutely continuous spectrum near 0) or spectrally localized estimates ([9, 39] where the evolution $e^{it(-\Delta_G)^\alpha}$ is replaced by $e^{it(-\Delta_G)^\alpha}\psi(\Delta_G)$ with $\psi \equiv 0$ near 0). We should also mention the recent results for asymptotically flat space times [35] where a (sharp) pointwise energy decay is obtained for long range perturbations, which are radial up to short range terms (see also [14] for a purely radial situation).

Using mainly Theorem 1.2 and the known results on high energy estimates [8], we obtain the following general result.

Theorem 1.3. *Let $d \geq 3$ and assume (1.1). For all positive real numbers $s > 0$, $\nu > 0$ there exists $C > 0$ such that, for the Schrödinger equation (1.8),*

$$\left|\left|\langle x\rangle^{-\nu} u(t)\right|\right|_{L^2} \leq C\left(1+\log\langle t\rangle\right)^{-s} ||\langle x\rangle^{\nu} u_0||_{H^s},$$

and, for the wave and Klein-Gordon equations (1.9),

$$\left|\left|\langle x\rangle^{-\nu}\partial_t w(t)\right|\right|_{L^2} + \left|\left|\langle x\rangle^{-\nu} w(t)\right|\right|_{H^1} \leq C\left(1+\log\langle t\rangle\right)^{-s}\left(||\langle x\rangle^{\nu} f||_{H^{1+s}} + ||\langle x\rangle^{\nu} g||_{H^s}\right).$$

We emphasize that the main novelty in this result is that no spectral cutoff is needed on the initial data and that the metric G is long range. Furthermore, we don't use any spherical symmetry. The time decay is very weak, but on the other hand there is no assumption on the geodesic flow. This is an analogue of a result of Burq [6], obtained initially for compactly supported perturbations of the Laplacian and then generalized to long range perturbations but with spectrally localized initial data in [9].

Under the non trapping condition, we obtain the following stronger decay which we shall prove for the Schrödinger equation only. Here we use the notation (1.4).

Theorem 1.4. *Let $d \geq 3$, assume (1.1) and that the geodesic flow is non trapping. For all real numbers $\nu > 2(\bar{r}(d)+1)$ and $0 \leq s < \nu$, there exists $C > 0$ such that, for the Schrödinger equation (1.8),*

$$\left|\left|\langle x\rangle^{-\nu} u(t)\right|\right|_{L^2} \leq C\langle t\rangle^{-\bar{r}(d)}||\langle x\rangle^{\nu} u_0||_{H^{-s}}. \tag{1.12}$$

Notice that, in addition to the time decay, (1.12) also means that we have a smoothing effect (as is naturally expected for the Schrödinger equation with a non trapping metric).

As in Theorem 1.3, the main point in Theorem 1.4 is again the (non radial) long range assumption and the absence of spectral localization on the initial data. Besides we note that if one avoids the low frequencies, ie replaces $u(t)$ by $\Phi(\Delta_G)u(t)$ with $\Phi \equiv 0$ near 0 and smooth, one can show that $\langle x\rangle^{-\nu}\Phi(\Delta_G)u(t)$ decays as $t^{\epsilon-\nu}$, ie with time decay rate growing with the spatial decay rate. Thus we have a fast decay in time if $\langle x\rangle^{-\nu}$ is replaced by a Schwartz function. This illustrates the fact that, in the non trapping case, the time decay is governed by the low frequency part of the spectrum. From the free case, we also know that this decay cannot be more than $\langle t\rangle^{-d/2}$ and we note that, in odd dimension, $\langle t\rangle^{-d/2} = \langle t\rangle^{-\bar{r}(d)-\frac{1}{2}}$.

Remark. We comment that, in principle, our method would show for the wave and Klein-Gordon equations with non trapping metrics that

$$\left|\left|\langle x\rangle^{-\nu}\partial_t w(t)\right|\right|_{L^2} + \left|\left|\langle x\rangle^{-\nu} w(t)\right|\right|_{H^1} \leq C\langle t\rangle^{-\bar{r}(d)} \left(||\langle x\rangle^{\nu} f||_{H^1} + ||\langle x\rangle^{\nu} g||_{L^2}\right).$$

We leave this as a remark since its proof would require bounds of the form, for $\varphi \in C_0^\infty(0,+\infty)$,

$$||\langle x\rangle^{-\nu} e^{it(m^2-\Delta_G)^{1/2}} \varphi(-h^2\Delta_G)\langle x\rangle^{-\nu}|| \leq C_\epsilon \langle t\rangle^{\epsilon-\nu}, \qquad t\in\mathbb{R},\ h\in(0,1],$$

for all $\epsilon > 0$ (see the proof of Theorem 1.4 in Subsection 6.3). The point in this estimate is that it is uniform with respect to h and cannot be clearly deduced from resolvent estimates only. It could however certainly be obtained as the similar one for the Schrödinger equation proved by Wang [40] using the Isozaki-Kitada parametrix.

2 Model operators

In this section, we introduce a class of second order differential operators which are small perturbations of the flat Laplacian. They will serve as models at infinity, in the sense that the Laplacians (1.2) will be unitarily equivalent to compactly supported perturbations of such operators (see Subsection 5.2). These model operators are of the form

$$P = a_{jk}(x)D_jD_k + b_k(x)D_k, \tag{2.1}$$

where we use the summation convention and $D_k = i^{-1}\partial/\partial x_k$. We also assume that

$$P \text{ is formally symmetric on } L^2(\mathbb{R}^d) \text{ with respect to the Lebesgue measure.} \tag{2.2}$$

By formally symmetric, we mean symmetric when tested against functions of the Schwartz space $\mathcal{S}$. The coefficients will be chosen in the following spaces. For integers such that

$$N\geq 0, \qquad 0\leq o\leq 1, \qquad r\geq 0 \qquad \text{and} \qquad o+r\leq \bar{r}(d),$$

where $\bar{r}(d)$ is defined by (1.4), we introduce the norm

$$||a||_{o,r,N} = \sum_{\substack{n\leq N,\\ |\alpha|\leq r}} ||\partial^\alpha (x\cdot\nabla)^n a||_{L^{\frac{d}{o+|\alpha|}}},$$

and define

$$S^{o,r,N} = \{a\in C_b^\infty(\mathbb{R}^d) \mid ||a||_{o,r,N} < \infty\}.$$

These are essentially spaces of functions which are conormal at the origin (see [18] for a general definition of conormal distributions). To make it closer to usual definitions of conormal distributions, we note that, in dimension 3 for instance, it is not hard to check that

$$\{a\in C_b^\infty(\mathbb{R}^3) \mid (x\cdot\nabla)^n a\in H^2(\mathbb{R}^3),\ n\leq N\} \subset S^{0,1,N},$$

since $H^2(\mathbb{R}^3)\subset L^\infty(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)\subset L^3(\mathbb{R}^3)$.

In most statements of Sections 2 and 3, we shall also assume that P is a small perturbation of $-\Delta$ in the sense that $a_{jk}-\delta_{jk}$ (δ_{jk} = Kronecker symbols) and b_k will be small in appropriate spaces, explicitly given in each proposition. A first example of such a statement is the following.

Proposition 2.1. *For all $a_{jk}, b_k \in C_b^\infty$ such that $\sum_{j,k} ||a_{jk} - \delta_{jk}||_{L^\infty}$ is small enough, the operator $P : \mathcal{S} \to L^2$ has a bounded closure $\overline{P} : H^2 \to L^2$ and $\overline{P}$ is self-adjoint on L^2 with domain H^2.*

The proof of this proposition is completely standard and the existence of a self-adjoint realization holds under much more general assumptions. The smallness of $\sum_{j,k} ||a_{jk} - \delta_{jk}||_{L^\infty}$ is only used to ensure that the operator is uniformly elliptic. The assumption that the coefficients belong to C_b^∞ guarantees the existence of the closure of P to H^2 and the fact that the domain of $\overline{P}^*$ is also H^2 by elliptic regularity. This proposition has to be considered as an algebraic preliminary, which is convenient for it gives explicitly the domain of $\overline{P}$. But, as far as estimates are concerned, all the bounds obtained below will be given in terms of $S^{o,r,N}$ norms only, so the condition $a_{jk}, b_k \in C_b^\infty$ is essentially irrelevant.

Most of our estimates rely on the following elementary proposition.

Proposition 2.2. *There exists $C > 0$ depending only on the dimension d such that*

$$||a\partial_j\partial_k u||_{L^2} \leq C||a||_{0,1,0}||\Delta u||_{L^2}, \tag{2.3}$$

$$||b\partial_k u||_{L^2} \leq C||b||_{1,0,0}||\Delta u||_{L^2}, \tag{2.4}$$

and

$$||a\partial_j\partial_k u||_{H^{-1}} \leq C||a||_{0,1,0}||\nabla u||_{L^2}, \tag{2.5}$$

$$||b\partial_k u||_{H^{-1}} \leq C||b||_{1,0,0}||\nabla u||_{L^2}, \tag{2.6}$$

for all $u \in \mathcal{S}$, all $a \in S^{0,1,0}$ and $b \in S^{1,0,0}$.

Proof. We consider first (2.3) and (2.4). By a standard limiting argument we may assume that the Fourier transform of u vanishes near 0. Then

$$a(x)\partial_j\partial_k u = a(x)\frac{\partial_j\partial_k}{|D^2|}|D|^2 u,$$

so (2.3) follows from the bound $||a||_{L^\infty} \leq ||a||_{0,1,0}$ and the L^2 boundedness of $\partial_j\partial_k/|D|^2$. To prove (2.4), we use the Hölder inequality

$$||\varphi\psi||_{L^2} \leq ||\varphi||_{L^d}||\psi||_{L^{2^*}}, \tag{2.7}$$

and the Poincaré-Sobolev inequality

$$||\psi||_{L^{2^*}} \leq C||\nabla\psi||_{L^2}, \tag{2.8}$$

using the standard notation

$$2^* = 2d/(d-2).$$

Indeed, by writing

$$b_k(x)\partial_k u = b_k(x)\frac{1}{|D|}\frac{\partial_k}{|D|}|D|^2 u,$$

we get the result since on one hand $\partial_k/|D|$ is bounded on L^2 and on the other hand (2.7) and (2.8) yield

$$||b_k|D|^{-1}\psi||_{L^2} \leq ||b_k||_{L^d}|||D|^{-1}\psi||_{L^{2^*}} \leq C||b_k||_{L^d}||\psi||_{L^2},$$

for all $\psi \in \mathcal{S}$ with Fourier transform vanishing near 0. We now prove (2.6). The latter simply follows from the fact that

$$|(\psi, b\partial_k u)| \leq ||\bar{b}\psi||_{L^2}||\partial_k u||_{L^2} \leq C||b||_{L^d}||\nabla\psi||_{L^2}||\nabla u||_{L^2},$$

using again (2.7) and (2.8). Finally for (2.5), we write

$$a(x)\partial_j\partial_k = \partial_j(a(x)\partial_k) - (\partial_j a)(x)\partial_k,$$

for which the contribution of the first term follows from an integration by part, and the contribution of the second term follows from (2.6) since $\partial_j a \in S^{1,0,0}$. □

A first consequence is the following.

Proposition 2.3. *For all P satisfying (2.2) and such that*

$$\sum_{j,k} ||a_{jk} - \delta_{jk}||_{0,1,0} + ||b_k||_{1,0,0}$$

is small enough, we have

$$\frac{1}{2}||\nabla u||_{L^2}^2 \leq (\overline{P}u, u) \leq 2||\nabla u||_{L^2}^2, \tag{2.9}$$

$$\frac{1}{2}||\Delta u||_{L^2} \leq ||\overline{P}u||_{L^2} \leq 2||\Delta u||_{L^2}^2. \tag{2.10}$$

for all $u \in H^2$. In particular,

$$\overline{P} \geq 0. \tag{2.11}$$

Proof. It suffices to prove the result when $u \in \mathcal{S}$. To prove (2.9), we write first

$$P = D_j a_{jk}(x) D_k + (b_k(x) - i\partial_j a_{jk}(x))\, D_k.$$

Then

$$(Pu, u) = ||\nabla u||^2 + \big((a_{jk} - \delta_{jk})D_k u, D_j u\big) + \big((\overline{b_k} + i\partial_j \overline{a_{jk}})u, D_k u\big),$$

so that

$$(Pu, u) - ||\nabla u||^2 \leq \sum_{j,k} ||a_{jk} - \delta_{jk}||_{L^\infty}||\nabla u||_{L^2}^2 + ||b_k - i\partial_j a_{jk}||_{L^d}||u||_{L^{2^*}}||\nabla u||_{L^2},$$

using the Hölder inequality (2.7). We conclude (2.9) with the Poincaré-Sobolev inequality (2.8). The estimate (2.10) follows simply from the fact that

$$Pu = -\Delta u + (a_{jk}(x) - \delta_{jk})D_j D_k u + b_k(x) D_k u,$$

and Proposition 2.2. □

We next recall the definition and some elementary properties of A the generator of L^2 dilations

$$e^{i\tau A}\varphi = e^{\frac{d}{2}\tau}\varphi\big(e^\tau x\big),$$

which is given by (1.6). We have the identities

$$||e^{i\tau A}\varphi||_{L^p} = e^{\tau\left(\frac{d}{2}-\frac{d}{p}\right)}||\varphi||_{L^p}, \qquad p\in[1,\infty], \tag{2.12}$$
$$||\partial^\alpha e^{i\tau A}\varphi||_{L^2} = e^{\tau|\alpha|}||\partial^\alpha\varphi||_{L^2}, \tag{2.13}$$

which are convenient to prove the $L^p\to L^p$ or $H^s\to H^s$ boundedness of

$$(\kappa A-\zeta)^{-1}=\frac{1}{i}\int_0^{\pm\infty}e^{-i\tau\zeta}e^{i\tau\kappa A}d\tau, \qquad \pm\mathrm{Im}(\zeta)<0, \tag{2.14}$$

for suitable parameters κ,ζ,s and p. For instance, if $s\geq 0$ is an integer, we have the useful estimate

$$||e^{i\tau\kappa A}\varphi||_{H^s}\leq C\left(1+e^{\tau s\kappa}\right)||\varphi||_{H^s}, \qquad \varphi\in\mathcal{S}. \tag{2.15}$$

Lemma 2.4. *There exists $C>0$ such that, for all $r\in\mathbb{R}$, $\kappa>0$ and $\zeta\in\mathbb{C}\setminus\mathbb{R}$ such that $|\mathrm{Im}(\zeta)|>\kappa|r|$, one has*

$$\left|\left|(\kappa A-\zeta)^{-1}\varphi\right|\right|_{H^r}\leq\frac{C}{|\mathrm{Im}(\zeta)|-\kappa|r|}||\varphi||_{H^r},$$

for all $\varphi\in H^r\cap L^2$.

Proof. We may assume that $r\geq 0$, otherwise we consider the adjoint. We consider the norm

$$||u||_{H^r}=||u||_2+\left|\left||D|^r u\right|\right|_2,$$

and recall that

$$|D|^r(\kappa A-\zeta)^{-1}=(\kappa A-\zeta-i\kappa r)^{-1}|D|^r, \tag{2.16}$$

which follows from (2.14) (see [2]). This formula and the self-adjointness of A give

$$\left|\left||D|^r(\kappa A-\zeta)^{-1}\varphi\right|\right|_{L^2}\leq\frac{1}{|\mathrm{Im}(\zeta)|-\kappa r}\left|\left||D|^r\varphi\right|\right|_{L^2},$$

so the result follows using $\left|\left|(\kappa A-\zeta)^{-1}\varphi\right|\right|_{L^2}\leq|\mathrm{Im}(\zeta)|^{-1}||\varphi||_{L^2}$ and $|\mathrm{Im}(\zeta)|^{-1}\leq(|\mathrm{Im}(\zeta)|-\kappa r)^{-1}$. □

We now consider commutators with A. We recall the following standard notation,

$$\mathrm{ad}_A^0 P=P, \qquad \mathrm{ad}_A P=[P,A], \qquad \mathrm{ad}_A^n P=\left[\mathrm{ad}_A^{n-1}P,A\right].$$

Here the commutators are defined in the sense of differential operators acting on Schwartz functions. One easily checks that

$$i^n\mathrm{ad}_A^n P=a_{jk}^{(n)}(x)D_jD_k+b_k^{(n)}(x)D_k, \tag{2.17}$$

with

$$a_{jk}^{(n)}=\left(2-x\cdot\nabla\right)^n a_{jk}, \qquad b_k^{(n)}=\left(1-x\cdot\nabla\right)^n b_k. \tag{2.18}$$

Proposition 2.5. *For all $n \geq 0$, there exists C_n such that*

$$||\mathrm{ad}_A^n P u||_{L^2} \leq C_n \left(\sum_{jk} ||a_{jk}||_{0,1,n} + ||b_k||_{1,0,n} \right) ||\Delta u||_{L^2}, \tag{2.19}$$

for all $u \in \mathcal{S}$, $a_{jk} \in S^{0,1,n}$ and $b_k \in S^{1,0,n}$. In particular,

1. *If $\sum_{j,k} ||a_{jk} - \delta_{jk}||_{0,1,0} + ||b_k||_{1,0,0}$ is small enough, then*

$$||\mathrm{ad}_A^n P u||_{L^2} \leq C_n \left(\sum_{jk} ||a_{jk}||_{0,1,n} + ||b_k||_{1,0,n} \right) ||Pu||_{L^2}, \tag{2.20}$$

2. *if $\sum_{j,k} ||a_{jk} - \delta_{jk}||_{0,1,1} + ||b_k||_{1,0,1}$ is small enough, then*

$$\big(u, i[P,A]u\big) \geq \frac{1}{2}(u, Pu), \tag{2.21}$$

for all $u \in \mathcal{S}$.

Proof. The estimate (2.19) follows from Proposition 2.2 and (2.18). If $\sum_{j,k} ||a_{jk} - \delta_{jk}||_{0,1,0} + ||b_k||_{1,0,0}$ is small then we may replace $||\Delta u||_{L^2}$ by $||Pu||_{L^2}$ in (2.19) using (2.10), which proves (2.20). To prove (2.21), we proceed similarly to (2.9) to show that $\big(u, i[P,A]u\big) \geq ||\nabla u||_{L^2}^2$ (note that $i[P,A]$ is close to -2Δ) and use (2.9) to conclude. $\square$

The estimate (2.21) is a positive commutator estimate which holds uniformly for all a_{jk}, b_k in bounded subsets of $S^{0,1,n}$ and $S^{1,0,n}$ respectively and satisfying the smallness condition of item 2. In the same spirit, the estimate (2.20) means that $\mathrm{ad}_A^n P$ is relatively bounded with respect to P with a fairly explicit dependence on the coefficients a_{jk}, b_k. In the next proposition, we derive some useful related estimates.

Proposition 2.6. *For all n, $\mathrm{ad}_A^n P : \mathcal{S} \to L^2$ has a bounded closure*

$$\overline{\mathrm{ad}_A^n P} : H^2 \to L^2,$$

provided that $a_{jk} \in S^{0,1,n}$, $b_k \in S^{1,0,n}$. If in addition (2.2) holds and

$$\sum_{j,k} ||a_{jk} - \delta_{jk}||_{0,1,0} + ||b_k||_{1,0,0} \tag{2.22}$$

is small enough, then for all integer $j \geq 0$,

$$\left|\left| \overline{\mathrm{ad}_A^n P} (\kappa A + i)^{-j} \big(\overline{P} - z\big)^{-1} \right|\right|_{L^2 \to L^2} \leq \frac{C_{n,j}}{(1-2\kappa)^j} \left(\sum_{jk} ||a_{jk}||_{0,1,n} + ||b_k||_{1,0,n} \right) \frac{\langle z \rangle}{\mathrm{dist}(z, [0,\infty))},$$

for all $z \in \mathbb{C} \setminus [0,\infty)$ and all $0 < \kappa < 1/2$. In particular, if the coefficients a_{kj}, b_k belong to bounded subsets of $S^{0,1,n}$ and $S^{1,0,n}$ respectively, and if (2.22) is small enough, then $\overline{\mathrm{ad}_A^n P}$ is $\overline{P}$ bounded and there is a constant C such that

$$\left|\left| \overline{\mathrm{ad}_A^n P} \big(\overline{P} + 1\big)^{-1} \right|\right|_{L^2 \to L^2} \leq C, \tag{2.23}$$

uniformly with respect to these coefficients.

Proof. The existence of the closure follows from (2.19). The prove the estimate, we write

$$\begin{aligned}\overline{\mathrm{ad}_A^n P}(\kappa A+i)^{-j}\big(\overline{P}-z\big)^{-1} &= \overline{\mathrm{ad}_A^n P}(1-\Delta)^{-1}\Big((\kappa A+i)^{-j}\big(\overline{P}-z\big)^{-1}\\ &\quad -(\kappa A+i-2i\kappa)^{-j}\Delta\big(\overline{P}-z\big)^{-1}\Big).\end{aligned}$$

using (2.16). We conclude by using (2.19) and the lower bound in (2.10) which shows that, for $k=0,1$,

$$\big|\big|\Delta^k(\overline{P}-z)^{-1}\big|\big|_{L^2\to L^2} \leq C_1\big|\big|\overline{P}^k(\overline{P}-z)^{-1}\big|\big|_{L^2\to L^2} \leq C_1 \sup_{\lambda\in[0,\infty)}\left|\frac{\lambda^k}{\lambda-z}\right| \leq C_2\frac{\langle z\rangle^k}{\mathrm{dist}(z,[0,\infty))},$$

the second estimate following from the Spectral Theorem and the fact that $\mathrm{spec}(\overline{P})\subset[0,\infty)$ by (2.11). $\square$

3 Weighted functional calculus

In this section, we investigate the $L^2\to L^2$ (and sometimes $H^{-1}\to H^1$) boundedness of operators of the form

$$(\kappa A+i)^{-n}\chi\big(\overline{P}\big)(\kappa A+i)^n,$$

with P as in Section 2 and where χ may be a bump function in C_0^∞, or corresponds to $(\overline{P}-z)^{-1}$ ie $\chi(\alpha)=(\alpha-z)^{-1}$. The expression above is not well defined on L^2 for it can be applied only to functions in $\mathrm{Dom}(A^n)$, but we shall see that it has a bounded closure. This kind of result is well known, but the additional point we want to stress here is to which extent the norms of these operators are uniform with respect to the coefficients a_{jk},b_k defining P in (2.1). Since this is a crucial tool in the proof of Theorem 1.2 (or, more precisely, of Theorem 5.3 below), we devote a section to this topic.

The following lemma will be of constant use in the sequel.

Lemma 3.1. *For all $n\geq 0$, all $u\in H^s$, with $s\geq 0$, and all $\kappa>0$ such that $\kappa s<1$, there exists a sequence θ_j in C_0^∞ such that, for $k=0,\dots,n$,*

$$(\kappa A+i)^k\theta_j\to(\kappa A+i)^{k-n}u, \qquad \text{in } H^s,$$

as $j\to\infty$.

Proof. Let first $T_j=\frac{1}{i}\int_0^j e^{-\tau}e^{i\tau\kappa A}dt$. By (2.15),

$$T_j\to(\kappa A+i)^{-1}, \qquad \text{strongly on } H^s,$$

and using $\kappa Ae^{it\kappa A}=-i\frac{d}{dt}e^{it\kappa A}$,

$$(\kappa A+i)T_j=I-e^{-j}e^{ij\kappa A} \qquad \text{on Dom}(A),$$

where

$$I-e^{-j}e^{ij\kappa A}\to I, \qquad \text{strongly on } H^s.$$

We can now prove the existence of θ_j by induction on n. The result is clear if $n=0$. For $n\geq 1$, the induction assumption allows to pick φ_j in C_0^∞ such that, for $k\leq n-1$,

$$(\kappa A+i)^k\varphi_j\to(\kappa A+i)^{k-n+1}u$$

in H^s. We then define

$$\theta_j = T_j \varphi_j.$$

Clearly, θ_j belongs to C_0^∞ since the integral defining T_j is over a bounded interval. Furthermore, since $(\kappa A + i)^k$ commutes with T_j, we have, for $k \leq n-1$,

$$\begin{aligned}
(\kappa A + i)^k \theta_j &= T_j (\kappa A + i)^k \varphi_j \\
&= T_j (\kappa A + i)^{k-n+1} u + o(1) \longrightarrow (\kappa A + i)^{k-n} u.
\end{aligned}$$

Further, for $k = n$,

$$\begin{aligned}
(\kappa A + i)^n \theta_j &= (\kappa A + i) T_j (\kappa A + i)^{n-1} \varphi_j, \\
&= (\kappa A + i)^{n-1} \varphi_j - e^{-j} e^{ij\kappa A} (\kappa A + i)^{n-1} \varphi_j \longrightarrow u,
\end{aligned}$$

and the result follows. □

Definition 3.2. *If $B : L^2 \to L^2$ is a bounded operator and $n \geq 0$ an integer such that*

$$(\kappa A + i)^{-n} B (\kappa A + i)^n : \mathcal{S} \to L^2,$$

has a bounded closure to L^2, we denote by

$$B_{\kappa,A,n} = \overline{(\kappa A + i)^{-n} B (\kappa A + i)^n},$$

this $L^2 \to L^2$ closure.

Proposition 3.3. *Let B be a bounded operator such that $B_{\kappa,A,n}$ exists. Then*

1.
$$B_{\kappa,A,n} (\kappa A + i)^{-n} = (\kappa A + i)^{-n} B.$$

2. *If C is another bounded operator such that $C_{\kappa,A,n}$ exist then $(BC)_{\kappa,A,n}$ exists as well and*
$$B_{\kappa,A,n} C_{\kappa,A,n} = (BC)_{\kappa,A,n}.$$

Item *2* gives a rigorous sense to the formally trivial identity

$$(\kappa A + i)^{-n} B (\kappa A + i)^n (\kappa A + i)^{-n} C (\kappa A + i)^n = (\kappa A + i)^{-n} BC (\kappa A + i)^n.$$

Proof. To prove that *1* holds when applied to any $u \in L^2$, we use Lemma 3.1 to pick $\theta_j \in C_0^\infty$ which approaches $(\kappa A + i)^{-n} u$ and such that $(\kappa A + i)^n \theta_j$ approaches u, both in L^2. To prove *2*, it suffices to show that, for all $\psi \in \mathcal{S}$,

$$(\kappa A + i)^{-n} BC (\kappa A + i)^n \psi = B_{\kappa,A,n} C_{\kappa,A,n} \psi. \tag{3.1}$$

Fix such a ψ and let

$$u = C_{\kappa,A,n} \psi = (\kappa A + i)^{-n} C (\kappa A + i)^n \psi \in L^2.$$

Choose θ_j as in Lemma 3.1 so that $\theta_j \to u$ and $(\kappa A + i)^n \theta_j \to C (\kappa A + i)^n \psi$ in L^2. Then, on one hand

$$B_{\kappa,A,n} \theta_j \to B_{\kappa,A,n} C_{\kappa,A,n} \psi, \tag{3.2}$$

and on the other hand,

$$(\kappa A + i)^n B_{\kappa,A,n}\theta_j = B(\kappa A + i)^n\theta_j \to BC(\kappa A + i)^n\psi, \qquad \text{in } L^2,$$

so we get

$$B_{\kappa,A,n}\theta_j \to (\kappa A + i)^{-n} BC(\kappa A + i)^n\psi, \qquad \text{in } L^2,$$

which, together with (3.2), implies (3.1). $\square$

For future purposes, we also record the following straightforward lemma which gives a precise meaning to the formal expression

$$(\kappa A + i)^{-n} B(\kappa A + i)^n = (\kappa A + i)^{-1}\left((\kappa A + i)^{1-n} B(\kappa A + i)^{n-1}\right)(\kappa A + i).$$

Lemma 3.4. *Let B be such that $B_{\kappa,A,n}$ and $B_{\kappa,A,n-1}$ exist. Then*

$$B_{\kappa,A,n} = \overline{(\kappa A + i)^{-1} B_{\kappa,A,n-1}(\kappa A + i)}, \tag{3.3}$$

the right hand side denoting the $L^2 \to L^2$ closure of the corresponding operator defined on $\mathcal{S}$.

We shall also need the following result.

Proposition 3.5. *Let Q be a second order differential operator with smooth coefficients such that Q and $[Q, A]$, defined on $\mathcal{S}$, have bounded closures*

$$\overline{Q},\ \overline{[Q, A]} : H^2 \to L^2.$$

Then, for all $0 < \epsilon < 1/2$ and $u \in H^2$, we have

$$(\epsilon A + i)^{-1}\overline{Q}u = \overline{Q}(\epsilon A + i)^{-1}u - \epsilon(\epsilon A + i)^{-1}\overline{[Q, A]}(\epsilon A + i)^{-1}u.$$

Note that, by Proposition 2.6, any Q of the form $\mathrm{ad}_A^j P$ satisfies the assumptions of this proposition.

Recall also Lemma 2.4 which shows that $(\epsilon A + i)^{-1}$ is bounded on H^2 so that $\overline{Q}(\epsilon A + i)^{-1}$ and $\overline{[Q, A]}(\epsilon A + i)^{-1}$ are well defined on H^2.

Proof. Choose θ_j as in Lemma 3.1, such that $(\epsilon A + i)\theta_j \to u$ and $\theta_j \to (\epsilon A + i)^{-1}u$ in H^2. Observe that

$$Q(\epsilon A + i)\theta_j = (\epsilon A + i)Q\theta_j + \epsilon[Q, A]\theta_j,$$

and apply $(\epsilon A + i)^{-1}$ to this equality. The result follows by letting $j \to \infty$. $\square$

Applying Proposition 3.5 with $Q = P$, and applying $(\overline{P} - z)^{-1}$ to the left of the corresponding identity we get:

Lemma 3.6. *For all $0 < \epsilon < 1/2$ and $z \notin [0, \infty)$,*

$$(\epsilon A + i)^{-1}\big(\overline{P} - z\big)^{-1} = \big(\overline{P} - z\big)^{-1}(\epsilon A + i)^{-1} - \epsilon\big(\overline{P} - z\big)^{-1}(\epsilon A + i)^{-1}\overline{[P, A]}(\epsilon A + i)^{-1}\big(\overline{P} - z\big)^{-1},$$

as operators from H^2 to L^2.

This lemma is useful to prove the following identity (note that, compared to the formula in Lemma 3.6, we swap the resolvents of $\overline{P}$ and A).

Proposition 3.7. *For all $\psi \in C_0^\infty$, all $\kappa > 0$ and all $z \notin [0, \infty)$, we have*

$$(\kappa A + i)^{-1}\big(\overline{P} - z\big)^{-1}(\kappa A + i)\psi \;\; = \;\; \big(\overline{P} - z\big)^{-1}\psi + \kappa(\kappa A + i)^{-1}\big(\overline{P} - z\big)^{-1}\overline{[P, A]}\big(\overline{P} - z\big)^{-1}\psi.$$

Proof. By the Spectral Theorem, we have

$$i(\epsilon A + i)^{-1} \to I, \qquad \epsilon \to 0,$$

in the strong sense on L^2 but also in H^2 by (2.16). On the other hand, one easily checks that

$$i(\epsilon A + i)^{-1}(\kappa A + i) = \frac{i\kappa}{\epsilon} I + \left(1 - \frac{\kappa}{\epsilon}\right)(\epsilon A + i)^{-1},$$

so using Lemma 3.6,

$$\begin{aligned} \big(\overline{P} - z\big)^{-1}(\kappa A + i)i(\epsilon A + i)^{-1}\psi \;\; &= \;\; (\kappa A + i)i(\epsilon A + i)^{-1}\big(\overline{P} - z\big)^{-1}\psi + \\ &\quad\;\; (\kappa - \epsilon)\big(\overline{P} - z\big)^{-1}(\epsilon A + i)^{-1}\overline{[P, A]}(\epsilon A + i)^{-1}\big(\overline{P} - z\big)^{-1}\psi. \end{aligned}$$

Applying $(\kappa A + i)^{-1}$ to this identity and letting $\epsilon \to 0$, we get the result. □

Corollary 3.8. *For all $z \notin [0, \infty)$ and all $\kappa > 0$, $\big(\overline{P} - z\big)^{-1}_{\kappa, A, 1}$ exists and is given by*

$$\big(\overline{P} - z\big)^{-1}_{\kappa, A, 1} = \big(\overline{P} - z\big)^{-1} + \kappa(\kappa A + i)^{-1}\big(\overline{P} - z\big)^{-1}\overline{[P, A]}\big(\overline{P} - z\big)^{-1}.$$

Notice that we do not need κ to be small since this identity makes sense for operators on L^2. If we want this result to hold in the sense of operators from L^2 to H^2 we have to restrict to $0 < \kappa < 1/2$.

We next will prove more generally that $\big(\overline{P} - z\big)^{-1}_{\kappa, A, n}$ exists for any n. We will proceed by induction using Lemma 3.4.

Proposition 3.9. *For all $z \notin [0, \infty)$, all $0 < \kappa < 1/2$ and all $n \geq 0$, $\big(\overline{P} - z\big)^{-1}_{\kappa, A, n}$ exists and is a linear combination of operators of the form*

$$(\kappa A + i)^{-i_l}\big(\overline{P} - z\big)^{-1}\prod_{\nu=1}^{l}\left((\kappa A + i)^{-j_\nu}\overline{\mathrm{ad}_A^{j_\nu} P}(\kappa A + i)^{-k_\nu}\big(\overline{P} - z\big)^{-1}\right),$$

where the product means composition of operators, from the left to the right increasingly in ν (it is I if $l = 0$), and

$$0 \leq l \leq n, \qquad 0 \leq i_l, j_\nu, k_\nu \leq n.$$

The coefficients of this combination are non negative powers of κ times complex numbers which are independent of κ, z and P.

Proof. We proceed by induction on n, the result being trivial if $n = 0$. To go from step $n-1$ to n, using Proposition 3.3 and Lemma 3.4, we have to show that $B_{\kappa,A,1}$ exists for operators B of the form

$$(\kappa A + i)^{-k}, \qquad \big(\overline{P} - z\big)^{-1}, \qquad \overline{\mathrm{ad}_A^j P}(\kappa A + i)^{-k}\big(\overline{P} - z\big)^{-1}.$$

This is trivial for the first one and follows from Corollary 3.8 for the second one. We thus consider the third one, which requires $\kappa < 1/2$ to ensure that $(\kappa A + i)^{-k}$ maps H^2 to H^2. By Proposition 3.5 (with $\epsilon = \kappa$), we have

$$(\kappa A + i)^{-1}\overline{\mathrm{ad}_A^j P} = \left(\overline{\mathrm{ad}_A^j P} - \kappa(\kappa A + i)^{-1}\overline{\mathrm{ad}_A^{j+1} P}\right)(\kappa A + i)^{-1},$$

which, by Proposition 3.3 and Corollary 3.8, shows that

$$\left(\overline{\mathrm{ad}_A^j P}(\kappa A + i)^{-k}\big(\overline{P} - z\big)^{-1}\right)_{\kappa,A,1} = \left(\overline{\mathrm{ad}_A^j P} - \kappa(\kappa A + i)^{-1}\overline{\mathrm{ad}_A^{j+1} P}\right)(\kappa A + i)^{-k}\big(\overline{P} - z\big)^{-1}_{\kappa,A,1},$$

which is a linear combination of products of operators of the expected form. □

We summarize the result obtained so far and derive somes estimates in the following proposition.

Proposition 3.10. *There exists $\epsilon > 0$ such that, for all integer $n \geq 0$ and all $M > 0$, there exists $C > 0$ such that for all coefficients $a_{jk} \in S^{0,1,n}$, $b_k \in S^{1,0,n}$ such that*

1. *(2.2) is satisfied*
2. $\sum_{j,k} ||a_{jk} - \delta_{jk}||_{0,1,0} + ||b_k||_{1,0,0} < \epsilon$
3. $\sum_{j,k} ||a_{jk}||_{0,1,n} + ||b_k||_{1,0,n} \leq M$

and for all $z \notin [0, \infty)$, all $0 < \kappa \leq 1/4$, we have

$$\big|\big|(\overline{P} - z)^{-1}_{\kappa,A,n}\big|\big|_{L^2 \to L^2} \leq C \left(\frac{\langle z \rangle}{\mathrm{dist}(z, [0, \infty))}\right)^{n+1}. \tag{3.4}$$

Notice that we could consider $0 < \kappa < 1/2$, but we restrict to the case $\kappa \leq 1/4$ to get a κ independent estimate in (3.4). We would otherwise get some positive power of $(1 - 2\kappa)^{-1}$ in the right hand side.

Proof. The result follows from the form of $(\overline{P} - z)^{-1}_{\kappa,A,n}$ described in Proposition 3.9 combined with the estimates of Lemma 2.4 and Proposition 2.6. □

Corollary 3.11. *Fix $n \geq 0$ integer, $M \geq 0$ and $\chi \in C_0^\infty(\mathbb{R})$. Then there exists $C > 0$ such that for all coefficients a_{jk}, b_k satisfying 1,2 and 3 in Proposition 3.10, and for all $0 < \kappa \leq 1/4$, the operator $\chi(\overline{P})_{\kappa,A,n}$ exists and we have*

$$\big|\big|\chi(\overline{P})_{\kappa,A,n}\big|\big|_{L^2 \to L^2} \leq C. \tag{3.5}$$

Proof. It is a simple consequence of Proposition 3.10 and the following Helffer-Sjöstrand formula (see for instance [13])

$$\chi\big(\overline{P}\big) = \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial}\widetilde{\chi}(z)\big(\overline{P} - z\big)^{-1} L(dz),$$

where $L(dz)$ is the Lebesgue measure on $\mathbb{C} \simeq \mathbb{R}^2$ and $\widetilde{\chi} \in C_0^\infty(\mathbb{C})$ is an almost analytic extension of χ, ie such that $\overline{\partial}\widetilde{\chi}(z) = \mathcal{O}(|\mathrm{Im}(z)|^\infty)$ and $\widetilde{\chi}_{|\mathbb{R}} = \chi$. □

We shall also need the following proposition.

Proposition 3.12. *There exists $\epsilon > 0$ such that for all $n \geq 0$ integer and all $M > 0$, there exists $C > 0$ such that for all coefficients a_{jk}, b_k satisfying 1,2,3 in Proposition 3.10, we have*

$$\left|\left|\left(\overline{P}+1\right)^{-1}_{\kappa,A,n} u\right|\right|_{H^1} \leq C||u||_{H^{-1}}, \tag{3.6}$$

for all $u \in L^2$ and all $0 < \kappa \leq 1/4$. Furthermore, if $\chi \in C_0^\infty$, we have

$$\left|\left|\chi\left(\overline{P}\right)_{\kappa,A,n} u\right|\right|_{H^1} \leq C||u||_{H^{-1}}. \tag{3.7}$$

Proof. It suffices to prove (3.6) since (3.7) would then follow from (3.5) and (3.6) using the identity

$$\chi\left(\overline{P}\right)_{\kappa,A,n} = \left(\overline{P}+1\right)^{-1}_{\kappa,A,n} \widetilde{\chi}\left(\overline{P}\right)_{\kappa,A,n} \left(\overline{P}+1\right)^{-1}_{\kappa,A,n},$$

with $\widetilde{\chi}(\alpha) = (\alpha+1)^2\chi(\alpha)$, which is justified by Proposition 3.3. Let us prove (3.6). Using the form of $\left(\overline{P}+1\right)^{-1}_{\kappa,A,n}$ given by Proposition 3.9, the result would follow from the estimates

$$\begin{aligned} \left|\left|\left(\overline{P}+1\right)^{-1} u\right|\right|_{H^1} &\leq C||u||_{H^{-1}}, \qquad u \in L^2, && (3.8)\\ \left|\left|\overline{\mathrm{ad}^j_A P} v\right|\right|_{H^{-1}} &\leq C||v||_{H^1}, \qquad v \in H^2,\ j \leq n, && (3.9)\\ \left|\left|(\kappa A + i)^{-1} w\right|\right|_{H^{\pm 1}} &\leq C||w||_{H^{\pm 1}}, \qquad w \in H^{1 \pm 1}, && (3.10) \end{aligned}$$

for some C independent of the coefficients of P and κ. The estimate (3.9) follows from (2.5) and (2.6). The estimate (3.10) is given in Lemma 2.4 in the $+$ case, and is the adjoint of the $H^1 \to H^1$ bound on $(\kappa A - i)^{-1}$ in the $-$ case. Finally (3.8) follows from the bound

$$||(\overline{P}+1)^{-1/2} u||_{H^1} \leq C||u||_{L^2}, \qquad u \in L^2, \tag{3.11}$$

(and the adjoint one) which follows in a standard fashion from (2.9). □

4 Elliptic estimates

In this section, we prove some elementary elliptic regularity estimates for $\left(\overline{P}+1\right)^{-1}_{\kappa,A,n}$ (recall Definition 3.2). Everywhere we set

$$r = \bar{r}(d),$$

where $\bar{r}(d)$ is defined by (1.4). We start with the following result.

Proposition 4.1. *Let $o \in \{0,1\}$ and s be an integer such that $0 \leq s \leq r$. Then there exists C such that*

$$||au||_{H^{s-o}} \leq C||a||_{o,r-o,0}||u||_{H^s}, \tag{4.1}$$

for all $a \in S^{o,r-o,0}$ and $u \in \mathcal{S}$.

The estimate (4.1) means that the multiplication by a behaves like a differential operator of order o.

Proof. We consider first the case when $s = 0$ and $o = 1$. In this case, the result follows from

$$||au||_{H^{-1}} \leq C||au||_{L^{\frac{2d}{d+2}}} \leq C||a||_{L^d}||u||_{L^2},$$

by the Hölder inequality. In the other cases, we have $s-o \geq 0$ and we proceed as follows. Observe that, for any $0 \leq k \leq d/2$,

$$||\varphi\psi||_{L^2} \leq ||\varphi||_{L^{\frac{d}{k}}} ||\psi||_{L^{\frac{2d}{d-2k}}}, \tag{4.2}$$

by the Hölder inequality. Let $|\alpha| \leq s-o$. By the Leibniz rule,

$$||\partial^\alpha(au)||_{L^2} \leq C \sum_{\gamma \leq \alpha} ||(\partial^\gamma a)(\partial^{\alpha-\gamma} u)||_{L^2}.$$

Since $u \in H^s$, we have

$$\partial^{\alpha-\gamma} u \in H^{s-|\alpha|+|\gamma|} \subset H^{o+|\gamma|} \subset L^{\frac{2d}{d-2(o+|\gamma|)}},$$

the last inclusion being the usual Sobolev embedding (here we use that $r < d/2$). By the continuity of this embedding and (4.2), we have

$$||(\partial^\gamma a)(\partial^{\alpha-\gamma} u)||_{L^2} \leq C ||\partial^\gamma a||_{L^{\frac{d}{o+|\gamma|}}} ||u||_{H^s},$$

from which the result follows (recall that $o+|\gamma| \leq o+|\alpha| \leq s \leq r$). □

Using the self-adjointness of P, we obtain the following result for Sobolev spaces of positive or negative order.

Corollary 4.2. *For all integer $n \geq 0$ and $-r \leq s \leq r$ integer, there exists C such that*

$$||\mathrm{ad}_A^n(P+\Delta)u||_{H^{s-1}} \leq C \left(\sum_{jk} ||a_{jk} - \delta_{jk}||_{0,r,n} + ||b_k||_{1,r-1,n} \right) ||u||_{H^{s+1}}, \tag{4.3}$$

for all $u \in \mathcal{S}$ and all $a_{jk} \in S^{0,r,n}$, $b_k \in S^{1,r-1,n}$ such that (2.2) holds.

Proof. For non negative s, the result follows from Proposition 4.1 and (2.17)-(2.18). For negative s, one takes the adjoint since $i^n \mathrm{ad}_A^n(P+\Delta)$ is (formally) self-adjoint. □

We next prove the following proposition which will be crucial in Subsection 5.1.

Proposition 4.3. *Fix an integer $n \geq 0$ and $M > 0$. There exists $\epsilon > 0$, $\kappa_0 > 0$ and $C > 0$ such that if*

1. *$||a_{jk} - \delta_{jk}||_{0,r,0} + ||b_k||_{1,r-1,0} < \epsilon$,*
2. *$||a_{jk}||_{0,r,n} + ||b_k||_{1,r,n} \leq M$,*
3. *$0 < \kappa \leq \kappa_0$,*
4. *$-r \leq s \leq r$ integer,*

then the operator

$$(\kappa A + i)^{-n}(P+1)(\kappa A + i)^n : \mathcal{S} \to H^{s-1} \cap L^2,$$

has a bounded closure $H^{s+1} \to H^{s-1}$ denoted by $(P+1)_{\kappa,A,n,s}$ which is an isomorphism between H^{s+1} and H^{s-1} and such that

$$||u||_{H^{s+1}}/C \leq ||(P+1)_{\kappa,A,n,s} u||_{H^{s-1}} \leq C ||u||_{H^{s+1}}, \qquad u \in H^{s+1}. \tag{4.4}$$

Furthermore, with the notation of Definition 3.2,

$$\left|\left|(\overline{P}+1)_{\kappa,A,n}^{-1} u\right|\right|_{H^{s+1}} \leq C ||u||_{H^{s-1}}, \qquad u \in L^2 \cap H^{s-1}. \tag{4.5}$$

Proof of Proposition. Observe first that

$$(\kappa A+i)^{-n}P(\kappa A+i)^{n}-P$$

is a linear combination, with coefficients which are universal constants, of operators of the form

$$\kappa^{m}(\kappa A+i)^{-m_1}\mathrm{ad}_A^{m}(P), \qquad m_1\geq 0,\ 1\leq m\leq n.$$

Therefore, using Corollary 4.2 and Lemma 2.4 (with $\kappa(|r|+1)<1/2$) we have

$$||(\kappa A+i)^{-n}P(\kappa A+i)^{n}u+\Delta u||_{H^{s-1}}\leq C\left(\epsilon+\kappa M\right)||u||_{H^{s+1}}.$$

By choosing ϵ and κ small enough, we obtain the existence of the closure $(P+1)_{\kappa,A,n,s}$ and the fact that it is close to $1-\Delta$ in the $H^{s+1}\to H^{s-1}$ topology, hence is an isomorphism. We also get (4.4). To prove (4.5) it suffices to show that

$$(\overline{P}+1)^{-1}_{\kappa,A,n}=\left((P+1)_{\kappa,A,n,s}\right)^{-1} \qquad \text{on } \ L^2\cap H^{s-1}, \tag{4.6}$$

and then use the lower bound in (4.4). One sees that (4.6) holds by checking that

$$\begin{aligned}(\overline{P}+1)^{-1}_{\kappa,A,n}(P+1)_{\kappa,A,n,s}w=w, &\qquad s-1\geq 0,\\ (P+1)_{\kappa,A,n,s}(\overline{P}+1)^{-1}_{\kappa,A,n}w=w, &\qquad s-1<0,\end{aligned}$$

for all $w\in\mathcal{S}$. This follows from Lemma 3.1 by approximating $u=(P+1)(\kappa A+i)^{n}w$ in H^{s-1} (hence in L^2) in the first case, and $u=(\overline{P}+1)^{-1}(\kappa A+i)^{n}w$ in H^2 (hence in H^{s+1}) in the second case. $\square$

5 Resolvent estimates

The purpose of this section is to prove Theorem 1.2. The proof will be divided into two steps. In Subsection 5.1, we shall prove resolvent estimates for operators of the form (2.1) which are small perturbations of $-\Delta$, using a scale invariant analysis. In Subsection 5.2, we will prove Theorem 1.2 by combining a compactness argument and the estimates of Subsection 5.1, by reducing $-\Delta_G$ to a compactly supported perturbation of an operator of the form (2.1).

5.1 Small perturbations

Throughout this subsection, P denotes an operator of the form (2.1) and, as before, $\overline{P}$ denotes its $H^2\to L^2$ closure which is self-adjoint on L^2 with domain H^2. We shall basically prove weighted estimates on $(\overline{P}-z)^{-n}$ seen as an operator from $H^{-n}\to H^{n}$. The first step is to get $L^2\to L^2$ estimates and is the purpose of the following proposition.

Proposition 5.1 (Jensen-Mourre-Perry estimates [22])**.** *There exists $\epsilon>0$ such that, for all integer $N\geq 0$, all $M\geq 0$ and all relatively compact interval $I\Subset(0,\infty)$, there exists $C>0$ such that, for all $n\leq N$,*

$$\left|\left|(A+i)^{-n}\left(\overline{P}-z\right)^{-n}(A+i)^{-n}\right|\right|_{L^2\to L^2}\leq C, \qquad \mathrm{Re}(z)\in I,\ \mathrm{Im}(z)\neq 0,$$

for all coefficients $a_{jk}\in S^{0,1,N+1}$, $b_k\in S^{1,0,N+1}$ such that

1. *(2.2) is satisfied*

2. $\sum_{j,k} ||a_{jk} - \delta_{jk}||_{0,1,1} + ||b_k||_{1,0,1} < \epsilon$

3. $\sum_{j,k} ||a_{jk}||_{0,1,N+1} + ||b_k||_{1,0,N+1} \leq M$.

This result follows by tracking the uniform dependence on the coefficients of P in the proofs of [22]. We simply point out that the smallness of $||a_{jk} - \delta_{jk}||_{0,1,1}$ and $||b_k||_{1,0,1}$ guarantees the positive commutator estimate

$$\chi(\overline{P}) i\overline{[P, A]} \chi(\overline{P}) \geq \frac{\inf \operatorname{supp}(\chi)}{2} \chi^2(\overline{P}),$$

for any $\chi \in C_0^\infty(\mathbb{R}^+)$, which follows from (2.21). The other ingredient is the uniform $\overline{P}$ boundedness estimate (2.23).

In the sequel, we shall use the notation

$$a_\tau(x) := a(e^\tau x), \qquad \tau \in \mathbb{R}, \ x \in \mathbb{R}^d, \tag{5.1}$$

namely

$$a_\tau = e^{i\tau A} a e^{-i\tau A}.$$

Here is the main property of the spaces $S^{o,r-o,N}$.

Proposition 5.2 (Scaling homogeneity)**.** *Let $o \in \{0, 1\}$ and $o \leq r \leq \bar{r}(d)$. For all $a \in S^{o,r-o,N}$ and $\tau \in \mathbb{R}$,*

$$e^{o\tau} ||a_\tau||_{o,r-o,N} = ||a||_{o,r-o,N}.$$

Proof. Observe first that

$$((x \cdot \nabla)^n a)_\tau = (x \cdot \nabla)^n (a_\tau), \tag{5.2}$$

either by a trivial direct computation, or by remarking that dilations commute with their generator. Then

$$(x \cdot \nabla)^n (\partial^\beta a_\tau) = e^{\tau|\beta|} \left((x \cdot \nabla)^n \partial^\beta a \right)_\tau,$$

and we see that

$$e^{\tau o} ||(x \cdot \nabla)^n \partial^\beta (a_\tau)||_{L^{\frac{d}{|\beta|+o}}} = ||(x \cdot \nabla)^n \partial^\beta a||_{L^{\frac{d}{|\beta|+o}}},$$

by an elementary change of variable in the integral when $|\beta| + o \neq 0$, and trivially if $|\beta| + o = 0$. □

We are now ready to prove the following theorem which is our main technical result.

Theorem 5.3. *Let $N := \bar{r}(d) + 1$. Fix a constant $M > 0$. Then, there exist $\epsilon > 0$ and $\kappa > 0$ such that, for all $a_{jk} \in S^{0,\bar{r}(d),N+1}$, $b_k \in S^{1,\bar{r}(d)-1,N+1}$ such that*

1. *(2.2) is satisfied*

2. $\sum_{j,k} ||a_{jk} - \delta_{jk}||_{0,1,1} + ||b_k||_{1,0,1} < \epsilon$,

3. $\sum_{j,k} ||a_{jk} - \delta_{jk}||_{0,\bar{r}(d),0} + ||b_k||_{1,\bar{r}(d)-1,0} < \epsilon$,

4. $\sum_{j,k} ||a_{jk}||_{0,\bar{r}(d),N+1} + ||b_k||_{1,\bar{r}(d)-1,N+1} < M$,

we have the following estimates,

- *if* $1 \leq n \leq \bar{r}(d)$

$$\big|\big|(\kappa A+i)^{-n}(\overline{P}-z)^{-n}(\kappa A-i)^{-n}\varphi\big|\big|_{L^{q(n)}} \leq C||\varphi||_{L^{p(n)}}, \tag{5.3}$$

- *if* $n=N$ *and* d *is odd,*

$$\big|\big|(\kappa A+i)^{-N}(\overline{P}-z)^{-N}(\kappa A-i)^{-N}\varphi\big|\big|_{L^{\infty}} \leq C\mathrm{Re}(z)^{-1/2}||\varphi||_{L^{1}}, \tag{5.4}$$

- *if* $n=N$ *and* d *is even, then for all* $2d<q<\infty$,

$$\big|\big|(\kappa A+i)^{-N}(\overline{P}-z)^{-N}(\kappa A-i)^{-N}\varphi\big|\big|_{L^{q}} \leq C_q\mathrm{Re}(z)^{-\frac{2d}{q}}||\varphi||_{L^{\frac{q}{q-1}}}, \tag{5.5}$$

all these estimates holding for

$$\varphi \in \mathcal{S}, \qquad \mathrm{Re}(z)>0, \qquad \mathrm{Im}(z)\neq 0.$$

This result can be viewed as a version of Theorem 1.1 for small perturbations of the flat metric. Notice that the coefficients of the perturbation are taken in the classes $S^{o,r-o,N}$, which is a more general condition than being in $S^{-\rho-o}$, as we shall see in Proposition 5.4. Note also that Theorem 5.3 holds for $\mathrm{Re}(z)$ small, which is its main interest, but actually for all $\mathrm{Re}(z)>0$ hence for large ones too.

Proof. It is based on an scaling argument. Let $\lambda=\mathrm{Re}(z)$ and write

$$P-z=\lambda\big(\lambda^{-1}P-1-i\delta\big),$$

where $\lambda\delta=\mathrm{Im}(z)$. Then, by setting

$$\lambda^{-1/2}=e^{\tau},$$

and

$$P_\tau=a_{jk,\tau}(x)D_jD_k+e^{\tau}b_{k,\tau}(x)D_k,$$

where we use the notation (5.1), we have

$$\lambda^{-1}P=e^{-i\tau A}P_\tau e^{i\tau A},$$

and thus

$$\big(\overline{P}-z\big)^{-1}=\lambda^{-1}e^{-i\tau A}\big(\overline{P}_\tau-1-i\delta\big)^{-1}e^{i\tau A}. \tag{5.6}$$

By Proposition 5.2, the conditions *2*, *3* and *4* hold for $a_{jk,\tau}$ and $e^{\tau}b_{k,\tau}$, uniformly with respect to $\tau\in\mathbb{R}$. In particular, using Proposition 5.1, we have

$$\big|\big|(A+i)^{-n}\big(\overline{P}_\tau-1-i\delta\big)^{-n}(A-i)^{-n}\big|\big|_{L^2\to L^2}\leq C_M, \tag{5.7}$$

for all $\tau\in\mathbb{R}$, $\delta\in\mathbb{R}\setminus 0$ and $1\leq n\leq N$. Since this is an $L^2\to L^2$ estimate, $(A\pm i)^{-n}$ can be replaced by $(\kappa A\pm i)^{-n}$ therein, for any $\kappa>0$, up to the replacement of C_M by a κ dependent constant. This will be useful to consider $H^{-n}\to H^{n}$ estimates as follows. Introduce $\chi\in C_0^\infty(\mathbb{R})$ which is real valued and equal to 1 near 1. We then split the resolvent as

$$\begin{aligned}\big(\overline{P}_\tau-1-i\delta\big)^{-n} &= \big(\overline{P}_\tau-1-i\delta\big)^{-n}(1-\chi^2)\big(\overline{P}_\tau\big)+\chi\big(\overline{P}_\tau\big)\big(\overline{P}_\tau-1-i\delta\big)^{-n}\chi\big(\overline{P}_\tau\big),\\ &= \mathrm{I}(\tau,\delta)+\mathrm{II}(\tau,\delta).\end{aligned} \tag{5.8}$$

We consider first $\mathrm{I}(\tau,\delta)$. By setting $\Phi_\delta(\alpha) = (1-\chi^2(\alpha))(\alpha+1)^n/(\alpha-1-i\delta)^n$, we can write

$$\mathrm{I}(\tau,\delta) = \big(\overline{P}_\tau + 1\big)^{-n/2}\Phi_\alpha(\overline{P}_\tau)\big(\overline{P}_\tau + 1\big)^{-n/2}.$$

Since Φ_δ is bounded in $L^\infty([0,\infty)_\alpha)$ as δ varies, the Spectral Theorem yields

$$||\Phi_\delta(\overline{P}_\tau)||_{L^2\to L^2} \leq C, \qquad \tau\in\mathbb{R}, \quad \delta \in \mathbb{R}\setminus 0.$$

On the other hand, using (4.5), and also (3.11) if n is odd, we have

$$||(\overline{P}_\tau + 1)^{-n/2}\psi||_{H^n} \leq C||\psi||_{L^2}, \qquad \psi\in L^2, \ \tau\in\mathbb{R}.$$

We also have the dual $H^{-n}\to L^2$ bound and we conclude, using Lemma 2.4, that

$$\big|\big|(\kappa A + i)^{-n}\mathrm{I}(\tau,\delta)(\kappa A - i)^{-n}\psi\big|\big|_{H^n} \leq C||\psi||_{H^{-n}},$$

for all $\psi\in L^2$, $\tau\in\mathbb{R}$ and $\delta\neq 0$. We next consider the second term of (5.8). By Proposition 3.3 and Corollary 3.11, we can write

$$(\kappa A+i)^{-n}\mathrm{II}(\tau,\delta)(\kappa A-i)^{-n} = \chi\big(\overline{P}_\tau\big)_{\kappa,A,n}(\kappa A+i)^{-n}\big(\overline{P}_\tau - 1 - i\delta\big)^{-n}(\kappa A-i)^{-n}\chi\big(\overline{P}_\tau\big)^*_{\kappa,A,n}.$$

We then observe that we have the estimate

$$||\chi\big(\overline{P}_\tau\big)_{\kappa,A,n}||_{L^2\to H^n} \leq C_n, \qquad \tau\in\mathbb{R}. \tag{5.9}$$

This is obtained by writting $\chi(\alpha) = (\alpha+1)^{-[n/2]}\psi(\alpha)$ with $[n/2]$ the integer part of $n/2$ so that

$$\chi\big(\overline{P}_\tau\big)_{\kappa,A,n} = \big(\overline{P}_\tau + 1\big)^{-[n/2]}_{\kappa,A,n}\psi\big(\overline{P}_\tau\big)_{\kappa,A,n} : L^2 \to H^n,$$

by (4.5) and Proposition 3.3 if n is even or (4.5) and (3.7) if n is odd. Similarly, we have a $H^{-n}\to L^2$ bound for $\chi\big(\overline{P}_\tau\big)^*_{\kappa,A,n}$. Thus, using the $L^2\to L^2$ bound (5.7), we deduce that

$$\big|\big|(\kappa A+i)^{-n}\mathrm{II}(\tau,\delta)(\kappa A-i)^{-n}\psi\big|\big|_{H^n} \leq C||\psi||_{H^{-n}},$$

and conclude that

$$\big|\big|(\kappa A+i)^{-n}\big(\overline{P}_\tau - 1 - i\delta\big)^{-n}(\kappa A-i)^{-n}\psi\big|\big|_{H^n} \leq C||\psi||_{H^{-n}},$$

for all $\psi\in L^2$, $\tau\in\mathbb{R}$ and $\delta\neq 0$. In terms of $\overline{P}$, this estimate reads

$$\big|\big|e^{i\tau A}(\kappa A+i)^{-n}\big(\overline{P} - z\delta\big)^{-n}(\kappa A-i)^{-n}\varphi\big|\big|_{H^n}, \leq C\lambda^{-n}||e^{i\tau A}\varphi||_{H^{-n}}, \tag{5.10}$$

where we recall that $\lambda = \mathrm{Re}(z)$. We can get rid of the negative powers of λ as follows. If $n\leq N-1$, we have on one hand the Sobolev embeddings

$$L^{p(n)}\subset H^{-n}, \qquad H^n \subset L^{q(n)}.$$

On the other hand, using (2.12) and the fact that $\lambda^{-1} = e^{2\tau}$, we have

$$\lambda^{-n}\ ||e^{i\tau A}||_{L^{p(n)}\to L^{p(n)}}||e^{i\tau A}||^{-1}_{L^{q(n)}\to L^{q(n)}} = 1. \tag{5.11}$$

Thus, by turning (5.10) into a $L^{p(n)}\to L^{q(n)}$ estimate and by using (5.11), we obtain (5.3). If $n = N$, the same argument applies using the Sobolev embeddings with

$$q(N) = \infty \ \text{ if } d \text{ is odd}, \qquad q(N) = q \text{ with an arbitrary } q > 2d \ \text{ if } d \text{ is even},$$

the only difference being that the left hand side of (5.11) becomes either $\lambda^{-1/2}$ or $\lambda^{-2d/q}$. $\square$

To apply Theorem 5.3 to perturbations of the Laplacian with coefficients in $S^{-\rho}$, we need the following result.

Proposition 5.4 (Symbol classes embeddings). *For all $1 \leq r \leq \bar{r}(d)$, $N \geq 0$ integers and $\mu > 0$ real, we have the continuous embeddings*

$$\begin{array}{rcl} S^{-\mu-1} & \subset & S^{1,r-1,N}, \\ S^{-\mu} & \subset & S^{0,r,N}. \end{array}$$

These embeddings are very convenient since the seminorms of the spaces $S^{-\mu}$ behave badly under scaling, unlike $S^{o,r-o,N}$ by Proposition 5.2.

Proof. We note first that

$$\langle x \rangle^{-\mu-|\beta|-o} \in L^{\frac{d}{|\beta|+o}}.$$

Furthermore, by an elementary induction, one checks that $(x \cdot \nabla)^n$ is a linear combination of $x^\alpha \partial^\alpha$ with $|\alpha| \leq n$. Therefore, we have the estimates

$$||(x \cdot \nabla)^n \partial^\beta a||_{L^{\frac{d}{|\beta|+o}}} \leq C ||\langle x \rangle^{\mu+o+|\beta|} (x \cdot \nabla)^n \partial^\beta a||_{L^\infty} \leq C \max_{|\alpha| \leq n} ||\langle x \rangle^{\mu+o+|\beta|+|\alpha|} \partial^{\alpha+\beta} a||_{L^\infty},$$

which lead easily to the result. □

By this proposition, we see that Theorem 5.3 holds if the coefficients of P are such that $a_{jk} - \delta_{jk}$ and b_k are small enough respectively in $S^{-\rho}$ and $S^{-1-\rho}$. We may also replace the weights $(\kappa A \pm i)^{-1}$ by powers of $\langle x \rangle^{-1}$ according to a classical procedure. This is the purpose of the following.

Corollary 5.5. *Assume (2.2) and that $a_{jk} - \delta_{jk} \in S^{-\rho}$ and $b_k \in S^{-1-\rho}$. Assume also that*

$$\left| \langle x \rangle^{\rho+|\alpha|} \partial^\alpha (a_{jk}(x) - \delta_{jk}) \right| + \left| \langle x \rangle^{1+\rho+|\alpha|} \partial^\alpha b_k(x) \right| \leq \epsilon, \tag{5.12}$$

for $|\alpha| \leq \bar{r}(d) + 1$. If ϵ is small enough, then for $1 \leq n \leq N := \bar{r}(d) + 1$

- *if $1 \leq n \leq \bar{r}(d)$*

$$\left| \left| \langle x \rangle^{-n} (\overline{P} - z)^{-n} \langle x \rangle^{-n} \varphi \right| \right|_{L^{q(n)}} \leq C ||\varphi||_{L^{p(n)}},$$

- *if $n = N$ and d is odd,*

$$\left| \left| \langle x \rangle^{-N} (\overline{P} - z)^{-N} \langle x \rangle^{-N} \varphi \right| \right|_{L^\infty} \leq C \mathrm{Re}(z)^{-1/2} ||\varphi||_{L^1},$$

- *if $n = N$ and d is even, then for all $2d < q < \infty$,*

$$\left| \left| \langle x \rangle^{-N} (\overline{P} - z)^{-N} \langle x \rangle^{-N} \varphi \right| \right|_{L^q} \leq C_q \mathrm{Re}(z)^{-\frac{2d}{q}} ||\varphi||_{L^{\frac{q}{q-1}}},$$

for

$$\varphi \in \mathcal{S}, \qquad 0 < \mathrm{Re}(z) < 1, \qquad \mathrm{Im}(z) \neq 0.$$

In particular, we may replace all L^p, L^q spaces above by L^2 if we change $\langle x \rangle^{-n}$ into $\langle x \rangle^{-2n-\varepsilon}$, for any $\varepsilon > 0$.

Proof. This kind of result is standard so we briefly recall the proof. Note first that (5.12) implies that items 2 and 3 of Theorem 5.3 are satisfied. Fix $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi \equiv 1$ near $[0, 1]$. By the Spectral Theorem and elliptic regularity

$$(1 - \chi(\overline{P}))(\overline{P} - z)^{-n},$$

maps H^{-n} to H^{n}, uniformly with respect to z, hence the appropriate Lebesgue spaces to their duals by Sobolev embeddings. Thus, it suffices to consider

$$\langle x\rangle^{-n}\big(\overline{P}-z\big)^{-n}\chi(\overline{P})\langle x\rangle^{-n}, \qquad \mathrm{Re}(z)\in(0,1),\ \mathrm{Im}(z)\neq 0.$$

By possibly choosing χ of the form φ^2 the result follows by writting

$$\langle x\rangle^{-n}\varphi(\overline{P})=\big(\langle x\rangle^{-n}\varphi(\overline{P})(\kappa A+i)^n\big)\,(\kappa A+i)^{-n},$$

where we observe that, for all $q\in[1,\infty]$,

$$\langle x\rangle^{-n}\varphi(\overline{P})(\kappa A+i)^n : L^q\to L^q,$$

for it is a pseudo-differential operator with symbol in $S^{-\infty}$ (see *e.g.* [1, Prop. 2.1]). The replacement of L^q spaces by L^2 after the replacement of $\langle x\rangle^{-n}$ by $\langle x\rangle^{-2n-\epsilon}$ follows from

$$||\langle x\rangle^{-n-\varepsilon}v||_{L^2}\leq C||v||_{L^{q(n)}}, \tag{5.13}$$

by the Hölder inequality (note that this works even for $n=N$ and $q(N)\in(2d,\infty]$). $\square$

In the next paragraph, we will also need the following result for $\mathrm{Re}(z)<0$.

Proposition 5.6. *Under the same assumptions as in Theorem 5.3, we have the following estimates:*

- *if* $1\leq n\leq\bar{r}(d)$
$$\big|\big|(\overline{P}-z)^{-n}\varphi\big|\big|_{L^{q(n)}}\leq C||\varphi||_{L^{p(n)}}, \tag{5.14}$$
- *if* $N=\bar{r}(d)+1$ *and* d *is odd,*
$$\big|\big|(\overline{P}-z)^{-N}\varphi\big|\big|_{L^\infty}\leq C|\mathrm{Re}(z)|^{-1/2}||\varphi||_{L^1}, \tag{5.15}$$
- *if* $N=\bar{r}(d)+1$ *and* d *is even, then for all* $2d<q<\infty$*,*
$$\big|\big|(\overline{P}-z)^{-N}\varphi\big|\big|_{L^q}\leq C_q|\mathrm{Re}(z)|^{-\frac{2d}{q}}||\varphi||_{L^{\frac{q}{q-1}}}, \tag{5.16}$$

all these estimates holding for

$$\varphi\in\mathcal{S}, \qquad \mathrm{Re}(z)<0, \qquad \mathrm{Im}(z)\neq 0.$$

Remark Since $(\kappa A\pm i)^{-1}$ preserve all L^p spaces for κ small enough, we may replace $(\overline{P}-z)^{-n}$ by

$$(\kappa A+i)^{-n}(\overline{P}-z)^{-n}(\kappa A-i)^{-n},$$

for $1\leq n\leq N$, in the estimates (5.14), (5.15) and (5.16). In particular, this shows that the estimates of Theorem 5.3 actually hold for $\mathrm{Re}(z)\in\mathbb{R}$. Also, using (5.13), we may clearly turn all the estimates of Proposition 5.6 into $L^2\to L^2$ estimates with weights.

Proof of Proposition 5.6. It is based on the same scaling argument as the proof of Theorem 5.3, from which we borrow the notation. We write $\mathrm{Re}(z)=-\lambda$ so that

$$\big(\overline{P}-z\big)^{-1}=\lambda^{-1}\big(\lambda^{-1}P+1+i\delta\big)^{-1}$$

the left hand side of which we write as

$$\lambda^{-1} e^{i\tau A} \big(P_\tau + 1 + i\delta\big)^{-1} e^{-i\tau A}.$$

We may then write

$$\big(P_\tau + 1\big)^{-n/2} \left(\frac{P_\tau + 1}{P_\tau + 1 + i\delta} \right)^n \big(P_\tau + 1\big)^{-n/2}.$$

This operator is bounded from H^{-n} to H^n, uniformly with respect to τ and δ (see (5.9)) and we conclude as in the proof of Theorem 5.3. $\square$

5.2 Non small perturbations

The purpose of this paragraph is to prove Theorems 1.1 and 1.2. We shall actually prove Theorem 1.2 first and then Theorem 1.1.

We start by doing some reductions.

We first choose suitable coordinates on $\mathbb{R}^d$ such that we may assume that $\det G(x) = 1$ outside a compact set. This is explained in Appendix A. We next conjugate in the usual way our Laplacian to get an operator which is self-adjoint with respect to the Lebesgue measure: the map $u \mapsto \det G(x)^{1/4} u$ is unitary from $L^2(\mathbb{R}^d, d_G x)$ onto $L^2(\mathbb{R}^d, dx)$ so $-\Delta_G$ is unitarily equivalent to the operator

$$P = -\det G(x)^{-1/4} \frac{\partial}{\partial x_j} \left(\det G(x)^{1/2} G_{jk}(x) \frac{\partial}{\partial x_k} \right) \det G(x)^{-1/4},$$

which has a self-adjoint closure $\overline{P}$, with domain H^2.

One may then clearly write

$$P = P_0 + W,$$

with

$$P_0 = a_{jk}(x) D_j D_k + b_k(x) D_k,$$

and

$$W = \chi_{jk}(x) D_j D_k + \theta_k(x) D_k + V(x). \tag{5.17}$$

such that

$$a_{jk} - \delta_{jk} \text{ is small enough in } S^{-\frac{\rho}{2}}, \qquad b_k \text{ is small enough in } S^{-1-\frac{\rho}{2}}, \tag{5.18}$$

and

$$\chi_{jk}, \theta_k, V \in C_0^\infty.$$

By small enough, we mean in (5.18) that we may assume that the estimates of Corollary 5.5 and Proposition 5.6 hold for $\overline{P}_0$. Here and in the sequel we denote by $\overline{P}_0$, $\overline{P}$ and $\overline{W}$ the $H^2 \to L^2$ closures of the corresponding differential operators which are a priori defined on $\mathcal{S}$. In particular, $\overline{P}_0$ and $\overline{P}$ are self-adjoint with domain H^2 and, by unitary equivalence with $-\Delta_G$, we have

$$\overline{P} \geq 0 \qquad \text{and} \qquad 0 \text{ is not an eigenvalue of } \overline{P}.$$

By the Spectral Theorem, it is sufficient to prove Theorem 1.2 with $(\overline{P} - z)^{-n}$ replaced by

$$R_\psi^n(z) := \psi\big(\overline{P}\big)\big(\overline{P} - z\big)^{-n}, \tag{5.19}$$

for some

$$\psi \in C_0^\infty(\mathbb{R}), \qquad \psi \equiv 1 \ \text{ near } 0.$$

It is also convenient to introduce $\Psi \in C_0^\infty(\mathbb{R})$ such that

$$\Psi\psi = \psi.$$

Both are chosen with values in $[0,1]$.

Proposition 5.7. *Let us set*

$$S_\Psi(z) = \overline{W}\big(\overline{P}_0 - z\big)^{-1}\Psi\big(\overline{P}\big),$$

and

$$B^1(z) = \Psi\big(\overline{P}\big)\big(\overline{P}_0 - z\big)^{-1}\psi\big(\overline{P}\big) - \Psi\big(\overline{P}\big)\big(\overline{P}_0 - z\big)^{-1}\overline{W}\psi\big(\overline{P}\big)\big(\overline{P}_0 - z\big)^{-1}\Psi\big(\overline{P}\big).$$

Then

$$R^1_\psi(z) = B^1(z) + S_\Psi(\bar{z})^* R^1_\psi(z) S_\Psi(z), \tag{5.20}$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$.

Proof. It is based on the resolvent identity, namely

$$\begin{aligned} \big(\overline{P} - z\big)^{-1} &= \big(\overline{P}_0 - z\big)^{-1} - \big(\overline{P}_0 - z\big)^{-1}\overline{W}\big(\overline{P} - z\big)^{-1}, \qquad (5.21)\\ &= \big(\overline{P}_0 - z\big)^{-1} - \big(\overline{P} - z\big)^{-1}\overline{W}\big(\overline{P}_0 - z\big)^{-1}. \qquad (5.22)\end{aligned}$$

The identity (5.20) is obtained by applying first $\psi(\overline{P})$ to the right of both sides of (5.21), then by inserting (5.22) on the right hand side of the resulting identity and finally by applying $\Psi(\overline{P})$ to the left and right. □

Our strategy is to show that one can make $S_\Psi(z)$ small enough (in operator norm on suitable weighted L^2 spaces) by choosing Ψ (and hence ψ) with a small enough support around 0 and by choosing z close enough to 0. To this end, we denote

$$z = \lambda + i\epsilon, \tag{5.23}$$

and introduce the decomposition

$$S_\Psi(z) \;=\; \overline{W}\big(\overline{P}_0 - i\epsilon\big)^{-1}\Psi\big(\overline{P}\big) \;+\; \overline{W}\left(\big(\overline{P}_0 - z\big)^{-1} - \big(\overline{P}_0 - i\epsilon\big)^{-1}\right)\Psi\big(\overline{P}\big). \tag{5.24}$$

Proposition 5.8. *Fix $M > 0$ and $\nu > 1$. If* $\operatorname{supp}(\Psi)$ *is contained in a sufficiently small neighborhood of* 0*, then*

$$\big|\big|\langle x\rangle^M \overline{W}\big(\overline{P}_0 - i\epsilon\big)^{-1}\Psi\big(\overline{P}\big)\langle x\rangle^{-\nu}\big|\big|_{L^2\to L^2} \le \frac{1}{4}, \qquad \epsilon > 0.$$

Proof. It suffices to show that, for some $\delta > 0$ as small as we want,

$$\big|\big|\langle x\rangle^M \overline{W}\big(\overline{P}_0 - i\epsilon\big)^{-1}\Psi\big(\overline{P}\big)\langle x\rangle^{-\nu}\big|\big|_{L^2\to L^2} \le C\big|\big|\Psi(\overline{P})\langle x\rangle^{-\delta}\big|\big|_{L^2\to L^2}, \tag{5.25}$$

since the norm in the right hand side goes to zero as the support of Ψ shrinks to $\{0\}$, for 0 is not an eigenvalue of $\overline{P}$. Recall that the classical trick is to choose a fixed smooth $\Psi_1 \equiv 1$ near 0 compactly supported and to write

$$\Psi(\overline{P})\langle x\rangle^{-\delta} = \Psi(\overline{P})\left(\Psi_1(\overline{P})\langle x\rangle^{-\delta}\right)$$

where the bracket is compact and $\Psi(\overline{P})$ goes weakly to zero as the support of Ψ shrinks to 0, which follows from the Spectral Theorem since $\{0\}$ has zero spectral measure. The second order and first order term of (5.17), namely $\overline{W}-V$, have a rather simple contribution. Indeed, we note that

$$||\langle x\rangle^M\big(\overline{W}-V\big)u||_{L^2}\leq C||\overline{P}_0 u||_{L^2},\qquad u\in H^2,$$

using (2.3), (2.4) and (2.10) for P_0. By the Spectral Theorem $\overline{P}_0\big(\overline{P}_0-i\epsilon\big)^{-1}$ is uniformly bounded on L^2 and thus

$$||\langle x\rangle^M\big(\overline{W}-V\big)\big(\overline{P}_0-i\epsilon\big)^{-1}\Psi\big(\overline{P}\big)\langle x\rangle^{-\nu}||_{L^2}\leq C||\Psi\big(\overline{P}\big)\langle x\rangle^{-\nu}||_{L^2\to L^2}.$$

We now consider V alone. Since $\langle x\rangle^M V$ has compact support, the Poincaré-Sobolev inequality and (2.9) for $\overline{P}_0$ yield

$$||\langle x\rangle^M Vu||_{L^2}\leq C_1||u||_{L^{2^*}}\leq C_2||\nabla u||_{L^2}\leq C_3\big|\big|\overline{P}_0^{1/2}u\big|\big|_{L^2}. \tag{5.26}$$

We also observe that

$$||\overline{P}^{1/2}u||_{L^2}\leq C||\overline{P}_0^{1/2}u||_{L^2}, \tag{5.27}$$

since, by the compact support of $\nabla\det G(x)$ and the Poincaré-Sobolev inequality,

$$||\overline{P}^{1/2}u||_{L^2}^2\leq C_1\big|\big|\nabla\big(\det G(x)^{-1/4}u\big)\big|\big|_{L^2}^2\leq C_2\left(||\nabla u||_{L^2}+||u||_{L^{2^*}}\right)^2\leq C_3||\nabla u||_{L^2}^2.$$

Therefore, by (5.26) and (5.27), we have

$$||\langle x\rangle^M V\big(\overline{P}_0-i\epsilon\big)^{-1}\overline{P}^{1/2}u||_{L^2}\leq C||u||_{L^2},\qquad \epsilon>0, \tag{5.28}$$

for all $u\in H^2$. On the other hand, by approximating $(\overline{P})^{-1/2}$ by

$$S_n:=\frac{1}{\sqrt{\pi}}\int_0^n e^{-t\overline{P}}\frac{dt}{t^{1/2}},$$

in the sense that $\overline{P}^{1/2}S_n\to I$ strongly on L^2, when applied to an H^2 function (see for instance [2]), we deduce from (5.28) that

$$||\langle x\rangle^M V\big(\overline{P}_0-i\epsilon\big)^{-1}\Psi\big(\overline{P}\big)\langle x\rangle^{-\nu}||_{L^2\to L^2}\leq C\sup_n||S_n\Psi\big(\overline{P}\big)\langle x\rangle^{-\nu}||_{L^2\to L^2}.$$

Since S_n commutes with $\Psi\big(\overline{P}\big)$, we shall obtain (5.25) if we show that, for some $\delta>0$,

$$\sup_n||\langle x\rangle^\delta S_n\langle x\rangle^{-\nu}||_{L^2\to L^2}<\infty.$$

By the usual heat kernel bounds for Δ_G (*e.g.* [11] and references therein) and the fact that the Euclidean distance $|x-y|$ is bounded from above and below by the geodesic distance $d_G(x,y)$, we have

$$[e^{-t\overline{P}}](x,y)\leq Ct^{-d/2}\exp(-|x-y|^2/Ct).$$

By integrating this estimate in t, we obtain that the kernel of $\langle x\rangle^\delta S_n\langle x\rangle^{-\delta}$ satisfies

$$\begin{aligned}0\leq\big[\langle x\rangle^\delta S_n\langle x\rangle^{-\delta}\big](x,y)&\leq C\langle x-y\rangle^\delta|x-y|^{1-d},\\&\leq C|x-y|^{1-d+\delta}+f(x-y),\end{aligned}$$

with $f \in L^1$. The convolution with f is bounded on L^2 hence so is the operator with kernel $f(x-y)\langle y\rangle^{\delta-\nu}$, if $\delta \leq \nu$. We now consider the first term in the last line. By the Hardy-Littlewood-Sobolev inequality, the operator with kernel

$$|x-y|^{1-d+\delta}\langle y\rangle^{\delta-\nu},$$

is continuous on L^2 if $\delta > 0$ is small enough, since the convolution by $|\cdot|^{1-d+\delta}$ maps $L^{\frac{2d}{d+2+2\delta}}$ into L^2 and the multiplication by $\langle\cdot\rangle^{\delta-\nu}$ maps L^2 into $L^{\frac{2d}{d+2+2\delta}}$. This shows that $\langle x\rangle^{\delta} S_n \langle x\rangle^{-\delta+(\delta-\nu)}$ is uniformly bounded on L^2 and the result follows. $\square$

We consider now the second term of (5.24).

Proposition 5.9. *Fix $M > 0$, $\nu > 4$ and $\Psi \in C_0^\infty$. Then, if ϵ_0 is small enough, we have*

$$\left|\left|\langle x\rangle^M \overline{W}\left(\left(\overline{P}_0 - z\right)^{-1} - \left(\overline{P}_0 - i\epsilon\right)^{-1}\right)\Psi\left(\overline{P}\right)\langle x\rangle^{-\nu}\right|\right|_{L^2\to L^2} \leq \frac{1}{4},$$

for $0 < |z| < \epsilon_0$ (recall also the notation (5.23)).

Proof. Recall first the standard fact that $\Psi\left(\overline{P}\right)$ preserves $\langle x\rangle^{-\nu}$, ie $\langle x\rangle^{\nu}\Psi\left(\overline{P}\right)\langle x\rangle^{-\nu}$ has a bounded closure on L^2 (this follows for instance from [1, Prop. 2.1]). Similarly $\langle x\rangle^M \overline{W}(\overline{P_0}+1)^{-1}\langle x\rangle^{\nu}$ is bounded on L^2 since $\langle x\rangle^M \overline{W}$ has compact support and $(\overline{P_0}+1)^{-1}$ preserves polynomial decay. It is thus sufficient to show that

$$\left|\left|\langle x\rangle^{-\nu}(\overline{P}_0+1)\left(\left(\overline{P}_0 - z\right)^{-1} - \left(\overline{P}_0 - i\epsilon\right)^{-1}\right)\langle x\rangle^{-\nu}\right|\right|_{L^2\to L^2} \to 0, \qquad |z| \to 0.$$

By writing

$$\begin{aligned}\left(\overline{P}_0+1\right)\left(\left(\overline{P}_0 - z\right)^{-1} - \left(\overline{P}_0 - i\epsilon\right)^{-1}\right) &= \left(\overline{P}_0 - z\right)^{-1} - \left(\overline{P}_0 - i\epsilon\right)^{-1} + \frac{z}{\overline{P}_0 - z} - \frac{i\epsilon}{\overline{P_0} - i\epsilon} \\ &= \int_0^{\lambda} (\overline{P}_0 - \mu - i\epsilon)^{-2} d\mu + \frac{z}{\overline{P}_0 - z} - \frac{i\epsilon}{\overline{P_0} - i\epsilon},\end{aligned}$$

the result follows from the bounds in Corollary 5.5 for $\overline{P}_0$ with $n = 1, 2$, using in particular the integrability in μ of $||\langle x\rangle^{-\nu}(\overline{P}_0 - \mu - i\epsilon)^{-2}\langle x\rangle^{-\nu}||_{L^2\to L^2}$. $\square$

Proof of Theorem 1.2. We first prove the theorem for some large enough ν independent of n, namely $\nu > 2N$. We shall see in the end of the proof how this implies the full result. So let us assume that $\nu > 2N$. By Propositions 5.8 and 5.9, by choosing Ψ with support close enough to 0 and by restricting z to the region $0 < |z| \leq \epsilon_0$ with ϵ_0 small enough, we may assume that

$$\left|\left|\langle x\rangle^{\nu} S_{\Psi}(z)\langle x\rangle^{-\nu}\right|\right|_{L^2\to L^2} \leq 1/2.$$

Therefore, by (5.20), we have

$$||\langle x\rangle^{-\nu} R_{\psi}^1(z)\langle x\rangle^{-\nu}||_{L^2\to L^2} \leq \frac{4}{3}||\langle x\rangle^{-\nu} B^1(z)\langle x\rangle^{-\nu}||_{L^2\to L^2},$$

where we observe that the right hand side is bounded with respect to z: indeed, if we set more generally

$$B^n(z) = \partial_z^{n-1} B^1(z), \qquad 1 \leq n \leq N,$$

Corollary 5.5 and Proposition 5.6 for $\overline{P}_0$ show that we have

- if $1 \leq n \leq \bar{r}(d)$

$$\left|\left|\langle x\rangle^{-\nu} B^n(z) \langle x\rangle^{-\nu}\right|\right|_{L^2 \to L^2} \leq C$$

- if $n = N = \bar{r}(d) + 1$ and d is odd,

$$\left|\left|\langle x\rangle^{-\nu} B^N(z) \langle x\rangle^{-\nu}\right|\right|_{L^2 \to L^2} \leq C |\mathrm{Re}(z)|^{-1/2},$$

- if $n = N$ and d is even, then for all $2d < q < \infty$,

$$\left|\left|\langle x\rangle^{-\nu} B^N(z) \langle x\rangle^{-\nu}\right|\right|_{L^2 \to L^2} \leq C_q |\mathrm{Re}(z)|^{-\frac{2d}{q}},$$

for $\mathrm{Re}(z) \neq 0$ and $\mathrm{Im}(z) \neq 0$. We also have the same estimates for $\langle x\rangle^{\nu} \partial_z^{n-1} S_\Psi(z) \langle x\rangle^{-\nu}$. In particular for $n = 1$, this shows that

$$||\langle x\rangle^{-\nu} R_\psi^1(z) \langle x\rangle^{-\nu}||_{L^2 \to L^2} \leq C, \qquad 0 < |z| < \epsilon_0.$$

For $n \geq 2$, we proceed as follows. By applying ∂_z^{n-1} to (5.20), we obtain

$$\langle x\rangle^{-\nu} R_\psi^n(z) \langle x\rangle^{-\nu} = \langle x\rangle^{-\nu} \widetilde{B}^n(z) \langle x\rangle^{-\nu} + \langle x\rangle^{-\nu} S_\Psi(\bar{z})^* R_\psi^n(z) S_\Psi(z) \langle x\rangle^{-\nu},$$

where, by an elementary induction, we see that $\widetilde{B}^n(z)$ satisfy the same estimates as $B^n(z)$. Therefore

$$||\langle x\rangle^{-\nu} R_\psi^n(z) \langle x\rangle^{-\nu}||_{L^2 \to L^2} \leq \frac{4}{3} ||\langle x\rangle^{-\nu} \widetilde{B}^n(z) \langle x\rangle^{-\nu}||_{L^2 \to L^2},$$

where the right hand side satisfies the expected estimates. We thus get the result with $\nu > 2N$ for all $n = 1, \ldots, N$. To see that one can choose $\nu > 2n$, we proceed as follows. Fix $M > 2N$ and write, by (5.20),

$$\langle x\rangle^{-\nu} R_\psi^1(z) \langle x\rangle^{-\nu} = \langle x\rangle^{-\nu} B^1(z) \langle x\rangle^{-\nu} + \langle x\rangle^{-\nu} S_\Psi(\bar{z})^* \langle x\rangle^{M} \left(\langle x\rangle^{-M} R_\psi^1(z) \langle x\rangle^{-M}\right) \langle x\rangle^{M} S_\Psi(z) \langle x\rangle^{-\nu}.$$

We observe in this identity that $\langle x\rangle^{-\nu} B^1(z) \langle x\rangle^{-\nu}$ is bounded with respect to z, by the resolvent estimates for $\overline{P}_0$. The same holds for $\langle x\rangle^M S_\Psi(z) \langle x\rangle^{-\nu}$ since $\langle x\rangle^M$ is harmless for $\overline{W}$ has compactly supported coefficients. Therefore, the boundedness of $\langle x\rangle^{-M} R_\psi^1(z) \langle x\rangle^{-M}$ proved above gives the result for $n = 1$. For $n \geq 2$, we differentiate $n - 1$ times with respect to z and proceed as before. □

Proof of Theorem 1.1. We may again replace $(\overline{P} - z)^{-n}$ by its spectrally localized version (5.19) since $(1 - \psi)(\overline{P})(\overline{P} - z)^{-n}$ maps H^{-n} to H^n, with bound independent of z for small z, and thus satisfies the expected $L^p \to L^{p'}$ boundedness. Let us consider first $n = 1$. Then $p(1) = 2_*$ and $q(1) = 2^*$ (recall that $2_* = 2d/(d+2)$). We start with (5.20) in which we observe that

$$||(\kappa A + i)^{-1} B^1(z) (\kappa A - i)^{-1}||_{L^{2_*} \to L^{2^*}} \leq C, \qquad 0 < |z| \leq 1. \tag{5.29}$$

The estimate (5.29) follows from

$$||(\kappa A + i)^{-1} (\overline{P}_0 - z)^{-1} (\kappa A - i)^{-1}||_{L^{2_*} \to L^{2^*}} \leq C, \qquad |\mathrm{Re}(z)| \leq 1, \tag{5.30}$$

by Theorem 5.3 and Proposition 5.6 for P_0, and from

$$||(\kappa A + i)^{-1} \Psi(\overline{P}) (\kappa A + i)||_{L^{2^*} \to L^{2^*}} < \infty, \tag{5.31}$$

$$||(\kappa A - i) \overline{W} \psi(\overline{P}) (\kappa A + i)||_{L^{2^*} \to L^{2_*}} < \infty, \tag{5.32}$$

as well as the adjoint estimates or similar ones with ψ instead of Ψ. The estimates (5.31) and (5.32) follow easily from the fact that $\Psi(\overline{P})$ and $\psi(\overline{P})$ are pseudo-differential operators (see [1]) with symbols in $S^{-\infty}(\mathbb{R}^d \times \mathbb{R}^d)$ and the fact that W has compactly supported coefficients. We also have, for any $\nu > 2$,

$$||\langle x\rangle^{\nu}\Psi(\overline{P})S_{\Psi}(z)(\kappa A - i)^{-1}||_{L^{2*}\to L^2} \leq C, \qquad |\mathrm{Re}(z)| < 1, \tag{5.33}$$

by (5.30) and the estimates

$$||\langle x\rangle^{\nu}\Psi(\overline{P})\overline{W}(\kappa A + i)||_{L^{2^*}\to L^2} < \infty, \tag{5.34}$$

$$||(\kappa A - i)\Psi(\overline{P})(\kappa A - i)^{-1}||_{L^{2*}\to L^{2*}} < \infty, \tag{5.35}$$

which follow again from the fact that $\Psi(\overline{P})$ is a pseudo-differential operator of order $-\infty$ and the compact support of the coefficients of W. Therefore, by (5.20) where one can replace $R^1_{\psi}(z)$ by $\Psi(\overline{P})R^1_{\psi}(z)\Psi(\overline{P})$ in the right hand side, and by (5.29) and (5.33), we obtain

$$||(\kappa A + i)^{-1}R^1_{\psi}(z)(\kappa A - i)^{-1}||_{L^{2*}\to L^{2*}} \leq C\left(1 + ||\langle x\rangle^{-\nu}R^1_{\psi}(z)\langle x\rangle^{-\nu}||_{L^2\to L^2}\right),$$

so the result follows from Theorem 1.2. For $n \geq 2$, we proceed by induction as in the proof of Theorem 1.2 by applying ∂_z^{n-1} to (5.20). We omit the details but rather point out that the analogues of the estimates (5.31), (5.32), (5.34) and (5.35) associated to $q(n)$ don't cause any trouble when $q(n) = q(N) = \infty$ since they involve pseudo-differential operators of order $-\infty$ (but no zero order pseudo-differential operator) which are bounded on all L^p spaces for $p \in [1, \infty]$. $\square$

6 Local energy decay

The purpose of this section is to prove Theorems 1.3 and 1.4. For convenience, we work with the self-adjoint realization $\overline{P}$ on $L^2(\mathbb{R}^d, dx)$ of

$$P = -\det\, G(x)^{-1/4}\frac{\partial}{\partial x_j}\left(\det\, G(x)^{1/2}G_{jk}(x)\frac{\partial}{\partial x_k}\right)\det\, G(x)^{-1/4},$$

which is unitarily equivalent to $-\Delta_G$ on $L^2(\mathbb{R}^d, d_G x)$.

6.1 Spectral localization

Let $m \geq 0$ be a real number and $\alpha = 1$ or $1/2$. In this paragraph, we define $U(t)$ by

$$U(t) = e^{it(\overline{P}+m^2)^{\alpha}},$$

which will allow to cover simultaneously the Schrödinger ($m = 0$, $\alpha = 1$), Klein-Gordon ($m > 0$, $\alpha = 1/2$) and wave equations ($m = 0$, $\alpha = 1/2$). Our purpose here is to reduce estimates on such flows to spectrally localized estimates. Actually, the result of this subsection only uses that $U(t)$ is some bounded function of $\overline{P}$ and nothing else.

Consider a dyadic partition of unity

$$\begin{aligned} 1 &= \Phi_0(\lambda) + \sum_{k\geq 0}\varphi(2^{-k}\lambda) && (6.1)\\ &= \Phi_0(\lambda) + \Phi(\lambda), && (6.2)\end{aligned}$$

defined for λ near $[0, \infty)$, with

$$\Phi_0 \in C_0^\infty(\mathbb{R}), \qquad \varphi \in C_0^\infty(0, +\infty).$$

We also select ψ such that

$$\psi \in C_0^\infty(0, +\infty), \qquad \psi \equiv 1 \ \text{ near } \operatorname{supp}(\varphi). \tag{6.3}$$

It will be convenient to denote

$$E_\nu(h, t) = \langle x \rangle^{-\nu} U(t) \varphi(h^2 \overline{P}) \langle x \rangle^{-\nu},$$

and

$$e_\nu(h, t) = \big|\big| E_\nu(h, t) \big|\big|_{L^2 \to L^2}. \tag{6.4}$$

Our main purpose here is to show the following proposition.

Proposition 6.1. *For all $\nu \geq 0$ and $M > 0$ there exists $C > 0$ such that*

$$\big|\big| \langle x \rangle^{-\nu} \Phi(\overline{P}) U(t) \langle x \rangle^{-\nu} u \big|\big|_{L^2}^2 \leq C \sum_{h^2 = 2^{-k}} e_\nu(h, t)^2 \left(\big|\big| \psi(h^2 \overline{P}) u \big|\big|_{L^2}^2 + h^M \big|\big| (1 + \overline{P})^{-M/2} u \big|\big|_{L^2}^2 \right),$$

for all $t \in \mathbb{R}$ and $u \in \mathcal{S}(\mathbb{R}^d)$. Here Φ is defined in (6.2).

As a corollary, we obtain the following estimate which we shall use in Subsection 6.3.

Corollary 6.2. *For all $\nu \geq 0$ and $s \in \mathbb{R}$, one has*

$$\big|\big| \langle x \rangle^{-\nu} \Phi(\overline{P}) U(t) \langle x \rangle^{-\nu} u \big|\big|_{L^2} \leq C_{\nu, s} \left(\sup_{h \in (0,1]} h^s e_\nu(h, t) \right) ||u||_{H^s},$$

for all $t \geq 0$ and $u \in \mathcal{S}(\mathbb{R}^d)$.

Proof of Corollary 6.2. By the Spectral Theorem, we have

$$h^{-s} ||\psi(h^2 \overline{P}) u||_{L^2} \leq C ||\psi(h^2 \overline{P})(1 + \overline{P})^{s/2} u||_{L^2},$$

for all $u \in L^2$. Since $||(1 + \overline{P})^{s/2} u||_{L^2} \leq C ||u||_{H^s}$ by classical elliptic estimates, we obtain, by almost orthogonality,

$$\sum_{h^2 = 2^{-k}} h^{-2s} ||\psi(h^2 \overline{P}) u||_{L^2}^2 \leq C ||u||_{H^s}^2. \tag{6.5}$$

On the other hand, by Proposition 6.1, we have

$$\big|\big| \langle x \rangle^{-\nu} \Phi(\overline{P}) U(t) \langle x \rangle^{-\nu} u \big|\big|_{L^2}^2 \leq C \sum_{h^2 = 2^{-k}} e_\nu(h, t)^2 h^{2s} \left(h^{-2s} \big|\big| \psi(h^2 \overline{P}) u \big|\big|_{L^2}^2 + h^{M - 2s} \big|\big| u \big|\big|_{H^{-M}}^2 \right).$$

Choosing $M > 2|s|$, we have $\sum_h h^{M-2s} < \infty$, $||u||_{H^{-M}} \leq ||u||_{H^s}$ and we conclude using (6.5). $\square$

We now consider the proof of Proposition 6.1. Write first

$$\langle x \rangle^{-\nu} \Phi(\overline{P}) U(t) \langle x \rangle^{-\nu} u = \sum_{h^2 = 2^{-k}} E_\nu(h, t) u,$$

where the sum converges weakly (and actually in L^2 by the analysis below). We will need the following result.

Lemma 6.3. *For all $M \geq 0$, one has*

$$\varphi(h^2\overline{P})\langle x\rangle^{-\nu}(1-\psi(h^2\overline{P})) = h^M\varphi(h^2\overline{P})\langle x\rangle^{-\nu}R_{M,\nu}(h),$$

with

$$\big|\big|R_{M,\nu}(h)(1+\overline{P})^{M/2}\big|\big|_{L^2\to L^2} \leq C, \qquad h\in(0,1].$$

Proof. By (6.3), we can select $\widetilde{\varphi}\in C_0^\infty(0,+\infty)$ such that

$$\varphi\widetilde{\varphi} = \varphi \qquad \text{and} \qquad \psi\equiv 1 \ \text{ near } \operatorname{supp}(\widetilde{\varphi}),$$

and thus write

$$\varphi(h^2\overline{P})\langle x\rangle^{-\nu} = \varphi(h^2\overline{P})\langle x\rangle^{-\nu}\big(\langle x\rangle^{\nu}\widetilde{\varphi}(h^2\overline{P})\langle x\rangle^{-\nu}\big).$$

The result follows then from the fact that, for all M,

$$\big|\big|\big(\langle x\rangle^{\nu}\widetilde{\varphi}(h^2\overline{P})\langle x\rangle^{-\nu}\big)(1-\psi(h^2\overline{P}))(1+\overline{P})^{M/2}\big|\big|_{L^2\to L^2} \leq C_{M,\nu}h^M, \qquad h\in(0,1],$$

by pseudo-differential functional calculus (*e.g.* [1]), since all terms of the pseudo-differential expansion cancel because $\widetilde{\varphi}$ and $1-\psi$ have disjoint supports. □

Proof of Proposition 6.1. By Lemma 6.3, we have

$$E_\nu(h,t) = \psi(h^2\overline{P})E_\nu(h,t)\psi(h^2\overline{P}) + h^M R_{M,\nu}(h)^*E_\nu(h,t)\psi(h^2\overline{P}) + h^M E_\nu(h,t)R_{M,\nu}(h),$$

and the result will follow from the estimates on each term given below.

1. *1st term.* By almost orthogonality, we have

$$\begin{aligned}\Bigg|\Bigg|\sum_{h^2=2^{-k}}\psi(h^2\overline{P})E_\nu(h,t)\psi(h^2\overline{P})u\Bigg|\Bigg|_{L^2}^2 &\leq C\sum_{h^2=2^{-k}}||E_\nu(h,t)\psi(h^2\overline{P})u||_{L^2}^2,\\ &\leq C\sum_{h^2=2^{-k}}e_\nu(h,t)^2||\psi(h^2\overline{P})u||_{L^2}^2.\end{aligned}$$

2. *2nd term.* Since $||R_{M,\nu}(h)^*||_{L^2\to L^2}\leq C$ by Lemma 6.3, we also have

$$\begin{aligned}\big|\big|\sum_{h^2=2^{-k}}h^M R_{M,\nu}(h)^*E_\nu(h,t)\psi(h^2\overline{P})u\big|\big|_{L^2} &\leq C\sum_{h^2=2^{-k}}h^M\big|\big|E_\nu(h,t)\psi(h^2\overline{P})u\big|\big|_{L^2},\\ &\leq C\left(\sum_{h^2=2^{-k}}e_\nu(h,t)^2\big|\big|\psi(h^2\overline{P})u\big|\big|_{L^2}^2\right)^{1/2},\end{aligned}$$

by the Cauchy-Schwarz inequality since $\sum_{h^2=2^{-k}}h^{2M}<\infty$.

3. *3rd term.* By Lemma 6.3,

$$\begin{aligned}\big|\big|\sum_{h^2=2^{-k}}h^M E_\nu(h,t)R_{M,\nu}(h)u\big|\big| &\leq C\sum_{h^2=2^{-k}}h^M e_\nu(h,t)||(1+\overline{P})^{-M/2}u||_{L^2},\\ &\leq C\left(\sum_{h^2=2^{-k}}h^M e_\nu(h,t)^2||(1+\overline{P})^{-M/2}u||_{L^2}^2\right)^{1/2},\end{aligned}$$

again by the Cauchy-Schwartz inequality since $\sum_{h^2=2^{-k}}h^M<\infty$.

The proof is complete. □

6.2 Semiclassical estimates

To prove quantitative decay rates for the Schrödinger group, we shall use integration by parts in the Stone formula. For this purpose, we need to estimate powers of the resolvent. In this subsection, we show that, if one has semiclassical estimates for the resolvent, then one has estimates for its powers. For simplicity, we will only consider the square of the resolvent, but higher powers can be treated similarly.

We introduce the usual notation

$$R(z,h) = (h^2\overline{P} - z)^{-1}.$$

Throughout this subsection, $J_0 \Subset (0,\infty)$ will be a relatively compact interval satisfying the following condition.

Assumption A. *There exist a real number $\nu_0 \geq 0$ and a function $F : (0,1] \to (0,+\infty)$ satisfying*

$$F(h) \gtrsim h^{-1}, \tag{6.6}$$

such that, for all $\nu > \nu_0$ and all open interval $J \Subset J_0$,

$$||\langle x\rangle^{-\nu} R(z,h) \langle x\rangle^{-\nu}||_{L^2\to L^2} \leq C_{\nu,J} F(h), \qquad h \in (0,1],\ \mathrm{Re}(z) \in J. \tag{6.7}$$

Without any condition on G, such estimates holds with $F(h) = Ce^{C/h}$ ([6, 7] and [8]). When the geodesic flow is non trapping, one can choose $F(h) = C/h$ [32, 40, 16, 31, 38]. In some cases where one has weak trapping one may take $F(h) = C|\log h|/h$ or polynomial powers of h^{-1} [26, 28].

Our purpose here is to prove the following.

Proposition 6.4. *If Assumption A holds then, for all $\nu > \nu_0$ and all interval $J \Subset J_0$, there exists $C > 0$ such that*

$$||\langle x\rangle^{-\nu-1} R(z,h)^2 \langle x\rangle^{-\nu-1}||_{L^2\to L^2} \leq CF(h)^2, \tag{6.8}$$

for all $h \in (0,1]$ and all z such that $\mathrm{Re}(z) \in J$.

The principle of the proof below is well known (see [20] and [21]) but we recall the main steps to emphasize the behaviour with respect to h (the previous works addressed either the case $h = 1$ or the high energy limit for potentials, which is a non trapping case). The approach is based on microlocal parametrices of the semiclassical Schrödinger group $e^{-ith\overline{P}}$, from which we recover the resolvent by

$$R(z,h) = \frac{i}{h}\int_0^{\pm\infty} e^{itz/h} e^{-ith\overline{P}} dt, \qquad \pm\mathrm{Im}(z) > 0. \tag{6.9}$$

It is convenient to record the following elementary lemma.

Lemma 6.5. *Let $A(t), B(t)$ be bounded operators on $L^2(\mathbb{R}^d)$, strongly continuous with respect to t and such that, for some $N \geq 0$,*

$$||A(t)||_{L^2\to L^2} + ||B(t)||_{L^2\to L^2} \leq C\langle t\rangle^N, \qquad t \in \mathbb{R}.$$

Then

$$\int_0^{\pm\infty} e^{it\zeta}\left(\int_0^t A(t-s)B(s)ds\right)dt = \left(\int_0^{\pm\infty} e^{it\zeta}A(t)dt\right)\left(\int_0^{\pm\infty} e^{it\zeta}B(t)dt\right),$$

provided that

$$\pm\mathrm{Im}(\zeta) > 0.$$

We will use the well known Isozaki-Kitada parametrix, introduced first for potential scattering (see [19]). Here we need it in the metric case with a semiclassical parameter. In this context, we refer for instance to [1] for the details or proofs of the statements quoted below, in particular Lemma 6.6. We recall only what is necessary for the proof of Proposition 6.4.

Denote by $S_{\rm scat}(\mu,-\infty)$ the set of smooth functions a on $\mathbb{R}^{2d}$ such that, for all $M>0$,

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leq C_{\alpha\beta M} \langle x\rangle^{\mu-|\alpha|}\langle \xi\rangle^{-M},$$

where the best constants $C_{\alpha\beta M}$ are seminorms for which it is a Fréchet space.

Given real numbers $R>0$, $\sigma\in(-1,1)$ and any interval $I\Subset(0,+\infty)$, one defines the outgoing $(+)$ and incoming $(-)$ areas by

$$\Gamma^{\pm}(R,I,\sigma) := \left\{ (x,\xi)\in\mathbb{R}^{2d} \ | \ |x|>R, \ |\xi|^2\in I, \ \pm x\cdot\xi > \sigma|x||\xi| \right\}.$$

It turns out that, for any I and σ as above, one can choose R large enough so that one can solve the following eikonal equations

$$\nabla_x \varphi^{\pm}(x,\xi)\cdot G(x)^{-1}\nabla_x\varphi^{\pm}(x,\xi) = |\xi|^2,$$

for $(x,\xi)\in\Gamma^{\pm}(R,I,\sigma)$ with solutions which are close to the free phase $x\cdot\xi$ (*i.e.* the solution if $G\equiv I$) in the sense that

$$\left|\partial_x^\alpha\partial_\xi^\beta(\varphi^{\pm}(x,\xi)-x\cdot\xi)\right| \leq C_{\alpha\beta}\langle x\rangle^{1-\rho-|\alpha|}, \qquad (x,\xi)\in\Gamma^{\pm}(R,I,\sigma),$$

where $\rho>0$ is the same as in (1.1). One can then define the following Fourier integral operators

$$J_{\pm}(a^{\pm})u(x,h) = (2\pi h)^{-d}\iint e^{\frac{i}{h}(\varphi^{\pm}(x,\xi)-y\cdot\xi)}a^{\pm}(x,\xi)u(y)dyd\xi,$$

for symbols such that

$$a^{\pm}\in S_{\rm scat}(0,-\infty), \qquad \operatorname{supp}(a^{\pm})\Subset\Gamma^{\pm}(R,I,\sigma).$$

We can now give a form of the Isozaki-Kitada parametrix.

Lemma 6.6 (Isozaki-Kitada parametrix)**.** *Fix two intervals $I\Subset I'\Subset(0,+\infty)$. Then, for all R large enough and all*

$$\chi^{\pm}\in S_{\rm scat}(0,-\infty), \qquad \operatorname{supp}(\chi^{\pm})\subset\Gamma^{\pm}(R,I,-1/2),$$

we can find, for all $M\geq 0$, symbols

$$\begin{array}{rcll} a_M^{\pm}(h) & \in & S_{\rm scat}(0,-\infty), & \operatorname{supp}(a_M^{\pm}(h))\subset\Gamma^{\pm}\big(R^{1/4},I',-9/10\big), \\ b_M^{\pm}(h) & \in & S_{\rm scat}(0,-\infty), & \operatorname{supp}(b_M^{\pm}(h))\subset\Gamma^{\pm}\big(R^{1/2},I',-3/4\big), \\ r_M^{\pm}(h) & \in & S_{\rm scat}(-2M,-\infty), & \end{array}$$

bounded with respect to h in their classes, such that

1.

$$\begin{aligned} e^{-ith\overline{P}}\chi^{+}(x,hD) &= J_{+}\big(a_M^{+}(h)\big)e^{ith\Delta}J_{+}\big(b_M^{+}(h)\big)^{*} + h^M e^{-ith\overline{P}}r_M^{+}(x,hD,h) \\ &\quad + h^M\int_0^t e^{-i(t-s)h\overline{P}}B_M^{+}(s,h)ds, \end{aligned}$$

with $B_M^{+}(s,h)$ strongly continuous with respect to s and such that

$$\left|\left|\langle x\rangle^M B_M^{+}(s,h)\langle x\rangle^M\right|\right|_{L^2\to L^2} \leq C\langle s\rangle^{-M}, \qquad s\geq 0, \ h\in(0,1].$$

2. *(Adjoint case)*

$$\begin{aligned}\chi^-(x,hD)e^{-ith\overline{P}} &= J_-\big(b_M^-(h)\big)e^{ith\Delta}J_-\big(a_M^-(h)\big)^* + h^M r_M^-(x,hD,h)e^{-ith\overline{P}} \\ &\quad + h^M\int_0^t B_M^-(-s,h)e^{-i(t-s)h\overline{P}}ds,\end{aligned}$$

with $B_M^-(-s,h)$ strongly continuous with respect to s and such that

$$\big|\big|\langle x\rangle^M B_M^-(-s,h)\langle x\rangle^M\big|\big|_{L^2\to L^2} \leq C\langle s\rangle^{-M}, \qquad s\geq 0,\ h\in(0,1].$$

We simply point out that this lemma gives good approximations for $t\geq 0$ only, which will be sufficient for us. There is of course a similar statement for negative times by exchanging $+$ and $-$ everywhere.

We also mention that the symbols $a_M^{\pm}(h)$ and $b_M^{\pm}(h)$ are finite sums of the form $\sum h^j c_j^{\pm}$ with $c_j\in S_{\mathrm{scat}}(-j,-\infty)$ independent of h. The following lemma will thus be useful to estimate the leading terms of the parametrix. Again, we consider only positive times.

Lemma 6.7 (Free propagation estimates)**.** *Let $\mu_1\geq\mu_2\geq 0$ be real numbers, $I\Subset(0,+\infty)$ an interval and $\sigma\in(-1,1)$. Then, for all R large enough and all symbol $c^{\pm}$ satisfying,*

$$c^{\pm}\in S_{\mathrm{scat}}(0,-\infty), \qquad \mathrm{supp}(c^{\pm})\subset\Gamma^{\pm}(R,I,\sigma),$$

we have

1. $$\big|\big|\langle x\rangle^{-\mu_1}e^{ith\Delta}J_+(c^+)^*\langle x\rangle^{\mu_2}\big|\big| \leq C\langle t\rangle^{\mu_2-\mu_1}, \qquad t\geq 0,\quad h\in(0,1],$$

2. *(Adjoint case)*

$$\big|\big|\langle x\rangle^{\mu_2}J_-(c^-)e^{ith\Delta}\langle x\rangle^{-\mu_1}\big|\big| \leq C\langle t\rangle^{\mu_2-\mu_1}, \qquad t\geq 0,\quad h\in(0,1].$$

We refer for instance to [20] or [1] for a proof of this lemma, which is fairly elementary and follows from integrations by parts in the (explicit) kernel of the operators for integers μ_1,μ_2 and then by an interpolation argument for real ones.

Proof of Proposition 6.4. We may assume that $\mathrm{Im}(z)>0$, otherwise one takes the adjoint. By the Spectral Theorem, it is sufficient to prove a $\mathcal{O}(F(h)^2)$ upper bound for

$$\phi(h^2\overline{P})R(z,h)^2 = R(z,h)\phi(h^2\overline{P})R(z,h),$$

with $\phi\in C_0^\infty(0,+\infty)$ which is equal to 1 near the interval J where $\mathrm{Re}(z)$ lives. Let $\chi\in C_0^\infty(\mathbb{R}^d)$ such that $\chi(x)=1$ for $|x|\leq R$, with R to be chosen below according to Lemmas 6.6 and 6.7. Then

$$\phi(h^2\overline{P})R(z,h)^2 = R(z,h)\phi(h^2\overline{P})\chi R(z,h) + R(z,h)\phi(h^2\overline{P})(1-\chi)R(z,h). \tag{6.10}$$

Since, for any $M>0$, there exists $C>0$ such that

$$\big|\big|\langle x\rangle^M\phi(h^2\overline{P})\chi\langle x\rangle^M\big|\big|_{L^2\to L^2}\leq C, \qquad h\in(0,1],$$

the $F(h)^2$ upper bound for the first term in the right hand side of (6.10), weighted on both sides by $\langle x\rangle^{-\nu-1}$, follows easily from (6.7). Note that the extra power $\langle x\rangle^{-1}$ is useless for this term. In the second term, we use the following pseudo-differential expansion (see [1]): for all $M\geq 1$,

$$\phi(h^2\overline{P})(1-\chi) = \sum_{j<M}h^j\chi_j^+(x,hD) + \sum_{j<M}h^j\chi_j^-(x,hD) + h^M R_M(h),$$

where, if $I \Subset (0,+\infty)$ is a neighborhood of $\mathrm{supp}(\phi)$ and R is large enough,

$$\chi_j^{\pm} \in S_{\mathrm{scat}}(-j,-\infty), \qquad \mathrm{supp}\big(\chi_j^{\pm}\big) \in \Gamma^{\pm}(R,I,-1/2),$$

and

$$\big|\big|\langle x\rangle^{M/2} R_M(h) \langle x\rangle^{M/2}\big|\big|_{L^2\to L^2} \leq C, \qquad h\in(0,1].$$

By choosing M large enough, the contribution of $R_M(h)$ is treated similarly to the one of $\phi(h^2\overline{P})\chi$ above, so we are left with the study of terms of the form

$$\langle x\rangle^{-\nu-1} R(z,h)\chi^{\pm}(x,hD)R(z,h)\langle x\rangle^{-\nu-1}.$$

The idea is to use Lemma 6.6 for

$$R(z,h)\chi^{+}(x,hD) \qquad \text{and} \qquad \chi^{-}(x,hD)R(z,h),$$

by expanding $R(z,h)$ via (6.9), with $t\geq 0$ since $\mathrm{Im}(z)>0$. We consider χ^+. By Lemma 6.5 and item 1 of Lemma 6.6, we have

$$R(z,h)\chi^+(x,hD) = J_+\big(a_M^+(h)\big)(-h^2\Delta - z)^{-1} J_+\big(b_M^+(h)\big)^* + h^{M-1}R(z,h)R_M^+(h),$$

where

$$R_M^+(h) = hr_M^+(x,hD,h) + \int_0^{+\infty} e^{-itz/h} B_M^+(t,h)dt,$$

satisfies

$$\big|\big|\langle x\rangle^{M} R_M^+(h)\langle x\rangle^{M/8}\big|\big|_{L^2\to L^2} \leq C, \qquad h\in(0,1].$$

The contribution of $R_M^+(h)$ is thus similar to the one of $R_M(h)$ and $\phi(h^2\overline{P})\chi$ above. We then consider

$$\langle x\rangle^{-\nu-1}\left(J_+\big(a_M^+(h)\big)(-h^2\Delta-z)^{-1}J_+\big(b_M^+(h)\big)^*\right) R(h,z)\langle x\rangle^{-\nu-1}.$$

Choose ν' such that

$$\nu_0 < \nu' < \nu.$$

Then, by Assumption A,

$$\big|\big|\langle x\rangle^{-\nu'}R(h,z)\langle x\rangle^{-\nu-1}\big|\big| \leq CF(h), \qquad h\in(0,1].$$

On the other hand, using item 1 of Lemma 6.7, (6.9) for $-\Delta$ and (6.6), we have

$$\begin{aligned}\left|\left|\langle x\rangle^{-\nu-1}\left(J_+\big(a_M^+(h)\big)(-h^2\Delta-z)^{-1}J_+\big(b_M^+(h)\big)^*\right)\langle x\rangle^{\nu'}\right|\right|_{L^2\to L^2} &\leq Ch^{-1},\\ &\leq CF(h).\end{aligned}$$

Here we use the additional fact that $\langle x\rangle^{-\nu-1}J_+(a_M^+(h))\langle x\rangle^{\nu+1}$ is bounded on L^2, uniformly in h. All this shows that $||\langle x\rangle^{-\nu-1}R(z,h)\chi^+(x,hD)R(z,h)\langle x\rangle^{-\nu-1}||_{L^2\to L^2}$ is bounded by $CF(h)^2$. The same analysis holds for χ^- using the Adjoint Cases in Lemma 6.6 and Lemma 6.7 and this completes the proof. $\square$

6.3 Time decay

In this paragraph, we prove Theorem 1.3 and Theorem 1.4.

The following proposition will give the contribution of the low frequencies.

Proposition 6.8. *Let $m > 0$ and $\chi \in C_0^\infty(\mathbb{R}, \mathbb{R})$. For each $t \in \mathbb{R}$, let $\varphi_t(\lambda)$ denote any of the following functions*

$$\chi(\lambda)e^{-it\lambda}, \qquad \chi(\lambda)e^{-it(|\lambda|+m^2)^{1/2}}, \qquad \chi(\lambda)\cos\left(t|\lambda|^{1/2}\right), \qquad \chi(\lambda)\frac{\sin\left(t|\lambda|^{1/2}\right)}{|\lambda|^{1/2}}.$$

Then, for all $\nu > 2(\bar{r}(d)+1)$, there exists C such that

$$\left|\left|\langle x\rangle^{-\nu}\varphi_t(\overline{P})\langle x\rangle^{-\nu}\right|\right|_{L^2\to L^2} \leq C\langle t\rangle^{-\bar{r}(d)}, \qquad t \in \mathbb{R}. \tag{6.11}$$

Remark. We note that the $L^2 \to L^2$ estimate of this proposition can be turned into a $H^{-s} \to H^s$ estimate for all $s \geq 0$. Indeed, one can write

$$\varphi_t(\overline{P}) = \widetilde{\chi}(\overline{P})\varphi_t(\overline{P})\widetilde{\chi}(\overline{P}),$$

with $\widetilde{\chi} \in C_0^\infty$ such that $\widetilde{\chi}\chi = \chi$, and use the fact that for any $\nu \geq 0$

$$\langle x\rangle^{-\nu}\widetilde{\chi}(\overline{P})\langle x\rangle^{\nu} : L^2 \to H^s,$$

is bounded (which follows for instance from the form of $\widetilde{\chi}(\overline{P})$ given in [1]).

We quote the following result whose proof can be found in [30].

Lemma 6.9 (Stone's formula)**.** *For all compactly supported continuous function $\varphi \in C_0^0(\mathbb{R})$, one has*

$$\varphi(\overline{P}) = \lim_{\delta\downarrow 0}\frac{1}{\pi}\int_{\mathbb{R}}\varphi(\lambda)\ \mathrm{Im}(\overline{P}-\lambda-i\delta)^{-1}d\lambda,$$

the limit being taken in the strong sense. Here $\mathrm{Im}B = \frac{B-B^*}{2i}$.

Proof of Proposition 6.8. Denote for simplicity $r = \bar{r}(d)$. In the first case, r integrations by part in the integral yield

$$t^r\int_{\mathbb{R}}\varphi_t(\lambda)\mathrm{Im}(\overline{P}-\lambda-i\delta)^{-1}d\lambda = (-i)^r\int_{\mathbb{R}}e^{-it\lambda}\partial_\lambda^r\left(\chi(\lambda)\mathrm{Im}(\overline{P}-\lambda-i\delta)^{-1}\right)d\lambda. \tag{6.12}$$

By Theorem 1.2, the first r derivatives with respect to λ of $(\overline{P}-\lambda\pm i\delta)^{-1}$ are integrable near 0, in the suitable weighted spaces, with uniform bounds in δ. Thus the right hand side of (6.12) is bounded uniformly with respect to δ and t and the result follows by using Lemma 6.9. The second case is similar, once we have noticed the following points. Since $\overline{P}$ is non negative, we may modify χ as we wish on $(-\infty, 0)$ without changing the operator $\varphi_t(\overline{P})$. In particular, we may assume that χ is supported in $\{\lambda > -m^2/2\}$ and then

$$\varphi_t(\overline{P}) = \chi(\overline{P})e^{it(\overline{P}+m^2)^{1/2}} = \widetilde{\varphi}_t(\overline{P}),$$

with $\widetilde{\varphi}_t(\lambda) = \chi(\lambda) e^{-it(\lambda+m^2)^{1/2}}$. The result follows again by integrating by part, using

$$te^{-it(\lambda+m^2)^{1/2}} = 2i(\lambda+m^2)^{1/2}\partial_\lambda e^{-it(\lambda+m^2)^{1/2}}$$

on the support of χ where $\lambda + m^2 > m^2/2$. In the last two cases, we need to work a little bit more since we shall have boundary terms in the integrations by part. We treat the last case, the third one being similar. By setting

$$B_\delta(\lambda) = \chi(\lambda)(\overline{P} - \lambda - i\delta)^{-1},$$

and by the change of variables $\lambda = \pm\mu^2$ on $\mathbb{R}^\pm$, we have

$$\int_{\mathbb{R}} \frac{\sin(t|\lambda|^{1/2})}{|\lambda|^{1/2}} \chi(\lambda) \mathrm{Im}(\overline{P} - \lambda - i\delta)^{-1} d\lambda = 2\int_0^\infty \sin(t\mu) \mathrm{Im}\left(B_\delta(\mu^2) + B_\delta(-\mu^2)\right) d\mu. \tag{6.13}$$

By r integrations by part as before, t^r times the right hand side of (6.13) is a linear combination of boundary terms of the form

$$t^m \mathrm{Im}\big((\overline{P} - i\delta)^{-k}\big)\chi^{(j)}(0), \qquad 0 \leq m + k \leq r, \tag{6.14}$$

and of integrals of the form

$$\int_0^\infty e^{\pm it\mu} \mu^l \chi^{(j)}(\pm\mu^2) \mathrm{Im}\left((\overline{P} \pm \mu^2 - i\delta)^{-k-1}\right) d\mu, \tag{6.15}$$

with everywhere

$$0 \leq j, k \leq r, \quad l \geq 0, \qquad \text{and} \quad l \geq 1 \quad \text{when} \quad k = r.$$

By Theorem 1.2, the integrals (6.15) are uniformly bounded with respect to δ and t since the resolvents are bounded (once weighted according to Theorem 1.2, as will be implicit throughout this proof), except perhaps when $k = r$ in which case they are at most of order $|\mu|^{-1}$, but this is controlled by the term μ^l with $l \geq 1$. To complete the proof, it suffices to show that the boundary terms (6.14) are bounded with respect to δ and t. This is clear if $m = 0$ since $k \leq r$ then, and the resolvent to this power is bounded near the origin. It remains to show that (6.14) goes to zero as $\delta \to 0$ if $m \geq 1$. Indeed, by writing

$$\mathrm{Im}\big((\overline{P} - i\delta)^{-k}\big) = \frac{\delta}{2}\int_{-\pi/2}^{\pi/2} (\overline{P} - \delta e^{i\theta})^{-k-1} e^{i\theta} d\theta,$$

and using that $k \leq r - 1$, we see that the limit is zero as $\delta \to 0$ since we have a uniform bound for the resolvent inside the integral, since $k + 1 \leq r$. The result follows. $\square$

Proof of Theorem 1.3. We study first the Schrödinger equation. We consider the second half of the partition of unity (6.2). Using Proposition 6.4 and the same integration by parts trick in the Stone formula as in the proof of Proposition 6.8 (which is now simpler since we have no boundary term and no singularity), we see that if N is large enough, then

$$\big|\big|\langle x\rangle^{-N} e^{-it\overline{P}} \varphi(h^2\overline{P})\langle x\rangle^{-N}\big|\big|_{L^2\to L^2} \leq C\langle t\rangle^{-1} e^{C/h}, \qquad t \in \mathbb{R},\ h \in (0,1].$$

By interpolation between this bound and the trivial bound $\big|\big|e^{-it\overline{P}}\varphi(h^2\overline{P})\big|\big|_{L^2\to L^2} \leq C$, we see that, for any $\theta \in (0,1)$,

$$\big|\big|\langle x\rangle^{-\theta N} e^{-it\overline{P}} \varphi(h^2\overline{P})\langle x\rangle^{-\theta N}\big|\big|_{L^2\to L^2} \leq C\langle t\rangle^{-\theta} e^{C\theta/h}, \qquad t \in \mathbb{R},\ h \in (0,1].$$

Fix $\nu > 0$ and choose θ such that $\nu = N\theta$, we then have the following alternative:

1. in the region where $e^{C/h}\langle t\rangle^{-\theta/2} \le 1$, we have

$$\big|\big|\langle x\rangle^{-\nu} e^{-it\overline{P}}\varphi(h^2\overline{P})\langle x\rangle^{-\nu}\big|\big|_{L^2\to L^2} \le C\langle t\rangle^{-\theta/2},$$

2. in the region $e^{C/h}\langle t\rangle^{-\theta/2} > 1$, we have $\log\langle t\rangle < 2C/\theta h$ so we obtain

$$h^s \le C(1+\log\langle t\rangle)^{-s},$$

and have anyway the trivial bound

$$\big|\big|\langle x\rangle^{-\nu} e^{-it\overline{P}}\varphi(h^2\overline{P})\langle x\rangle^{-\nu}\big|\big|_{L^2\to L^2} \le C.$$

This discussion shows that

$$h^s\big|\big|\langle x\rangle^{-\nu} e^{-it\overline{P}}\varphi(h^2\overline{P})\langle x\rangle^{-\nu}\big|\big|_{L^2\to L^2} \le C(1+\log\langle t\rangle)^{-s}, \qquad t\in\mathbb{R}, \quad h\in(0,1],$$

and we conclude using Corollary 6.2 and Proposition 6.8 to handle the low frequency part. More precisely, for the latter, we interpolate between (6.11) for $\chi = \Phi_0$ (see (6.2)) and the trivial bound $||\Phi_0(\overline{P})e^{-it\overline{P}}||_{L^2\to L^2} \le C$ to be able to use the weight $\langle x\rangle^{-\nu}$, in which case we still have a polynomial time decay rate hence a logarithmic one.

The proof is completely similar for the wave and Klein-Gordon equations, using only the additional fact that, if $\widetilde{\Phi}\Phi = \Phi$ and $\widetilde{\Phi} \equiv 0$ near 0,

$$\langle x\rangle^{-\nu}\widetilde{\Phi}(\overline{P})(\overline{P}+m^2)^{-1/2}\langle x\rangle^{\nu}$$

is a bounded operator from H^s to H^{s+1} for any fixed $\nu \ge 0$, $m \ge 0$ and $s \in \mathbb{R}$.

□

Proof of Theorem 1.4. By the non trapping assumption, we have the semiclassical estimates (see [40, 31])

$$\big|\big|\langle x\rangle^{-\nu}\varphi(h^2\overline{P})e^{-ith\overline{P}}\langle x\rangle^{-\nu}\big|\big|_{L^2\to L^2} \le C\langle t\rangle^{-s}, \qquad t\in\mathbb{R},\ h\in(0,1],$$

provided that

$$0 \le s < \nu.$$

In the non semiclassical time scaling, this gives

$$\big|\big|\langle x\rangle^{-\nu}\varphi(h^2\overline{P})e^{-it\overline{P}}\langle x\rangle^{-\nu}\big|\big|_{L^2\to L^2} \le C\langle t/h\rangle^{-s} \le Ch^s\langle t\rangle^{-s},$$

which, using the notation (6.4), shows that

$$h^{-s}e_\nu(h,t) \le C\langle t\rangle^{-s}, \qquad t\in\mathbb{R},\ h\in(0,1].$$

Using Corollary 6.2, we obtain

$$\big|\big|\langle x\rangle^{-s}\Phi(\overline{P})e^{-it\overline{P}}\langle x\rangle^{-s}\big|\big|_{H^{-s}\to L^2} \le C\langle t\rangle^{-s}.$$

Since we may assume that $2(\bar{r}(d)+1) < s < \nu$, the right hand side decays faster than $\langle t\rangle^{-\bar{r}(d)}$ so the conclusion follows from Proposition 6.8 and the remark thereafter. □

A Change of coordinates

In this appendix, we recall how to choose a smooth diffeomorphism $\chi : \mathbb{R}^d \to \mathbb{R}^d$ such that χ^*G has determinant 1 outside a compact set and is still a long range perturbation of the Euclidean metric. Recall that, if $G = (G^{jk})$ then,

$$\chi^*G = \big(\widetilde{G}^{jk}(y)\big) = {}^t\mathrm{Jac}_x(\chi)^{-1}(G^{jk}(x))\mathrm{Jac}_x(\chi)^{-1}, \qquad y = \chi(x), \tag{A.1}$$

where $\mathrm{Jac}_x(\chi)$ is the Jacobian matrix of χ at x. We shall show the following.

Proposition A.1. *Assume that $0 < \rho < 1$. One can choose a smooth function $\phi : \mathbb{R}^d \to \mathbb{R}$ such that, for some $C > 0$*

1. *for all $x \in \mathbb{R}^d$,*
$$C^{-1} \leq \phi(x) \leq C,$$
2. *$\phi - 1$ is a symbol of order $-\rho$, ie*
$$|\partial_x^\alpha(\phi(x) - 1)| \leq C_\alpha \langle x\rangle^{-\rho-|\alpha|},$$
3. *The map χ defined below is diffeomorphism from $\mathbb{R}^d$ to $\mathbb{R}^d$,*
$$\chi(x) = \phi(x)x,$$
4. *for all $|x| \geq C$,*
$$\begin{aligned} \det(G^{jk}(x))^{1/2} &= \phi(x)^{n-1}\left(\phi(x) + x\cdot\nabla\phi(x)\right), && \text{(A.2)}\\ &= \det\left(\mathrm{Jac}_x(\chi)\right). && \text{(A.3)}\end{aligned}$$
5. *The metric $\widetilde{G}$ defined by (A.1) is a long range perturbation of the Euclidean metric, ie*
$$\left|\partial_y^\alpha\big(\widetilde{G}^{jk}(y) - \delta_{jk}\big)\right| \leq C_\alpha\langle y\rangle^{-\rho-|\alpha|}.$$

Proof. We first solve (A.2) for $|x| \geq R$, for some $R > 0$ to be chosen. Since G is a long range perturbation of the Euclidean metric, we have $\det(G)^{1/2} = 1 + \delta$, with $\delta \in S^{-\rho}$. By the change of unknown function $\phi^n = \varphi$, (A.2) reads

$$\varphi + \frac{x}{n}\cdot\nabla\varphi(x) = 1 + \delta(x), \qquad |x| \geq R. \tag{A.4}$$

This is a transport equation which is easily solved using polar coordinates; the solution which is equal to 1 on $|x| = R$ is given by

$$\varphi(x) = 1 + n\int_1^{|x|/R}\delta(x/\tau)\frac{d\tau}{\tau^{n+1}}, \qquad |x| \geq R. \tag{A.5}$$

It is well defined for any $R > 0$ and is clearly smooth. We shall choose R large enough to guarantee that φ is close enough to 1 and thus that $\varphi^{1/n}$ is still smooth. Indeed, we have $|x|/\tau \geq R$ on the interval of integration, thus

$$|\varphi(x) - 1| \leq \sup_{|z|\geq R}|\delta(z)|, \qquad |x| \geq R,$$

where the right hand side goes to zero as $R \to \infty$. By multiplying φ by a smooth cutoff with values in $[0,1]$ which equals 1 near infinity and 0 near $\{|x| \leq R\}$, we obtain a new function φ defined on $\mathbb{R}^d$, such that $|\varphi - 1| \leq 1/2$ everywhere which satisfies (A.4) for a larger R. Furthermore, by choosing R large enough, we may even assume that

$$\frac{1}{2} \leq \varphi(x) + \frac{x}{n} \cdot \nabla \varphi(x) \leq \frac{3}{2}. \tag{A.6}$$

We next check that $\varphi - 1 \in S^{-\rho}$. Since this is a condition at infinity, it is sufficient to consider the expression (A.5). Using that $\delta(z) \leq C|z|^{-\rho}$ we have,

$$|\varphi(x) - 1| \leq n \int_1^\infty |\delta(x/\tau)| \frac{d\tau}{\tau^{n+1}} \leq \frac{C}{|x|^\rho} \int_1^\infty \frac{\tau^\rho}{\tau^{n+1}} d\tau,$$

where the last integral is finite since $\rho < 1$. It is then not hard to check that

$$|\partial_x^\alpha (\varphi(x) - 1)| \leq C_\alpha |x|^{-\rho - |\alpha|}, \qquad |x| > R,$$

by showing by induction that $\partial_x^\alpha(\varphi(x) - 1)$ is a linear combination of a symbol of order $-|\alpha| - \rho$ and of terms of the form $s_\gamma(x) \int_1^{|x|/R} (\partial^\gamma \delta)(x/\tau) \tau^{-n-1-|\gamma|} d\tau$ with s_γ a symbol of order $|\gamma| - |\alpha|$, for $|\gamma| \leq |\alpha|$.

Setting $\phi = \varphi^{1/n}$, we get a function satisfying the items 1 and 2, as well as (A.2). We now prove item 3. It is not hard to check that χ is a diffeomorphism if and only if, given $\omega \in \mathbb{S}^{d-1}$, the map $r \mapsto r\phi(r\omega)$ is diffeomorphism from $\mathbb{R}^+$ onto itself. The derivative of this function is

$$\phi(r\omega) + r\omega \cdot \nabla \phi(r\omega) = \varphi(x)^{\frac{1}{n} - 1} \left(\varphi(x) + \frac{x}{n} \cdot \nabla \varphi(x) \right)_{x = r\omega},$$

so the result follows from (A.6) and item 1. To prove (A.3), one simply observes that

$$\mathrm{Jac}_x(\chi) = \phi(x) I_d + \nabla \phi(x)^T x,$$

where the first matrix in the right hand side is scalar and the second matrix has rank one. The only possible non zero eigenvalue of the latter is given by its trace which is $x \cdot \nabla \phi(x)$. The calculation of the determinant is thus easy and shows that (A.3) coincides with the right hand side of (A.2). Finally, since $\mathrm{Jac}_x(\chi) - I_d$ is a matrix with entries in $S^{-\rho}$ then so is the right hand side of (A.1) and item 5 follows from a simple induction on $|\alpha|$ using that $\langle x \rangle^{-\rho} \approx \langle \chi(x) \rangle^{-\rho}$ (ie their quotient is bounded from above and below) and $|\partial^\alpha \chi(x)| \leq C_\alpha \langle x \rangle^{1-|\alpha|}$ for $|\alpha| \geq 1$. □

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