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Propagation estimates for the Schrödinger equation

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Workshop on Harmonic Analysis and Spectral Theory

Consider a differential operator in divergence form, on $\mathbb{R}^d$, $d \geq 3$,

$$P = -\text{div}\left(G(x)\nabla\right),$$

with $G(x)$ a real, positive definite matrix, such that,

$$c \leq G(x) \leq C, \qquad x \in \mathbb{R}^d,$$

for some $C, c > 0$.

Under weak regularity assumptions on G, P has a selfadjoint realization on $L^2(\mathbb{R}^d)$ and one may define its *resolvent*

$$R(z) = (P - z)^{-1} : \text{Dom}(P) \to L^2(\mathbb{R}^d), \qquad z \in \mathbb{C} \setminus \mathbb{R},$$

which is bounded on L^2:

$$||R(z)||_{L^2 \to L^2} \leq |\text{Im}(z)|^{-1}.$$

One may (and will) more generally consider powers of the resolvent

$$R(z)^k = (P - z)^{-k} = \partial_z^{k-1}(P - z)^{-1}/(k-1)!$$

In this talk, we are interested in the limit $\boxed{\mathrm{Im}(z) \to 0}$ of (powers of) the resolvent.

1. <u>If $\mathrm{Re}(z) < 0$</u>: no problem ! $R(z)$ is bounded on L^2 since

$$\mathrm{Re}\left(u, (P-z)u\right)_{L^2} \geq c||\nabla u||_{L^2}^2 - \mathrm{Re}(z)||u||_{L^2}^2,$$

hence

$$||R(z)f||_{L^2} \leq -\frac{1}{\mathrm{Re(z)}}||f||_{L^2}.$$

2. <u>If $\mathrm{Re}(z) = 0$</u>. The situation is more difficult but, under very general conditions one may define

$$P^{-1} = \int_0^{+\infty} e^{-tP} dt : L^{\frac{2d}{d+2}}(\mathbb{R}^d) \to L^{\frac{2d}{d-2}}(\mathbb{R}^d).$$

One uses heat kernel bounds

$$0 \leq \left[e^{-tP}\right](x,y) \lesssim t^{-\frac{d}{2}} e^{-c\frac{|x-y|^2}{t}}, \qquad t > 0,$$

which imply

$$\left[P^{-1}\right](x,y) \lesssim |x-y|^{2-d},$$

and then concludes with Hardy-Littlewood-Sobolev inequality.

If $\mathrm{Re}(z) > 0$?

One needs much stronger assumptions on G. Here we will assume that, for some $\rho > 0$,

$$|\partial^\alpha (G(x) - I_d)| \lesssim \langle x \rangle^{-\rho-|\alpha|}. \qquad (1)$$

This is a flatness assumption at infinity: P is a *long range* perturbation of $-\Delta$ (*short range* $\leftrightarrow$ $\rho > 1$).

The spectrum of P is then the half line $[0, +\infty)$.

Absence of embedded eigenvalues *Any $u \in L^2$ such that*

$$Pu = \lambda u,$$

for some $\lambda \geq 0$, is identically 0. (Most general proof by Koch-Tataru '06; previous results by Froese-Herbst-Hoffmann-Ostenhoff and Cotta-Ramuniso-Krüger-Schrader)

Consider the *generator of dilations* (on L^2)

$$A = \frac{x \cdot \nabla + \nabla \cdot x}{2i},$$

ie the selfadjoint generator of the unitary group

$$e^{i\tau A}\varphi(x) = e^{\tau \frac{d}{2}}\varphi(e^{\tau}x).$$

One controls the behavior of the resolvent as $\mathrm{Im}(z) \to 0$ as follows.

Jensen-Mourre-Perry weighted estimates *For any* $I \Subset (0, +\infty)$ *and any* $k \geq 1$,

$$\sup_{\mathrm{Re}(z) \in I} ||(A+i)^{-k} R(z)^k (A-i)^{-k}||_{L^2 \to L^2} < \infty.$$

Furthermore, the limits

$$R(\lambda \pm i0)^k = \lim_{\epsilon \to 0^+} R(\lambda \pm i\epsilon)^k, \qquad \lambda > 0,$$

exist (in weighted spaces) and

$$R(\lambda \pm i0)^k = \partial_\lambda^{k-1} R(\lambda \pm i0)/(k-1)!.$$

Here the weights $(A \pm i)^{-1}$ may be replaced by $\langle x \rangle^{-1}$.

A formal computation

Consider the time dependent Schrödinger equation

$$i\partial_t u - Pu = 0, \qquad u_{|t=0} = u_0 \in L^2,$$

ie $u(t) = e^{-itP}u_0$. By the Spectral Theorem

$$e^{-itP} = \int e^{-it\lambda} dE_\lambda,$$

where the spectral measure can be (formally) written as

$$2i\pi \frac{dE_\lambda}{d\lambda} = (P - \lambda - i0)^{-1} - (P - \lambda + i0)^{-1}.$$

Thus, by (formal) integrations by parts

$$t^k e^{-itP} = c_k \int_{\mathbb{R}} e^{-it\lambda} \Big((P - \lambda - i0)^{-k-1} - (P - \lambda + i0)^{-k-1} \Big) d\lambda.$$

Conclusion. If the R.H.S. is bounded in t, then we get a time decay for e^{-itP}.

Problem. To justify the integrations by parts, we need to know the behaviour of $(P - \lambda \pm i0)^{-k-1}$ at the thresholds: $\lambda \to 0$, $\lambda \to +\infty$.

Behavior of the resolvent as $\lambda \to \infty$ *Under the non trapping condition, one has for all* $k \geq 1$,

$$||\langle x\rangle^{-k}(P-\lambda \pm i0)^{-k}\langle x\rangle^{-k}||_{L^2 \to L^2} \lesssim \lambda^{-\frac{k}{2}}, \qquad \lambda \to \infty.$$

From such well known estimates and the integrations by parts trick, one gets spectrally localized estimates of the form

$$||\langle x\rangle^{-k} e^{-itP}(1-\varphi)(P)\langle x\rangle^{-k}||_{L^2 \to L^2} \leq C_{\varphi,k}\langle t\rangle^{-k},$$

if $\varphi \in C_0^\infty(\mathbb{R})$ satisfies $\varphi \equiv 1$ near 0, and $k \geq 0$.

To avoid the spectral cutoffs, we need to study the regime $\lambda \to 0$.

Results

Let $N(d)$ be the largest even integer $< \frac{d}{2} + 1$.

Theorem 1 *If* $\nu > \frac{d}{2} + N(d)$, *then, as* $|\lambda| \to 0$

$$||\langle x\rangle^{-\nu}(P-\lambda\pm i0)^{-N(d)}\langle x\rangle^{-\nu}||_{L^2\to L^2} \lesssim \begin{cases} |\lambda|^{-1/2} & \text{if } d \equiv 3 \text{ mod } 4, \\ |\lambda|^{-\varepsilon} \text{ for any } \varepsilon & \text{if } d \equiv 0 \text{ mod } 4 \\ 1 & \text{otherwise.} \end{cases}$$

Theorem 2 *Under the non trapping condition,*

$$||\langle x\rangle^{-\nu}e^{-itP}\langle x\rangle^{-\nu}||_{L^2\to L^2} \lesssim \langle t\rangle^{1-N(d)}.$$

Main steps of the proof

Assume for simplicity that $G - I_d$ is small everywhere.

1 - Scaling

$$P - \lambda - i\epsilon = \lambda e^{i\tau A} \left(P_\lambda - 1 - i\mu \right) e^{-i\tau A}$$

with $\mu = \epsilon / \lambda$,

$$P_\lambda = -\mathrm{div} \left(G_\lambda(x) \nabla \right), \qquad G_\lambda(x) = G \left(\frac{x}{\lambda^{1/2}} \right),$$

and τ such that

$$\left(e^{-i\tau A} \varphi \right)(x) = \lambda^{-d/4} \varphi(x/\lambda^{1/2}).$$

Interest: prove estimates for the resolvent of P_λ near energy 1 (ie away from the 0 threshold).

Problem: behavior of the coefficients of P_λ as $\lambda \to 0$ (the condition (1) for G_λ is not uniform with respect to λ).

2- Jensen-Mourre-Perry estimates. We obtain, for any $k \in \mathbb{N}$,

$$\sup_{\substack{\mu \in \mathbb{R}\setminus 0, \\ \lambda > 0}} ||(A+i)^{-k}(P_\lambda - 1 - i\mu)^{-k}(A-i)^{-k}||_{L^2 \to L^2} < \infty.$$

These estimates rely on the positive commutator estimate

$$\begin{aligned} i[P_\lambda, A] &= -\mathrm{div}\,(2G_\lambda(x) - (x \cdot \nabla G_\lambda)(x))\,\nabla \\ &\geq -\Delta, \end{aligned}$$

if $||G_\lambda - I_d||_\infty + ||x \cdot \nabla G_\lambda||_\infty = ||G - I_d||_\infty + ||x \cdot \nabla G||_\infty$ is small enough, and on the fact that higher commutators

$$\mathrm{ad}_A^k(P_\lambda) = \left[A, \mathrm{ad}_A^{k-1}(P_\lambda)\right] \qquad \mathrm{ad}_A^0(P_\lambda) = P_\lambda,$$

are bounded from H^{-1} to H^1.

The uniformity of the bounds w.r.t. λ is simply due to the fact that we only need to control the scale invariant norms

$$||(x \cdot \nabla)^j G_\lambda||_\infty = ||(x \cdot \nabla)^j G||_\infty.$$

3- Elliptic estimates. Let $N = N(d)$. We show that we can improve L^2 bounds into

$$\sup_{\substack{\mu\in\mathbb{R}\setminus 0,\\ \lambda>0}} ||(hA+i)^{-N}(P_\lambda - 1 - i\mu)^{-N}(hA-i)^{-N}||_{H^{-N}\to H^N} < \infty.$$

for some fixed $h > 0$ small enough.

1. Choose h small to guarantee that $(hA \pm i)^{-1}$ is bounded on $H^{\pm N}$.

2. Pick $\phi \in C_0^\infty(0,\infty)$, $\phi \equiv 1$ near 1. Then

$$\begin{aligned}(P_\lambda - z)^{-N} &= \phi(P_\lambda)(P_\lambda - z)^{-N}\phi(P_\lambda) + (1-\phi^2(P_\lambda))(P_\lambda - z)^{-N} \\ &= I + II.\end{aligned}$$

By the Spectral Theorem,

$$II = (P_\lambda + 1)^{-N/2} B_\lambda(z)(P_\lambda + 1)^{-N/2},$$

with $B_\lambda(z)$ bounded in L^2 uniformly w.r.t. $\lambda > 0$ and $\mathrm{Re}(z) = 1$.

Lemma. *If the scale invariant norms* $||\partial^\alpha(G - I_d)||_{L^{d/|\alpha|}}$ *are small enough for* $|\alpha| < d/2$, then

$$\sup_{\lambda>0} ||(P_\lambda + 1)^{-N/2}||_{H^{-N}\to L^2} \lesssim 1.$$

3. By setting

$$K_\lambda^- = (hA - i)^N \phi(P_\lambda), \qquad K_\lambda^+ = \left(K_\lambda^-\right)^*,$$

observe that

$$I = K_\lambda^+ (hA + i)^{-N} (P_\lambda - 1 - i\mu)^{-N} (hA - i)^{-N} K_\lambda^-.$$

Lemma. *If the scale invariant norms*

$$||(x \cdot \nabla)^j \partial^\alpha (G - I_d)||_{L^{d/|\alpha|}}, \qquad |\alpha| < \frac{d}{2}, \quad j \leq N(d),$$

are small enough, then

$$\sup_{\lambda>0} ||K_\lambda^- (hA - i)^{-N}||_{H^{-N}\to L^2} < \infty.$$

4- Conclusion: Sobolev imbeddings . We obtain: for some $h > 0$,

$$\sup_{\substack{\lambda>0,\\ \mu\in\mathbb{R}\setminus\{0\}}} ||(hA+i)^{-N}(P_\lambda - 1 - i\mu)^{-N}(hA-i)^{-N}||_{L^p\to L^{p'}} =: C_N < \infty,$$

with $N = N(d)$ and

$$p = \frac{2d}{d+2s} \qquad \text{with} \quad s = \begin{cases} \frac{d}{2} & \text{if } d \equiv 3 \text{ mod } 4, \\ \text{any } s < \frac{d}{2} & \text{if } d \equiv 0 \text{ mod } 4, \\ N & \text{otherwise.} \end{cases}$$

But

$$(P - \lambda - i\epsilon)^{-N} = \lambda^{-N} e^{i\tau A}(P_\lambda - 1 - i\mu)^{-N} e^{-i\tau A}$$

and

$$||e^{i\tau A}||_{L^{p'}\to L^{p'}} = e^{\tau\left(\frac{d}{2}-\frac{d}{p'}\right)} = \lambda^{\frac{d}{4}\left(1-\frac{2}{p'}\right)} = \lambda^{\frac{s}{2}},$$

thus

$$||(hA+i)^{-N}(P-\lambda-i\epsilon)^{-N}(hA-i)^{-N}||_{L^p\to L^{p'}} \le C_N \lambda^{-N+s}.$$