The Nori fundamental gerbe

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Plan

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  The étale fundamental group
  The Nori fundamental group

Main results
  Reminder on gerbes
  Existence of the profinite fundamental gerbe

An application
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  Proof

Beyond the profinite fundamental gerbe
  Pro-algebraic fundamental gerbe
  \( C \)-fundamental gerbe
**Definition**

Let $X$ be a connected scheme, endowed with a geometric point $\overline{x} : \text{spec } \Omega \to X$. One defines the profinite fundamental group

$$\pi^\text{et}_1( X, \overline{x} ) = \text{Aut}(F_{\overline{x}})$$

where

$$F_{\overline{x}} : \text{Cov } X \to \text{Sets}$$

is the fiber functor associated to $\overline{x}$. If $G$ is a finite group, then

$$\{ \pi^\text{et}_1( X, \overline{x} ) \to G \} \longleftrightarrow \{ \text{étale } G\text{-covers } (Y, \overline{y}) \to (X, \overline{x}) \}$$
Base-point free versions

First formulation (SGA1): if $X$ is a connected scheme, the category of fiber functors

$$F : \text{Cov } X \rightarrow \text{Sets}$$

is a transitive groupoid.

More elaborate description (Deligne): if $X$ is defined over $k$, and $Y \rightarrow X$ is a Galois cover, then

$$P_Y = \text{Isom}_{X \times_k X}^{\text{Aut} Y} (\text{pr}_2^* Y, \text{pr}_1^* Y) \rightrightarrows X$$

is an étale transitive groupoid acting on $X$, and $\hat{P}_X = \varprojlim_Y P_Y$ is Deligne fundamental (absolute) groupoid.

Note that if $\overline{x}_i : \text{spec } \Omega \rightarrow X$ are geometric points, $i = 1, 2$, then

$$\left(\hat{P}_X\right)_{(\overline{x}_1, \overline{x}_2)} = \text{Isom}(F_{\overline{x}_1}, F_{\overline{x}_2}).$$
Definition and existence statement

Let $X/k$ be a scheme, endowed with a rational point $x \in X(k)$. Nori considers triples $(G, Y \to X, y)$, where $G$ is a finite group-scheme, $Y \to X$ is a $G$-torsor, and $y \in Y(k)$ is a lift of $x$.

**Definition**

*$X$ admits a fundamental group scheme if there is a triple $(\pi(X, x), \tilde{X}_x, \tilde{x})$ which is pro-universal among such triples.*

**Theorem (Nori)**

*If $X$ is reduced and connected, then $X$ admits a fundamental group scheme.*

**Sketch of a proof.**

Show that the category of triples is co-filtered and put

$$\pi(X, x) = \lim_{\leftarrow} (G, Y \to X, y) G.$$
The Nori fundamental groupoid

Let $X/k$ be proper and reduced. Esnault-Hai: even if no $x \in X(k)$ is given, the category $\text{EF Vect}_X$ of essentially finite vector bundles on $X$ is a (non neutral) tannakian category.

One can use the universal fiber functor:

$$F : \text{EF Vect}_X \rightarrow \text{Vect}_X$$

and consider the transitive groupoid:

$$P_X = \text{Isom}_{X \times_k X}(\text{pr}_2^* F, \text{pr}_1^* F) \rightrightarrows X$$

For any $x \in X(k)$, one recovers Nori’s group in the cartesian diagram:

$$\begin{align*}
\pi(X, x) & \xrightarrow{\sim} P_X \\
\downarrow & \\
\text{spec } k & \xrightarrow{(x,x)} X \times_k X
\end{align*}$$
Affine gerbes

Any scheme $X/k$ is characterized by its functor of points

$$h_X : \text{Sch} / k^{op} \rightarrow \text{Sets}$$

$$S \rightarrow \text{Hom}_k(S, X)$$

One can consider more generally pseudo-functors:

$$\text{Sch} / k^{op} \rightarrow \text{Groupoids}$$

The key example: the classifying stack of a group-scheme:

$$B G : S \rightarrow \{G \text{ torsors over } S\}$$

Definition

A pseudo functor $\Gamma : \text{Sch} / k^{op} \rightarrow \text{Groupoids}$ is an affine gerbe if $\Gamma$ is a (fpqc) stack, $\Gamma \neq \emptyset$, two objects are locally isomorphic, and the automorphism group of any object is affine.

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More examples

Fix a central extension:

\[ 0 \to A \to G \to H \to 1 \]

of algebraic \( k \)-groups, and \( T/k \) a \( H \)-torsor. Then

\[ \Gamma : S \xrightarrow{\sim} \{ \text{reductions of structure group of } T_S \text{ to } G \} \]

is an affine gerbe, banded by \( A \).

Gerbes endowed with a chart \( \leftrightarrow \) transitive groupoids

\[(S \to \Gamma) \to (S \times_{\Gamma} S \rightrightarrows S)\]

Affine gerbes with a given object over \( k \) \( \leftrightarrow \) affine groups/\( k \)

\[(\Gamma, \xi \in \Gamma(k)) \to \text{Aut}_k(\xi) \]

\[ B \to G \]
Even more examples

The definition of a gerbe makes sense over any topology. A typical example from deformation theory:
Fix $i : X_0 \to X$ a thickening of order one defined by an ideal $\mathcal{I}$ of square 0, and $\mathcal{E}_0$ a vector bundle on $X_0$. Then:

$$
\Gamma : (X'_0 \to X_0) \mapsto \left\{ \text{liftings of } \mathcal{E}_0|_{X'_0} \text{ to } X' \right\}
$$

is a gerbe, banded by $\mathcal{I} \otimes \text{End}(\mathcal{E}_0)$. It is neutral if and only if the deformation problem is unobstructed, and in this case the stack of obstructions is isomorphic, after choosing a lifting, to $B(\mathcal{I} \otimes \text{End}(\mathcal{E}_0))$. 

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Construction as an inverse limit

Theorem (B.-Vistoli)

Let $X/k$ be geometrically (connected and reduced) scheme. There exists a morphism $X \to \pi_{X/k}$ universal among morphisms to (pro)finite affine gerbes over $k$.

Sketch of a proof.

Say a morphism $X \to \Gamma$ is Nori-reduced if for any factorisation $X \to \Gamma' \to \Gamma$ where $\Gamma' \to \Gamma$ is faithful, then $\Gamma' \simeq \Gamma$.

Then the category of Nori-reduced morphisms $X \to \Gamma$ is (equivalent to) a co-filtered category, so we can put:

$$\pi_{X/k} = \lim_{\leftarrow} \Gamma \quad \text{Nori reduced}$$

that by construction has the wanted universal property.
**Tannakian interpretation**

The Tannaka correspondence, in Rivano’s form, reads:

\[
\text{Affine gerbes over } k \leftrightarrow (\text{Non neutral) tannakian categories}/
\\Gamma \rightarrow \text{Vect}_k \Gamma
\]

\[
(S \rightarrow \{ \text{fibre functors of } C_S \}) \leftrightarrow C
\]

**Theorem (B.-Vistoli)**

*Let $X/k$ be a proper scheme.*

1. *If $X$ is geometrically (connected and reduced), then the pullback along $X \rightarrow \pi_{X/k}$ identifies the non-neutral tannakian category $\text{Vect}_{\pi_{X/k}}$ with $\text{EF Vect}_X$.*

2. *$\text{EF Vect}_X$ is tannakian if and only if every morphism $X \rightarrow F$, where $F$ is a finite stack, factors as $X \rightarrow \Gamma \rightarrow F$, where $\Gamma$ is a finite gerbe, and $\Gamma \rightarrow F$ is a closed immersion.*
Section conjecture

One is interested in producing curves whose homotopy exact sequence

\[ 1 \rightarrow \pi_1(X_k, \bar{x}) \rightarrow \pi_1(X, \bar{x}) \rightarrow \text{Gal}_k \rightarrow 1 \]

does not split. Indeed, these sections are conjecturally related to rational points.
By functoriality, \( x \in X(k) \) induces a section \( s_X(x) \), well defined up to conjugacy by an element of \( \pi_1(X_k, \bar{x}) \). One thus get:

\[ s_X : X(k) \rightarrow \mathcal{S}_{\pi_1(X/k)} \]

Conjecture (Section conjecture, Grothendieck 1983)

If \( X \) a smooth, proper curve of genus greater than 2 over a field \( k \) of finite type over \( \mathbb{Q} \), the application \( s_X \) is one to one.
Curves with non trivial homotopy exact sequences

Let $P/k$ be a Brauer-Severi variety (that is, $P_{k^{\text{sep}}} \cong \mathbb{P}^n_{k^{\text{sep}}}$). Denote by $r$ its period:

$$r = \min \{ l \in \mathbb{N}^* / \mathcal{O}(l) \text{ is defined over } k \}$$

**Theorem (B.-Vistoli)**

Assume that the characteristic of $k$ is 0, that there is a given morphism $f : X \to P$, where $P$ is a Brauer-Severi variety of period $r$. Assume moreover that there exists a prime divisor $p$ of $r$, an invertible sheaf $\Lambda$ on $X$, and an isomorphism $\Lambda \otimes^p f^* \mathcal{O}_P(r)$. Then the fundamental exact sequence

$$1 \to \pi_1^{et}(X_k, \bar{x}) \to \pi_1^{et}(X, \bar{x}) \to \text{Gal}_k \to 1$$

*does not split.*
Sketch of a proof

The first step of the proof relies on the correspondence

$$\{ \text{Isomorphism classes in } \pi_{X/k}(k) \} \leftrightarrow \mathcal{L}_{\pi_1(X/k)}$$

So to show that $\mathcal{L}_{\pi_1(X/k)} = \emptyset$, it is enough to produce a morphism $X \to \Gamma$, where $\Gamma$ is a non-neutral finite gerbe. Indeed by the universal property, such a morphism factors as $X \to \pi_{X/k} \to \Gamma$, and the hypothesis on $\Gamma$ means that $\Gamma(k) = \emptyset$. To construct such a gerbe, consider the dual Brauer-Severi variety $P^\vee$, and put

$$\Gamma(S) = \left\{ (\mathcal{L}, \psi), \mathcal{L} \text{ invertible sheaf on } P^\vee_S, \psi : \mathcal{L} \otimes p \cong \mathcal{O}_{P^\vee_S}(r) \right\}$$

A morphism $X \to \Gamma$ exists by construction.
A pro-algebraic fundamental gerbe?

Let $X/k$ be a proper geometrically (connected and reduced) scheme. Does there exists a morphism $X \to \pi^\text{alg}_{X/k}$ universal among morphisms $X \to \Gamma$, where $\Gamma$ is an algebraic gerbe? The answer is no, even for very standard $k$-schemes $X$, for instance $X = \mathbb{P}^1$. One reason is the following: if $G$ is an algebraic group, and $H$ is a subgroup, then the diagram

$$
\begin{array}{ccc}
\text{spec } k & \longrightarrow & B G \\
\uparrow & & \uparrow \\
G/H & \longrightarrow & B H
\end{array}
$$

is cartesian. Assume that $H$ is a Borel, so that $G/H$ is projective, and fix a non constant morphism $X \to G/H$. Such a morphism cannot factor through a gerbe over $k$, since the scheme-theoretic image of a gerbe is a point.
Pro-$\mathcal{C}$ gerbe

If $\mathcal{C}$ is a class of algebraic groups over $k$, we say that a gerbe $\Gamma$ is a $\mathcal{C}$-gerbe if it is banded by an element of $\mathcal{C}$. So we can speak of unipotent gerbes, abelian gerbes, and so on. A $\mathcal{C}$-fundamental gerbe $X \to \pi^C_{X/k}$ is a morphism (pro-)universal among morphisms $X \to \Gamma$ to a $\mathcal{C}$-gerbe.

In a work in progress with A. Vistoli, we have worked out a technical condition on $\mathcal{C}$ that ensures the existence of a $\mathcal{C}$-fundamental gerbe under mild assumptions on $X/k$ (geometrically reduced, concentrated, $H^0(X, \mathcal{O}_X) = k$). The two largest classes for which we can show the condition is respected are the classes of virtually abelian (resp. virtually unipotent) groups. We say that an affine group scheme of finite type $G$ over a perfect field $k$ is virtually abelian (resp. virtually unipotent) if $G^0_{\text{red}}$ is abelian (resp. unipotent).
Tannakian interpretation

The (virtually-)unipotent fundamental gerbe has a natural and standard tannakian interpretation. Nori has proved in his Phd that $\text{Vect} \pi^\text{uni}_{X/k}$ is equivalent to the category of vector bundles on $X$ that are successive extensions of the structure sheaf $\mathcal{O}_X$. Similarly, it should not be too hard to show that $\text{Vect} \pi^\text{virt-uni}_{X/k}$ is equivalent to the category of vector bundles on $X$ that are successive extensions of essentially finite bundles.

The (virtually)-abelian fundamental gerbe is much more mysterious. The main issue is that the pullback functor $\text{Vect} \pi^\text{ab}_{X/k} \rightarrow \text{Vect} X$ is not fully faithful. We have no idea of a tannakian interpretation at the moment.