

# WARING'S PROBLEM FOR POLYNOMIALS IN TWO VARIABLES

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ABSTRACT. We prove that all polynomials in several variables can be decomposed as the sums of  $k$ th powers:  $P(x_1, \dots, x_n) = Q_1(x_1, \dots, x_n)^k + \dots + Q_s(x_1, \dots, x_n)^k$ , provided that elements of the base field are themselves sums of  $k$ th powers. We also give bounds for the number of terms  $s$  and the degree of the  $Q_i^k$ . We then improve these bounds in the case of two variables polynomials of large degree to get a decomposition  $P(x, y) = Q_1(x, y)^k + \dots + Q_s(x, y)^k$  with  $\deg Q_i^k \leq \deg P + k^3$  and  $s$  that depends on  $k$  and  $\ln(\deg P)$ .

## 1. INTRODUCTION

For any domain  $A$  and any integer  $k \geq 2$ , let  $W(A, k)$  denote the subset of  $A$  formed by all finite sums of  $k$ th powers  $a^k$  with  $a \in A$ . Let  $\underline{w}_A(k)$  denote the least integer  $s$ , if it exists, such that for every element  $a \in W(A, k)$ , the equation

$$a = a_1^k + \dots + a_s^k$$

admits solutions  $(a_1, \dots, a_s) \in A^s$ .

The case of polynomial rings  $K[t]$  over a field  $K$  is of particular interest (see [10], [7]). The similarity between the arithmetic of the ring  $\mathbb{Z}$  and the arithmetic of the polynomial rings in a single variable  $F[t]$  over a finite field  $F$  with  $q$  elements led to investigate a restricted variant of Waring's problem over  $F[t]$ , namely the strict Waring problem. For  $P \in F[t]$ , a representation

$$P = Q_1^k + \dots + Q_s^k \quad \text{with } \deg Q_i^k < \deg P + k,$$

and  $Q_i \in F[t]$  is a *strict representation*.

For the strict Waring problem, analog to the classical numbers  $g_{\mathbb{N}}(k)$  and  $G_{\mathbb{N}}(k)$  have been defined as follows. Let  $g_{F[t]}(k)$  (resp.  $G_{F[t]}(k)$ ) denote the least integer  $s$ , if it exists, such that every polynomial in  $W(F[t], k)$  (resp. every polynomial in  $W(F[t], k)$  of sufficiently large degree) may be written as a sum satisfying the strict degree condition.

General results about Waring's problem for the ring of polynomials over a finite field may be found in [9], [10], [11], [12], [14] for the unrestricted

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problem and in [13], [8], [5], [3], [7] for the strict Waring problem. Gallardo's method introduced in [6] and performed in [4] to deal with Waring's problem for cubes was generalized in [3] and [7] where bounds for  $g_{F[t]}(k)$  and  $G_{F[t]}(k)$  were established when  $q$  and  $k$  satisfy some conditions.

The goal of this paper is a study of Waring's problem for the ring  $F[x, y]$  of polynomials in two variables over a field  $F$ . As for the one variable case, two variations of Waring's problem may be considered. The first one, is the unrestricted Waring's problem; the second one takes degree conditions in account.

In Section 2 we start by some relations between Waring's problem for polynomials in one variable and Waring's problem for polynomials in  $n \geq 2$  variables. In Section 3, we prove that, provided all elements of the field  $F$  are sums of  $k$ th powers, there exists a positive integer  $s$  (depending on  $F$  and  $k$ ) such that every polynomial  $P \in F[x, y]$  may be written as a sum

$$(\dagger) \quad P = Q_1^k + \cdots + Q_s^k,$$

where for  $i = 1, \dots, s$ ,  $Q_i$  is a polynomial of  $K[x, y]$  such that  $\deg Q_i \leq \deg P$ . We then prove various improvements, the goal being to have in representations  $(\dagger)$  a decomposition with the following properties: the first priority is to have the lowest possible degree for the polynomials  $Q_i$  and the second priority is a small number of terms. In Section 5, we prove that  $(\dagger)$  is possible for polynomials of large degree with  $\deg Q_i^k \leq \deg P + k^3$ , the number  $s$  of terms depending on  $F$ ,  $k$  and  $\deg P$ . To do that, in Section 4, we introduce the notion of approximate root.

Let  $F$  be a field such that:  $F$  has more than  $k$  elements, the characteristic of  $F$  does not divide  $k$  and each element of  $F$  can be written as a sum of  $w_F(k)$   $k$ th powers of elements of  $F$ . We summarize in the tabular below the different bounds we get for a decomposition of a polynomial  $P(x, y)$  of degree  $d$  as a sum  $P = \sum_{i=1}^s Q_i^k$ .

	$\deg Q_i^k$	$s$
Corollary 4	$kd$	$kw_F(k)$
Proposition 5	$d + 2(k - 1)^2$	$\frac{1}{2}k(d + 1)(d + 2)w_F(k)$
Proposition 6	$2d + 4k^2$	$k^2(2k - 1)w_F(k)$
Theorem 8	$d + k^3$	$2k^3 \ln(\frac{d}{k} + 1) \ln(2k) + 7k^4 \ln(k)w_F(k)^2$

The two basic results are Corollary 4 that give a decomposition with very few terms of high degree and Proposition 5 with many terms of low degree. Our first main result is Proposition 6, that provides a decomposition with terms of medium degree, but the number of terms depends only on  $k$  and not on the degree of  $P$ . Then Theorem 8 decomposes  $P$ , of sufficiently large degree  $d \geq 2k^4$ , into a sum of few terms of low degree.

For instance, let a field with  $w_F(k) = 1$  (that is to say each element of  $F$  is a  $k$ th power), set  $d = 200$  and  $k = 3$ , then each polynomial  $P(x, y)$  of degree 200 can be written  $P = \sum_{i=1}^s Q_i^3$  with<sup>1</sup>

	$\deg Q_i^k$	$s$
Corollary 4	600	3
Proposition 5	208	60903
Proposition 6	436	45
Theorem 8	227	812

## 2. THE UNRESTRICTED WARING'S PROBLEM

If  $A$  is a domain, we denote by  $W(A, k, s)$  the set of elements  $a \in A$  that can be written as a sum  $a = a_1^k + \cdots + a_s^k$  with  $a_1, \dots, a_s \in A$ ; if  $A = W(A, k, s)$  for an integer  $s$ , then for any integer  $s' \geq s$ , we have  $A = W(A, k, s')$ . Let  $w_A(k)$  denote the least integer  $s$  such that  $A = W(A, k, s)$ . If such a  $s$  does not exist, let  $w_A(k) = \infty$ . Observe that  $w_A(k) \geq \underline{w}_A(k)$  and in the case that  $A = W(A, k)$  then  $w_A(k) = \underline{w}_A(k)$ . In this section we are concerned with rings of polynomials in  $n \geq 1$  variables.

**Lemma 1.** *Let  $A$  be a domain and let  $s$  be a positive integer.*

- (1) *If  $A[t] = W(A[t], k, s)$ , then  $A = W(A, k, s)$ , so that  $w_A(k) \leq w_{A[t]}(k)$ .*
- (2)  *$A[t] = W(A[t], k, s)$  if and only if  $A[x_1, \dots, x_n] = W(A[x_1, \dots, x_n], k, s)$ , so that  $w_{A[x_1, \dots, x_n]}(k) = w_{A[t]}(k)$ .*

A kind of reciprocal to (1) will be discussed later in Proposition 3.

*Proof*

- (1) Suppose  $A[t] = W(A[t], k, s)$ . Every  $a \in A$  is a sum  $a = Q_1^k + \cdots + Q_s^k$  for some  $Q_i \in A[t]$ . Specializing  $t$  at 1 for instance, gives  $a = Q_1(1)^k + \cdots + Q_s(1)^k$ , a sum in  $A$ . Therefore,  $w_{A[t]}(k) \geq w_A(k)$ .
- (2) (a) If  $A[t] = W(A[t], k, s)$ , then there exist  $Q_1, \dots, Q_s \in A[t]$  such that  $t = Q_1(t)^k + \cdots + Q_s(t)^k$ . Pick  $P \in A[x_1, \dots, x_n]$  and substitute  $P$  for  $t$ , we get:  $P(x_1, \dots, x_n) = Q_1(P(x_1, \dots, x_n))^k + \cdots + Q_s(P(x_1, \dots, x_n))^k$ . Hence  $w_{A[x_1, \dots, x_n]}(k) \leq w_{A[t]}(k)$ .
- (b) If  $A[x_1, \dots, x_n] = W(A[x_1, \dots, x_n], k, s)$  then any  $P(t) \in A[t]$  can be written  $P(t) = Q_1(t, x_2, \dots, x_n)^k + \cdots + Q_s(t, x_2, \dots, x_n)^k$ . By the specialization  $x_2 = \cdots = x_n = 1$  we get that  $P(t) \in W(A[t], k, s)$ . Therefore  $w_{A[x_1, \dots, x_n]}(k) \geq w_{A[t]}(k)$ .

*Remark.* It is also true that  $A[t] = W(A[t], k, s)$  if and only if  $t \in W(A[t], k, s)$ .

This remark motivates the fact that we consider Waring's problem for a polynomial ring  $F[x_1, \dots, x_n]$  where  $F$  is a field satisfying the condition

<sup>1</sup>In fact the last bound comes from a sharper bound obtained in the proof of Theorem 8.

$F = W(F, k)$ . Such a field is called a *Waring field for the exponent  $k$* , or briefly, a  *$k$ -Waring field*.

Let us give some examples. An algebraically closed field  $F$  is a  $k$ -Waring field with  $w_F(k) = 1$  for every positive integer  $k$ . If  $F$  is a finite field of characteristic  $p$ , for every positive integer  $n$ ,  $F$  is a  $p^n$ -Waring field with  $w_F(p^n) = 1$ . It is known, c.f. [1], [5], that for a finite field  $F$  of characteristic  $p$  that does not divide  $k$  and order  $q = p^m$ ,  $F$  is a Waring field for the exponent  $k$  if and only if for all  $d \neq m$  dividing  $m$ ,  $(q-1)/(p^d-1)$  does not divide  $k$ .

When  $F$  has prime characteristic  $p$ , it is sufficient to consider Waring's problem for exponents  $k$  coprime with  $p$ . Indeed, we have

**Proposition 2.** *Let  $k \geq 2$  be coprime with  $p$ . Then, for any positive integer  $\nu$  and for any positive integer  $s$ , we have*

$$W(F[x_1, \dots, x_n], kp^\nu, s) = \{Q^{p^\nu} \mid Q \in W(F[x_1, \dots, x_n], k, s)\},$$

$$w_{F[x_1, \dots, x_n]}(kp^\nu) = w_{F[x_1, \dots, x_n]}(k).$$

The proof is similar to that of [3, Theorem 2.1] and relies on the relation  $(Q_1^k + \dots + Q_s^k)^p = Q_1^{pk} + \dots + Q_s^{pk}$ .

### 3. VANDERMONDE DETERMINANTS

**3.1. Sum with high degree.** Let us recall that for  $(\alpha_1, \dots, \alpha_n) \in L^n$ , where  $L$  is a field containing  $F$ , Vandermonde's determinant  $V(\alpha_1, \dots, \alpha_n)$  verifies:

$$(1) \quad V(\alpha_1, \dots, \alpha_n) := \begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ \vdots & & & & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-1} \end{vmatrix} = \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j).$$

**Proposition 3.** *Let  $F$  be a field with more than  $k$  elements, whose characteristic does not divide  $k$ , such that each element of  $F$  can be written as a sum of  $k$ th powers of elements of  $F$ . Then any polynomial  $P(x_1, \dots, x_n)$  with coefficients in the field  $F$  is a sum of  $k$ th powers. In other words, for any positive integer  $n$ ,*

$$F[x_1, \dots, x_n] = W(F[x_1, \dots, x_n], k).$$

*Proof.* The proof follows ideas from [7]. Let  $\alpha_1, \dots, \alpha_k$  be distinct elements of  $F$ . First notice that by formula (1), if  $t$  is any transcendental element over  $F$ ,  $V(\alpha_1, \dots, \alpha_k) = V(t + \alpha_1, \dots, t + \alpha_k)$ . By expanding the determinant  $V(t + \alpha_1, \dots, t + \alpha_k)$  along the last column we get (a term marked  $\tilde{x}_i$  means

that it is omitted):

$$\begin{aligned}
 V(\alpha_1, \dots, \alpha_k) &= V(t + \alpha_1, \dots, t + \alpha_k) \\
 &= \pm \sum_{i=1}^k (-1)^i (t + \alpha_i)^{k-1} V(t + \alpha_1, \dots, \overbrace{t + \alpha_i}^{\vee}, \dots, t + \alpha_k) \\
 &= \pm \sum_{i=1}^k (-1)^i (t + \alpha_i)^{k-1} V(\alpha_1, \dots, \check{\alpha}_i, \dots, \alpha_k).
 \end{aligned}$$

The constant  $\gamma = V(\alpha_1, \dots, \alpha_k)$  is non-zero since the  $\alpha_i$  are distinct elements of  $F$ . We write

$$\sum_{i=1}^k \frac{(t + \alpha_i)^{k-1}}{\beta_i} = \gamma,$$

where  $\beta_i$  are non-zero constants in  $F$ . This formula proves that the function  $C(t) = \sum_{i=1}^k \frac{(t + \alpha_i)^k}{\beta_i} - \gamma kt$  has an identically null derivative; since the characteristic of  $F$  does not divide  $k$ , it implies that  $C(t)$  is a constant. So that, for some  $\delta \in F$ :

$$(2) \quad \sum_{i=1}^k \frac{(t + \alpha_i)^k}{\beta_i} = \gamma kt + \delta.$$

Let  $P(x_1, \dots, x_n) \in F[x_1, \dots, x_n]$ . By substitution of  $t$  by  $(P - \delta)/(\gamma k)$  in equality (2) we get  $P = \sum_{i=1}^k \frac{(P - \delta + \alpha_i \gamma k)^k}{\beta_i (\gamma k)^k}$ . But by assumption  $1/\beta_i (\gamma k)^k$  is a sum of  $k$ th powers of elements of  $F$ . So that  $P(x_1, \dots, x_n)$  is also a sum of  $k$ th powers of elements of  $F[x_1, \dots, x_n]$ .  $\square$

**Corollary 4.** *Let  $F$  have more than  $k$  distinct elements such that its characteristic does not divide  $k$ . Every polynomial  $P(x_1, \dots, x_n) \in F[x_1, \dots, x_n]$  of degree  $d$  can be written as a sum*

$$P(x_1, \dots, x_n) = \delta_1 Q_1(x_1, \dots, x_n)^k + \dots + \delta_k Q_k(x_1, \dots, x_n)^k,$$

where  $\delta_1, \dots, \delta_k \in F$  and  $Q_1, \dots, Q_k$  are polynomials in  $F[x_1, \dots, x_n]$  such that  $\deg Q_i^k \leq kd$ . If moreover each element of  $F$  is a sum of  $w_F(k)$   $k$ th powers, then

$$P(x_1, \dots, x_n) = Q_1(x_1, \dots, x_n)^k + \dots + Q_s(x_1, \dots, x_n)^k$$

where  $Q_1, \dots, Q_s \in F[x_1, \dots, x_n]$  such that  $\deg Q_i^k \leq kd$  for some  $s \leq k \cdot w_F(k)$ .

*Proof.* It comes from formula (2) and the discussion below it.  $\square$

In the sequel, we consider polynomials in two variables.

### 3.2. Low degree, many terms.

**Proposition 5.** *Let  $F$  be a field with more than  $k$  distinct elements such that its characteristic does not divide  $k$ . Every polynomial  $P \in F[x, y]$  of degree  $d$  admits a decomposition:*

$$P(x, y) = \delta_1 Q_1(x, y)^k + \cdots + \delta_s Q_s(x, y)^k,$$

where  $\delta_1, \dots, \delta_s \in F$  and  $Q_1, \dots, Q_s$  are polynomials in  $F[x, y]$  such that  $\deg Q_i^k \leq d + 2(k-1)^2$  and  $s \leq k \cdot \frac{(d+1)(d+2)}{2}$ .  
If moreover each element of  $F$  is a sum of  $k$ th powers then  $P$  admits a decomposition:

$$P(x, y) = Q_1(x, y)^k + \cdots + Q_s(x, y)^k,$$

where  $Q_1, \dots, Q_s \in F[x, y]$  with  $\deg Q_i^k \leq d + 2(k-1)^2$  and  $s \leq kw_F(k) \frac{(d+1)(d+2)}{2}$ .

*Proof.* Let  $P(x, y) = \sum a_{i,j} x^i y^j$ . We make the Euclidean divisions:  $i = pk + a$  and  $j = qk + b$  with  $0 \leq a, b < k$ . Each monomial  $x^i y^j$  can now be written  $x^i y^j = (x^p y^q)^k \cdot x^a y^b$ . By Corollary 4,  $x^a y^b$  can be written  $x^a y^b = \delta_1 Q_1(x, y)^k + \cdots + \delta_k Q_k(x, y)^k$  with  $\delta_1, \dots, \delta_k \in F$ ,  $Q_1, \dots, Q_k \in F[x, y]$  and  $\deg Q_i \leq \deg(x^a y^b)$ , so that

$$x^i y^j = \delta_1 (x^p y^q Q_1(x, y))^k + \cdots + \delta_k (x^p y^q Q_k(x, y))^k.$$

Moreover  $\deg((x^p y^q Q_i(x, y))^k) = k(p + q + \deg Q_i) \leq kp + kq + ka + kb = i + j + (k-1)(a + b) \leq i + j + 2(k-1)^2 \leq d + 2(k-1)^2$ .

As  $\deg P = d$  the number of monomials  $x^i y^j$  is less or equal than  $\frac{(d+1)(d+2)}{2}$ , so that  $P$  admits a decomposition  $P(x, y) = \delta_1 Q_1(x, y)^k + \cdots + \delta_s Q_s(x, y)^k$  with  $\deg Q_i^k \leq d + 2(k-1)^2$  and  $s \leq k \frac{(d+1)(d+2)}{2}$ . Thus we can find a decomposition  $P(x, y) = Q_1(x, y)^k + \cdots + Q_s(x, y)^k$  for some  $s \leq kw_F(k) \frac{(d+1)(d+2)}{2}$ .  $\square$

**3.3. Medium degree, few terms.** We improve this method to get fewer terms in the sum but the degree of each term is higher.

**Proposition 6.** *Let  $F$  be a field with more than  $k$  elements, such that its characteristic does not divide  $k$  and each element of  $F$  is a sum of  $k$ th powers. Any  $P \in F[x, y]$  admits a decomposition:*

$$P(x, y) = Q_1(x, y)^k + \cdots + Q_s(x, y)^k,$$

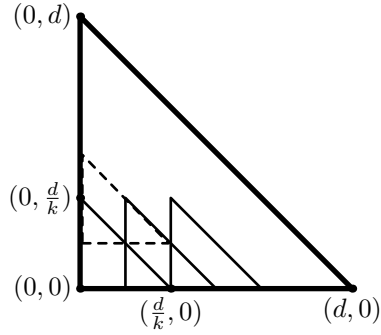
where  $Q_1, \dots, Q_s$  are polynomials in  $F[x, y]$  with  $\deg Q_i^k \leq 2 \deg P + 4k^2$  and  $s \leq k^2(2k-1)w_F(k)$ .

Observe that the bound for  $s$  does not depend on the degree of the polynomial  $P$ .

*Proof.*

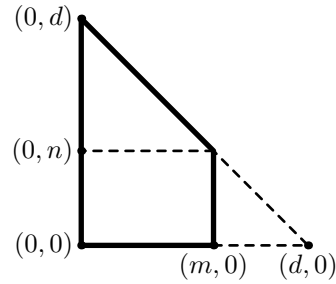
Let  $d$  be the least multiple of  $2k^2$  such that  $d \geq \deg P$ . The Newton polygon of  $P$  is included in the triangle  $ABC$  with  $A(0,0)$ ,  $B(0,d)$ ,  $C(d,0)$ .

We cover this triangle  $ABC$  by  $k(2k-1)$  small triangles that are translations (by  $\frac{d}{2k}$ -units) of  $A'B'C'$  with  $A'(0,0)$ ,  $B'(0, \frac{d}{k})$ ,  $C'(\frac{d}{k}, 0)$ . This covering means that we can write  $P(x, y)$  as a sum of  $k(2k-1)$  polynomials of the form  $x^{i\frac{d}{2k}}y^{j\frac{d}{2k}}P_{i,j}(x, y)$  with  $\deg P_{i,j} \leq \frac{d}{k}$  and  $0 \leq i+j \leq 2k-2$  (so that  $\deg x^{i\frac{d}{2k}}y^{j\frac{d}{2k}} < d$ ). As  $2k^2$  divides  $d$  then  $x^{i\frac{d}{2k}}y^{j\frac{d}{2k}}$  is a  $k$ th power. Furthermore, by Corollary 4, we can write each  $P_{i,j}$  as a sum of  $kw_F(k)$  powers, each power being of degree at most  $k\frac{d}{k} = d$ . Hence we get a decomposition  $P(x, y) = Q_1(x, y)^k + \dots + Q_s(x, y)^k$  with  $s \leq k^2(2k-1)w_F(k)$  terms and  $\deg Q_i^k < 2d$ .  $\square$

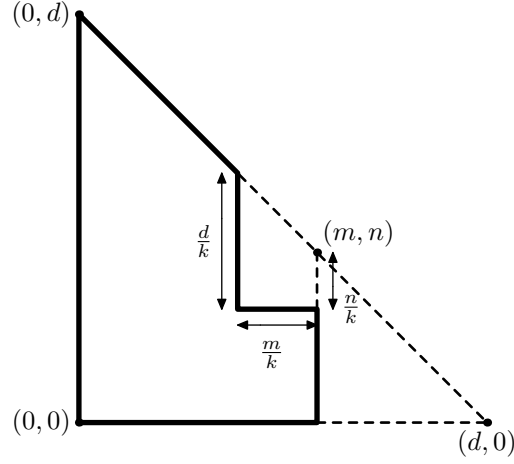


#### 4. APPROXIMATE ROOT

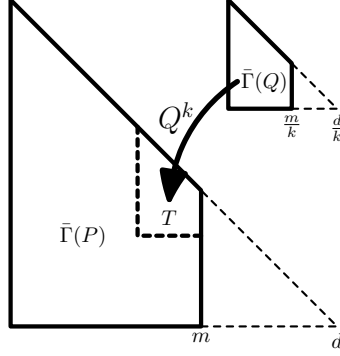
In this section  $F$  is a field whose characteristic does not divide  $k$ . Let  $P \in F[x, y]$  be a polynomial that verifies the following conditions:  $\deg P \leq d$ ,  $\deg_x P < m$ . So that the Newton polygon  $\Gamma(P)$  of  $P$  is (included in) the following polygon  $\bar{\Gamma}(P)$  (whose vertices are  $(0,0)$ ,  $(m,0)$ ,  $(m,n)$ ,  $(0,d)$ ). We set  $n = d - m$  and we suppose that  $k|m$ ,  $k|n$ ,  $k|d$ . We will look for a  $Q \in F[x, y]$  such that  $\deg Q \leq \frac{d}{k}$ ,  $\deg_x Q \leq \frac{m}{k}$ , so that  $\Gamma(Q^k) \subset \bar{\Gamma}(P)$ . In fact the Newton polygon of  $Q$  is homothetic to the one of  $P$  with a ratio  $\frac{1}{k}$ .



**Proposition 7.** *There exists a unique  $Q(x, y) \in F[x, y]$ , monic in  $x$ , such that  $P + x^m y^n - Q^k$  has no monomial  $x^i y^j$  with  $i \geq m - \frac{m}{k}$  and  $j \geq n - \frac{n}{k}$ . That is to say, the Newton polygon of  $P + x^m y^n - Q^k$  is (included in):*



It means that with two  $k$ th powers ( $x^m y^n$  and  $Q^k$ ) we “cancel” the trapezium  $T$  (defined by the vertices  $(m, n)$ ,  $(m, n - \frac{n}{k})$ ,  $(m - \frac{m}{k}, n - \frac{n}{k})$ ,  $(m - \frac{m}{k}, n + \frac{d}{k} - \frac{n}{k})$ ). This procedure is similar to the computation of the approximate  $k$ th root of a one variable polynomial, see [2]. The proof is sketched into the following picture:



Morally, the coefficients of  $Q$  provide a set of unknowns, which is chosen in order that  $Q^k$  and  $P$  can be identified into the trapezium area ( $T$ ).

*Proof.* We write  $P$  as the sum  $P = P_1 + P_2$  corresponding to the decomposition into two areas of  $\bar{\Gamma}(P) = T \cup (\bar{\Gamma}(P) \setminus T)$ : we write  $P_1$  as a polynomial in  $x$  whose coefficients are in  $F[y]$  so that  $P_1(x, y) = a_1(y)x^{m-1} + \dots + a_{\frac{m}{k}}(y)x^{m-\frac{m}{k}}$  with  $\deg a_i(y) \leq n + i$  and  $\text{val } a_i(y) \geq n - \frac{n}{k}$ . We denote by  $\text{val}$  the  $y$ -adic valuation:  $\text{val} \sum \alpha_i y^i = \min\{i \mid \alpha_i \neq 0\}$ .

We set  $P'_1(x, y) = y^n x^m + P_1(x, y)$  and  $a_0(y) = y^n$ . Notice that we have added a  $k$ th power since  $k|m$  and  $k|n$ .

We also write  $Q(x, y)$  as a polynomial in  $x$  with coefficients in  $F[y]$ :  $Q(x, y) = b_0(y)x^{\frac{m}{k}} + b_1(y)x^{\frac{m}{k}-1} + \dots + b_{\frac{m}{k}}(y)$ .



We now identify the monomials of  $P_1'(x, y) = x^m y^n + P_1(x, y)$  with the monomials of  $Q(x, y)^k$ , in the trapezium  $T$ . As we only want to identify the monomials of a sufficiently high degree we define the following equivalence:

$$a(y) \simeq b(y) \quad \text{if and only if} \quad \deg(a(y) - b(y)) < n - \frac{n}{k}.$$

It yields the following polynomial system of equations ( $a_i(y)$  are data, and  $b_i(y)$  unknowns):

$$(S) \quad \begin{cases} a_0 \simeq b_0^k \\ a_1 \simeq k b_0^{k-1} b_1 \\ a_2 \simeq k b_0^{k-1} b_2 + \binom{k}{2} b_0^{k-2} b_1^2 \\ \vdots \\ a_\ell \simeq k b_0^{k-1} b_\ell + \sum_{\substack{i_1+2i_2+\dots+(\ell-1)i_{\ell-1}=\ell \\ i_0+i_1+i_2+\dots+i_{\ell-1}=k}} c_{i_1\dots i_{\ell-1}} b_0^{i_0} b_1^{i_1} \cdots b_{\ell-1}^{i_{\ell-1}}, \quad 1 \leq \ell \leq \frac{m}{k}, \end{cases}$$

where the coefficients  $c_{i_1\dots i_{\ell-1}}$  are the multinomial coefficients defined by the following formula:

$$c_{i_1\dots i_{\ell-1}} = \binom{k}{i_1, \dots, i_{\ell-1}} = \frac{k!}{i_1! \cdots i_{\ell-1}! (k - i_1 - \cdots - i_{\ell-1})!}.$$

The first equation has a solution  $b_0(y) = y^{\frac{n}{k}}$ . Then, as  $\text{val } a_1(y) \geq n - \frac{n}{k}$ , we have  $b_1(y) = \frac{1}{k} \frac{a_1(y)}{b_0(y)^{k-1}} \in F[y]$  ( $k$  is invertible in  $F$ ). Next we compute  $b_2(y), \dots$  by induction using the fact that system (S) is triangular. Suppose that  $b_0(y), b_1(y), \dots, b_{\ell-1}(y)$  have been found. System (S) provides the relation:

$$a_\ell \simeq k b_0^{k-1} b_\ell + \sum c_{i_1\dots i_{\ell-1}} b_0^{i_0} b_1^{i_1} \cdots b_{\ell-1}^{i_{\ell-1}}.$$

As  $b_0(y) = y^{\frac{n}{k}}$  it means that the polynomials  $k y^{n - \frac{n}{k}} b_\ell(y)$  and  $a_\ell - \sum c_{i_1\dots i_{\ell-1}} b_0^{i_0} b_1^{i_1} \cdots b_{\ell-1}^{i_{\ell-1}}$  have equal coefficients associated to monomials  $y^i$  with  $i \geq n - \frac{n}{k}$ . Whence  $b_\ell(y)$  is uniquely determined. We have proved that system (S) has a unique solution  $(b_0(y), b_1(y), \dots, b_{\frac{m}{k}}(y))$ .

Finally, we need to prove that  $\deg b_i \leq \frac{n}{k} + i$  for  $0 \leq i \leq \frac{m}{k}$ . We have  $b_0(y) = y^{\frac{n}{k}}$ , so that  $\deg b_0 = \frac{n}{k}$  and  $b_1(y) = \frac{1}{k} \frac{a_1(y)}{(y^{\frac{n}{k}})^{k-1}}$ ; thus,  $\deg b_1 \leq \deg a_1 - n + \frac{n}{k} \leq n + 1 - n + \frac{n}{k} = \frac{n}{k} + 1$ . Then, by induction we get

$$\begin{aligned} \deg b_0^{i_0} b_1^{i_1} \cdots b_{\ell-1}^{i_{\ell-1}} &\leq i_0 \left( \frac{n}{k} + 0 \right) + i_1 \left( \frac{n}{k} + 1 \right) + \cdots + i_\ell \left( \frac{n}{k} + \ell \right) \\ &= \frac{n}{k} (i_0 + i_1 + \cdots + i_\ell) + i_1 + 2i_2 + \cdots + (\ell - 1)i_{\ell-1} \\ &= \frac{n}{k} k + \ell \\ &= n + \ell. \end{aligned}$$

We also find  $\deg a_\ell \leq n + \ell$  so that  $\deg b_\ell \leq \frac{n}{k} + \ell$ .  $\square$

5. STRICT SUM OF  $k$ TH POWERS

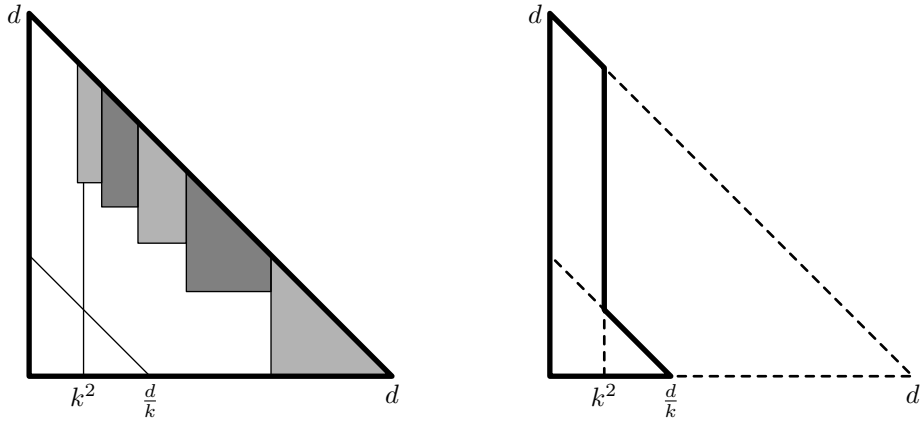
This section is devoted to the proof of the main theorem:

**Theorem 8.** *Let  $F$  be a field with more than  $k$  elements, whose characteristic does not divide  $k$ , such that each element of  $F$  can be written as a sum of  $w_F(k)$   $k$ th powers of elements of  $F$ . Each polynomial  $P(x, y) \in F[x, y]$  of degree  $d \geq 2k^4$  is the sum of  $k$ th powers*

$$P(x, y) = Q_1(x, y)^k + \cdots + Q_s(x, y)^k,$$

of polynomials  $Q_i \in F[x, y]$  with  $\deg Q_i^k \leq d + k^3$  and  $s \leq 2k^3 \ln(\frac{d}{k} + 1) \ln(2k) + 7k^4 \ln(k) w_F(k)^2$ .

The bound for  $s$  is derived from a sharper bound given at the end of the proof. We start by sketching the proof by pictures:



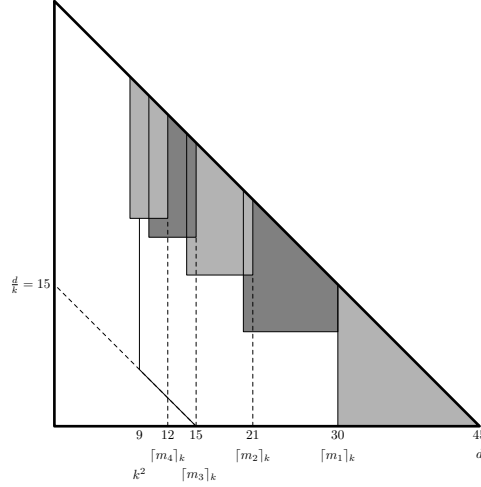
We consider the Newton polygon of  $P$ , it is included in a large triangle (see the left figure). We first cut off trapeziums, corresponding to monomials of higher degree. Each trapezium corresponds to a polynomial  $Q_i^k$  computed by an approximate  $k$ th root as explained in Section 4. It enables to lower the degree of  $P$ , except for monomials whose degree in  $x$  is less than  $k^2$  that will be treated at the end. We iterate this process until we get a polynomial of degree less than  $\frac{d}{k}$  (right figure) to which we will apply Corollary 4.

**Notation.** We will denote  $\lceil x \rceil_k = k \lceil \frac{x}{k} \rceil$  the least integer larger or equal to  $x$  and divisible by  $k$ .

**First step: lower the degree.** Set  $d = \deg P$ ,  $m_0 = \lceil d \rceil_k$  and  $P_0 := P$ . We apply Proposition 7 to  $P_0 = P$ , with  $P_0$  considered as a polynomial of total degree  $\leq m_0$  and  $m = m_0$ ,  $n = 0$ . It yields a polynomial  $Q_0(x, y)$  such that  $\deg_x(P + x^{m_0} - Q_0^k) < m_0 - \frac{m_0}{k}$ . That is to say we have canceled a trapezium, which is there the triangle  $(m_0, 0)$ ,  $(m_0 - \frac{m_0}{k}, 0)$ ,  $(m_0 - \frac{m_0}{k}, \frac{m_0}{k})$ .

We then set  $m_1 = \lceil m_0 \rceil_k - \frac{\lceil m_0 \rceil_k}{k}$  and  $P_1 = P_0 + x^{m_0} - Q_0^k$ . Note that  $\deg_x P_1 < m_1$  and we apply Proposition 7 to  $P_1$ .

To iterate the process, consider the decomposition  $P_i = P'_i + x^{m_i} \cdot P''_i$  with  $\deg_x P'_i < m_i$ . We apply Proposition 7 to  $P'_i$  (with  $m = \lceil m_i \rceil_k$  and  $n = n_i$  such that  $\lceil m_i \rceil_k + n_i = m_0$ ) that yields  $Q_i$  such that  $P'_i + x^{\lceil m_i \rceil_k} y^{n_i} - Q_i^k$  has no monomials in the corresponding trapezium whose  $x$ -coordinates are in between  $\lceil m_i \rceil_k$  and  $m_{i+1} := \lceil m_i \rceil_k - \frac{\lceil m_i \rceil_k}{k}$ . Notice that  $P_{i+1} := P'_i + x^{\lceil m_i \rceil_k} y^{n_i} - Q_i^k + x^{m_i} \cdot P''_i$  also does not have monomials in this trapezium. Here is an example, set  $d = 45$  and  $k = 3$  then we get  $m_0 = 45$ ,  $m_1 = 30$ ,  $m_2 = 20$ ,  $m_3 = 14$ ,  $m_4 = 10$ ,  $m_5 = 8$  and then we stop since  $m_5 < k^2$ . It implies that the first trapezium has its  $x$ -coordinates in between 45 and 30, the second one between 30 and 20,... The height of the left side of each trapezium is always  $\frac{d}{k} = 15$ . The picture is the following:



**End of iterations.** We iterate the process until we reach monomials whose degree in  $x$  is less than  $k^2$ . That is to say we look for  $\ell$  such that  $m_\ell \leq k^2$ . First notice that

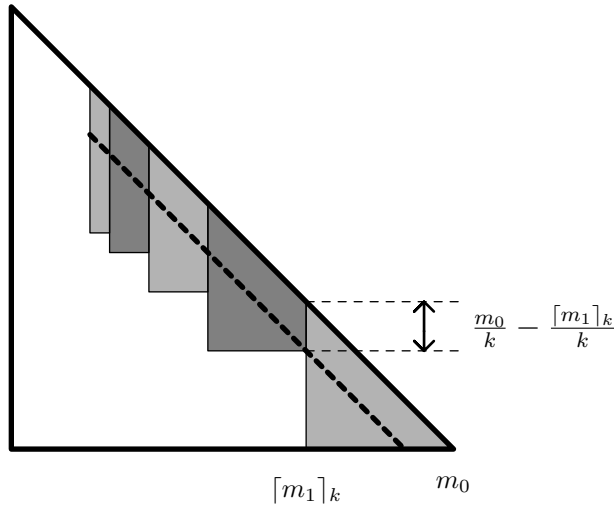
$$\begin{aligned} m_{i+1} &= \lceil m_i \rceil_k - \frac{\lceil m_i \rceil_k}{k} \\ &= (k-1) \left\lceil \frac{m_i}{k} \right\rceil \\ &\leq \left(1 - \frac{1}{k}\right) m_i + k - 1. \end{aligned}$$

Then, by induction

$$\begin{aligned}
 m_i &\leq \left(1 - \frac{1}{k}\right)^i m_0 + (k-1) \left(1 + \left(1 - \frac{1}{k}\right) + \left(1 - \frac{1}{k}\right)^2 + \cdots + \left(1 - \frac{1}{k}\right)^{i-1}\right) \\
 &\leq \left(1 - \frac{1}{k}\right)^i m_0 + k(k-1) \\
 &\leq (d+k)e^{-\frac{i}{k}} + k(k-1), \quad \text{since } \left(1 - \frac{1}{k}\right) \leq e^{-\frac{1}{k}}.
 \end{aligned}$$

Now, for  $\ell \geq k \ln\left(\frac{d}{k} + 1\right)$  we get  $m_\ell \leq k^2$ .

**Fall of the total degree.** At the end of the first series of iterations the total degree (of the monomials whose degree in  $x$  is more or equal to  $k^2$ ) falls (see the picture below).



We give a lower bound for this fall  $\delta_0$  of the degree (starting from degree  $m_0$ ):

$$\begin{aligned} \delta_0 &\geq \frac{m_0}{k} - \frac{\lceil m_1 \rceil_k}{k} \\ &= \left\lfloor \frac{d}{k} \right\rfloor - \left\lfloor \frac{k \left\lfloor \frac{d}{k} \right\rfloor - \left\lfloor \frac{d}{k} \right\rfloor}{k} \right\rfloor \quad (\text{since } d = \lceil m_0 \rceil_k) \\ &\geq \left\lfloor \frac{\left\lfloor \frac{d}{k} \right\rfloor}{k} \right\rfloor \\ &\geq \frac{d}{k^2} - 1. \end{aligned}$$

Therefore the total degree, starting now from degree  $d$ , of the monomials whose degree in  $x$  is more than  $k^2$  has fallen of more than  $\delta \geq \frac{d}{k^2} - k$ .

**Iteration of the fall.** Set  $d_0 = d$ . At each series of iterations the degree (of the monomials whose degree in  $x$  is more or equal to  $k^2$ ) falls from  $d_i$  to  $d_{i+1} := d_i - \left\lfloor \frac{d_i}{k^2} - k \right\rfloor \leq \left(1 - \frac{1}{k^2}\right) d_i + k$ , so that (by a computation similar to the one for  $m_i$  above)  $d_i \leq d e^{-\frac{i}{k^2}} + k^3$ . Suppose that  $d \geq 2k^4$ , so that  $\frac{d}{2k} + k^3 \leq \frac{d}{k}$ . Then for  $\ell \geq k^2 \ln(2k)$ , we get  $d_\ell \leq \frac{d}{k}$ . Each fall of the degree needs less than  $k \ln\left(\frac{d}{k} + 1\right)$  iterations, so that we need to apply Proposition 7 many times, to get a total of  $s_0 = 2k \ln\left(\frac{d}{k} + 1\right) \times k^2 \ln(2k)$   $k$ th powers.

**Monomials of low degree in  $x$ .** At this point, we have written  $P = \sum_{i=1}^{s_0} Q_i^k + P_1 + P_2$ , where  $Q_1, \dots, Q_{s_0}, P_1, P_2 \in F[x, y]$  are such that  $\deg Q_i^k \leq \lceil d \rceil_k$ ,  $\deg_x P_1 < k^2$ ,  $\deg P_2 \leq \frac{d}{k}$  (see the right picture below Theorem 8). By Corollary 4 we can write  $P_2$  as a sum  $P_2 = \sum_{i=1}^{s_2} Q_{i,2}^k$  of  $s_2 \leq k w_F(k)$  terms and  $\deg Q_{i,2}^k \leq k \left\lfloor \frac{d}{k} \right\rfloor = \lceil d \rceil_k < d + k$ .

Now write  $P_1(x, y) = \sum_{0 \leq j < k^2} x^j R_j(y)$ , where  $R_j \in F[y]$  with  $\deg R_j \leq d - j$ . By Corollary 4, write each  $x^j$  as the sum of  $k w_F(k)$  terms of degree  $\leq jk$ . Then, for each  $R_j(y)$  we apply the result in one variable [7, Theorem 1.4 (iii)] (or we can do a similar work as before) so that we can write (since  $d \geq 2k^4$ ):  $R_j(y) = \sum_{i=1}^s S_{ij}^k(y)$  with  $s \leq k(w_F(k) + 3 \ln(k)) + 2$  and  $\deg S_{ij}^k \leq \deg R_j + k - 1$ . We get  $x^j R_j(y)$  as the sum of  $s' \leq k w_F(k)(k(w_F(k) + 3 \ln(k)) + 2)$ ,  $k$ th powers of degree  $\leq jk + \deg R_j + k - 1 \leq d + k^3$  ( $j = 0, \dots, k^2 - 1$ ). Therefore,  $P_1 = \sum_{i=1}^{s_1} Q_{i,1}^k$  with  $s_1 \leq k^3 w_F(k)(k(w_F(k) + 3 \ln(k)) + 2)$  terms and  $\deg Q_{i,1}^k \leq d + k^3$ .

**Conclusion.** For  $d \geq 2k^4$  we can write  $P(x, y)$  as the sum

$$P(x, y) = \sum_{i=1}^s Q_i^k(x, y)$$

such that  $\deg Q_i^k \leq d + k^3$  and  $s \leq s_0 + s_2 + s_1$  that is to say<sup>2</sup>

$$s \leq 2k^3 \ln \left( \frac{d}{k} + 1 \right) \ln(2k) + kw_F(k) + k^3 w_F(k)(k(w_F(k) + 3 \ln(k)) + 2).$$

It yields the announced bound  $s \leq 2k^3 \ln(\frac{d}{k} + 1) \ln(2k) + 7k^4 \ln(k)w_F(k)^2$ .

**Question.** Is it possible to have a sum

$$P(x, y) = \sum_{i=1}^s Q_i^k(x, y)$$

such that  $\deg Q_i^k \leq \deg P + k^3$  and a bound  $s$  depending only on  $k$  and not on  $\deg P$ ?

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<sup>2</sup>This is the bound used to fill the numerical table of the introduction.