# REDUCIBILITY OF RATIONAL FUNCTIONS IN SEVERAL VARIABLES

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Abstract. We prove a analogous of Stein theorem for rational functions in several variables: we bound the number of reducible fibers by a formula depending on the degree of the fraction.

## 1. Introduction

Let $K$ be an algebraically closed field. Let $f = \frac{p}{q} \in K(\underline{x})$, with $\underline{x} = (x_1, \ldots, x_n)$, $n \geqslant 2$ and $\gcd(p,q) = 1$, the *degree* of $f$ is $\deg f = \max\{\deg p, \deg q\}$. We associate to a fraction $f = \frac{p}{q}$ the pencil $p - \lambda q$, $\lambda \in \hat{K}$ (where we denote $\hat{K} = K \cup \{\infty\}$ and by convention if $\lambda = \infty$ then $p - \lambda q = q$).

For each $\lambda \in \hat{K}$ write the decomposition into irreducible factors:

$$p - \lambda q = \prod_{i=1}^{n_\lambda} F_i^{r_i}.$$

The *spectrum* of $f$ is $\sigma(f) = \{\lambda \in \hat{K} \mid n_\lambda > 1\}$, and the *order of reducibility* is $\rho(f) = \sum_{\lambda \in \hat{K}} (n_\lambda - 1)$.

A fraction $f$ is *composite* if it is the composition of a univariate rational fraction of degree more than 1 with another rational function.

**Theorem 1.1.** *Let $K$ be an algebraically closed field of characteristic $0$. Let $f \in K(\underline{x})$ be non-composite then*

$$\rho(f) < (\deg f)^2 + \deg f.$$

A theorem of Bertini and Krull implies that if $f$ is non-composite then $\sigma(f)$ is finite and we should notice that $\#\sigma(f) \leqslant \rho(f)$. Later on, for an algebraically closed field of characteristic zero and for a polynomial $f \in K[x,y]$, Stein [St] proved the formula $\rho(f) < \deg f$. This formula has been generalized in several directions, see [Na1] for references. For a rational function $f \in \mathbb{C}(x,y)$ a consequence of the work of Ruppert [Ru] on pencil of curves, is that $\#\sigma(f) < (\deg f)^2$. For $K$ algebraically closed (of any characteristic) and $f \in K(x,y)$ Lorenzini

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[Lo] proved under geometric hypotheses on the pencil $(p - \lambda q)$ that $\rho(f) < (\deg f)^2$. This has been generalized by Vistoli [Vi] for a pencil in several variables for an algebraically closed field of characteristic 0.

Let us give an example extracted from [Lo]. Let $f(x,y) = \frac{x^3+y^3+(1+x+y)^3}{xy(1+x+y)}$, then $\deg(f) = 3$ and $\sigma(f) = \{1, j, j^2, \infty\}$ (where $\{1, j, j^2\}$ are the third roots of unity). For $\lambda \in \sigma(f)$, $(f = \lambda)$ is composed of three lines hence $\rho(f) = 8 = (\deg f)^2 - 1$. Then Lorenzini's bound is optimal in two variables.

The motivation of this work is that we develop the analogous theory of Stein for rational function: composite fractions, kernels of Jacobian derivatives, groups of divisors,... The method for the two variables case is inspired from the work of Stein [St] and the presentation of that work by Najib [Na1]. For completeness even the proofs similar to the ones of Stein have been included. Another motivation is that with a bit more effort we get the case of several variables by following the ideas of [Na1] (see the articles [Na2], [Na3]).

In §2 we prove that a fraction is non-composite if and only its spectrum is finite. Then in §3 we introduce a theory of Jacobian derivation and compute the kernel. Next in §4 we prove that for a non-composite fraction in two variables $\rho(f) < (\deg f)^2 + \deg f$. Finally in §5 we extend this formula to several variables and we end by stating a result for fields of any characteristic.

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## 2. Composite rational functions

Let $K$ be an algebraically closed field. Let $\underline{x} = (x_1, \ldots, x_n)$, $n \geqslant 2$.

*Definition* 2.1. A rational function $f \in K(\underline{x})$ is *composite* if there exist $g \in K(\underline{x})$ and $r \in K(t)$ with $\deg r \geqslant 2$ such that

$$f = r \circ g.$$

**Theorem 2.2.** *Let $f = \frac{p}{q} \in K(\underline{x})$. The following assertions are equivalent:*

1. *$f$ is composite;*
2. *$p - \lambda q$ is reducible in $K[\underline{x}]$ for all $\lambda \in \hat{K}$ such that* $\deg p - \lambda q = \deg f$*;*
3. *$p - \lambda q$ is reducible in $K[\underline{x}]$ for infinitely many $\lambda \in \hat{K}$.*

Before proving this result we give two corollaries.

**Corollary 2.3.** *$f$ is non-composite if and only if its spectrum $\sigma(f)$ is finite.*

One aim of this paper is to give a bound for $\sigma(f)$. The hard implication of this theorem (3) $\Rightarrow$ (1) is in fact a reformulation of a theorem of Bertini and Krull.

We also give a nice application pointed out to us by P. Débes:

**Corollary 2.4.** *Let $p \in K[\underline{x}]$ irreducible. Let $q \in K[\underline{x}]$ with $\deg q < \deg p$ and $\gcd(p,q) = 1$. Then for all but finitely many $\lambda \in K$, $p - \lambda q$ is irreducible in $K[\underline{x}]$.*

*Convention :* When we define a fraction $F = \frac{P}{Q}$ we will assume that $\gcd(P,Q) = 1$.

We start with the easy part of Theorem 2.2:

*Proof.* (2) $\Rightarrow$ (3) is trivial. Let us prove (1) $\Rightarrow$ (2). Let $f = \frac{p}{q}$ be a composite rational function. There exist $g = \frac{u}{v} \in K(\underline{x})$ and $r \in K(t)$ with $k = \deg r \geqslant 2$ such that $f = r \circ g$ . Let us write $r = \frac{a}{b}$. Let $\lambda \in \hat{K}$ such that $\deg a - \lambda b = \deg r$ and factorize $a(t) - \lambda b(t) = \alpha(t-t_1)(t-t_2)\cdots(t-t_k)$, $\alpha \in K^*, t_1, \dots, t_k \in K$. Then

$$p - \lambda q = q \cdot (f - \lambda) = q \cdot \left(\frac{a - \lambda b}{b}\right)(g) = \alpha q \frac{(g-t_1)\cdots(g-t_k)}{b(g)}.$$

Then by multiplication by $v^k$ at the numerator and denominator we get:

$$(p - \lambda q) \cdot (v^k b(g)) = \alpha q (u - t_1 v) \cdots (u - t_k v),$$

which is a polynomial identity. As $\gcd(a,b) = 1$, $\gcd(u,v) = 1$ and $\gcd(p,q) = 1$ then $u - t_1 v, \dots, u - t_k v$ divide $p - \lambda q$. Hence $p - \lambda q$ is reducible in $K[\underline{x}]$. □

Let us reformulate the Bertini-Krull theorem in our context from [Sc, Theorem 37]. It will enable us to end the proof of Theorem 2.2.

**Theorem 2.5** (Bertini, Krull)**.** *Let $F(\underline{x}, \lambda) = p(\underline{x}) - \lambda q(\underline{x}) \in K[\underline{x}, \lambda]$ an irreducible polynomial. Then the following conditions are equivalent:*

(1) *$F(\underline{x}, \lambda_0) \in K[\underline{x}]$ is reducible for all $\lambda_0 \in K$ such that $\deg_{\underline{x}} F(\underline{x}, \lambda_0) = \deg_{\underline{x}} F$.*

(2) (a) *either there exist $\phi, \psi \in K[\underline{x}]$ with $\deg_{\underline{x}} F > \max\{\deg \phi, \deg \psi\}$, and $a_i \in K[\lambda]$, such that*

$$F(\underline{x}, \lambda) = \sum_{i=0}^{n} a_i(\lambda) \phi(\underline{x})^{n-i} \psi(\underline{x})^i;$$

(b) *or $\mathrm{char}(K) = \pi > 0$ and $F(\underline{x}, \lambda) \in K[\underline{x}^\pi, \lambda]$, where $\underline{x}^\pi = (x_1^\pi, \dots, x_n^\pi)$.*

We now end the proof of Theorem 2.2:

*Proof.* (3) $\Rightarrow$ (1) Suppose that $p - \lambda_0 q$ is reducible in $K[\underline{x}]$ for infinitely many $\lambda_0 \in \hat{K}$; then it is reducible for all $\lambda_0 \in K$ such that $\deg_{\underline{x}} F(\underline{x}, \lambda_0) = \deg_{\underline{x}} F$ (see Corollary 3 of Theorem 32 of [Sc]). We apply Bertini-Krull theorem:

*Case (a):* $F(\underline{x}, \lambda) = p(\underline{x}) - \lambda q(\underline{x})$ can be written:

$$p(\underline{x}) - \lambda q(\underline{x}) = \sum_{i=0}^{n} a_i(\lambda)\phi(\underline{x})^{n-i}\psi(\underline{x})^i.$$

So we may suppose that for $i = 1, \ldots, n$, $\deg_\lambda a_i = 1$, let us write $a_i(\lambda) = \alpha_i - \lambda\beta_i$, $\alpha_i, \beta_i \in K$. Then

$$p(\underline{x}) = \sum_{i=0}^{n} \alpha_i \phi(\underline{x})^{n-i}\psi(\underline{x})^i = \phi^n \sum_{i=0}^{n} \alpha_i \Big(\frac{\psi}{\phi}\Big)^i(\underline{x}),$$

and

$$q(\underline{x}) = \sum_{i=0}^{n} \beta_i \phi(\underline{x})^{n-i}\psi(\underline{x})^i = \phi^n \sum_{i=0}^{n} \beta_i \Big(\frac{\psi}{\phi}\Big)^i(\underline{x}).$$

If we set $g(\underline{x}) = \frac{\psi(\underline{x})}{\phi(\underline{x})} \in K[\underline{x}]$, and $r(t) = \frac{\sum_{i=0}^n \alpha_i t^i}{\sum_{i=0}^n \beta_i t^i}$ then $\frac{p}{q}(\underline{x}) = r \circ g$. Moreover as $\deg_{\underline{x}} F > \max\{\deg\phi, \deg\psi\}$ this implies $n \geqslant 2$ so that $\deg r \geqslant 2$. Then $\frac{p}{q} = f = r \circ g$ is a composite rational function

*Case (b):* Let $\pi = \operatorname{char}(K) > 0$ and $F(\underline{x}, \lambda) = p(\underline{x}) - \lambda q(\underline{x}) \in K[\underline{x}^\pi, \lambda]$, For $\lambda = 0$ it implies that $p(\underline{x}) = P(\underline{x}^\pi)$, then there exists $p' \in K[\underline{x}]$ such that $p(\underline{x}) = (p'(\underline{x}))^\pi$. For $\lambda = -1$ we obtain $s' \in K[\underline{x}]$ such that $p(\underline{x}) + q(\underline{x}) = (s'(\underline{x}))^\pi$. Then $q(\underline{x}) = (p(\underline{x}) + q(\underline{x})) - p(\underline{x}) = (s'(\underline{x}))^\pi - (p'(\underline{x}))^\pi = (s'(\underline{x}) - p'(\underline{x}))^\pi$. Then if we set $q' = s' - p'$ we obtain $q(\underline{x}) = (q'(\underline{x}))^\pi$. Now set $r(t) = t^\pi$ and $g = \frac{p'}{q'}$ we get $f = \frac{p}{q} = \left(\frac{p'}{q'}\right)^\pi = r \circ g$. □

## 3. Kernel of the Jacobian derivation

We now consider the two variables case and $K$ is an uncountable algebraically closed field of characteristic zero.

3.1. **Jacobian derivation.** Let $f, g \in K(x, y)$, the following formula:

$$D_f(g) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x},$$

defines a derivation $D_f : K(x, y) \to K(x, y)$. Notice the $D_f(g)$ is the determinant of the Jacobian matrix of $(f, g)$. We denote by $C_f$ the kernel of $D_f$:

$$C_f = \{g \in K(x, y) \mid D_f(g) = 0\}.$$

Then $C_f$ is a subfield of $K(x,y)$. We have the inclusion $K(f) \subset C_f$. Moreover if $g^k \in C_f$, $k \in \mathbb{Z} \setminus \{0\}$ then $g \in C_f$.

**Lemma 3.1.** *Let $f = \frac{p}{q}$, $g \in K(x,y)$. The following conditions are equivalent:*

1. *$g \in C_f$;*
2. *$f$ and $g$ are algebraically dependent;*
3. *$g$ is constant on irreducible components of the curves $(p - \lambda q = 0)$ for all but finitely many $\lambda \in \hat{K}$;*
4. *$g$ is constant on infinitely many irreducible components of the curves $(p - \lambda q = 0)$, $\lambda \in \hat{K}$.*

**Corollary 3.2.** *If $g \in C_f$ is not a constant then $C_f = C_g$.*

*Proof.*

- (1) $\Leftrightarrow$ (2). We follow the idea of [Na1] instead of [St]. $f$ and $g$ are algebraically dependent if and only $\operatorname{transc}_K K(f,g) = 1$. And $\operatorname{transc}_K K(f,g) = 1$ if and only the rank of the Jacobian matrix of $(f,g)$ is less or equal to 1, which is equivalent to $g \in C_f$.
- (2) $\Rightarrow$ (3). Let $f$ and $g$ be algebraically dependent. Then there exists a two variables polynomial in $f$ and $g$ that vanishes. Let us write
$$\sum_{i=0}^{n} R_i(f) g^i = 0$$
where $R_i(t) \in K[t]$. Let us write $f = \frac{p}{q}$, $g = \frac{u}{v}$ and $R_n(t) = \alpha(t-\lambda_1)\cdots(t-\lambda_m)$. Then
$$\sum_{i=0}^{n} R_i\left(\frac{p}{q}\right)\left(\frac{u}{v}\right)^i = 0, \text{ hence } \sum_{i=0}^{n} R_i\left(\frac{p}{q}\right) u^i v^{n-i} = 0.$$
By multiplication by $q^d$ for $d = \max\{\deg R_i\}$ (in order that $q^d R_i(\frac{p}{q})$ are polynomials) we obtain
$$q^d R_n\left(\frac{p}{q}\right) u^n = v\left(-q^d R_{n-1}\left(\frac{p}{q}\right)u^{n-1} - \cdots\right).$$
As $\gcd(u,v) = 1$ then $v$ divides the polynomial $q^d R_n(\frac{p}{q})$, then $v$ divides $q^{d-m}(p-\lambda_1 q)\cdots(p-\lambda_m q)$. Then all irreducible factors of $v$ divide $q$ or $p - \lambda_i q$, $i = 1,\ldots,m$.

  Let $\lambda \notin \{\infty, \lambda_1, \ldots, \lambda_m\}$. Let $V_\lambda$ be an irreducible component of $p - \lambda q$, then $V_\lambda \cap Z(v)$ is zero dimensional (or empty). Hence

$v$ is not identically equal to 0 on $V_\lambda$. Then for all but finitely many $(x,y) \in V_\lambda$ we get:

$$\sum_{i=0}^{n} R_i(\lambda) g(x,y)^i = 0.$$

Therefore $g$ can only reach a finite number of values $c_1, \ldots, c_n$ (the roots of $\sum_{i=0}^n R_i(\lambda)t^i$). Since $V_\lambda$ is irreducible, $g$ is constant on $V_\lambda$.

- (3) $\Rightarrow$ (4). Clear.
- (4) $\Rightarrow$ (1). We first give a proof that if $g$ is constant along an irreducible component $V_\lambda$ of $(p-\lambda q = 0)$ then $D_f(g) = 0$ on $V_\lambda$ (we suppose that $V_\lambda$ is not in the poles of $g$). Let $(x_0, y_0) \in V_\lambda$ and $t \mapsto p(t)$ be a local parametrization of $V_\lambda$ around $(x_0, y_0)$. By definition of $p(t)$ we have $f(p(t)) = \lambda$, this implies that:

$$\left\langle \frac{dp}{dt} \mid \overline{\operatorname{grad} f} \right\rangle = \frac{d(f(p(t))}{dt} = 0$$

and by hypotheses $g$ is constant on $V_\lambda$ this implies $g(p(t))$ is constant and again:

$$\left\langle \frac{dp}{dt} \mid \overline{\operatorname{grad} g} \right\rangle = \frac{d(g(p(t))}{dt} = 0.$$

Then $\operatorname{grad} f$ and $\operatorname{grad} g$ are orthogonal around $(x_0, y_0)$ on $V_\lambda$ to the same vector, as we are in dimension 2 this implies that the determinant of Jacobian matrix of $(f, g)$ is zero around $(x_0, y_0)$ on $V_\lambda$. By extension $D_f(g) = 0$ on $V_\lambda$.

We now end the proof: If $g$ is constant on infinitely many irreducible components $V_\lambda$ of $(p - \lambda q = 0)$ this implies that $D_f(g) = 0$ on infinitely many $V_\lambda$. Then $D_f(g) = 0$ in $K(x,y)$.

□

3.2. **Group of the divisors.** Let $f = \frac{p}{q}$, let $\lambda_1, \ldots, \lambda_n \in \hat{K}$, we denote by $G(f; \lambda_1, \ldots, \lambda_n)$ the multiplicative group generated by all the divisors of the polynomials $p - \lambda_i q$, $i = 1, \ldots, n$.

Let

$$d(f) = (\deg f)^2 + \deg f.$$

**Lemma 3.3.** *Let $F_1, \ldots, F_r \in G(f; \lambda_1, \ldots, \lambda_n)$. If $r \geqslant d(f)$ then there exists a collection of integers $m_1, \ldots, m_r$ (not all equal to zero) such that*

$$g = \prod_{i=1}^{r} F_i^{m_i} \in C_f.$$

*Proof.* Let $\mu \notin \{\lambda_1, \ldots, \lambda_n\}$, and let $S$ be an irreducible component of $(p - \mu q = 0)$. Let $\bar{S}$ be the projective closure of $S$. The functions $F_i$ restricted to $\bar{S}$ have their poles and zeroes on the points at infinity of $S$ or on the intersection $S \cap Z(F_i) \subset Z(p) \cap Z(q)$.

Let $n : \tilde{S} \to \bar{S}$ be a normalization of $\bar{S}$. The inverse image under normalisation of the points at infinity are denoted by $\{\gamma_1, \ldots, \gamma_k\}$, their number verifies $k \leqslant \deg S \leqslant \deg f$.

At a point $\delta \in Z(p) \cap Z(q)$, the number of points of $n^{-1}(\delta)$ is the local number of branches of $S$ at $\delta$ then it is less or equal than $\operatorname{ord}_\delta(S)$, where $\operatorname{ord}_\delta(S)$ denotes the order (or multiplicity) of $S$ at $\delta$ (see e.g. [Sh], paragraph II.5.3). Then

$$\begin{aligned}\#n^{-1}(\delta) \leqslant \operatorname{ord}_\delta(S) &\leqslant \operatorname{ord}_\delta Z(p-\mu q) \leqslant \operatorname{ord}_\delta Z(p-\mu q) \cdot \operatorname{ord}_\delta Z(p)\\ &\leqslant \operatorname{mult}_\delta(p-\mu q, p) = \operatorname{mult}_\delta(p,q)\end{aligned}$$

where $\operatorname{mult}_\delta(p,q)$ is the intersection multiplicity (see e.g. [Fu]). Then by Bézout theorem:

$$\sum_{\delta \in Z(p)\cap Z(q)} \#n^{-1}(\delta) \leqslant \sum_{\delta \in Z(p)\cap Z(q)} \operatorname{mult}_\delta(p,q) \leqslant \deg p \cdot \deg q \leqslant (\deg f)^2.$$

Then the inverse image under normalisation of $\cup_{i=1}^r S \cap Z(F_i)$ denoted by $\{\gamma_{k+1}, \ldots, \gamma_\ell\}$ have less or equal than $(\deg f)^2$ elements. Notice that $\ell \leqslant \deg f + (\deg f)^2 = d(f)$.

Now let $\nu_{ij}$ be the order of $F_i$ at $\gamma_j$ ($i = 1, \ldots, r$; $j = 1, \ldots, \ell$). Consider the matrix $M = (\nu_{ij})$. Because the degree of the divisor $(F_i)$ (seen over $\tilde{S}$) is zero we get $\sum_{j=1}^\ell \nu_{ij} = 0$, for $i = 1, \ldots, r$, that means that columns of $M$ are linearly dependent. Then $\operatorname{rk} M < \ell \leqslant d(f)$, by hypothesis $r \geqslant d(f)$, then the rows of $M$ are also linearly dependent. Let $m_1(\mu, S), \ldots, m_r(\mu, S)$ such that $\sum_{i=1}^r m_i(\mu, S)\nu_{ij} = 0$, $j = 1, \ldots, \ell$.

Consider the function $g_{\mu,S} = \prod_{i=1}^r F_i^{m_i(\lambda,S)}$. Then this function is regular and does not have zeroes or poles at the points $\gamma_j$, because $\sum_{i=1}^r m_i(\mu, S)\nu_{ij} = 0$. Then $g_{\mu,S}$ is constant on $S$.

This construction gives a map $(\mu, S) \mapsto (m_1(\mu, S), \ldots, m_r(\mu, S))$ from $K$ to $\mathbb{Z}^r$. Since $K$ is uncountable, there exists infinitely many $(\mu, S)$ with the same $(m_1, \ldots, m_r)$. Then the function $g = \prod_{i=1}^r F_i^{m_i}$ is constant on infinitely many components of curves of $(p - \mu q = 0)$ and by Lemma 3.1 this implies $g \in C_f$. □

3.3. **Non-composite rational function.** Let $f = \frac{p}{q}$. Let $G(f)$ be the multiplicative group generated by all divisors of the polynomials

$p - \lambda q$ for all $\lambda \in \hat{K}$. In fact we have

$$G(f) = \bigcup_{(\lambda_1,\ldots,\lambda_n)\in K^n} G(f;\lambda_1,\ldots,\lambda_n).$$

*Definition* 3.4. A family $F_1,\ldots,F_r \in G(f)$ is *$f$-free* if $(m_1,\ldots,m_r) \in \mathbb{Z}^r$ is such that $\prod_{i=1}^r F_i^{m_i} \in C_f$ then $(m_1,\ldots,m_r) = (0,\ldots,0)$.

A $f$-free family $F_1,\ldots,F_r \in G(f)$ is *$f$-maximal* if for all $F \in G(f)$, $\{F_1,\ldots,F_r,F\}$ is not $f$-free.

**Theorem 3.5.** *Let $f \in K(x,y)$, $\deg f > 0$. Then the following conditions are equivalent:*

1. $\deg f = \min\{\deg g \mid g \in C_f \setminus K\}$*;*
2. $\sigma(f)$ *is finite;*
3. $C_f = K(f)$*;*
4. $f$ *is non-composite.*

*Remark* 3.6. This does not give a new proof of "$\sigma(f)$ is finite $\Leftrightarrow f$ is non-composite" because we use Bertini-Krull theorem.

*Remark* 3.7. The proof (1) $\Rightarrow$ (2) is somewhat easier than in [St], whereas (2) $\Rightarrow$ (3) is more difficult.

*Proof.*

- (1) $\Rightarrow$ (2). Let us suppose that $\sigma(f)$ is infinite. Set $f = \frac{p}{q}$, with $\gcd(p,q) = 1$. For all $\alpha \in \sigma(f)$, let $F_\alpha$ be an irreducible divisor of $p - \alpha q$, such that $\deg F_\alpha < \deg f$. By Lemma 3.3 there exists a $f$-maximal family $\{F_1,\ldots,F_r\}$ with $r \leqslant d(f)$. Moreover $r \geqslant 1$ because $\{F_\alpha\}$ is $f$-free: if not there exists $k \neq 0$ such that $F_\alpha^k \in C_f$ then $F_\alpha \in C_f$, but $\deg F_\alpha < \deg f$ that contradicts the hypothesis of minimality.

  Now the collection $\{F_1,\ldots,F_r,F_\alpha\}$ is not $f$-free, so that there exist integers $\{m_1(\alpha),\ldots,m_r(\alpha),m(\alpha)\}$, with $m(\alpha) \neq 0$, such that
  $$F_1^{m_1(\alpha)} \cdots F_r^{m_r(\alpha)} \cdot F_\alpha^{m(\alpha)} \in C_f.$$
  Since $\sigma(f)$ is infinite then is equal to $\hat{K}$ minus a finite number of values (see Theorem 2.2) then $\sigma(f)$ is uncountable and the map $\alpha \mapsto (m_1(\alpha),\ldots,m_r(\alpha),m(\alpha))$ is not injective. Let $\alpha \neq \beta$ such that $m_i(\alpha) = m_i(\beta) = m_i$, $i = 1,\ldots,r$ and $m(\alpha) = m(\beta) = m$. Then $F_1^{m_1} \cdots F_r^{m_r} \cdot F_\alpha^m \in C_f$ and $F_1^{m_1} \cdots F_r^{m_r} \cdot F_\beta^m \in C_f$, it implies that $(F_\alpha/F_\beta)^m \in C_f$, therefore $F_\alpha/F_\beta \in C_f$.

  Now $\deg \frac{F_\alpha}{F_\beta} < \deg f$, then by the hypothesis of minimality it proves $\frac{F_\alpha}{F_\beta}$ is a constant. Let $a \in K^*$ such that $F_\alpha = aF_\beta$, by definition $F_\alpha$ divides $p-\alpha q$, but moreover $F_\alpha$ divides $p-\beta q$ (as $F_\beta$ do). Then as $F_\alpha$ divides both $p-\alpha q$ and $p-\beta q$, $F_\alpha$ divides $p$ and $q$, that contradicts $\gcd(p,q)=1$.

- (2) $\Rightarrow$ (3). Let $f=\frac{p}{q}$, $\sigma(f)$ finite and $g\in C_f$, we aim at proving that $g\in K(f)$. The proof will be done in several steps:
  (a) *Reduction to the case $g=\frac{u}{q^\ell}$.* Let $g=\frac{u}{v}\in C_f$, then $f$ and $g$ are algebraically dependent, then there exists a polynomial in $f$ and $g$ that vanishes. As before let us write

$$\sum_{i=0}^{n} R_i(f)g^i=0$$

where $R_i(t)\in K[t]$. As $f=\frac{p}{q}$, $g=\frac{u}{v}$ then

$$\sum_{i=0}^{n} R_i\Big(\frac{p}{q}\Big)\left(\frac{u}{v}\right)^i=0,\text{ hence }\sum_{i=0}^{n} R_i\Big(\frac{p}{q}\Big)u^iv^{n-i}=0.$$

By multiplication by $q^d$ for $d=\max\{\deg R_i\}$ (in order that all $q^dR_i(\frac{p}{q})$ are polynomials) we get:

$$q^dR_n\Big(\frac{p}{q}\Big)u^n=v\left(-q^dR_{n-1}\Big(\frac{p}{q}\Big)u^{n-1}-\cdots\right).$$

As $\gcd(u,v)=1$ then $v$ divides the polynomial $q^dR_n(\frac{p}{q})$; we write $vu'=q^dR_n(\frac{p}{q})$ then

$$g=\frac{u}{v}=\frac{uu'}{q^dR_n(\frac{p}{q})}.$$

But $R_n(\frac{p}{q})\in K(\frac{p}{q})$ then $\frac{uu'}{q^d}\in C_f$, but also we have that $g\in K(f)$ if and only if $\frac{uu'}{q^d}\in K(f)$. This proves the reduction.

  (b) *Reduction to the case $g=qu$.* Let $g=\frac{u}{q^\ell}\in C_f$, $\ell\geqslant 0$. As $\sigma(f)$ is finite by Lemma 3.1 we choose $\lambda\in K$ such that $p-\lambda q$ is irreducible and $g\in C_f$ is constant (equal to $c$) on $p-\lambda q$. As $g=\frac{u}{q^\ell}$,we have $p-\lambda q$ divides $u-cq^\ell$. We can write:

$$u-cq^\ell=u'(p-\lambda q).$$

Then

$$\frac{u}{q^\ell}=\frac{u'}{q^{\ell-1}}\Big(\frac{p}{q}-\lambda\Big)+c.$$

As $\frac{u}{q^\ell}$ and $f = \frac{p}{q}$ are in $C_f$ we get $\frac{u'}{q^{\ell-1}} \in C_f$; moreover $\frac{u}{q^\ell} \in K(f)$ if and only if $\frac{u'}{q^{\ell-1}} \in K(f)$. By induction on $\ell \geqslant 0$ this prove the reduction.

(c) *Reduction to the case $g = q$.* Let $g = qu \in C_f$. $g$ is constant along the irreducible curve $(p - \lambda q = 0)$. Then $qu = u_1(p - \lambda q) + c_1$.
Let $\deg p = \deg q$. Then $q^h u^h = u_1^h(p^h - \lambda q^h)$ (where $P^h$ denotes the homogeneous part of higher degree of the polynomial $P$). Then $p^h - \lambda q^h$ divides $q^h u^h$ for infinitely many $\lambda \in K$. As $\gcd(p,q) = 1$ this gives a contradiction.
Hence $\deg p \neq \deg q$. We may assume $\deg p > \deg q$ (otherwise $qu \in C_f$ and $\frac{p}{q} \in C_f$ implies $pu \in C_f$). Then we write:

$$qu = qu_1\Big(\frac{p}{q} - \lambda\Big) + c_1,$$

that proves that $qu_1 \in C_f$ and that $qu \in K(f)$ if and only if $qu_1 \in K(f)$. The inequality $\deg p > \deg q$ implies that $\deg u_1 < \deg u$. We continue by induction, $qu_1 = qu_2(\frac{p}{q} - \lambda) + c_2$, with $\deg u_2 < \deg u_1$,..., until we get $\deg u_n = 0$ that is $u_n \in K^*$. Thus we have prove firstly that $qu_n \in C_f$, that is to say $q \in C_f$, and secondly that $qu \in K(f)$ if and only if $q \in K(f)$.

(d) *Case $g = q$.* If $q \in C_f$ then $q$ is constant along the irreducible curve $(p - \lambda q = 0)$ then $q = a(p - \lambda q) + c$, $a \in K^*$. Then

$$q = \frac{c}{1 - a(\frac{p}{q} - \lambda)} \in K\Big(\frac{p}{q}\Big) = K(f).$$

- (3) $\Rightarrow$ (4). Let us assume that $C_f = K(f)$ and that $f$ is composite, then there exist $r \in K(t)$, $\deg r \geqslant 2$ and $g \in K(x,y)$ such that $f = r \circ g$. By the formula $\deg f = \deg r \cdot \deg g$ we get $\deg f > \deg g$. Now if $r = \frac{a}{b}$ then we have a relation $b(g)f = a(g)$, then $f$ and $g$ are algebraically dependent, hence by Lemma 3.1, $g \in C_f$. As $C_f = K(f)$, there exists $s \in K(t)$ such that $g = s \circ f$. Then $\deg g \geqslant \deg f$. That yields to a contradiction.
- (4) $\Rightarrow$ (1). Assume that $f$ is non-composite and let $g \in C_f$ of minimal degree. By Corollary 3.2 we get $C_f = C_g$, then $\deg g = \min\{\deg h \mid h \in C_g \setminus K\}$. Then by the already proved implication (1) $\Rightarrow$ (3) for $g$, we get $C_g = K(g)$. Then $f \in C_f = C_g = K(g)$, then there exists $r \in K(t)$ such that $f = r \circ g$, but

as $f$ is non-composite then $\deg r = 1$, hence $\deg f = \deg g = \min\{\deg h \mid h \in C_f \setminus K\}$.

□

## 4. Order of reducibility of rational functions in two variables

Let $f = \frac{p}{q} \in K(x,y)$; for all $\lambda \in \hat{K}$, let $n_\lambda$ be the number of irreducible components of $p - \lambda q$. Let

$$\rho(f) = \sum_{\lambda \in \hat{K}} (n_\lambda - 1).$$

By Theorem 2.2, $\rho(f)$ is finite if and only if $f$ is non-composite. We give a bound for $\rho(f)$. Recall that we defined:

$$d(f) = (\deg f)^2 + \deg f.$$

**Theorem 4.1.** *Let $K$ be an algebraic closed field of characteristic 0. If $f \in K(x,y)$ is non-composite then*

$$\rho(f) < d(f).$$

*Proof.* First notice that $K$ can be supposed uncountable, otherwise it can be embedded into an uncountable field $L$ and the spectrum in $K$ would be included in the spectrum in $L$.

Let us assume that $f$ is non-composite, then by Theorem 2.2 and its corollary we have that $\sigma(f)$ is finite: $\sigma(f) = \{\lambda_1, \ldots, \lambda_r\}$. We suppose that $\rho(f) \geqslant d(f)$. Let $f = \frac{p}{q}$. We decompose the polynomials $p - \lambda_i q$ in irreducible factors, for $i = 1, \ldots, r$:

$$p - \lambda_i q = \prod_{j=1}^{n_i} F_{i,j}^{k_{i,j}},$$

where $n_i$ stands for $n_{\lambda_i}$. Notice that since $\gcd(p,q) = 1$ then $F_{i,j}$ divides $p - \lambda_i q$ but do not divides any of $p - \mu q$, $\mu \neq \lambda_i$. The collection $\{F_{1,1}, \ldots, F_{1,n_1-1}, \ldots, F_{r,1}, \ldots, F_{r,n_r-1}\}$, is included in $G(f, \lambda_1, \ldots, \lambda_r)$ and contains $\rho(f) \geqslant d(f)$ elements, then Lemma 3.3 provides a collections $\{m_{1,1}, \ldots, m_{1,n_1-1}, \ldots, m_{r,1}, \ldots, m_{r,n_r-1}\}$ of integers (not all equal to 0) such that

$$g = \prod_{i=1}^{r} \prod_{j=1}^{n_i-1} F_{i,j}^{m_{i,j}} \in C_f. \tag{1}$$

By Theorem 3.5 it implies that $g \in K(f)$, then $g = \frac{u(f)}{v(f)}$, where $u, v \in K[t]$. Let $\mu_1, \ldots, \mu_k$ be the roots of $u$ and $\mu_{k+1}, \ldots, \mu_\ell$ the roots

of $v$. Then
$$g = \frac{u(\frac{p}{q})}{v(\frac{p}{q})} = \alpha \frac{\prod_{i=1}^{k} \frac{p}{q} - \mu_i}{\prod_{i=k+1}^{\ell} \frac{p}{q} - \mu_i}$$
so that
$$g = \alpha q^{\ell-2k} \frac{\prod_{i=1}^{k} p - \mu_i q}{\prod_{i=k+1}^{\ell} p - \mu_i q}. \tag{2}$$

If $m_{i_0,j_0} \neq 0$ then by the definition of $g$ by equation (1) and by equation (2), we get that $F_{i_0,j_0}$ divides one of the $p - \mu_i q$ or divides $q$. If $F_{i_0,j_0}$ divides $p - \mu_i q$ then $\mu_i = \lambda_{i_0} \in \sigma(f)$. If $F_{i_0,j_0}$ divides $q$ then $\mu_i = \infty$, so that $\infty \in \sigma(f)$. In both cases $p - \lambda_{i_0} q$ appears in formula (2) at the numerator or at the denominator of $g$. Then $F_{i_0,n_{i_0}}$ should appears in decomposition (1), that gives a contradiction. Then $\rho(f) < d(f)$. □

## 5. Extension to several variables

We follows the lines of the proof of [Na3]. We will need a result that claims that the irreducibility and the degree of a family of polynomials remain constant after a generic linear change of coordinates. For $\underline{x} = (x_1, \ldots, x_n)$ and a matrix $B = (b_{ij}) \in Gl_n(K)$, we denote the new coordinates by $B \cdot \underline{x}$:
$$B \cdot \underline{x} = (\sum_{j=1}^{n} b_{1j} x_j, \ldots, \sum_{j=1}^{n} b_{nj} x_j).$$

**Proposition 5.1.** *Let $K$ be an infinite field. Let $n \geqslant 3$ and $p_1, \ldots, p_\ell \in K[x_1, \ldots, x_n]$ be irreducible polynomials. Then there exists a matrix $B \in Gl_n(K)$ such that for all $i = 1, \ldots, \ell$ we get:*

- *$p_i(B \cdot \underline{x})$ is irreducible in $\overline{K(x_1)}[x_2, \ldots, x_n]$;*
- *$\deg_{(x_2,\ldots,x_n)} p_i(B \cdot \underline{x}) = \deg_{(x_1,\ldots,x_n)} p_i$.*

The proof of this proposition can be derived from [Sm, Ch. 5, Th. 3D] or by using [FJ, Prop. 9.31]. See [Na3] for details.

Now we return to our main result.

**Theorem 5.2.** *Let $K$ be an algebraically closed field of characteristic 0. Let $f \in K(\underline{x})$ be non-composite then $\rho(f) < (\deg f)^2 + \deg f$.*

*Proof.* We will prove this theorem by induction on the number $n$ of variables. For $n = 2$, we proved in Theorem 4.1 that $\rho(f) < (\deg f)^2 + \deg f$.

Let $f = \frac{p}{q} \in K(\underline{x})$, with $\underline{x} = (x_1, \ldots, x_n)$. We suppose that $f$ is non-composite. For each $\lambda \in \sigma(f)$ we decompose $p - \lambda q$ into irreducible factors:

$$p - \lambda q = \prod_{i=1}^{n_\lambda} F_{\lambda,i}^{r_{\lambda,i}}. \tag{3}$$

We fix $\mu \notin \sigma(f)$. We apply Proposition 5.1 to the polynomials $p-\mu q$ and $F_{\lambda,i}$, for all $\lambda \in \sigma(f)$ and all $i = 1, \ldots, n_\lambda$. Then the polynomials $p(B\cdot\underline{x})-\mu q(B\cdot\underline{x})$ and $F_{\lambda,i}(B\cdot\underline{x})$ are irreducible in $\overline{K(x_1)}[x_2, \ldots, x_n]$ and their degrees in $(x_2, \ldots, x_n)$ are equals to the degrees in $(x_1, \ldots, x_n)$ of $p - \mu q$ and $F_{\lambda,i}$.

Let denote by $k = \overline{K(x_1)}$. This is an uncountable field, algebraically closed of characteristic zero. Now $p(B \cdot \underline{x}) - \mu q(B \cdot \underline{x})$ is irreducible, then $f(B \cdot \underline{x})$ is non-composite in $k(x_2, \ldots, x_n)$.

Now equation (3) become:

$$p(B \cdot \underline{x}) - \lambda q(B \cdot \underline{x}) = \prod_{i=1}^{n_\lambda} F_{\lambda,i}(B \cdot \underline{x})^{r_{\lambda,i}}.$$

Which is the decomposition of $p(B \cdot \underline{x}) - \lambda q(B \cdot \underline{x})$ into irreducible factors in $k(x_2, \ldots, x_n)$. Then

$$\sigma(f) \subset \sigma(f(B \cdot \underline{x})),$$

where $\sigma(f)$ is a subset of $K$, and $\sigma(f(B\cdot\underline{x}))$ is a subset of $k = \overline{K(x_1)}$. As $n_\lambda$ is also the number of distinct irreducible factors of $p(B\cdot\underline{x})-\lambda q(B\cdot\underline{x})$ we get:

$$\rho(f) \leqslant \rho(f(B \cdot \underline{x})).$$

Now suppose that the result is true for $n - 1$ variables. Then for $f(B \cdot \underline{x}) \in k(x_2, \ldots, x_n)$ we get:

$$\rho(f(B \cdot \underline{x})) < (\deg_{(x_2,\ldots,x_n)} f(B \cdot \underline{x}))^2 + (\deg_{(x_2,\ldots,x_n)} f(B \cdot \underline{x})).$$

Hence:

$$\begin{aligned}
\rho(f) &\leqslant \rho(f(B \cdot \underline{x})) \\
&< (\deg_{(x_2,\ldots,x_n)} f(B \cdot \underline{x}))^2 + (\deg_{(x_2,\ldots,x_n)} f(B \cdot \underline{x})) \\
&= (\deg_{(x_1,\ldots,x_n)} f)^2 + (\deg_{(x_1,\ldots,x_n)} f) \\
&= (\deg f)^2 + (\deg f)
\end{aligned}$$

□

If for $n = 2$ we start the induction with Lorenzini's bound $\rho(f) < (\deg f)^2$ we obtain with the same proof the following result for several variables, for $K$ of any characteristic $K$ and a better bound:

**Theorem 5.3.** *Let $K$ be an algebraically closed field. Let $f \in K(\underline{x})$ be non-composite then $\rho(f) < (\deg f)^2$.*

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