CLASSIFICATION OF POLYNOMIALS FROM $\mathbb{C}^2$ TO $\mathbb{C}$ WITH ONE CRITICAL VALUE

ARNAUD BODIN

1. Introduction

Let $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ be a polynomial map. The *bifurcation set* $\mathcal{B}$ is the minimal set of points of $\mathbb{C}$ such that $f : \mathbb{C}^2 \setminus f^{-1}(\mathcal{B}) \longrightarrow \mathbb{C} \setminus \mathcal{B}$ is a locally trivial fibration. We can describe $\mathcal{B}$ as follows: let $\mathcal{B}_{\mathit{aff}} = \big\{ f(x,y) \mid \operatorname{grad}_f(x,y) = (0,0) \big\}$ be the set of *affine critical values*. The set $\mathcal{B}_{\mathit{aff}}$ is a subset of $\mathcal{B}$ but is not necessarily equal to $\mathcal{B}$. The value $c \in \mathbb{C}$ is *regular at infinity* if there exists a disk D centered at c and a compact set K of $\mathbb{C}^2$ with a locally trivial fibration $f : f^{-1}(D) \setminus K \longrightarrow D$. There is only a finite number of non-regular values at infinity: the *critical values at infinity* collected in $\mathcal{B}_\infty$. The bifurcation set $\mathcal{B}$ is now:

$$\mathcal{B} = \mathcal{B}_{\mathit{aff}} \cup \mathcal{B}_\infty.$$

For $c \in \mathbb{C}$, we denote the fiber $f^{-1}(c)$ by $\mathcal{F}_c$. If $s \notin \mathcal{B}$, then the fiber $\mathcal{F}_s$ is called a *generic fiber* and is denoted $\mathcal{F}_{gen}$.

The aim of this paper is to describe the classification of reduced polynomial maps with one critical value, that is, for convenience, $\mathcal{B} = \{0\}$. The classification is given up to homeomorphisms: two polynomials f and g are *topologically equivalent* ($f \approx g$) if there exists homeomorphisms Φ and Ψ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{\ \Phi\ } & \mathbb{C}^2 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathbb{C} & \xrightarrow[\ \Psi\ ]{} & \mathbb{C}. \end{array}$$

Theorem. *Let $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ be a reduced polynomial. We denote by p and q two relatively prime natural numbers, $\varepsilon, \varepsilon' \in \{0,1\}$, $\sigma = \sigma(x,y) = x^s y + 1$, ($s > 0$). Let $n \geqslant 1$ and let $g(x)$ be the polynomial $g(x) = \prod_{i=1}^n (x-i)^{m_i}$ with $1 \leqslant m_1 \leqslant m_2 \leqslant \cdots \leqslant m_n$; and let g_{red} be the reduced polynomial associated to g, $g_{red}(x) = \prod_{i=1}^n (x-i)$.*

- *If $\mathcal{B}_{\mathit{aff}} = \mathcal{B}_\infty = \varnothing$ then $f \approx x$,*
- *if $\mathcal{B}_{\mathit{aff}} = \{0\}$ and $\mathcal{B}_\infty = \varnothing$ then*
  - *$f \approx y \cdot g_{red}(x)$,*
  - *or $f \approx x \prod_{i=1}^n (x^p - iy)$ (if $p = 1$ then $n \geqslant 2$),*
  - *or $f \approx x^\varepsilon y^{\varepsilon'} \prod_{i=1}^n (x^p - iy^q)$, ($1 < p < q$),*

- *if* $\mathcal{B}_{\mathit{aff}} = \varnothing$ *and* $\mathcal{B}_\infty = \{0\}$ *then*
  - $f \approx x \prod_{i=1}^{n} (x^p y^q - i)$
  - *or* $f \approx x\sigma \prod_{i=1}^{n} (x^p \sigma^q - i)$ *(*$p > 1$ *or* $q > 1$*),*
  - *or* $f \approx x\sigma^\varepsilon \prod_{i=1}^{n} (x^p - i\sigma^q)$ *(if* $\varepsilon = 0$ *then* $q > 1$*),*
  - *or* $f \approx g_{\mathit{red}}(x)(g(x)y + 1)$ *(*$n > 1$*),*
- *if* $\mathcal{B}_{\mathit{aff}} = \{0\}$ *and* $\mathcal{B}_\infty = \{0\}$ *then*
  - $f \approx xy \prod_{i=1}^{n}(x^p y^q - i)$ *(*$1 \leqslant p < q$*),*
  - *or* $f \approx g_{\mathit{red}}(x)k(x)(g(x)y + 1)$ *(*$k(x) = \prod_{i=1}^{n'}(x + i)$*,* $n' \geqslant 1$*).*

Moreover, two different polynomials of this classification are not topologically equivalent.

We define a stronger notion of equivalence. Two polynomials f and g are *algebraically equivalent* ($f \sim g$) if there exists an algebraic automorphism Φ of $\mathbb{C}^2$ and Ψ an automorphism of $\mathbb{C}$ with equation $\Psi(z) = az + b$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{\Phi} & \mathbb{C}^2 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathbb{C} & \xrightarrow[\Psi]{} & \mathbb{C}. \end{array}$$

The study of polynomials with one critical value is reduced to a few cases, up to algebraic equivalence. The case $\mathcal{B} = \varnothing$ is the famous Abhyankar-Moh theorem ([AM], see paragraph 2). A theorem of M. Zaĭdenberg and V. Lin [ZL] corresponds to the case $\mathcal{B}_{\mathit{aff}} = \{0\}$ and $\mathcal{B}_\infty = \varnothing$ for irreducible polynomials. We generalize this result to the reducible case by using methods from the proof of Zaĭdenberg-Lin theorem by W. Neumann and L. Rudolph [NR] (paragraph 3). The remaining cases ($\mathcal{B}_\infty = \{0\}$) are studied in paragraphs 4 and 5. The arguments are essentially topological: we find a smooth disk in the fiber $f^{-1}(0)$ and we argue with branched coverings in order to give equations that represent equivalent classes of polynomials up to algebraic equivalence. That enables us to recover the list obtained by M. Zaĭdenberg by the use of $\mathbb{C}^*$-action [Za].

The last part of the work (paragraph 6) is to deduce from the former results the topological classification. Resolution of singularities determines polynomials with one critical value up to topological equivalence. It gives a classification without redundancy. The algebraic and the topological classification for irreducible polynomials with $\mathcal{B}_\infty = \varnothing$ (and with $\mathcal{B}_{\mathit{aff}} = \varnothing$ or $\mathcal{B}_{\mathit{aff}} = \{0\}$) given by Abhyankar-Moh and Zaĭdenberg-Lin are the same. However this is not true in general: we give polynomials (with $\mathcal{B}_{\mathit{aff}} = \varnothing$ and $\mathcal{B}_\infty = \{0\}$) that are topologically equivalent but not algebraically equivalent.

2. Preliminaries

When there is no critical value, the situation has been completed by S. Abhyankar and T. Moh [AM]. Abhyankar-Moh theorem is formulated as follows:

Theorem 1. *If* $\mathcal{B} = \varnothing$ *then* $f \sim x$.

Recall that $\mathcal{F}_0 = f^{-1}(0)$. A polynomial is *primitive* if its generic fiber is connected. The link between Euler characteristic of the zero fiber and the inclusion $\mathcal{B} \subset \{0\}$ (that is to say $\mathcal{B} = \varnothing$ or $\mathcal{B} = \{0\}$) is explained in the lemma:

Lemma 2. *If* $\mathcal{B} \subset \{0\}$ *then* $\chi(\mathcal{F}_0) = +1$. *Moreover, if the polynomial* f *is primitive and* $\chi(\mathcal{F}_0) = +1$ *then* $\mathcal{B} \subset \{0\}$.

Proof. The decomposition $\mathbb{C} = \mathbb{C} \setminus \{0\} \cup \{0\}$ gives a partition $\mathbb{C}^2 = f^{-1}(\mathbb{C} \setminus \{0\}) \cup f^{-1}(0)$. By additivity of the Euler characteristic, [Fu, p. 95]

$$1 = \chi\big(f^{-1}(\mathbb{C} \setminus \{0\})\big) + \chi\big(f^{-1}(0)\big).$$

If $\mathcal{B} \subset \{0\}$ then f defines a locally trivial fibration onto $\mathbb{C} \setminus \{0\}$. Then

$$\chi\big(f^{-1}(\mathbb{C} \setminus \{0\})\big) = \chi\big(\mathbb{C} \setminus \{0\}\big) \times \chi\big(f^{-1}(1)\big).$$

The Euler characteristic of $\mathbb{C} \setminus \{0\}$ is zero. Hence $\chi\big(f^{-1}(\mathbb{C} \setminus \{0\})\big) = 0$ and $\chi\big(f^{-1}(0)\big) = 1$.

Conversely, if f is a primitive polynomial then by Suzuki formula [Su]:

$$1 - \chi(\mathcal{F}_{gen}) = \sum_{c \in \mathcal{B}} \big(\chi(\mathcal{F}_c) - \chi(\mathcal{F}_{gen})\big).$$

If $\chi(\mathcal{F}_0) = +1$ then $\sum_{c \in \mathcal{B} \setminus \{0\}} \big(\chi(\mathcal{F}_c) - \chi(\mathcal{F}_{gen})\big) = 0$, but if $c \in \mathcal{B}$ then $\chi(\mathcal{F}_c) - \chi(\mathcal{F}_{gen}) > 0$ (see [HL]), then $\mathcal{B} \subset \{0\}$. □

Remark. *For a primitive polynomial Suzuki formula proves the equivalence* $\chi\big(f^{-1}(0)\big) = +1 \Leftrightarrow \mathcal{B} \subset \{0\}$ *(see* [ZL], [GP] *for example). However the non primitive polynomial* $f(x,y) = xy(xy+1)$ *verifies* $\chi\big(f^{-1}(0)\big) = 1$ *but* $\mathcal{B} = \{0, -\frac{1}{4}\}$.

We denote $h(0)$ the algebraic monodromy induced in homology[1] on $H_1(\mathcal{F}_{gen})$ by a small circle $S^1_\varepsilon(0)$ of radius ε centered at 0. The key of this paper is the following simple remark: for all $S^1_r(0)$ $(r > 0)$ the induced monodromies are equal since 0 is the only critical value.

To compactify the situation we need resolution of singularities at infinity [LW1]:

$$\begin{array}{ccccc} \mathbb{C}^2 & \longrightarrow & \mathbb{C}P^2 & \xleftarrow{\ \pi\ } & \Sigma \\ {\scriptstyle f}\downarrow & & {\scriptstyle \tilde{f}}\downarrow & \swarrow{\scriptstyle \bar{f}} & \\ \mathbb{C} & \longrightarrow & \mathbb{C}P^1 & & \end{array}$$

[1] Homology with integer coefficients.

where $\tilde{f}$ is the natural —but not well-defined— map coming from the homogenization of f; π is the blow-up of some points on the line at infinity L_∞ of $\mathbb{C}P^2$ and of the affine singular points.

We denote $D_0 = \bar{f}^{-1}(0)$ and $D_\infty = \bar{f}^{-1}(\infty)$. The *dual graph* G_0 of D_0 is obtained as follows: one vertex for each irreducible component of D_0 and one edge between two vertices for one intersection of the corresponding components. A similar construction is done to obtain G_∞, we know that G_∞ is a tree [LW1].

The monodromy induced by a small circle $S^1_\varepsilon(\infty)$ centered at ∞ in $\mathbb{C}P^1$ is exactly the monodromy $h(0)$ with the reverse orientation:

$$h(\infty) = h(0)^{-1}.$$

This property allows us to prove the three following lemmas.

Lemma 3. *The fiber $\mathcal{F}_0 = f^{-1}(0)$ is rational, that is to say the union of punctured spheres.*

Proof. Let $B_1, \ldots, B_p$ be small 4-balls around the affine singularities of $\mathcal{F}_0$ and set $\mathcal{F}_0^\circ = \mathcal{F}_0 \cap B^4_R \setminus B_1 \cup \ldots \cup B_p$. Then $\mathcal{F}_0^\circ$ can be isotoped into $\mathcal{F}_{gen}$ and we denote $\ell_* : H_1(\mathcal{F}_0^\circ) \longrightarrow H_1(\mathcal{F}_{gen})$ the induced morphism. Then, by [Bo] or [MW], the invariant cycles for $h(0)$ are $\mathrm{Ker}(h(0) - \mathrm{id}) = \mathrm{Im}\,\ell_*$. Suppose that one of the components of $\mathcal{F}_0$ has genus, then $\mathcal{F}_{gen}$ has genus and the cycles corresponding to genus induced be $\mathcal{F}_0^\circ$ are invariant cycles.

On the other hand the cycles invariant by all the monodromies associated to elements of $\pi_1(\mathbb{C}\setminus\mathcal{B}, *)$ are cycles corresponding to the boundary[2] of $\mathcal{F}_{gen}$ (see [Bo] or [DN]). Here there is only one monodromy and invariant cycles by all the monodromies are exactly cycles invariant by $h(0)$. It provides a contradiction. □

Lemma 4. *There is no cycle in G_0: $H_1(G_0) = 0$.*

Proof. A theorem of F. Michel and C. Weber [MW] asserts firstly, that the cycles of G_0 correspond to Jordan 2-blocks $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ for the monodromy $h(0)$ and secondly, that $h(\infty)$ does not have any such blocks since G_∞ is a tree. Now, as $h(0) = h(\infty)^{-1}$, G_0 has no cycle. □

Lemma 5. *The tube $f^{-1}(S^1_r(0))$ is a Seifert manifold.*

Proof. Let us suppose that in the minimal Waldhausen decomposition[3] of $f^{-1}(S^1_r(0))$ there exists two distinct Seifert pieces. This decomposition can be obtained, as described in [LMW], from the boundary of a neighborhood of the divisor D_0; moreover a Dehn twist between two Seifert pieces can be calculated (see [MW]) and is non-positive. But the decomposition of $f^{-1}(S^1_r(0))$ can also be obtained as the boundary of a neighborhood of D_∞

[2]If $\bar{\mathcal{F}}_{gen}$ is the surface without boundary associated to $\mathcal{F}_{gen}$, $\iota : \mathcal{F}_{gen} \longrightarrow \bar{\mathcal{F}}_{gen}$ is the inclusion and $\iota_* : H_1(\mathcal{F}_{gen}) \longrightarrow H_1(\bar{\mathcal{F}}_{gen})$ is the induced morphism, then the "boundary cycles" are $\mathrm{Ker}\,\iota_*$.

[3]Or the Jaco-Shalen-Johannson decomposition.

(because $\mathcal{B} \subset \{0\}$). Then the same formula proves that the Dehn twist is non-negative since the orientation of the second boundary is the opposite of the first; now the Dehn twist is non-negative and non-positive, hence equal to zero. That contradicts the fact that the essential pieces were distinct. □

In other words, let us call a singularity that provides a Seifert piece in the decomposition of $f^{-1}(S^1_r(0))$ an *essential singularity.* We have proved that at most one essential singularity can occur. The non-essential affine singularities are ordinary quadratic singularities.

3. Generalization of Zaĭdenberg-Lin theorem

Let us recall Zaĭdenberg-Lin theorem [ZL]. For a proof using topological arguments see [NR].

Theorem 6. *Let f be an irreducible polynomial with the fiber $\mathcal{F}_0 = f^{-1}(0)$ simply connected then for some relatively prime natural numbers p and q (or for $p = 1$ and $q = 0$):*

$$f \sim x^p - y^q.$$

The following lemma links the topology of $\mathcal{F}_0$ to the case without critical value at infinity.

Lemma 7. *Let f be a reduced polynomial. Then $\mathcal{B}_{\mathit{aff}} \subset \{0\}$ and $\mathcal{B}_\infty = \varnothing$ if and only if $\mathcal{F}_0$ is a simply connected set.*

Proof. If $\mathcal{F}_0$ is simply connected then the irregular fiber is connected and it is reduced because f is a reduced polynomial, hence the generic fiber is also connected, then f is a primitive polynomial. Moreover $\chi(\mathcal{F}_0) = +1$ so $H_1(\mathcal{F}_0) = \{0\}$ and by lemma 2, $\mathcal{B} \subset \{0\}$. Let T_0 be the tube $f^{-1}(\Delta)$ where Δ is a small[4] disk centered at 0. Then, as in the proof of lemma 2, by additivity of the Euler characteristic we have $\chi(\mathcal{F}_0) = \chi(T_0) = 1$. Since the generic fiber is connected then T_0 is connected and $H_1(T_0) = \{0\}$. The morphism $j_0 : H_1(\mathcal{F}_0) \longrightarrow H_1(T_0)$ induced by inclusion, is an isomorphism if and only if 0 is a regular value at infinity (see [ACD] for the case where $\mathcal{F}_0$ is connected, and [Bo] for the general case). In our situation j_0 is an isomorphism since $H_1(\mathcal{F}_0) = H_1(T_0) = \{0\}$ hence $\mathcal{B}_\infty = \varnothing$.

Conversely, let suppose now $\mathcal{B}_{\mathit{aff}} \subset \{0\}$ and $\mathcal{B}_\infty = \varnothing$. As $\mathcal{B}_\infty = \varnothing$ then $\mathcal{F}_0 = f^{-1}(0)$ has the homotopy type of $f^{-1}(\Delta) \cap B^4_R$. But, always because there is no critical value at infinity, $f^{-1}(\Delta) \setminus B^4_R$ is just a product $\big(f^{-1}(\Delta) \cap S^3_R\big) \times]0, +\infty[$ (where $f^{-1}(\Delta) \cap S^3_R$ is a tubular neighborhood in S^3_R of the link $f^{-1}(0) \cap S^3_R$). Then $f^{-1}(\Delta) \cap B^4_R$ has the same homotopy type as $f^{-1}(\Delta)$. Now the polynomial f is primitive since f is reduced and $\mathcal{B} \subset \{0\}$, hence $f^{-1}(\Delta)$ is connected. So $\mathcal{F}_0$ is a connected set. As the Euler characteristic of the connected set $\mathcal{F}_0$ is $+1$, it implies that all the irreducible components of $\mathcal{F}_0$ are disks (possibly singular), crossing together, without cycle (lemma 4). As a conclusion $\mathcal{F}_0$ is a simply connected set. □

[4] Small enough in comparison to R that defines the link at infinity $f^{-1}(0) \cap S^3_R$.

Remark. *As a corollary if $\mathcal{B} = \{0\}$ with $\mathcal{B}_\infty = \{0\}$ then $\mathcal{F}_0$ is not connected: by contraposition if $\mathcal{F}_0$ is a connected set then, as $\chi(\mathcal{F}_0) = +1$, all irreducible components are disks (possibly singular). Since there is no cycle (lemma 4) then $\mathcal{F}_0$ is simply connected, thus $\mathcal{B}_\infty = \varnothing$.*

Zaĭdenberg-Lin theorem admits the following generalization when f is not irreducible.

Theorem 8. *Let f be a reduced polynomial with $\mathcal{B}_{\mathit{aff}} = \{0\}$ and $\mathcal{B}_\infty = \varnothing$ then*

$$f \sim xg(y) \qquad \text{or} \qquad f \sim x^\varepsilon y^{\varepsilon'} \prod_{i=1}^{n} \left(x^p - \alpha_i y^q\right),$$

with a non-constant polynomial $g \in \mathbb{C}[y]$, $\varepsilon, \varepsilon' \in \{0,1\}$, p, q relatively prime numbers, $n \geqslant 1$, and $\{\alpha_i\}_{i=1,\ldots,n}$ a family of distinct non-zero complex numbers.

This is stated in [ZL]. As there is no proof of this result in the literature, we will give one. Our proof uses ideas from [BF], [NR] and [Ru], particularly it uses Zaĭdenberg-Lin theorem for an irreducible component of the polynomial f. We need the following lemma which is a stronger version of Abhyankar-Moh theorem.

Lemma 9. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two smooth disks with equations $(f_1 = 0)$ and $(f_2 = 0)$.*

- *If $\mathcal{C}_1 \cap \mathcal{C}_2 = \varnothing$ then $f_1 f_2 \sim x(x+\alpha)$, $(\alpha \in \mathbb{C}^*)$.*
- *If $\mathcal{C}_1$ and $\mathcal{C}_2$ have one transversal intersection then $f_1 f_2 \sim xy$.*

Proof. By Abhyankar-Moh theorem an equation for $\mathcal{C}_1$ is $(x = 0)$. As in [NR] a parameterization of $\mathcal{C}_2$ is $(P(t), Q(t))$, with $P, Q \in \mathbb{C}[t]$. If $\mathcal{C}_2$ does not intersect $\mathcal{C}_1$ then $P(t)$ is a non-zero constant. If $\mathcal{C}_2$ intersects $\mathcal{C}_1$ transversally then, as in the proof of Abhyankar-Moh theorem by W. Neumann and L. Rudolph in [NR], polynomial automorphisms of type $(x,y) \mapsto (x, y + \lambda x^\mu)$ enable us to choose $(y = 0)$ as an equation of $\mathcal{C}_2$. □

Proof of the theorem. If there is no essential singularity, then singularities are ordinary quadratic singularities. As $\mathcal{F}_0$ is simply connected then it is the union of smooth disks $\mathcal{C}_1, \ldots, \mathcal{C}_r$. Let us suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect transversally. Then, by the lemma above, an equation of $\mathcal{C}_1 \cup \mathcal{C}_2$ is $(xy = 0)$, moreover another disk $\mathcal{C}_3$ can not intersect $\mathcal{C}_1$ and $\mathcal{C}_2$ otherwise there is a cycle in G_0 or an essential singularity. Then $\mathcal{C}_3$ has equation, for instance, $(y + \beta_3 = 0)$. The other disks $\mathcal{C}_i$, $i \geqslant 4$ are parallel to $\mathcal{C}_3$ otherwise there are cycles in G_0, thus $\mathcal{C}_i$ has equation $(y + \beta_i = 0)$. Then f is algebraically equivalent to $xy \prod_i (y + \beta_i)$.

Let us suppose that there is an essential singularity, then by lemma 5 there is only one essential singularity. All the other singularities are ordinary quadratic singularities. Moreover, as $\mathcal{B}_\infty = \varnothing$ and as the tube $f^{-1}(S^1_r(0))$ is a Seifert manifold (lemma 5), then the link at infinity $f^{-1}(0) \cap S^3_R$ (S^3_R is a 3-dimensional sphere with radius $R \gg 1$) is a Seifert link (that is to say $S^3_R \setminus f^{-1}(0)$ admits a Seifert fibration, and the components of the link are fibers for this fibration). By [NR, th. 2.7], as $\mathcal{B}_{aff} = \{0\}$ and $\mathcal{B}_\infty = \varnothing$, this link at infinity is the connected sum of the local links of the singularities of $f^{-1}(0)$, that is to say the link at infinity is the connected sum of the local link of the essential singularity with Hopf links. But a Seifert link can not have such a structure, then there is only one singularity. So the local link and the link at infinity are isotopic and are a sublink of

$$\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}(p,q) \cup \mathcal{O}(p,q) \cup \ldots$$

Let explain the notations: in the sphere S^3_r of $\mathbb{C}^2$, $\mathcal{O}_1$, $\mathcal{O}_2$ are unknots such that $\mathcal{O}_1 \cup \mathcal{O}_2$ is the Hopf link; $\mathcal{O}(p,q)$ denotes a torus knot of type (p,q) (p and q are relatively prime non-zero natural numbers) such that $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}(p,q) \cup \mathcal{O}(p,q) \cup \ldots$ is isotopic to the link

$$\big((x=0) \cup (y=0) \cup (x^p - y^q = 0) \cup (x^p - 2y^q = 0) \cup \ldots\big) \cap S^3_r.$$

We now prove that f can be written, up to algebraic equivalence, as required. We discuss according to the number of smooth disks in $\mathcal{F}_0$.

First case. We assume that, in $\mathcal{F}_0$, there are two smooth disks with transversal intersection at the essential singularity. By lemma 9, up to algebraic equivalence, an equation of f is $xyg_1(x,y)\ldots g_n(x,y)$. We have to prove that $g_i(x,y) = x^p - \alpha_i y^q$. Let us consider the polynomial $xyg_i(x,y)$ and let $(P(t),Q(t))$ be a polynomial injective parameterization of the curve $(g_i(x,y) = 0)$. The local link for the locally irreducible singularity is a link of type $\mathcal{O}(p,q)$ so this parameterization can be written:

$$\begin{cases} P(t) = a_q t^q + \cdots + a_N t^N \\ Q(t) = b_p t^p + \cdots + b_M t^M \end{cases}$$

with $N \geqslant q$ and $M \geqslant p$. As $(0,0)$ is the only point of intersection between $(xy=0)$ and $(g_i(x,y)=0)$ then $P(t)=0$ implies $Q(t)=0$ and then $t=0$. So P is monomial: $P(t) = a_p t^p$. For similar reasons $Q(t) = b_q t^q$, and then $g_i(x,y) = x^p - \alpha_i y^q$.

Second case. If $\mathcal{F}_0$ has only one smooth disk then for some coordinates an equation of f is $xg_1(x,y)\ldots g_n(x,y)$. As before we denote by $(P(t),Q(t))$ a parameterization of $(g_i(x,y)=0)$; we obtain again $P(t) = a_q t^q$ ($q \geqslant 2$) but with $Q(t) = b_p t^p + \cdots + b_M t^M$. We can conclude as in [NR, p. 434]: the parameterization $(a_q t^q, Q(t))$ is injective so $Q(t) - Q(\zeta t)$ has only one root $t=0$ for all q-th root ζ of unity. Hence $Q(t)$ is of the form $Q(t) = b_p t^p + F(t^q)$ and the polynomial change of coordinates $(x,y) \mapsto (x, y - F(x))$, that preserves the axis $(x=0)$, gives a parameterization $(a_q t^q, b_p t^p)$ in these new coordinates. So $g_i(x,y) = x^p - \alpha_i y^q$ in these coordinates.

Third case. If $\mathcal{F}_0$ has no smooth disk we can assume that for the decomposition $f = g_1 \ldots g_n$ we have $g_1(x,y) = x^p - y^q$ by Zaĭdenberg-Lin theorem for the irreducible component $(g_1 = 0)$. Let $(g_i = 0)$ be another component with parameterization $(P(t),Q(t)) = (a_q t^q + \cdots + a_N t^N, b_p t^p +$

$\cdots + b_M t^M)$. The link at infinity for $(g_i = 0)$ is an iterated torus knot of type $\mathcal{O}(m,n;m_2,n_2;\ldots;m_k,n_k)$ (see [Ru]), with $m = M/\gcd(M,N)$ and $n = N/\gcd(M,N)$. But the link at infinity for $(g_i = 0)$ is isotopic to local link of the affine singularity of $(g_i = 0)$ and then is of type $\mathcal{O}(p,q)$. As in [NR] either $\mathcal{O}(m,n) = \mathcal{O}(p,q)$ and then $\gcd(M,N) = 1$ so $M = p$, $N = q$ and the result is proved; or $\mathcal{O}(m,n)$ is the unknot and then M divides N or N divides M. It implies that $qM \neq pN$. As the components $(g_1 = 0)$ and $(g_i = 0)$ have only one intersection (at $(0,0)$) the one variable polynomial $g_1(P(t),Q(t))$ is equal to t^ℓ. For example if we assume that $qM > pN$ then $\ell = qM$. But the valuation of $g_1(P(t),Q(t))$ is the intersection multiplicity of g_1 and g_i at $(0,0)$ and it is equal to pq. Thereby $\ell = pq$ and $M = p$ and as before $N = q$. □

4. Case $\mathcal{B}_{aff} = \varnothing$ and $\mathcal{B}_\infty = \{0\}$

Let denote $f = f_1 \times \cdots \times f_r$ the decomposition of f into irreducible factors, let $\mathcal{C}_i = f_i^{-1}(0)$ be the plane algebraic curve associated to f_i. We firstly obtain an "abstract" classification: we describe the $\mathcal{C}_i$'s as punctured spheres.

Proposition 10. *In the case $\mathcal{B}_{aff} = \varnothing$ and $\mathcal{B}_\infty = \{0\}$, we can reorder the $(\mathcal{C}_i)_i$ so that*

- *either $\mathcal{C}_1$ is a disk and for $i = 2,\ldots,r$, $\mathcal{C}_i$ is an annulus;*
- *or $\mathcal{C}_1,\ldots,\mathcal{C}_{r-1}$ are disks and $\mathcal{C}_r$ is a r-punctured sphere.*

This proposition has been obtained independently in [GP].

Proof. Notice that, since $\mathcal{B}_{aff} = \varnothing$ the components $\mathcal{C}_i$ $(i = 1,\ldots,r)$ are disjoint, then

$$\chi(\mathcal{C}_1) + \cdots + \chi(\mathcal{C}_r) = \chi(\mathcal{F}_0) = 1,$$

and one of the component has positive Euler characteristic. But as $\chi(\mathcal{C}_i) \leqslant 1$ for all i, we can suppose that the components of Euler characteristic $+1$ are $\mathcal{C}_1,\ldots,\mathcal{C}_j$ $(j \geqslant 1)$.

We firstly assume that $j = 1$; all the other components verify $\chi(\mathcal{C}_i) \leqslant 0$ for $i \geqslant 2$, this implies that $\chi(\mathcal{C}_i) = 0$ $(i = 2,\ldots,r)$. As a conclusion the component $\mathcal{C}_1$ is a disk and the others are annuli.

Secondly we suppose that $j \geqslant 2$. Because of the Abhyankar-Moh theorem (lemma 9) we can assume that these disks $\mathcal{C}_1,\ldots,\mathcal{C}_j$ are parallel lines with equation $(x = \alpha_1),\ldots,(x = \alpha_j)$. All the other components $\mathcal{C}_i$ $(i > j)$ have at least $j+1$ branches at infinity because of the non-intersection with the lines: such a component $\mathcal{C}_i$ has j branches at infinity whose tangents at infinity are the j parallel lines that intersect the line at infinity at one point; if there is one other point at infinity for $\mathcal{C}_i$ then there exists one other branch, if there is no other point at infinity then the line at infinity is tangent to $\mathcal{C}_i$, that gives one more branch. Particularly we have $\chi(\mathcal{C}_i) \leqslant 2 - (j+1)$ for $i > j$;

then

$$\begin{aligned} 1 = \chi(\mathcal{F}_0) &= \chi(\mathcal{C}_1) + \cdots + \chi(\mathcal{C}_j) + \chi(\mathcal{C}_{j+1}) + \cdots + \chi(\mathcal{C}_r) \\ &\leqslant \qquad\qquad j \qquad\qquad + (1-j) + \cdots + (1-j) \leqslant 1. \end{aligned}$$

Thus this inequality is an equality; it implies that $j+1 = r$ and $\chi(\mathcal{C}_{j+1}) = 2-(j+1) = 2-r$, particularly there are exactly r branches at infinity. All the components are disks, except the last one which is a r-punctured sphere. This completes the proof. □

We now need a non-compact version of the Riemann-Hurwitz formula. Let $\bar{\mathcal{C}}$ and $\bar{\mathcal{C}}'$ be compact Riemann surfaces and let $\pi : \bar{\mathcal{C}} \longrightarrow \bar{\mathcal{C}}'$ be a surjective holomorphic map of degree n. For $\mathcal{S}$ a finite set of points in $\bar{\mathcal{C}}'$ we denote $\mathcal{C}' = \bar{\mathcal{C}}' \setminus \mathcal{S}$ and $\mathcal{C} = \bar{\mathcal{C}} \setminus \pi^{-1}(\mathcal{S})$. For any point $s \in \mathcal{C}'$, $\nu(s)$ is the multiplicity of π at $s \in \mathcal{C}'$, we have $\sum_{t\in\pi^{-1}(s)}(\nu(t)-1) = n - \#\pi^{-1}(s)$.

Theorem 11 (Riemann-Hurwitz formula)**.**

$$\chi(\mathcal{C}) = n.\chi(\mathcal{C}') - \sum_{s\in\mathcal{C}}(\nu(s)-1).$$

Proof. The proof is similar to the standard proof, see for example [Ki]. By abuse, we also denote $\mathcal{C}' = \bar{\mathcal{C}}' \setminus \mathcal{N}(\mathcal{S})$ and $\mathcal{C} = \bar{\mathcal{C}} \setminus \pi^{-1}(\mathcal{N}(\mathcal{S}))$ where $\mathcal{N}(\mathcal{S})$ is the union of small open disks around the points of $\mathcal{S}$. Let (V', E', F') be a triangulation of $\mathcal{C}'$ with ramification points contained in $\pi^{-1}(V')$, we denote $v' = \#V'$, $e' = \#E'$,... There exists a triangulation (V, E, F) of $\mathcal{C}$ above (V', E', F') such that $e = ne'$, $f = nf'$ and $v = nv' - \sum_{t\in V'}(n - \#\pi^{-1}(t))$. Then $\chi(\mathcal{C}) = f - e + v = n(f' - e' + v') - \sum_{t\in V'}(n - \#\pi^{-1}(t)) = n\chi(\mathcal{C}') - \sum_{s\in\mathcal{C}}(\nu(s)-1)$. □

We will use this formula for a component $\mathcal{C} = \mathcal{C}_i$ that is not a disk with the natural compactification $\bar{\mathcal{C}}$ of $\mathcal{C}$: $\bar{\mathcal{C}} = \mathcal{C} \cup \mathcal{C}_\infty$. We define $\bar{\mathcal{C}}' = \mathbb{C}P^1$ and if the disks $\mathcal{C}_1, \ldots, \mathcal{C}_j$ have equation $(x = \alpha_1), \ldots, (x = \alpha_j)$ we set $\mathcal{S} = \{\infty, \alpha_1, \ldots, \alpha_j\}$ and define $\mathcal{C}' = \mathbb{C}P^1 \setminus \mathcal{S} = \mathbb{C} \setminus \{\alpha_1, \ldots, \alpha_j\}$. The projection $\pi : \mathcal{C} \longrightarrow \mathcal{C}'$ is defined by $\pi(x,y) = x$. Then π can be continued to a holomorphic map on $\bar{\mathcal{C}}$. If we prove that $\pi^{-1}(\mathcal{S}) = \mathcal{C}_\infty$ then we can apply Riemann-Hurwitz formula.

We can give the algebraic classification.

Proposition 12. *Depending on the cases of proposition 10 above, for a reduced polynomial f with $\mathcal{B}_{\mathit{aff}} = \varnothing$ and $\mathcal{B}_\infty = \{0\}$ then*

- *either $f \sim x\sigma^\varepsilon \prod_{i=1}^{n}(x^p\sigma^q - \alpha_i)$ or $f \sim x\sigma^\varepsilon \prod_{i=1}^{n}(x^p - \alpha_i\sigma^q)$ with p and q relatively prime, $\varepsilon \in \{0,1\}$, $n \geqslant 1$, $\{\alpha_i\}_{i=1,\ldots,n}$ a finite family of distinct non-zero complex numbers. Moreover $\sigma = \sigma(x,y) = x^s y + \ell(x)$, with $s \geqslant 0$, $\ell \in \mathbb{C}[x]$ and $\deg \ell < s$ (if $s > 0$ then $\ell(0) \neq 0$, if $s = 0$ then $\ell = 0$). If $\varepsilon = 1$ then $s > 0$ and then in the first case p (or q) is greater than 1.*

- *or* $f \sim g_{red}(x)\big(g(x)y + h(x)\big)$ *with* $g, h \in \mathbb{C}[x]$*,* $\deg g \geqslant 2$*,* $\deg g > \deg h$ *and* $h(t) \neq 0$ *if* $g(t) = 0$*.*

The situation of the first case of proposition 10 has been studied by S. Kaliman, we sketch the beautiful proof of [Ka]. Let g be an equation of the algebraic curve $\mathcal{C}_1 \cup \mathcal{C}$ where $\mathcal{C}_1$ is a smooth disk and $\mathcal{C} = \mathcal{C}_i$ $(i \geqslant 2)$ is a disjoint annulus.

Lemma 13.

$$g \sim x\big(x^p - \sigma^q\big) \qquad \text{or} \qquad g \sim x\big(x^p \sigma^q - 1\big)$$

p, q are relatively prime natural numbers, and $\sigma = \sigma(x,y) = x^s y + \ell(x)$ *($s > 0$ in the first polynomial) with* $\ell \in \mathbb{C}[x]$*,* $\deg \ell < s$ *and* $\ell(0) \neq 0$ *if* $s > 0$*.*

Proof. By Abhyankar-Moh theorem we can assume that $(x = 0)$ is the equation for the disk $\mathcal{C}_1$, let $k(x,y)$ be an equation of $\mathcal{C}$ in these coordinates, there exists $m > 0$ such that $k(x, x^{-m}y) = x^e h(x,y)$ with $e < 0$, $h \in \mathbb{C}[x,y]$ and $h(0,y) = y^n$, $n \geqslant 1$.

If $\mathcal{C}'$ denotes the curve of equation $(h = 0)$ then the "blow-up" $(x,y) \mapsto (x, x^{-m}y)$ gives an isomorphism from $\mathcal{C}' \setminus \{(0,0)\}$ to $\mathcal{C}$, so $\mathcal{C}'$ is homeomorphic to a disk and according to Zaĭdenberg-Lin theorem the polynomial $xh(x,y)$ equal $u(u^{p'} - v^{q'})$, the new coordinates are given by $u = x$ and $v = y + \varphi(x)$, then $h(x,y) = x^{p'} - (y + \varphi(x))^{q'}$. Returning to k by $k(x,y) = x^e h(x, x^m y)$, and distinguishing the cases $e + p' = 0$ and $e + p' > 0$ leads to $k(x,y) = 1 - x^p \sigma^q$ and $k(x,y) = x^p - \sigma^q$, with $\sigma(x,y) = x^s y + \ell(x)$. By triangular automorphisms $(x,y) \mapsto (x, y + \lambda x^\mu)$ we can assume that $\deg \ell < s$. That ends the proof. □

The generalization to the case where there are several annuli, corresponds to the generalization of Zaĭdenberg-Lin theorem (theorem 8).

Proof of proposition 12. We deal with the second case of proposition 12, the disks are given by an equation $\prod_{i=1}^{r-1}(x - \alpha_i)$ and the equation of $\mathcal{C}_r$ is

$$\prod_{i=1}^{r-1}(x - \alpha_i)^{m_i}\big(a_m(x)y^m + \cdots + a_1(x)y\big) + h(x)$$

with $h(\alpha_i) \neq 0$, $m_i > 0$. The projection $\pi : \mathcal{C}_r \longrightarrow \mathcal{C}' = \mathbb{C} \setminus \{\alpha_1, \ldots, \alpha_{r-1}\}$ given by $\pi(x,y) = x$, is of degree m and verify the hypothesis of our Riemann-Hurwitz formula since the points at infinity of $\mathcal{C}_r$ correspond to $\alpha_1, \ldots, \alpha_{r-1}$. As $\chi(\mathcal{C}_r) = \chi(\mathcal{C}')$ then $m = 1$ and a_m is a constant. The equation of $\mathcal{C}_r$ is now $\prod_{i=1}^{r-1}(x - \alpha_i)^{m_i} y + h(x)$ and by some triangular automorphisms $(x,y) \mapsto (x, y + \lambda x^\mu)$ we can assume that $\deg h < \deg \prod_{i=1}^{r-1}(x - \alpha_i)^{m_i}$. □

5. Case $\mathcal{B}_{aff} = \{0\}$ and $\mathcal{B}_\infty = \{0\}$

Notations are those of the previous paragraph.

Proposition 14. *For a reduced polynomial f with $\mathcal{B}_{aff} = \{0\}$ and $\mathcal{B}_\infty = \{0\}$*

- *either $\mathcal{C}_1$ and $\mathcal{C}_2$ are disks, intersecting transversally, and $\mathcal{C}_i$ ($i = 3,\ldots,r$) are disjoint annuli;*
- *or $\mathcal{C}_1,\ldots,\mathcal{C}_j,\mathcal{C}_{j+1},\ldots,\mathcal{C}_{r-1}$ are disjoint disks and $\mathcal{C}_r$ is a $(j+1)$-punctured sphere. Moreover the curves $\mathcal{C}_{j+1},\ldots,\mathcal{C}_{r-1}$ intersect $\mathcal{C}_r$ transversally at one point.*

The corresponding algebraic list is

- *either $f \sim xy\prod_i \big(x^p y^q - \alpha_i\big)$ with $p > 1$ and q relatively prime, $\alpha_i \in \mathbb{C}^*$.*
- *or $f \sim g_{red}(x)k(x)\big(g(x)y + h(x)\big)$ with $g,h,k \in \mathbb{C}[x]$ (g and k non constant, k reduced), $\deg h < \deg g$ and if $g(t) = 0$ then $h(t) \neq 0$.*

Lemma 15. *One of the irreducible component of $\mathcal{F}_0 = f^{-1}(0)$ is a smooth disk.*

We will make the distinction between "smooth" and "smooth in $\mathcal{F}_0$": a smooth component is not necessarily smooth in $\mathcal{F}_0$, there may exist singularities on this component coming from intersection with other components.

Proof of the lemma. Let us recall that from lemma 5 we know that there is at most only one essential singularity, and affine non-essential singularities are ordinary quadratic singularities. The non-essential singularities at infinity correspond to a bamboo[5] for the divisor at infinity $\pi^{-1}(L_\infty) \cap \pi^{-1}(0)$ for the value 0 which intersects the compactification of some smooth disks and another component (possibly singular) of $\mathcal{F}_0$, moreover the multiplicities of $\bar{f}$ equal to 1 on all the components of the bamboo. A typical example is given by Broughton polynomial $f(x,y) = x(xy+1)$, another example is given in paragraph 6.

Let us notice that one of the components of $\mathcal{F}_0$ is a disk (possibly singular) because $\chi(\mathcal{F}_0) = +1$. We firstly suppose that no affine singularity is essential. Then affine singularities are ordinary quadratic singularities and a disk of $\mathcal{F}_0$ is smooth because it can not intersect itself as there is no cycle in G_0.

In a second time if there exists one essential affine singularity, it is unique and singularities at infinity are non-essential. As $\mathcal{B}_\infty \neq \varnothing$ such singularities do exist. Then one of the disks associated to a non-essential singularity at infinity is smooth. □

Proof of proposition 14. Let $\mathcal{C}_1$ be the disk of lemma 15. Let denote $\mathcal{C}_1,\ldots,\mathcal{C}_k$ the smooth disks parallel to $\mathcal{C}_1$. According to Abhyankar-Moh theorem, we can suppose that equations for these disks are $(x=\alpha_1),\ldots,(x=\alpha_k)$. Let $\mathcal{C}$ be one of the $\mathcal{C}_i$ ($i > k$) which does not intersect one of the $\mathcal{C}_1,\ldots,\mathcal{C}_k$: such

[5]Each component intersects at most two other components.

a $\mathcal{C}$ exists otherwise $\mathcal{F}_0$ is a connected set and as $\mathcal{B}_\infty = \{0\}$ this is impossible by the remark below lemma 7. After reordering the disks $(\mathcal{C}_i)_{i=1,\ldots,k}$ we denote by $\mathcal{C}_1, \ldots, \mathcal{C}_j$ $(1 \leqslant j \leqslant k)$ the disks that do not intersect $\mathcal{C}$. Then as above (proposition 10) $\chi(\mathcal{C}) = 1 - j$, and $\mathcal{C}$ has exactly $j+1$ branches at infinity. Components other than $\mathcal{C}, \mathcal{C}_1, \ldots, \mathcal{C}_j$ do not contribute to Euler characteristic: that is to say the other disks have intersection with one of the $\mathcal{C}, \mathcal{C}_1, \ldots, \mathcal{C}_j$ at one point and at exactly one point because G_0 has no cycle; components that are not disks are annuli.

For $\mathcal{C}' = \mathbb{C} \setminus \{\alpha_1, \ldots, \alpha_j\}$ we have $\chi(\mathcal{C}) = \chi(\mathcal{C}')$ and the points at infinity correspond to $\alpha_1, \ldots, \alpha_j$. The Riemann-Hurwitz formula for the covering $\pi : \mathcal{C} \longrightarrow \mathcal{C}'$ defined by $(x, y) \mapsto x$ proves that π is non-branched and that $\mathcal{C}$ is smooth.

Singularities coming from intersections with other components can only be transversal intersections of a smooth disk and another component: the key point is the topology of $\mathcal{F}_0$. First of all, to keep $\chi(\mathcal{F}_0) = +1$, two components with non-positive Euler characteristic can not intersect; in a second time a disk that intersects the disk $\mathcal{C}_1$ is smooth, otherwise it contradicts the configuration for non-essential singularities at infinity; and finally to avoid cycles in G_0, only two directions for disks (for example $(x = 0)$ and $(y = 0)$) can occur, hence there are no multiple points of order greater than 2. We have just proved that the affine singularities were ordinary quadratic singularities.

We end the classification as in proposition 12. The main difference comes from some lines that give ordinary quadratic singularities. If $j = 1$ and $\mathcal{C}_1$ is not smooth in $\mathcal{F}_0$ then, for Euler characteristic reasons, there can be only one more disk $\mathcal{C}_2$, and $\mathcal{C}_2$ intersects transversally $\mathcal{C}_1$ at one point. With the algebraic classification of annuli, we see that only one kind of annuli can occur:

$$f \sim xy \prod_i \big(x^p y^q - \alpha_i\big),$$

where $\{\alpha_i\}$ is a family of distinct non-zero complex numbers.

For similar reasons, if $j = 1$ and the disk $\mathcal{C}_1$ is smooth in $\mathcal{F}_0$, then only one annulus can occur, but disks $\mathcal{C}_i$ $(i = 2, \ldots, r-1)$ parallel to $\mathcal{C}_1$ can intersect this annulus. Then

$$f \sim x \prod_i \big(x - \beta_i\big)\big(x^s y + \ell(x)\big).$$

The case $j \geqslant 2$ is treated as in proposition 12 with parallel lines added:

$$f \sim \prod_i (x - \alpha_i) \prod_i (x - \beta_i) \Big(\prod_i (x - \alpha_i)^{m_i} y + h(x) \Big).$$

This completes the proof. □

The tabular summarizes the algebraic list of reduced polynomials with one critical value, notations are those of theorem 8, propositions 12 and 14.

	$\mathcal{B}_{\mathit{aff}} = \varnothing$	$\mathcal{B}_{\mathit{aff}} = \{0\}$
$\mathcal{B}_\infty = \varnothing$	x	$yg(x)$ or $x^\varepsilon y^{\varepsilon'} \prod_i (x^p - \alpha_i y^q)$
$\mathcal{B}_\infty = \{0\}$	$x\sigma^\varepsilon \prod_i (x^p\sigma^q - \alpha_i)$ or $x\sigma^\varepsilon \prod_i (x^p - \alpha_i\sigma^q)$ or $g_{red}(x)(g(x)y + h(x))$	$xy \prod_i (x^p y^q - \alpha_i)$ or $g_{red}(x)k(x)(g(x)y + h(x))$

6. Topological classification

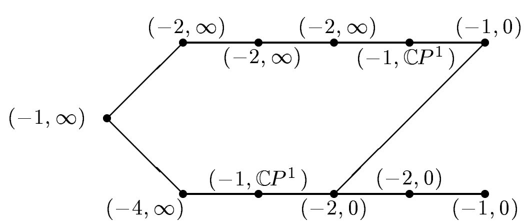
Recall that two polynomials f and g are *topologically equivalent* ($f \approx g$) if there exists homeomorphisms Φ and Ψ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{\Phi} & \mathbb{C}^2 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathbb{C} & \xrightarrow[\Psi]{} & \mathbb{C}. \end{array}$$

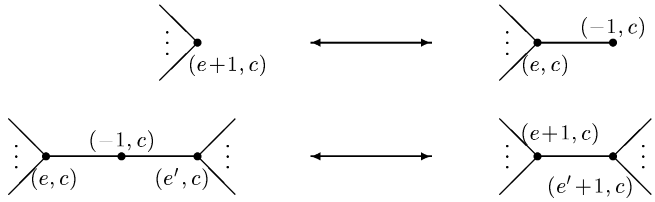
Two algebraically equivalent polynomials are topologically equivalent but the converse is false. For example $f(x,y) = x(x^2y+1)$ and $g(x,y) = x(x^2y+x+1)$ are topologically equivalent (they have the same colored graph, see below) but are not algebraically equivalent (an algorithm to determine if two polynomials are algebraically equivalent is given in [Wi]).

For a polynomial f with resolution map ϕ, we define the *colored graph* G_f. A vertex of the dual graph of the resolution of f is colored by the value of ϕ on the irreducible component associated to the vertex. In the case $\mathcal{B} = \{0\}$ the colors are ∞ (that corresponds to the subgraph G_∞), 0 (for G_0) and $\mathbb{C}P^1$ for the dicritical components. Moreover the vertices are weighted by the auto-intersection of the component. In our situation all the components are rational and we do not need to add the genus for each component.

For example, here is the graph for the polynomials f and g defined above.



Two colored graphs are *equivalent* if after a sequence of absorptions and blowing-ups (see the picture below) they are isomorphic (with respect to the colors and weights). We do not authorize dicritical components to disappear in this sequence, that is to say the color c is in $\{0, \infty\}$.



Proposition 16. *Let two reduced polynomials have only one critical value. If they have equivalent colored graphs then they are topologically equivalent*

This proposition can not be generalized to the case of several critical values, a counter-example is given in [Ar]. The converse is true: the main ideas for proving this are in the proof of the next proposition or refer to [Fo].

Proof. Let f and g be polynomials with just one critical value 0 and with equivalent colored graphs. Let $(\pi_f, \bar{f})$, $(\pi_g, \bar{g})$ come from the resolution of f and g. One can suppose, after some blowing-ups and absorptions, that their graphs are equal. We set $D_{f,0} = \bar{f}^{-1}(0)$ and $D_{g,0} = \bar{g}^{-1}(0)$.

By standard arguments ([A'C], [Du], [Fo]), a small neighborhood of $D_{f,0}$ is homeomorphic to a small neighborhood of $D_{g,0}$. As all the components of $D_{f,0}$ and $D_{g,0}$ are rational the monodromies for $\bar{f}$ and $\bar{g}$ induced by a small circle around the value 0 act equivalently: that is to say the following diagram commutes:

$$\begin{array}{ccc} \bar{f}^{-1}(\Delta) & \xrightarrow{\bar{\Phi}_0} & \bar{g}^{-1}(\Delta') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \Delta & \xrightarrow[\Psi_0]{} & \Delta' \end{array}$$

where $\bar{\Phi}_0$ and Ψ_0 are homeomorphisms and Δ and Δ' topological closed disks of $\mathbb{C}$ with $0 \in \operatorname{Int} \Delta \cap \operatorname{Int} \Delta'$.

Let $D^{\infty}_{f,0} = D_{f,0} \cap \pi_f^{-1}(L_\infty)$ be the part of $D_{f,0}$ that corresponds to the irregularity at infinity of the value 0 ($D^{\infty}_{g,0}$ is set in the same way). Then $\bar{\Phi}_0$ defines an homeomorphism between $D^{\infty}_{f,0}$ and $D^{\infty}_{g,0}$. Then the homeomorphism $\bar{\Phi}_0$ from $\bar{f}^{-1}(\Delta) \setminus D^{\infty}_{f,0}$ to $\bar{g}^{-1}(\Delta') \setminus D^{\infty}_{g,0}$ can be restricted to an homeomorphism Φ_0 that respects the fibration because $f \circ \pi_f = \bar{f}$ on the set $\bar{f}^{-1}(\Delta) \setminus D^{\infty}_{f,0}$. We have proved that f and g are topologically equivalent in a neighborhood of the zero fiber:

$$\begin{array}{ccc} f^{-1}(\Delta) & \xrightarrow{\Phi_0} & g^{-1}(\Delta') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \Delta & \xrightarrow[\Psi_0]{} & \Delta'. \end{array}$$

We now explain how to continue theses homeomorphisms. As the only critical value for f is in Δ the fibration $f : f^{-1}(\overline{\mathbb{C} \setminus \Delta}) \longrightarrow \overline{\mathbb{C} \setminus \Delta}$ is isomorphic to the fibration $f \times \mathrm{id} : f^{-1}(\partial\Delta) \times \mathbb{R}_+ \longrightarrow \partial\Delta \times \mathbb{R}_+$. It provides homeomorphisms ϕ_f and ψ_f (see diagrams). For g we obtain homeomorphisms ϕ_g and ψ_g. But the fibration $f \times \mathrm{id}$ above $\partial\Delta \times \mathbb{R}_+$ is isomorphic to the fibration $g \times \mathrm{id}$ above $\partial\Delta' \times \mathbb{R}_+$, the corresponding homeomorphisms are $\Phi_1 = \Phi_0 \times \mathrm{id}$ and $\Psi_1 = \Psi_0 \times \mathrm{id}$.

$$\begin{array}{ccccccc} f^{-1}(\overline{\mathbb{C}\setminus\Delta}) & \xrightarrow{\phi_f} & f^{-1}(\partial\Delta)\times\mathbb{R}_+ & \xrightarrow{\Phi_1} & g^{-1}(\partial\Delta')\times\mathbb{R}_+ & \xleftarrow{\phi_g} & g^{-1}(\overline{\mathbb{C}\setminus\Delta'}) \\ \downarrow f & & \downarrow f\times\mathrm{id} & & \downarrow g\times\mathrm{id} & & \downarrow g \\ \overline{\mathbb{C}\setminus\Delta} & \xrightarrow[\psi_f]{} & \partial\Delta\times\mathbb{R}_+ & \xrightarrow[\Psi_1]{} & \partial\Delta'\times\mathbb{R}_+ & \xleftarrow[\psi_g]{} & \overline{\mathbb{C}\setminus\Delta'}. \end{array}$$

Then Φ_0 can be continued by $\phi_g^{-1} \circ \Phi_1 \circ \phi_f$ and Ψ_0 by $\psi_g^{-1} \circ \Psi_1 \circ \psi_f$. □

Theorem 17. *A reduced polynomial with at most one critical value is topologically equivalent to one, and only one, of the following polynomials (notations are those of the introduction):*

	$\mathcal{B}_{aff} = \varnothing$	$\mathcal{B}_{aff} = \{0\}$
$\mathcal{B}_\infty = \varnothing$	x	$y g_{red}(x)$ *or* $x\prod_{i=1}^{n}(x^p - iy)$ *or* $x^\varepsilon y^{\varepsilon'}\prod_{i=1}^{n}(x^p - iy^q)$
$\mathcal{B}_\infty = \{0\}$	$x\prod_{i=1}^{n}(x^p y^q - i)$ *or* $x\sigma\prod_{i=1}^{n}(x^p\sigma^q - i)$ *or* $x\sigma^\varepsilon\prod_{i=1}^{n}(x^p - i\sigma^q)$ *or* $g_{red}(x)(g(x)y+1)$	$xy\prod_{i=1}^{n}(x^p y^q - i)$ *or* $g_{red}(x)k(x)(g(x)y+1)$

Proof. We firstly have to prove that the list of polynomials up to algebraic equivalence can be reduced, up to topological equivalence, to the list above. Finally we shall prove that two distinct polynomials of this list are not topologically equivalent.

For the cases with $\mathcal{B}_\infty = \varnothing$, replacing α_i by i does not change the polynomial, up to topological equivalence. Moreover the list, for theses cases, is not redundant.

Let study what happens for the case $\mathcal{B}_\infty = \{0\}$. Let f be one of the polynomials coming from the algebraic list, and let f' be the corresponding polynomial with the constant 1 instead of the polynomial $\ell(x)$ or $h(x)$ and with i instead of α_i. We may find $f \approx f'$ by proving that the graphs G_f and $G_{f'}$ are equivalent. As f and f' have the same behavior at finite distance, we just have to study what happens at infinity.

Let $F(x,y,z)$ be the homogeneous polynomial associated to f, $P_1 = (1:0:0)$ and $P_2 = (0:1:0)$ are the two points at infinity of f; we denote $f_1(y,z) = F(1,y,z)$, $f_2(x,z) = F(x,1,z)$ the local equations of F at the points P_1, P_2. To calculate the part of G_f at infinity, we have two —equivalent— choices.

Firstly we can calculate the irregular link at infinity $f^{-1}(0) \cap S^3_R$ (*resp.* $f'^{-1}(0) \cap S^3_R$), it is a sufficient condition since the (single) irregular link determines the regular links at infinity $f^{-1}(s) \cap S^3_R$ $(s \neq 0)$, see [NL].

Secondly, we can calculate the Puiseux expansions of the branches of f_1 (and f_2) and the intersection multiplicities between the branches of f_1 (and between the branches of f_2) by taking into account the line at infinity with local equation $(z = 0)$. It is a sufficient condition since if we know the topology of zf_i then one can recover the topology of the family $(f_i - tz^d)_{t \in \mathbb{C}P^1}$ (see [LW2]) as $t = 0$ and $t = \infty$ are the only critical values for this family.

We will use the second method: $f \approx f'$ if and only if f_1 and f'_1 (and f_2 and f'_2) have equivalent Puiseux expansions and the same intersection multiplicities.

We will detail the calculus for $f(x, y) = x\sigma \prod_{i=1}^n (x^p \sigma^q - \alpha_i)$ with $\sigma(x, y) = x^s y + \ell(x) = x^s y + a_{s-1}x^{s-1} + \cdots + a_0$, $a_0 \neq 0$ and $n > 1$, the calculus are similar for the other polynomials. Then $f'(x, y) = x\sigma' \prod_i (x^p \sigma'^q - i)$ and $\sigma'(x, y) = x^s y + 1$. The local equation of F at P_1 is

$$\begin{aligned} f_1(y, z) =&(y + a_{s-1}z^2 + \cdots + a_0 z^{s+1}) \times \\ &\prod_i \left((y + a_{s-1}z^2 + \cdots + a_0 z^{s+1})^q - \alpha_i z^{p+q(s+1)} \right). \end{aligned}$$

A similar formula holds for f'_1. The branches of f_1 and f'_1 are smooth and intersect the line at infinity $(z = 0)$ transversally. Moreover the intersection multiplicities for the branches of f_1 are independent of the coefficients $a_{s-1}, \ldots, a_1$, of $a_0 \neq 0$, and of the $\alpha_i \neq 0$: let $\ell_1(y, z) = y + a_{s-1}z^2 + \cdots + a_0 z^{s+1}$ then $\mathfrak{m}_0(\ell_1(y, z), \ell_1^q(y, z) - \alpha_i z^{p+q(s+1)}) = p + q(s + 1)$; for $i \neq j$, $\mathfrak{m}_0(\ell_1^q(y, z) - \alpha_j z^{p+q(s+1)}, \ell_1^q(y, z) - \alpha_i z^{p+q(s+1)}) = q(p + q(s + 1))$ (see how to calculate intersection multiplicities below), so f_1 and f'_1 have equivalent Puiseux expansions and the same intersection multiplicities.

The following lemma allows us to calculate intersection multiplicities; the first point is well-known (see [BK] or [Di]), the second point is a consequence of the first.

Lemma 18. *Let f, g, f_1, f_2 be irreducible plane curve germs at* 0.

- *Let K_f, K_g be the local links of f and g. Then the intersection multiplicity verify*

$$\begin{aligned} \mathfrak{m}_0(f, g) &= \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/(f(x, y), g(x, y)) \\ &= \mathrm{lk}(K_f, K_g) = \mathrm{val}_t(f \circ p(t)) \end{aligned}$$

  with lk *is the linking number,* val *is the valuation and* $p(t) = (t^n, \varphi(t))$ *is a Puiseux parameterization for the curve* $(g = 0)$ *(which is supposed not to contain* $(y = 0)$*).*

- *Let t, t' be complex numbers with $t \neq t'$ and $t' \neq 0$. Then*

$$\mathfrak{m}_0(f_1 f_2, g) = \mathfrak{m}_0(f_1, g) + \mathfrak{m}_0(f_2, g),$$
$$\mathfrak{m}_0(f + tg, f + t'g) = \mathfrak{m}_0(f, f + t'g).$$

For the second point at infinity P_2 the local equation of F is

$$f_2(x,z) = x(x^s + a_{s-1}x^{s-1}z^2 + \cdots + a_0 z^{s+1})\times \prod_i \left(x^p(x^s + a_{s-1}x^{s-1}z^2 + \cdots + a_0 z^{s+1})^q - \alpha_i z^{p+q(s+1)}\right).$$

All the branches intersect transversally the line at infinity, and the topology of each branch is given by one of the Puiseux expansions $x = 0$, $x = z^{\frac{s+1}{s}}$ and $x = z^{\frac{p+q(s+1)}{p+qs}}$ and is independent of $a_{s-1}, \ldots, a_1$, of $a_0 \neq 0$ and of the $\alpha_i \neq 0$. Moreover intersection multiplicities are also independent of the coefficients: let $\ell_2(x,z) = x^s + a_{s-1}x^{s-1}z^2 + \cdots + a_0 z^{s+1}$ then $\mathfrak{m}_0(x, \ell_2(x,z)) = s+1$, $\mathfrak{m}_0(x, x^p\ell_2(x,z)^q - \alpha_j z^{p+q(s+1)}) = p + q(s+1)$, $\mathfrak{m}_0(\ell_2(x,z), x^p\ell_2(x,z)^q - \alpha_j z^{p+q(s+1)}) = s(p+q(s+1))$, and for $i \neq j$, $\mathfrak{m}_0(x^p\ell_2(x,z)^q - \alpha_i z^{p+q(s+1)}, x^p\ell_2(x,z)^q - \alpha_j z^{p+q(s+1)}) = (p+qs)(p+q(s+1))$.

As a conclusion f_1, f_1' and f_2, f_2' have the same branches and the branches have the same tangency, so f and f' are topologically equivalent.

Finally, we shall prove that the list is non-redundant. As before, we detail the calculus for the polynomial $f(x,y) = x\sigma\prod_{i=1}^n(x^p\sigma^q - i)$ with $\sigma = x^s y + 1$; for the other polynomials the method is the same. Let suppose that another polynomial, f', of the topological list verify $f \approx f'$. Then f' has the same type as f, that is to say that $f'(x,y) = x\sigma'\prod_{i=1}^n(x^{p'}\sigma'^{q'} - i)$ with $\sigma' = x^{s'}y+1$. As $f \approx f'$, the localizations f_1 and f_1' (*resp.* f_2 and f_2') at P_1 (*resp.* P_2) have equivalent Puiseux expansions and the same intersection multiplicities. We deduce from the calculus of intersection multiplicities at P_2 that $s+1 = s'+1$ and at P_1 that $p + q(s+1) = p' + q'(s'+1)$ and $q(p+q(s+1)) = q'(p'+q'(s'+1))$. It implies that $s = s'$, $p = p'$, $q = q'$ and then $f = f'$. □

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Arnaud BODIN
Université Paul Sabatier Toulouse III, laboratoire Émile Picard,
118 route de Narbonne, 31062 Toulouse cédex 4, France.
bodin@picard.ups-tlse.fr