Irregular fibers of complex polynomials in two variables

Arnaud Bodin

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Introduction

Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a polynomial. The bifurcation set $\mathcal{B}$ for $f$ is the minimal set of points of $\mathbb{C}$ such that $f : \mathbb{C}^n \setminus f^{-1}(\mathcal{B}) \rightarrow \mathbb{C} \setminus \mathcal{B}$ is a locally trivial fibration. For $c \in \mathbb{C}$, we denote the fiber $f^{-1}(c)$ by $F_c$. The fiber $F_c$ is irregular if $c$ is in $\mathcal{B}$. If $s \notin \mathcal{B}$, then $F_s$ is a generic fiber and is denoted by $F_{gen}$. The tube $T_c$ for the value $c$ is a neighborhood $f^{-1}(\mathcal{D}_c^2(c))$ of the fiber $F_c$, where $\mathcal{D}_c^2(c)$ stands for a 2-disk in $\mathbb{C}$, centered at $c$, of radius $\varepsilon \ll 1$. We assume that affine critical singularities are isolated. The value $c$ is regular at infinity if there exists a compact set $K$ of $\mathbb{C}^n$ such that the restriction of $f$, $f : T_c \setminus K \rightarrow \mathcal{D}_c^2(c)$ is a locally trivial fibration.

Set $n = 2$. Let $j_c : H_1(F_c) \rightarrow H_1(T_c)$ be the morphism induced by the inclusion of $F_c$ in $T_c$. The first part of this work is the study of this morphism. Let $G_c$ the dual graph of $F_c = f^{-1}(c)$, and $\tilde{G}_c$ the dual graph of a compactification of the fiber $F_c$ obtained by a resolution at infinity of $f$. The value $c$ is acyclic if the dual graph $G_c$ and some dual graphs $G_{c, P}$ obtained by compactification have the same number of cycles (see the full definition later). This is a combinatoric condition, for example if the fiber $F_c$ is connected then $c$ is acyclic if and only if $H_1(G_c)$ is isomorphic to $H_1(\tilde{G}_c)$. Finally we define $j_\infty : H_1(F_c \setminus K) \rightarrow H_1(T_c \setminus K)$ induced by inclusion.

Theorem.

(A) $j_c$ is injective if and only if $F_c$ is connected and $c$ is acyclic.

(B) $j_c$ is surjective if and only if $j_\infty$ is surjective and $c$ is acyclic.

(C) $j_c$ is an isomorphism if and only if $c$ is a regular value at infinity.

E. Artal-Bartolo, Pi. Cassou-Noguès and A. Dimca have proved the part (C) in [ACD] for polynomials with a connected fiber $F_c$. In fact we have a stronger result for the part (A) because the rank of the kernel of $j_c$ is: $\text{rk Ker } j_c = n(F_c) - 1 + \text{rk } H_1(\tilde{G}_c) - \text{rk } H_1(G_c)$ where $n(F_c)$ is the number of connected components of $F_c$.

We apply these results to the study of neighborhoods of irregular fibers. Set $n \geq 2$. Let $F_c^\circ$ be the smooth part of $F_c$: $F_c^\circ$ is obtained by intersecting $F_c$ with a large $2n$-ball and cutting out a small neighborhood of the (isolated) singularities. Then $F_c^\circ$ can be embedded
in \( F_{\text{gen}} \). We study the following commutative diagram that links the three elements \( F_c^\ast \), \( F_{\text{gen}} \), and \( T_c \):

\[
\begin{array}{ccc}
H_q(F_c^\ast) & \xrightarrow{j_c^\ast} & H_q(T_c) \\
\downarrow{\ell_c} & & \uparrow{k_c} \\
H_q(F_{\text{gen}}) & & \\
\end{array}
\]

where \( \ell_c \) is the morphism induced in integral homology by the embedding; \( j_c^\ast \) and \( k_c \) are induced by inclusions. The morphism \( k_c \) is well-known and \( V_q(c) = \text{Ker} \ k_c \) are vanishing cycles for the value \( c \). Let \( h_c \) be the monodromy induced on \( H_q(F_{\text{gen}}) \) by a small circle around the value \( c \). Then we prove that the image of \( \ell_c \) are invariant cycles by \( h_c \):

\[
\text{Ker}(h_c - \text{id}) = \ell_c(H_q(F_c^\ast)).
\]

This formula for the case \( n = 2 \) has been obtained by F. Michel and C. Weber in [MW]. Finally we give a description of vanishing cycles with respect to eigenvalues of \( h_c \) for homology with complex coefficients. For \( \lambda \neq 1 \) and \( p \) a large integer the characteristic space \( E_\lambda = \text{Ker}(h_c - \lambda \text{id})^p \) is composed of vanishing cycles for the value \( c \). For \( \lambda = 1 \) the situation is different. If \( K_q(c) = V_q(c) \cap \text{Ker} \ (h_c - \text{id}) \) are invariant and vanishing cycles we have

\[
K_q(c) = \ell_c(\text{Ker} j_c^\ast).
\]

And for \( n = 2 \) we get the formula

\[
\text{rk} K_1(c) = r(F_c^\ast) - 1 + \text{rk} H_1(\tilde{G}_c).
\]

In the view of [DN], vanishing cycles are important: the monodromy \( h_\infty : H_1(F_{\text{gen}}) \longrightarrow H_1(F_{\text{gen}}) \) induces by a large circle around the set \( \mathcal{B} \) and Broughton’s decomposition \( H_1(F_{\text{gen}}) = \bigoplus_{c \in \mathcal{B}} V_1(c) \) determine the monodromy representation \( \pi_1(\mathbb{C} \setminus \mathcal{B}) \longrightarrow \text{Aut} H_1(F_{\text{gen}}) \).

The former formula for \( \text{rk} K_1(c) \) enables us to describe where the vanishing cycles are with respect to a decomposition of the homology of the generic fiber given by the resolution of singularities.

## 1 Irregular fibers and tubes

### 1.1 Bifurcation set

We can describe the bifurcation set \( \mathcal{B} \) as follows: let \( \text{Sing} = \{z \in \mathbb{C}^n \mid \text{grad}_f(z) = 0\} \) be the set of affine critical points and let \( \mathcal{B}_{\text{aff}} = f(\text{Sing}) \) be the set of affine critical values. The set \( \mathcal{B}_{\text{aff}} \) is a subset of \( \mathcal{B} \). The value \( c \in \mathbb{C} \) is regular at infinity if there exists a disk \( D \) centered at \( c \) and a compact set \( K \) of \( \mathbb{C}^n \) with a locally trivial fibration \( f : f^{-1}(D) \setminus K \longrightarrow D \). The non-regular values at infinity are the critical values at infinity and are collected in \( \mathcal{B}_\infty \). The finite set \( \mathcal{B} \) of critical values is now:

\[
\mathcal{B} = \mathcal{B}_{\text{aff}} \cup \mathcal{B}_\infty.
\]

In this article we always assume that affine singularities are isolated, that is to say that \( \text{Sing} \) is an isolated set in \( \mathbb{C}^n \). For \( n = 2 \) this hypothesis implies that the generic fiber is a connected set.
1.2 Preliminaries

In this paragraph $n = 2$. The inclusion of $F_c$ in $T_c$ induces a morphism $j_c : H_1(F_c) \rightarrow H_1(T_c)$. We firstly recall notations and results from [ACD].

Let denote $F_{\text{aff}} = F_c \cap B_R^1 (R \gg 1)$ and $F_{\infty} = F_c \setminus F_{\text{aff}}$, thus $F_{\text{aff}} \cap F_{\infty} = K_c = f^{-1}(c) \cap S_R^3$ is the link at infinity for the value $c$. Similarly $T_{\text{aff}} = T_c \cap B_R^1$ and $T_{\infty} = T_c \setminus T_{\text{aff}}$. We denote $j_\infty : H_1(F_{\infty}) \rightarrow H_1(T_{\infty})$ the morphism induced by inclusion. The morphism $j_{\text{aff}} : H_1(F_{\text{aff}}) \rightarrow H_1(T_{\text{aff}})$ is an isomorphism. $H_1(F_{\text{aff}} \cap F_{\infty})$ and $H_1(T_{\text{aff}} \cap T_{\infty})$ are isomorphic.

Mayer-Vietoris exact sequences for the decompositions $F_c = F_{\text{aff}} \cup F_{\infty}$ and $T_c = T_{\text{aff}} \cup T_{\infty}$ give the commutative diagram ($\mathcal{D}$):

$$
\begin{array}{cccc}
0 & \rightarrow & H_1(F_{\infty} \cap F_{\text{aff}}) & \rightarrow \\
& & \rightarrow & H_1(F_{\infty}) \oplus H_1(F_{\text{aff}}) \rightarrow \leftarrow \\
& & \rightarrow & H_1(F_c) \rightarrow \rightarrow 0
\end{array}
$$

$$
\begin{array}{cccc}
0 & \rightarrow & H_1(T_{\infty} \cap T_{\text{aff}}) & \rightarrow \\
& & \rightarrow & H_1(T_{\infty}) \oplus H_1(T_{\text{aff}}) \rightarrow \rightarrow \\
& & \rightarrow & H_1(T_c) \rightarrow H_0(T_{\infty} \cap T_{\text{aff}})
\end{array}
$$

The 0 at the upper-right corner is provided by the injectivity of $H_0(F_{\infty} \cap F_{\text{aff}}) \rightarrow H_0(F_{\infty})$ ($F_c$ need not to be a connected set) hence $H_0(F_{\infty} \cap F_{\text{aff}}) \rightarrow H_0(F_{\infty}) \oplus H_0(F_{\text{aff}})$ is injective.

1.3 Resolution of singularities

To compactify the situation, for $n = 2$, we need resolution of singularities at infinity [LW]:

$$
\begin{array}{cccc}
\mathbb{C}^2 & \rightarrow & \mathbb{CP}^2 & \rightarrow \\
\downarrow & \rightarrow & \downarrow & \rightarrow \\
\Sigma & \rightarrow & \mathbb{CP}^1 & \rightarrow
\end{array}
$$

$\tilde{j}$ is the map coming from the homogenization of $j$; $\pi$ is the minimal blow-up of some points on the line at infinity $L_\infty$ of $\mathbb{CP}^2$ in order to obtain a well-defined morphism $\phi_w : \Sigma_w \rightarrow \mathbb{CP}^1$: this is the weak resolution. We denote $\phi_w(\infty)$ by $D_\infty$, and let $D_{\text{dic}}$ be the set of components $D$ of $\pi_w^{-1}(L_\infty)$ that verify $\phi_w(D) = \mathbb{CP}^1$. Such a $D$ is a dicritical component. The degree of a dicritical component $D$ is the degree of the branched covering $\phi_w : D \rightarrow \mathbb{CP}^1$. For the weak resolution the divisor $\phi_w^{-1}(\infty) \cap \pi_w^{-1}(L_\infty)$, $c \in \mathbb{C}$, is a union of bamboos (possibly empty) (a bamboo is a divisor whose dual graph is a linear tree). The set $B_\infty$ is the set of values of $\phi_w$ on non-empty bamboos with the set of critical values of the restriction of $\phi_w$ to the dicritical components.

We can blow-up more points to obtain the total resolution, $\phi : \Sigma_\infty \rightarrow \mathbb{CP}^1$, such that all fibers of $\phi$ are normal crossing divisors that intersect the dicritical components transversally; moreover we blow-up affine singularities. Then $D_\infty = \phi^{-1}(\infty)$ is the same as above and for $c \in B$ we denote $D_c$ the divisor $\phi_c^{-1}(c)$.

The dual graph $\tilde{G}_c$ of $D_c$ is obtained as follows: one vertex for each irreducible component of $D_c$ and one edge between two vertices for one intersection of the corresponding components. A similar construction is done for $D_\infty$, we know that $\tilde{G}_\infty$ is a tree [LW]. The multiplicity of a component is the multiplicity of $\phi_c$ on this component.
1.4 Study of $j_{\infty}$

See [ACD]. Let $\phi$ be the weak resolution map for $f$. Let denote by $\text{Dic}_c$ the set of points $P$ in the dicritical components, such that $\phi(P) = c$. To each $P \in \text{Dic}_c$ is associated one, and only one, connected component $T_P$ of $T_{\infty}$; $T_P$ is the place at infinity for $P$. We have $T_{\infty} = \bigsqcup_{P \in \text{Dic}_c} T_P$ and we set $F_P = T_P \cap F_{\infty} = T_P \cap F_c$ and $K_P = \partial F_P$, finally $n(F_P)$ denotes the number of connected components of $F_P$. Let $\bar{F}_P$ be the strict transform of $c$ by $\phi$, intersected with $T_P$. The study of $j_{\infty}$ follows from the study of $j_P : H_1(F_P) \rightarrow H_1(T_P)$. Let $m_P$ be the intersection multiplicity of $\bar{F}_P$ with the divisor $\pi^{-1}_w(L_{\infty})$ at $P$.

Case of $P \in \bar{F}_P$. The group $H_1(T_P)$ is isomorphic to $\mathbb{Z}$ and is generated by $[M_P]$, $M_P$ being the boundary of a small disk with transversal intersection with the dicritical component. Moreover if $F_P = \bigsqcup_{i=1}^{n(F_P)} F_P^i$ then $j_P([F_P^i]) = j_P([K_P^i]) = m_P[I_i[M_P]]$.

Case of $P$ being a bamboo. The group $H_1(T_P)$ is also isomorphic to $\mathbb{Z}$ and is generated by $[M_P]$, $M_P$ being the boundary of a small disk, with transversal intersection with the last component of the bamboo. Then $j_P[F_P^i] = j_P[K_P^i] = m_P[I_i[M_P]]$. The integer $\ell_i$ only depends of the position where $F_P^i$ intersects the bamboo, moreover $\ell_i \geq 1$ and $\ell_i = 1$ if and only if $F_P^i$ intersects the bamboo at the last component. For a computation of $\ell_i$, refer to [ACD].

As a consequence $j_P$ is injective if and only if $n(F_P) = 1$ and $j_{\infty}$ is injective if and only if $n(F_P) = 1$ for all $P$ in $\text{Dic}_c$. In fact the rank of the kernel of $j_{\infty}$ is the sum of the ranks of the kernels of $j_P$ then

$$\text{rk ker } j_{\infty} = \sum_{P \in \text{Dic}_c} (n(F_P) - 1).$$

Finally $j_{\infty}$ is surjective if and only if for all $P \in \text{Dic}_c$, $j_P$ is surjective.

1.5 Acyclicity

The value $c$ is acyclic if the morphism $\psi : H_0(T_{\infty} \cap T_{aff}) \rightarrow H_0(T_{\infty}) \oplus H_0(T_{aff})$ given by the Mayer-Vietoris exact sequence is injective.

Let give some interpretations of the acyclicity condition.

1. The injectivity of $\psi$ can be view as follows: two branches at infinity that intersect the same place at infinity have to be in different connected components of $F_c$.

2. Let $G_c$ be the dual graph of $F_c$ (one vertex for an irreducible component of $F_c$, two vertices are joined by an edge if the corresponding irreducible components have non-empty intersection, if a component has auto-intersection it provides a loop) and let $G_{c,P}$ be the graph obtained from $G_c$ by adding edges to vertices that correspond to the same place at infinity $T_P$. In other words $c$ is acyclic if and only if there is no new cycles in $G_{c,P}$, that is to say $H_1(G_c) \cong H_1(G_{c,P})$ for all $P$ in $\text{Dic}_c$. 
3. Another interpretation is the following: \( c \) is acyclic if and only if the morphism \( h' \) of the diagram \((D)\) is surjective. This can be proved by the exact sequence:

\[
H_1(T_\infty) \oplus H_1(T_{\text{aff}}) \xrightarrow{h'} H_1(T_c) \xrightarrow{\varphi} H_0(T_\infty \cap T_{\text{aff}}) \xrightarrow{\psi} H_0(T_\infty) \oplus H_0(T_{\text{aff}}) \xrightarrow{\tilde{\psi}} H_0(T_c).
\]

4. Let consider the above Mayer-Vietoris exact sequence in reduced homology, the morphism \( \tilde{\psi} : \tilde{H}_0(T_\infty \cap T_{\text{aff}}) \rightarrow \tilde{H}_0(T_\infty) \oplus \tilde{H}_0(T_{\text{aff}}) \) is surjective because \( \tilde{H}_0(T_c) = \{0\} \). Moreover \( \tilde{\psi} \) is injective if and only if \( \psi \) is injective. As \( \tilde{\psi} \) is surjective, \( \tilde{\psi} \) is injective if and only if \( \text{rk } \tilde{H}_0(T_\infty \cap T_{\text{aff}}) = \text{rk } \tilde{H}_0(T_\infty) + \text{rk } \tilde{H}_0(T_{\text{aff}}) \), that is to say \( c \) is acyclic if and only if

\[
\sum_{P \in \text{Dic}_c} n(F_P) - 1 = \#\text{Dic}_c - 1 + n(F_c) - 1. \tag{*}
\]

This implies the lemma:

**Lemma 1.** \( j_\infty \) is injective \( \iff \) \( F_c \) is a connected set and \( c \) is acyclic.

**Proof.** If \( j_\infty \) is injective then \( n(F_P) = 1 \) for all \( P \) in \( \text{Dic}_c \), then \( H_0(T_\infty \cap T_{\text{aff}}) \cong H_0(T_\infty) \) and \( \psi \) is injective, hence \( c \) is acyclic and from equality \((*)\), we have \( n(F_c) = 1 \) i.e. \( F_c \) is a connected set. Conversely, if \( c \) is acyclic and \( n(F_c) = 1 \) then equality \((*)\) gives \( n(F_P) = 1 \) for all \( P \) in \( \text{Dic}_c \), thus \( j_\infty \) is injective. \( \square \)

Let us define a stronger notion of acyclicity. Let \( \tilde{G}_c \) be the dual graph of \( \phi^{-1}(c) \). The graph \( \tilde{G}_c \) can be obtained from \( G_c \) by adding edges between vertices that belong to the same place at infinity for all \( P \) in \( \text{Dic}_c \). The value \( c \) is strongly acyclic if \( H_1(\tilde{G}_c) \cong H_1(G_c) \). Strong acyclicity implies acyclicity, but the converse can be false. However if \( F_c \) is a connected set (that is to say \( G_c \) is a connected graph) then both conditions are equivalent. This is implicitly expressed in the next lemma, which is just a result involving graphs.

**Lemma 2.** \( \text{rk } H_1(\tilde{G}_c) - \text{rk } H_1(G_c) = \sum_{P \in \text{Dic}_c} (n(F_P) - 1) - (n(F_c) - 1) \).

### 1.6 Surjectivity

**Part (B).** \( j_c \) surjective \( \iff j_\infty \) surjective and \( c \) acyclic.

**Proof.** Let us suppose that \( j_c \) is surjective then a version of the five lemma applied to diagram \((D)\) proves that \( j_\infty \) is surjective. As \( j_c \) and \( j_\infty \) are surjective, diagram \((D)\) implies that \( h' : H_1(T_\infty) \oplus H_1(T_{\text{aff}}) \rightarrow H_1(T_c) \) is surjective, that means that \( c \) is acyclic. Conversely if \( j_\infty \) is surjective and \( c \) is acyclic then \( h' \) is surjective and diagram \((D)\) implies that \( j_c \) is surjective. \( \square \)
1.7 Injectivity

**Part (A).** $j_c$ is injective $\iff F_c$ is a connected set and $c$ is acyclic.

It follows from lemma 1 and from the next lemma.

**Lemma 3.** $j_c$ injective $\iff j_\infty$ injective.
Moreover the rank of the kernel is:

$$\text{rk} \ker j_c = \text{rk} \ker j_\infty = \sum_{P \in \text{Dic}} (n(F_P) - 1) = n(F_c) - 1 + \text{rk} H_1(\overline{G}_c) - \text{rk} H_1(G_c).$$

*Proof.* The first part of this lemma can be proved by a version of the five lemma. However we shall only prove the equality of the ranks of $\ker j_c$ and $\ker j_\infty$. It will imply the lemma because we already know that $\text{rk} \ker j_\infty = \sum_{P \in \text{Dic}} (n(F_P) - 1)$ and from lemma 2 we then have $\text{rk} \ker j_\infty = n(F_c) - 1 + \text{rk} H_1(\overline{G}_c) - \text{rk} H_1(G_c)$.

The study of the morphism $j_c : H_1(F_c) \rightarrow H_1(T_c)$ is equivalent to the study of the morphism $H_1(T_{\text{off}}) \rightarrow H_1(T_c)$ induced by inclusion that, by abuse, will also be denoted by $j_c$. To see this, it suffices to remark that $F_c$ is obtained from $F_{\text{off}} = F_c \cap B^4_R$ by gluing $F_c \cap S^3_R \times [0, +\infty]$ to its boundary $F_c \cap S^3_R$. Then the morphism $H_1(F_{\text{off}}) \rightarrow H_1(F_c)$ induced by inclusion is an isomorphism; finally $j_{\text{off}} : H_1(F_{\text{off}}) \rightarrow H_1(T_{\text{off}})$ is also an isomorphism. The long exact sequence for the pair $(T_c, T_{\text{off}})$ is:

$$H_2(T_c) \rightarrow H_2(T_c, T_{\text{off}}) \rightarrow H_1(T_{\text{off}}) \rightarrow H_1(T_c)$$

but $H_2(T_c) = 0$ (see [ACD] for example) then the rank of $\ker j_c$ is the rank of $H_2(T_c, T_{\text{off}})$. On the other hand, the study of $j_\infty : H_1(F_\infty) \rightarrow H_1(T_\infty)$ is the same as the study of $H_1(\partial T_\infty) \rightarrow H_1(T_\infty)$ induced by inclusion (and denoted by $j_\infty$) because the morphisms $H_1(\partial F_\infty) \rightarrow H_1(F_\infty)$ and $H_1(\partial F_\infty) \rightarrow H_1(\partial T_\infty)$ induced by inclusions are isomorphisms. The long exact sequence for $(T_\infty, \partial T_\infty)$ is:

$$H_2(T_\infty) \rightarrow H_2(T_\infty, \partial T_\infty) \rightarrow H_1(\partial T_\infty) \rightarrow H_1(T_\infty).$$

As $H_2(T_\infty) = 0$ (see [ACD]), then the rank of $\ker j_\infty$ is the same as $H_2(T_\infty, \partial T_\infty)$. Finally the groups $H_2(T_\infty, \partial T_\infty)$ and $H_2(T_c, T_{\text{off}})$ are isomorphic by excision, and then the ranks of $\ker j_c$ and of $\ker j_\infty$ are equal. That completes the proof. \qed

1.8 Bijectivity

**Part (C).** $j_c$ is an isomorphism $\iff c \notin \mathcal{B}_\infty$

*Proof.* If $c \notin \mathcal{B}_\infty$, then the isomorphism $j_{\text{off}} : H_1(F_{\text{off}}) \rightarrow H_1(T_{\text{off}})$ implies that $j_c$ is an isomorphism. Let suppose that $c$ is a critical value at infinity and that $j_c$ is injective. We have to prove that $j_c$ is not surjective. As $j_c$ is injective then by lemma 3, $j_\infty$ is injective. By the part (B) it suffices to prove that $j_\infty$ is not surjective. Let $P$ be a point of Dic, that provides irregularity at infinity for the value $c$, then $n(F_P) = 1$ because $j_\infty$ is injective. Let us prove that the morphism $j_P$ is not surjective. For the case of $P \in \overline{F}_P$, the
intersection multiplicity $m_P$ is greater than 1, then $j_P$ is not surjective. For the second case, in which $P$ belongs to a bamboo, then $m_P \ell_i > 1$ except for the situation where only one strict transform intersects the bamboo at the last component. This is exactly the situation excluded by the lemma “bamboo extremity fiber” of [MW]. Hence $j_\infty$ is not surjective and $j_c$ is not an isomorphism.

1.9 Examples

We apply the results to two classical examples.

Broughton polynomial. Let $f(x, y) = x(xy + 1)$, then $B_{\emptyset} = \emptyset$, $B = B_\infty = \{0\}$. Then for $c \neq 0$, $j_c$ is an isomorphism. The value 0 is acyclic since $H_1(G_0) \cong H_1(\tilde{G}_0)$. The fiber $F_0$ is not connected hence $j_0$ is not injective. As the new component of $\tilde{G}_0$ is of multiplicity 1 the corresponding morphism $j_\infty$ is surjective, hence $j_0$ is surjective.

\[
\begin{align*}
\bullet & \quad \bullet & \quad \bullet & \quad \bullet \\
G_0 & & & & & \tilde{G}_0
\end{align*}
\]

Briançon polynomial. Let $f(x, y) = yp^3 + y^2 + a_1 ps + a_0 s$ with $s = xy + 1$, $p = x(xy + 1) + 1$, $a_1 = -\frac{5}{3}$, $a_0 = -\frac{1}{3}$. The bifurcation set is $B = B_\infty = \{0, c = -\frac{16}{9}\}$, moreover all fibers are smooth and irreducible. The value 0 is not acyclic then $j_0$ is neither injective nor surjective (but $j_\infty$ is surjective).

\[
\begin{align*}
\bullet & \quad \bullet & \quad \bullet \\
G_0 & & & & & \tilde{G}_0
\end{align*}
\]

The value $c$ is acyclic, and $F_c$ is connected (since irreducible) then $j_c$ is injective. The morphism $j_c$ is not surjective: $j_\infty$ is not surjective because the compactification of $F_c$ does not intersect the bamboo at the last component.

\[
\begin{align*}
+2 & \quad +6 & \quad +3 \\
G_c & \quad \bullet & \quad \tilde{G}_c
\end{align*}
\]

2 Situation around an irregular fiber

For $f : \mathbb{C}^n \to \mathbb{C}$ we study the neighborhood of an irregular fiber.

2.1 Smooth part of $F_c$

Let fix a value $c \in \mathbb{C}$ and let $B_{R}^{2n}$ be a large closed ball ($R \gg 1$). Let $B_{1}^{2n}, \ldots, B_{p}^{2n}$ be small open balls around the singular points (which are supposed to be isolated) of $F_c$:
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\( F_c \cap \text{Sing} \). We denote \( B_1^{2n} \cup \ldots \cup B_p^{2n} \) by \( B_1 \). Then the smooth part of \( F_c \) is

\[
F_c^s = F_c \cap B_{2n}^R \setminus B_1.
\]

It is possible to embed \( F_c^s \) in the generic fiber \( F_{\text{gen}} \) (see [MW] and [NN]). We now explain the construction of this embedding by W. Neumann and P. Norbury. As \( F_c \) has transversal intersection with the balls of \( B_1 \) and with \( B_{2n}^R \), then there exists a small disk \( D_2^2(c) \) such that for all \( s \) in this disk, \( F_s \) has transversal intersection with these balls. According to Ehresmann fibration theorem, \( f \) induces a locally trivial fibration

\[
f_1 : f^{-1}(D_2^2(c)) \cap B_{2n}^R \setminus B_1 \rightarrow D_2^2(c).
\]

In fact, as \( D_2^2(c) \) is null homotopic, this fibration is trivial. Hence \( F_c^s \times D_2^2(c) \) is diffeomorphic to \( f^{-1}(D_2^2(c)) \cap B_{2n}^R \setminus B_1 \). That provides an embedding of \( F_c^s \) in \( F_s \) for all \( s \) in \( D_2^2(c) \); and for such a \( s \) with \( s \neq c \), \( F_s \) is a generic fiber. The morphism induced in homology by this embedding is denoted by \( \ell_c \). Let \( j_c^s \) be the morphism induced by the inclusion of \( F_c^s \) in \( T_c = f^{-1}(D_2^2(c)) \). Similarly \( k_c \) denotes the morphism induced by the inclusion of the generic fiber \( F_{\text{gen}} = F_s \) (for \( s \in D_2^2(c), s \neq c \)) in \( T_c \). As all morphisms are induced by natural maps we have the lemma:

**Lemma 4.** The following diagram commutes:

\[
\begin{array}{ccc}
H_q(F_c^s) & \xrightarrow{j_c^s} & H_q(T_c) \\
\ell_c & \downarrow & \uparrow k_c \\
H_q(F_{\text{gen}}) & &
\end{array}
\]

### 2.2 Invariant cycles by \( h_c \)

Invariant cycles by the monodromy \( h_c \) can be recovered by the following property.

**Proposition 5.**

\[
\ker (h_c - \text{id}) = \ell_c(H_q(F_c^s)).
\]

For \( n = 2 \), there is a similar formula in [MW], even for non-isolated singularities.

**Proof.** The proof uses a commutative diagram due to W. Neumann and P. Norbury [NN]:

\[
\begin{array}{ccc}
H_q(F_{\text{gen}}, F_c^s) & \xrightarrow{\varphi} & V_q(c) \\
\uparrow \psi & & \uparrow i \\
H_q(F_{\text{gen}}) & \xrightarrow{i - h_c} & H_q(F_{\text{gen}})
\end{array}
\]

The morphism \( i \) is the inclusion and \( \psi \) is an isomorphism, so \( \ker(h_c - \text{id}) \) equals \( \ker \varphi \). The long exact sequence for the pair \( (F_{\text{gen}}, F_c^s) \) is:

\[
\cdots \rightarrow H_q(F_c^s) \xrightarrow{\ell_c} H_q(F_{\text{gen}}) \xrightarrow{\varphi} H_q(F_{\text{gen}}, F_c^s) \rightarrow \cdots
\]

So \( \text{Im} \ell_c = \ker \varphi = \ker(h_c - \text{id}) \).
We are able to apply this result to the calculus of the rank of $\ker(h_c - \text{id})$ for $n = 2$. Let denote the number of irreducible components in $F_c$ by $r(F_c)$, and let $\text{Sing}_c$ be $\text{Sing} \cap \tilde{F}_c$; the affine singularities on $F_c$. Then $H_2(F_{p\text{gen}}^c, F_c^c)$ has rank the cardinal of $\text{Sing}_c$, which is also the rank of $\ker \ell_c$. Moreover $\text{rk} \ H_1(F_c^c) = r(F_c) - \chi(F_c) + \# \text{Sing}_c$.

$$\text{rk} \ker (h_c - \text{id}) = \text{rk} \text{Im} \ell_c$$

$$= \text{rk} H_1(F_c^c) - \text{rk} \ker \ell_c$$

$$= r(F_c) - \chi(F_c) + \# \text{Sing}_c$$

$$= r(F_c) - \chi(F_c).$$

**Remark.** We obtain the following fact (see [MW]): if the fiber $F_c$ ($c \in \mathcal{B}$) is irreducible then $h_c \neq \text{id}$. The proof is as follows: if $r(F_c) = 1$ and $h_c = \text{id}$ then from one hand $\text{rk} \ker(h_c - \text{id}) = \text{rk} H_1(F_{p\text{gen}}^c) = 1 - \chi(F_{p\text{gen}}^c)$ and from the other hand $\text{rk} \ker(h_c - \text{id}) = 1 - \chi(F_c)$; thus $\chi(F_c) = \chi(F_{p\text{gen}}^c)$ which is absurd for $c \in \mathcal{B}$ by Suzuki formula.

### 2.3 Vanishing cycles

Now and until the end of this paper homology is homology with complex coefficients.

**Vanishing cycles for eigenvalues** $\lambda \neq 1$. Let $E_\lambda$ be the space $E_\lambda = \ker(h_c - \lambda \text{id})^p$ for a large integer $p$.

**Lemma 6.** If $\lambda \neq 1$ then $E_\lambda \subset V_q(c)$.

**Proof.** If $\sigma \in H_q(F_{p\text{gen}})$ then $h_c(\sigma) - \sigma \in V_q(c)$. This is just the fact that the cycle $h_c(\sigma) - \sigma$ corresponds to the boundary of a “tube” defined by the action of the geometrical monodromy. We remark this fact can be generalized for $j \geq 1$ to

$$h_c^j(\sigma) - \sigma \in V_q(c).$$

Let $p$ be an integer that defines $E_\lambda$, then for $\sigma \in E_\lambda$:

$$0 = (h_c - \lambda \text{id})^p(\sigma) = \sum_{j=0}^{p} \binom{p}{j} (-\lambda)^{p-j} h_c^j(\sigma)$$

$$= \sum_{j=0}^{p} \binom{p}{j} (-\lambda)^{p-j} (h_c^j(\sigma) - \sigma) + \sum_{j=0}^{p} \binom{p}{j} (-\lambda)^{p-j} \sigma$$

$$= \sum_{j=0}^{p} \binom{p}{j} (-\lambda)^{p-j} (h_c^j(\sigma) - \sigma) + (1 - \lambda)^p \sigma.$$  

Each $h_c^j(\sigma) - \sigma$ is in $V_q(c)$, and a sum of such elements is also in $V_q(c)$, then $(1 - \lambda)^p \sigma \in V_q(c)$. As $\lambda \neq 1$, then $\sigma \in V_q(c)$. \qed
Vanishing cycles for the eigenvalue $\lambda = 1$. We study what happens for cycles associated to the eigenvalue 1. Let recall that vanishing cycles $V_q(c) = \text{Ker } k_c$ for the value $c$, are cycles that “disappear” when the generic fiber tends to the fiber $F_c$. Hence cycles that will not vanish are cycles that already exist in $F_c$. From the former paragraph these cycles are associated to the eigenvalue 1.

Let $(\tau_1, \ldots, \tau_p)$ be a family of $H_q(F_{\text{gen}})$ such that the matrix of $h_c$ in this family is:

$$
\begin{pmatrix}
1 & 1 & (0) \\
1 & 1 & \\
1 & \\
(0) & \\
1 & \\
1 & 
\end{pmatrix}
$$

Then, the cycles $\tau_1, \ldots, \tau_{p-1}$ are vanishing cycles. It is a simple consequence of the fact that $h_c(\sigma) - \sigma \in V_q(c)$, because for $i = 1, \ldots, p - 1$, we have $h(\tau_{i+1}) - \tau_{i+1} = \tau_i$, and then $\tau_i$ is a vanishing cycle. It remains the study of the cycle $\tau_p$ and the particular case of Jordan blocks (1) of size $1 \times 1$. We will start with the second part.

Vanishing and invariant cycles. Let $K_q(c)$ be invariant and vanishing cycles for the value $c$. $K_q(c) = \text{Ker}(h_c - \text{id}) \cap V_q(c)$. Let us remark that the space $K_q(c) \oplus \bigoplus_{c' \neq c} V_q(c')$ is not equal to $\text{Ker}(h_c - \text{id})$. But equality holds in cohomology.

**Lemma 7.** $K_q(c) = \ell_c(\text{Ker } j_c^*)$.

This lemma just follows from the description of invariant cycles (proposition 5) and from the diagram of lemma 4. For $n = 2$ we can calculate the dimension of $K_1(c)$.

**Proposition 8.** For $n = 2$, $\text{rk } K_1(c) = r(F_c) - 1 + \text{rk } H_1(G_c)$.

**Proof.** The proof will be clear after the following remarks:

1. $K_1(c) = \ell_c(\text{Ker } j_c^*)$, by lemma 7.

2. $j_c^* = j_c \circ i_c$ with $i_c : H_1(F_c) \to H_1(F_c^c)$ the morphism induced by inclusion. It is consequence of the commutative diagram:

$$
\begin{array}{ccc}
H_1(F_c) & \xrightarrow{i_c} & H_1(F_c^c) \\
\downarrow j_c & & \downarrow j_c^* \\
H_1(T_c) & \to & H_1(T_c)
\end{array}
$$

3. $\text{rk } \text{Ker } j_c^* = \text{rk } \text{Ker } i_c + \text{rk } \text{Ker } j_c \cap \text{Im } i_c$, which is general formula for the kernel of the composition of morphisms.

4. $\text{Ker } j_c \cap \text{Im } i_c = \text{Ker } j_c$, because cycles of $H_1(F_c)$ that do not belong to $\text{Im } i_c$ are cycles corresponding to $H_1(G_c)$, so they already exist in $F_c$ and are not vanishing cycles.

5. $\text{rk } \text{Ker } i_c = \sum_{z \in \text{Sing}_c} r(F_c, z)$, where $F_{c, z}$ denotes the germ of the curve $F_c$ at $z$. 
6. \( \text{rk} \ker j_c = \text{rk} \ker j_\infty = \sum_{P \in \text{Disc}} (n(F_P) - 1) = n(F_c) + \text{rk} H_1(\tilde{G}_c) - \text{rk}(G_c), \) it has been proved in lemma 3.

7. \( r(F_c) + \text{rk} H_1(G_c) = n(F_c) + \sum_{z \in \text{Sing}_c} (r(F_c, z) - 1). \) This a general formula for the graph \( G_c, \) the number of vertices of \( G_c \) is \( r(F_c) \), the number of connected components is \( n(F_c) \), the number of loops is \( \text{rk} H_1(G_c) \) and the number of edges for a vertex that correspond to an irreducible component \( F_{i_{\tau}} \) of \( F_c \) is: \( \sum_{z \in F_{i_{\tau}}} (r(F_{i_{\tau}}, z) - 1). \)

8. \( \text{rk} K_1(c) = \text{rk} \ker j_c^\ast - \# \text{Sing}_c \) because \( \ker i_c \) is a subspace of \( \ker \ell_c \) so \( \text{rk} K_1(c) = \text{rk} \ker j_c^\ast - \text{rk} \ker \ell_c \) and the dimension of \( \ker \ell_c \) is \( \# \text{Sing}_c \) (see paragraph 2.2).

We complete the proof:

\[
\begin{align*}
\text{rk} K_1(c) &= \text{rk} \ell_c(\ker j_c^\ast) \\
&= \text{rk} \ker j_c^\ast - \text{rk} \ker \ell_c \\
&= \text{rk} \ker j_c \circ i_c - \# \text{Sing}_c \tag{8} \\
&= \text{rk} \ker i_c + \text{rk} \ker j_c \cap \text{Im} i_c - \# \text{Sing}_c \tag{3} \\
&= \text{rk} \ker i_c - \# \text{Sing}_c + \text{rk} \ker j_c \tag{4} \\
&= \sum_{z \in \text{Sing}_c} (r(F_c, z) - 1) + n(F_c) + \text{rk} H_1(\tilde{G}_c) - \text{rk}(G_c) \tag{5} \\
&= r(F_c) - 1 + \text{rk} H_1(\tilde{G}_c). \tag{7}
\end{align*}
\]

\( \square \)

**Filtration.** Let \( \phi \) be the map provided by the total resolution of \( f \). The divisor \( \phi^{-1}(c) \) is denoted by \( D = \sum m_i D_i \) where \( m_i \) stands for the multiplicity of \( D_i \). We associate to \( D_i \) a part of the generic fiber denoted by \( F_i \). We briefly recall this construction (see [MW]), let \( V = \phi^{-1}(D^2(c)) \) be a tubular neighborhood of \( D \), we will identify the generic fiber \( F_{gen} \) with \( \phi^{-1}(s) \setminus \pi^{-1}(L_{\infty}) \) for a generic value \( s \in \partial D^2(c) \), \( \pi \) is the blow-up. There is a natural deformation retraction \( R : V \longrightarrow D \), and we set \( F_i = R^{-1}(D_i) \cap F_{gen} \). The filtration of the homology of the generic fiber is the sequence of inclusions:

\[
W_{-1} \subset W_0 \subset W_1 \subset W_2 = H_1(F_{gen}).
\]

with

- **\( W_{-1} \):** the boundary cycles, that is to say, if \( \bar{F}_{gen} \) is the compactification of \( F_{gen} \) and \( i_s : H_1(F_{gen}) \longrightarrow H_1(\bar{F}_{gen}) \) is induced by inclusion then \( W_{-1} = \ker i_s \).

- **\( W_0 \):** these are gluing cycles: the homology group on the components of \( F_i \cap F_j \) \((i \neq j)\).

- **\( W_1 \):** the direct sum of the \( H_1(F_i) \).

- **\( W_2 \):** \( H_1(F_{gen}) \).

The subspaces \( W_0 \) and \( W_1 \) depend on the value \( c \).
Jordan blocks for \( n = 2 \). For polynomials in two variables, the size of Jordan blocks for the monodromy \( h_c \) is less or equal to 2. Let denote by \( \sigma \) and \( \tau \) cycles of \( H_1(F_{\text{gen}}) \) such that \( h(\sigma) = \sigma \) and \( h(\tau) = \sigma + \tau \). The matrix of \( h_c \) for the family \((\sigma, \tau) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \). We already know that the cycle \( \sigma \) vanishes.

A large cycle is a cycle of \( W_2 = H_1(F_{\text{gen}}) \) that has a non-trivial class in \( W_2/W_1 \). According to [MW] \( \tau \) is large cycle; moreover large cycles associated to the eigenvalue \( 1 \) are the embedding of \( H_1(\tilde{G}_c) \) in \( H_1(F_{\text{gen}}) \).

So large cycles are not vanishing cycles. The number of classes of large cycles in \( W_2/W_1 \) is \( \text{rk} \, H_1(\tilde{G}_c) \), this is also the number of Jordan 2-blocks for the eigenvalue \( 1 \).

Vanishing cycles. We are now able to describe vanishing cycles. For all the spaces \( W_{-1}, W_0/W_{-1}, W_1/W_0 \) and \( W_2/W_1 \) the cycles associated to eigenvalues different from 1 are vanishing cycles.

**Proposition 9.** Vanishing cycles for the eigenvalue \( 1 \) are dispatch as follows:

- for \( W_{-1} \): \( r(F_c) - 1 \) cycles,
- for \( W_0 \): \( \text{rk} \, H_1(\tilde{G}_c) \) other cycles,
- \( W_1, W_2 \): no cycle.

**Proof.** We have already remark that large cycles associated to \( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \) are not vanishing cycles, so vanishing cycles in \( W_2 \) are in \( W_1 \). Moreover there is \( \text{rk} \, H_1(\tilde{G}_c) \) Jordan 2-blocks for the eigenvalue \( 1 \) that provide \( \text{rk} \, H_1(\tilde{G}_c) \) vanishing cycles (like \( \sigma \)) in \( W_0 \). The other vanishing cycles for the eigenvalue \( 1 \) are invariant cycles by \( h_c \), in other words they belong to \( K_1(c) \). We have \( W_1 \cap K_1(c) = W_0 \cap K_1(c) \) because invariant cycles for \( W_1 \) that are not in \( W_0 \) correspond to the genus of the smooth part \( F_c^* \) of \( F_c \) (this is due to the equality \( \text{Ker}(h_c - \text{id}) = \tau_c(H_1(F_c^*)) \)). As they already appear in \( F_c \), these cycles are not vanishing cycles for the value \( c \). Finally, if we have two distinct cycles \( \sigma \) and \( \sigma' \) in \( W_0 \cap K_1(c) \), with the same class in \( W_0/W_{-1} \), then \( \sigma' = \sigma + \pi, \pi \in W_{-1} \); this implies that \( \pi = \sigma' - \sigma \) is a vanishing cycle of \( K_1(c) \). We can choose the \( r(F_c) - 1 \) remaining cycles of \( K_1(c) \) in \( W_{-1} \). \( \square \)

**References**


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Arnaud Bodin
Centre de Recerca Matemàtica, Apartat 50, 08193 Bellatera, Spain
abodin@crm.es