Milnor fibration and fibred links at infinity

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Introduction

Let $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ be a polynomial function. By definition $c \in \mathbb{C}$ is a *regular value at infinity* if there exists a disc $\mathcal{D}$ centred at c and a compact set $\mathcal{C}$ of $\mathbb{C}^2$ such that the map $f : f^{-1}(\mathcal{D}) \setminus \mathcal{C} \longrightarrow \mathcal{D}$ is a locally trivial fibration. There are only a finite number of *critical* (or *irregular*) *values at infinity.* For $c \in \mathbb{C}$ and a sufficiently large real number R, the *link at infinity* $K_c = f^{-1}(c) \cap S^3_R$ is well-defined.

In this paper we sketch the proof of the following theorem which gives a characterization of fibred multilinks at infinity.

Theorem. *A multilink $K_0 = f^{-1}(0) \cap S^3_R$ is fibred if and only if all the values $c \neq 0$ are regular at infinity.*

We first obtain theorem 1, a version of this theorem was proved by A. Némethi and A. Zaharia in [NZ] (with “semitame” as a hypothesis). Here we give a new proof using resolution of singularities at infinity. This method enables us to describe the fibre and the monodromy of the Milnor fibration in terms of combinatorial invariants of a resolution of f.

Theorem 1. *If there is no critical value at infinity outside $c = 0$ then in the homotopy class of*

$$\frac{f}{|f|} : S^3_R \setminus f^{-1}(0) \longrightarrow S^1$$

there exists a fibration.

The value 0 may be regular or not. One may specify what kind of fibration it is; if f is a reduced polynomial, then this is an open book decomposition, otherwise it is a multilink fibration of $K_0 = f^{-1}(0) \cap S^3_R$ (see paragraph 1). The weights of K_0 are given by the multiplicities of the factorial decomposition of f.

If 0 is a regular value at infinity and $c \neq 0$ is a critical value at infinity, W. Neumann and L. Rudolph proved in [NR] that the link $f^{-1}(0) \cap S^3_R$ is not fibred. In the following theorem 2 we do not have any hypothesis on the value 0, in particular 0 can be a critical value at infinity.

Theorem 2. *Suppose that $c \neq 0$ is a critical value at infinity for f, then the multilink $K_0 = f^{-1}(0) \cap S^3_R$ is not a fibred multilink.*

We begin with definitions, the second part is devoted to the proof of theorem 1. We conclude with the proof of theorem 2.

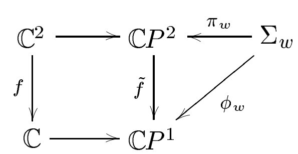
1 Definitions

As in [EN], a *multilink* $L(\mathbf{m})$ ($\mathbf{m} = (m_1, \dots, m_k)$) is a link having each component L_i weighted by the integer m_i.

The multilink $L(\mathbf{m})$ is a *fibred multilink* if there exists a differentiable fibration $\theta : S^3_R \setminus L \longrightarrow S^1$ such that m_i is the degree of the restriction of θ on a meridian of L_i. A fibre $\theta^{-1}(x)$ is a *Seifert surface* for the multilink. The link $K_0 = f^{-1}(0) \cap S^3_R$ is a multilink, the weights being naturally given by the multiplicities of the factorial decomposition of f.

A *fibred link* is a fibred multilink having all its components weighted by $+1$. Then θ is called an *open book decomposition*.

Next we give definitions and results about resolutions, see [LW]. Let n be the degree of f and F be the corresponding homogeneous polynomial with the same degree. The map $\tilde{f} : \mathbb{C}P^2 \longrightarrow \mathbb{C}P^1$, $\tilde{f}(x:y:z) = (F(x,y,z) : z^n)$ is not everywhere defined, nevertheless there exists a minimal composition of blowing-ups $\pi_w : \Sigma_w \longrightarrow \mathbb{C}P^2$ such that $\tilde{f} \circ \pi_w$ extends to a well-defined morphism ϕ_w from Σ_w to $\mathbb{C}P^1$. This is the *weak resolution*.



For an irreducible component D of $\pi_w^{-1}(L_\infty)$ (L_∞ is the line of $\mathbb{C}P^2$ having the equation $(z=0)$), we distinguish three cases:

1. $\phi_w(D) = \infty$, we denote $D_\infty = \phi_w^{-1}(\infty)$.
2. $\phi_w(D) = \mathbb{C}P^1$, D is a *dicritical component*, the restriction of ϕ_w to D is a ramified covering, the *degree* of D is the degree of this restriction. The divisor which contains all these components is the *dicritical divisor* D_{dic}.
3. $\phi_w(D) = c \in \mathbb{C}$, there is a finite number of such components, collected in $D_{crit} = D_{c_1} \cup \ldots \cup D_{c_g}$.

The irregular values at infinity for f are the values $c_1, \dots, c_g$ and the critical values of the map ϕ_w restricted to D_{dic}; moreover each divisor D_{c_i} is a disjoint union of bamboos.

We now increase the number of blowing-ups of π_w in a minimal way, in order to obtain $\pi_p : \Sigma_p \longrightarrow \mathbb{C}P^2$ and $\phi_p = \tilde{f} \circ \pi_p : \Sigma_p \longrightarrow \mathbb{C}P^1$ such that the fibre

$\phi_p^{-1}(0)$ cuts the divisor D_{dic} transversally and is a normal crossing divisor. This is the *partial resolution* for the value $c = 0$.

We continue with blowing-ups in order to obtain π_t, Σ_t, ϕ_t such that each fibre of ϕ_t cuts the divisor D_{dic} transversally and all the fibres of ϕ_t are normal crossing divisors. This is the *total resolution*.

For the total resolution the values $c_1, \dots, c_{g'}$ coming from the components D of the new D_{crit} with $\phi_t(D) = c_i$ are the critical values at infinity.

2 Milnor fibration at infinity

Until the end of this section, we suppose that the only irregular value at infinity for f can be the value 0. Let $\phi = \phi_t$ coming from the total resolution. In Σ_t the sphere $\pi_t^{-1}(S_R^3)$ is diffeomorphic to the boundary S of a neighbourhood of $\pi_t^{-1}(L_\infty)$ (see [D]).

Instead of studying $f/|f|$ restricted to $S_R^3 \setminus f^{-1}(0)$ we study $\phi/|\phi|$ restricted to $S \setminus \phi^{-1}(0)$. Let θ be the restriction of $\phi/|\phi|$ to $S \setminus \phi^{-1}(0)$. By changing the sphere $\pi_t^{-1}(S_R^3)$ into S we only know that θ is in the homotopy class of $f/|f|$.

As in [LMW] there is a correspondence between the irreducible components of $\pi_t^{-1}(L_\infty)$ and a Waldhausen decomposition of $S \setminus \phi^{-1}(0)$ into Seifert three-manifolds. We will prove that the restriction of θ to the Seifert manifold $\sigma(D)$ associated to any irreducible component D of $\pi_t^{-1}(L_\infty)$ is a fibration. If $D \subset D_\infty \cup D_0$, the equations are similar to the local case; we thus have to look at what happens with the components of the dicritical divisor.

Lemma 1. *The smooth points in $\pi_t^{-1}(L_\infty)$ of each dicritical component with non-empty intersection with $D_{crit} = D_0$ is an annulus.*

In other words the intersection of D_0 with each dicritical component is empty or reduced to a single point.

Proof. This is a consequence of the fact that above $\mathbb{C}P^1 \setminus \{0, \infty\}$, ϕ is a regular covering. □

With similar arguments, one can prove:

Lemma 2. *Each dicritical component D with $D \cap D_{crit} = \varnothing$ is of degree* 1.

2.1 Fibration on $\sigma(D)$ for $D \subset D_{dic}$

Let D be a dicritical component and let U be the simple points of D in $\pi_t^{-1}(L_\infty) \cup \phi^{-1}(0)$. By lemmas 1 and 2 we know that U is an annulus and $\phi_{|U} : U \longrightarrow \mathbb{C}P^1 \setminus \{0, \infty\}$ is a regular covering of order d.

Let $u \in \mathbb{C}^*$ be a parametrisation of U. For each point of U we choose local coordinates (u, v) such that ϕ can be written $\phi(u, v) = u$. We choose S so that S is locally given by $(|v| = \varepsilon)$ where ε is a small positive real number.

With these facts one can calculate that the restriction of θ to the Seifert component $\sigma(D)$ associated to D is a fibration whose fibres consist of d annuli.

2.2 Fibration in a neighbourhood of a non-simple point

In a neighbourhood V of a non-simple point, i.e. a point belonging to a dicritical component D and another component $D' \in \pi_t^{-1}(L_\infty) \cup \phi^{-1}(0)$, ϕ is defined in appropriate local coordinates by $(u,v) \mapsto u^d$.

Let T be the tubular neighbourhood of $D \cap V$ given by $(|v| \leqslant \varepsilon)$. $\theta_{|T}$ defines a fibration whose fibres consist of d annuli:

$$\theta^{-1}(e^{i\alpha}) \cap T = \Big\{(u,v) \in T; |v| = \varepsilon, u \neq 0 \ \text{ and } \ u^d/|u|^d = e^{i\alpha}\Big\}.$$

For T' a tubular neighbourhood of $D' \cap V$ given by $(|u| \leqslant \varepsilon)$, the fibre $\theta^{-1}(e^{i\alpha}) \cap T'$ is also a union of d annuli.

These different pieces fit nicely on the torus $\partial T \cap \partial T'$. So with a plumbing of T and T', θ is a fibration on V.

2.3 Fibration in a neighbourhood of the strict transform

Let F be an irreducible component of $\phi^{-1}(0) \setminus D_0$ (which corresponds to the affine set $f^{-1}(0)$). F can intersect D_0 or D_{dic}. If $F \cap D_0 \neq \varnothing$ then locally in a neighbourhood V, $\phi(u,v) = u^p v^q$ with $(v = 0)$ is an equation for D_0. The associated component of the link is $\phi^{-1}(0) \cap S \cap V$. Then $\theta_{|V}$ is a fibration whose fibres consist of $\gcd(p,q)$ annuli:

$$\theta^{-1}(e^{i\alpha}) \cap V = \Big\{(u,v) \in V; |v| = \varepsilon, u \neq 0 \ \text{ and } \ u^p v^q/|u^p v^q| = e^{i\alpha}\Big\}.$$

Moreover this fibration is a multilink fibration, because on a torus $D^2_\delta \times S^1_\varepsilon \setminus \{0\}$, the trace of the fibre at $v = cst$ is p radii of the annulus $D^2_\delta \setminus \{0\} \times v$. If f is a reduced polynomial function then $p = 1$ and θ is an open book decomposition.

Similarly, θ is still locally a fibration if $F \cap D_{dic} \neq \varnothing$.

We now conclude by collecting and gluing previous results. $\phi/|\phi|$ is a fibration in a neighbourhood of $S \cap \phi^{-1}(0)$ and on all $V \cap S$ which cover $S \setminus \phi^{-1}(0)$, so $\phi/|\phi| : S \setminus \phi^{-1}(0) \longrightarrow S^1$ is a fibration. Furthermore with the discussion above $\phi/|\phi|$ is an open book decomposition or a multilink fibration depending on f being reduced or not.

3 Non-fibred multilinks

Under the hypotheses of theorem 2 and without loss of generality we suppose that $\{\lambda c \text{ with } \lambda < 0\}$ does not contain critical values of f at infinity. The surface $\mathcal{F} = (f/|f|)^{-1}(-c/|c|) \cap S^3_R$ is a Seifert surface for the multilink $K_0 = f^{-1}(0) \cap S^3_R$. Moreover, for complex numbers ω with $0 \leqslant |\omega - c| \ll |c|$ the links $f^{-1}(\omega) \cap S^3_R$ do not cut $\mathcal{F}$.

We choose ω as a regular value at infinity. For the partial resolution $\phi = \phi_p$ at infinity for f and the value 0, there exists one dicritical component with a valency at least 3 in $\pi_p^{-1}(L_\infty) \cup \phi^{-1}(0)$: let D be a dicritical component where

c is a critical value at infinity. If the intersection $\phi^{-1}(0) \cap D$ has more than two points or if there is a bamboo of D_c that cuts D then we can easily conclude. But no other case is possible because $\phi_{|D}$, with the critical values 0 and c, has more than two zeroes. So the manifold $\sigma(D)$ induces a Seifert manifold of the minimal decomposition of $S^3_R \setminus K_0$; by crossing this component, $f^{-1}(\omega)$ defines a *virtual component* of $M = S^3_R \setminus f^{-1}(0)$ (see [LMW]): that is to say a regular fibre of the minimal Waldhausen decomposition of the manifold M.

According to [EN, th. 11.2], since $\mathcal{F}$ and a virtual component of M have empty intersection, K_0 is not a fibred multilink, if we exclude the case where M is $S^1 \times S^1 \times [0,1]$. This case is studied in the following lemma.

Lemma 3. *If the underlying link associated to $K_0 = f^{-1}(0) \cap S^3_R$ is the Hopf link then $c \neq 0$ is a regular value at infinity for f.*

Proof. We suppose first that f is a reduced polynomial function. Then K_0 is the Hopf link, and since K_0 is an iterated torus link around Neumann's multilink L [N, §2], this multilink can only be the trivial knot or the Hopf link.

Case of L being the trivial knot: There is only one dicritical component. If f is not a primitive polynomial (i.e. with connected generic fibre) then with the use of the Stein factorisation, let $h \in \mathbb{C}[t]$ and let $g \in \mathbb{C}[x,y]$ be a primitive polynomial with $f = h \circ g$. By the Abhyankar-Moh theorem (see [A]), there exists an algebraic automorphism Θ of $\mathbb{C}^2$ with $g \circ \Theta(x,y) = x$ and then $f \circ \Theta(x,y) = h(x)$.

Let $x_1, \dots, x_n$ be the zeroes of h; $x_1 \times \mathbb{C}, \dots, x_n \times \mathbb{C}$ are the solutions of $f \circ \Theta(x,y) = 0$. Therefore the link K_0 is a union of trivial knots with zero linking numbers, so K_0 is not the Hopf link.

Case of L being the Hopf link: K_0 and the multilink L are isotopic. On the one hand in the weak resolution for f, the restriction of $\phi = \phi_w$ to D_{dic} cannot have the critical value 0 without a bamboo. If so, one component of K_0 would be a true iterated torus knot around a component of L, in contradiction with the hypothesis. On the other hand, each component of the multilink L can be represented by a disc which crosses transversally the last component of each bamboo (start counting at the dicritical component). If there exists a bamboo for the value 0, the component C of $\phi^{-1}(0) \setminus D_0$ with $C \cap D_0 \neq \varnothing$ must be irreducible, reduced and cross D_0 transversally at the last component; this configuration is excluded by lemma 8.24 of [MW]. So 0 is a regular value at infinity and since K_0 is isotopic to L, all the dicritical components have degree one and there is no value having a bamboo, so c is a regular value at infinity for f.

If f is not reduced, let g be the reduced polynomial function associated to f. Then the link $g^{-1}(0) \cap S^3_R$ is the Hopf link and from the discussion above we know that 0 is a regular value at infinity for g. From the classification of regular algebraic annuli [N, §8], there exists an algebraic automorphism Θ of $\mathbb{C}^2$ with $\Theta(0,0) = (0,0)$ such that $g \circ \Theta(x,y) = xy + \lambda$, $\lambda \in \mathbb{C}$. So $f \circ \Theta(x,y) = (xy+\lambda)^l$ if $\lambda \neq 0$ and $f \circ \Theta(x,y) = x^p y^q$ if $\lambda = 0$. In both cases, c is a regular value at infinity for f. $\square$

In conclusion, whether 0 is a regular value at infinity or not, the multilink $K_0 = f^{-1}(0) \cap S^3_R$ is not fibred when $c \neq 0$ is a critical value at infinity.

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