BILIPSCHITZ EQUIVALENCE OF POLYNOMIALS

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ABSTRACT. We study a family of two variables polynomials having moduli up to bilipschitz equivalence: two distinct polynomials of this family are not bilipschitz equivalent. However any level curve of the first polynomial is bilipschitz equivalent to a level curve of the second.

1. GLOBAL BILIPSCHITZ EQUIVALENCE

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. For polynomial maps $f, g : \mathbb{K}^n \to \mathbb{K}$ we introduce two notions of bilipschitz equivalence: a level equivalence (a hypersurface $(f = c)$ is sent to a hypersurface $(g = c')$) and a global equivalence (any level $(f = c)$ is sent to another level $(g = c')$).

- $\mathbb{K}^n$ is endowed with the Euclidean canonical metric.
- A map $\Phi : \mathbb{K}^n \to \mathbb{K}^n$ is *Lipschitz* if there exists $K > 0$ such that for all $x, y \in \mathbb{K}^n$:

$$\|\Phi(x) - \Phi(y)\| \leqslant K\|x - y\|.$$

- A map $\Phi : \mathbb{K}^n \to \mathbb{K}^n$ is *bilipschitz* if it is a homeomorphism, Lipschitz and Φ^{-1} is also Lipschitz. Equivalently, Φ is bijective and there exists $K > 0$ such that $\frac{1}{K}\|x - y\| \leqslant \|\Phi(x) - \Phi(y)\| \leqslant K\|x - y\|$.
- Two sets $\mathcal{C}$ and $\mathcal{C}'$ of $\mathbb{K}^2$ are *bilipschitz equivalent* if there exists a bilipschitz map $\Phi : \mathbb{K}^n \to \mathbb{K}^n$ such that $\Phi(\mathcal{C}) = \mathcal{C}'$.
- Two functions $f, g : \mathbb{K}^n \to \mathbb{K}$ are *right-bilipschitz equivalent* if there exists a bilipschitz map $\Phi : \mathbb{K}^n \to \mathbb{K}^n$ such that $g \circ \Phi = f$.
- Two functions $f, g : \mathbb{K}^n \to \mathbb{K}$ are *left-right-bilipschitz equivalent* if there exist a bilipschitz map $\Phi : \mathbb{K}^n \to \mathbb{K}^n$ and a bilipschitz map $\Psi : \mathbb{K} \to \mathbb{K}$ such that $g \circ \Phi = \Psi \circ f$.

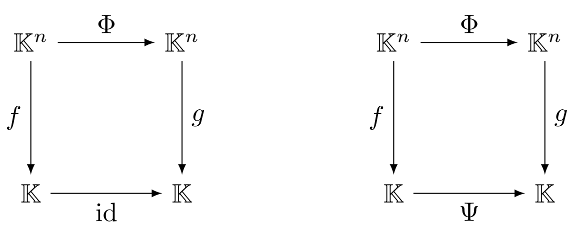

FIGURE 1. Two commutative diagrams. On the left: right-bilipschitz equivalence. On the right: left-right-bilipschitz equivalence.

Remarks:

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– A map $\Phi : \mathbb{K}^n \to \mathbb{K}^n$ can be C^1 but not Lipschitz. Hence (bi-)Lipschitz is *not* an intermediate case between smooth and continuous. This is due to the non-compactness: for instance $\Phi : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ is C^1 but not Lipschitz.
– For similar reasons an algebraic automorphism of $\mathbb{K}^n$ does not necessarily provide a bilipschitz equivalence. For instance $f(x,y) = y$ and $g = y + x^2$ are algebraically equivalent using the map $\Phi : (x,y) \mapsto (x, y - x^2)$, but Φ is not bilipschitz.

It is clear that bilipschitz equivalence implies topological equivalence (i.e. when Φ and Ψ are only homeomorphisms). The main question is: does topological equivalence implies bilipschitz equivalence? The answer is negative.

We will actually prove more. A theorem of Fukuda asserts that in a family of polynomials there is only a finite number of different types, up to topological equivalence, see [4], [3]. However the following theorem proves that the family of polynomials $f_s(x,y) = x(x^2y^2 - sxy - 1)$ has moduli for bilipschitz equivalence, i.e. any two polynomials in this family are not right-bilipschitz equivalent.

Theorem 1. *Consider the family of polynomial in* $\mathbb{K}[x,y]$*:*

$$f_s(x,y) = x(x^2y^2 - sxy - 1).$$

– $\mathbb{K} = \mathbb{R}$. *Any two polynomials* f_s *and* $f_{s'}$ *with* $s, s' \in \mathbb{R}$, $s \neq s'$ *are not right-bilipschitz equivalent. However the special levels* $(f_0 = 0)$ *and* $(f_1 = 0)$ *are bilipschitz equivalent and the generic levels* $(f_0 = 1)$ *and* $(f_1 = 1)$ *are bilipschitz equivalent.*
– $\mathbb{K} = \mathbb{C}$. *Fix* $s \in \mathbb{C}$, *with* $s^2 + 3 \neq 0$. *For all but finitely many* $s' \in \mathbb{C}$, f_s *and* $f_{s'}$ *are not right-bilipschitz equivalent. However, if* $s^2 + 4 \neq 0$ *and* $s'^2 + 4 \neq$, *the polynomials* f_s *and* $f_{s'}$ *are topologically equivalent.*

This is a version at infinity of a result by Henry and Parusiński, [5]. Our polynomials f_s have only one special level $(f_s = 0)$ which plays the role of the singular level of the local examples of [5]. We recall that for a polynomial map $f : \mathbb{K}^n \to \mathbb{K}$ there is a notion of *generic levels* $(f = c)$ and a finite number of *special levels* whose topology is not the generic one. Special levels can be due to the presence of a singular point or to singularity at infinity as this the case in our examples. We will in fact prove a non bilipschitz equivalence "at infinity", after defining that two functions are bilipschitz equivalent at infinity if they are bilipschitz equivalent outside some compact sets.

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2. Levels are bilipschitz equivalent

Lemmas 2 and 3 in this section will prove the bilipschitz real equivalence of theorem 1. Let

$$f_s(x,y) = x(x^2y^2 - sxy - 1)$$

which, in this section, is considered as a family of polynomials in $\mathbb{R}[x,y]$.

Lemma 2. *The levels* $(f_0 = 0)$ *and* $(f_1 = 0)$ *are bilipschitz equivalent, that is to say there exists a bilipschitz map* $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ *such that* $\Phi((f_0 = 0)) = (f_1 = 0)$.

In other words, the (unique) special fibers of f_0 and f_1 are bilipschitz equivalent.

Proof.

Definition of Φ.

- Let $\sigma = \frac{\sqrt{5}+1}{2}$ be the positive root of $z^2 - z - 1 = 0$. Let $\tau = \frac{\sqrt{5}-1}{2}$ be the positive root of $z^2 + z - 1 = 0$.
- We define a map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ by the following formulas:
  - For $(x, y) \in (xy = 1)$ we define:
  $$\Phi(x, y) = (ax, by) \qquad \text{with } ab = \sigma,$$
  such that (a, b) depends on (x, y) in the following way:
  $$\begin{cases} (a, b) = (\sigma, 1) & \text{if } |x| \leqslant \frac{1}{2} \\ (a, b) = (1, \sigma) & \text{if } |x| \geqslant 2 \end{cases}$$
  and extended to a smooth map for $\frac{1}{2} \leqslant |x| \leqslant 2$ so that the relation $ab = \sigma$ is always satisfied on $(xy = 1)$.
  - For $(x, y) \in (xy = -1)$ we similarly define $\Phi(x, y) = (ax, by)$ with $ab = \tau$, and $(a, b) = (\tau, 1)$ for $|x| \leqslant \frac{1}{2}$, $(a, b) = (1, \tau)$ for $|x| \geqslant 2$ and extended in a smooth map for $\frac{1}{2} \leqslant |x| \leqslant 2$.
  - $\Phi(0, y) = (0, y)$ for all $y \in \mathbb{R}$.
  - $\Phi(x, y) = (x, y)$ for (x, y) outside a neighborhood $\mathcal{N}$ of radius 1 of $(xy = 1) \cup (xy = -1)$.
  - Φ is extended on $\mathcal{N}$ to a bilipschitz homeomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$.

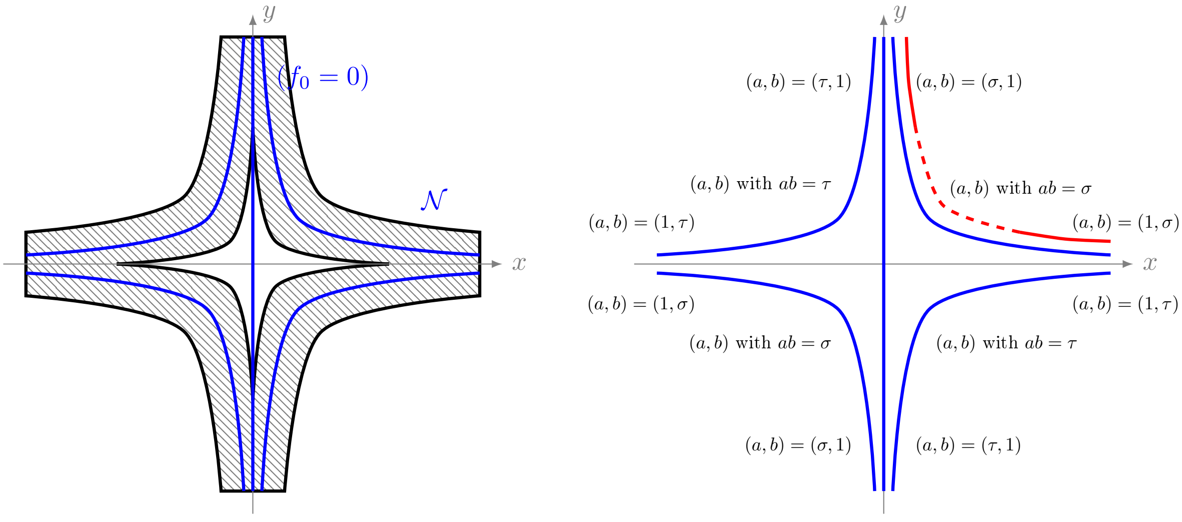

FIGURE 2. The definition of Φ. Left: the level, a neighborhood of the level. Right: the values (a, b) for the definition of $\Phi(x, y) = (ax, by)$ on the level.

- The only point to prove is that the formulas actually yield a bilipschitz map around the axis. For instance let $(x_1, y_1) \in (xy = 1)$ with $x_1 > 2$, so that $\Phi(x_1, y_1) = (x_1, \sigma y_1)$ and $(x_2, y_2) \in (xy = -1)$ with $x_2 > 2$ and $\Phi(x_2, y_2) = (x_2, \tau y_2)$. Then
$$\begin{aligned} \|\Phi(x_1, y_1) - \Phi(x_2, y_2)\| &= \|(x_1 - x_2, \sigma y_1 - \tau y_2)\| \\ &\leqslant \|(x_1 - x_2, 2\sigma(y_1 - y_2)\| \\ &\leqslant 2\sigma \|(x_1 - x_2, y_1 - y_2)\| \end{aligned}$$
(using that $y_1 - y_2 = |y_1| + |y_2|$). A similar bound holds for Φ^{-1} on this branch.

Then $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is a bilipschitz homeomorphism.

Equivalence.

- Let $f(x,y) = f_0(x,y) = x(x^2y^2-1)$ and $g(x,y) = f_1(x,y) = x(x^2y^2-xy-1)$.
- By definition of Φ, $\Phi(0,y) = (0,y)$ so that the component $(x=0) \subset (f=0)$ is sent by Φ to $(x=0) \subset (g=0)$.
- Let $(x,y) \in (xy=1) \subset (x^2y^2=1) \subset (f=0)$. For such (x,y), $\Phi(x,y) = (ax,by)$ with $ab=\sigma$.
- Let $\tilde{g}(x,y) = x^2y^2-xy-1$:
$$\tilde{g}\circ\Phi(x,y) = \tilde{g}(ax,by) = a^2b^2x^2y^2 - abxy - 1 = \sigma^2(xy)^2 - \sigma xy - 1.$$
As $xy=1$ we get:
$$\tilde{g}\circ\Phi(x,y) = \sigma^2-\sigma-1 = 0,$$
by definition of σ. Then $\Phi(x,y) \subset (\tilde{g}=0) \subset (g=0)$. A similar reasoning holds for $(xy=-1)$.

□

We now prove that two generic fibers are also bilipschitz equivalent.

Lemma 3. *The levels $(f_0=1)$ and $(f_1=1)$ are bilipschitz equivalent, that is to say there exists a bilipschitz map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Phi((f_0=1)) = (f_1=1)$.*

Proof.

- **Parameterization of $(f_0=1)$.** The curve $(f_0=1)$ has equation $x^3y^2-x-1=0$ and a parameterization (x,y) is given by
$$y_+ = \sqrt{\frac{1}{x^2}+\frac{1}{x^3}} \quad \text{or} \quad y_- = -\sqrt{\frac{1}{x^2}+\frac{1}{x^3}} \quad \text{for } x \in]-\infty,-1] \cup]0,+\infty[.$$

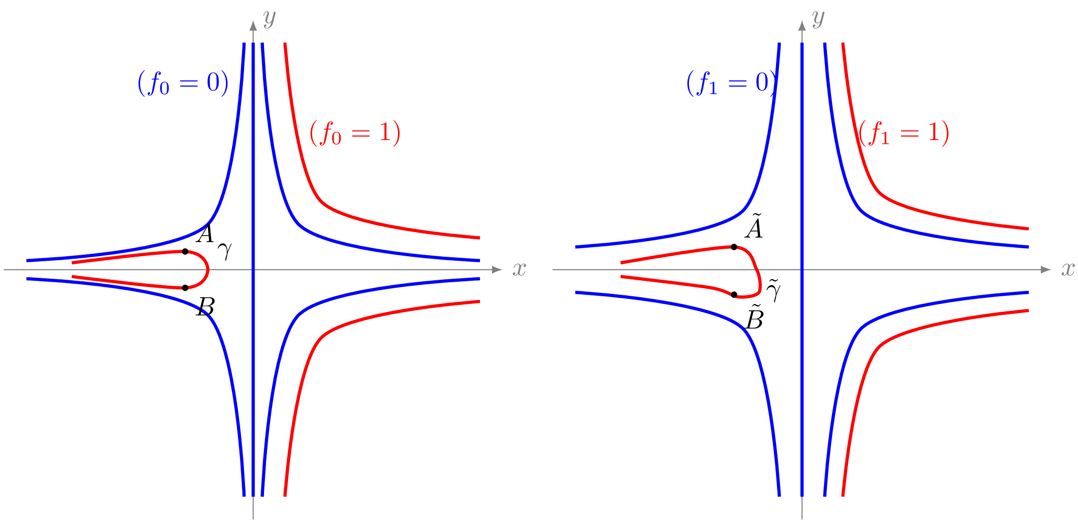

FIGURE 3. The levels $(f_0=1)$ and $(f_1=1)$.

- **Parameterization of $(f_1=1)$.** The curve $(f_1=1)$ has equation $x^3y^2-x^2y-x-1=0$, a parameterization is given by:
$$Y_+ = \frac{1}{2x}+\frac{1}{2}\sqrt{\frac{5}{x^2}+\frac{4}{x^3}} \quad \text{or} \quad Y_- = \frac{1}{2x}-\frac{1}{2}\sqrt{\frac{5}{x^2}+\frac{4}{x^3}} \quad \text{for } x \in]-\infty,-\tfrac{5}{4}] \cup]0,+\infty[.$$
- **Definition of Φ.**

  - Case $x > 0$. Φ is defined on $(f_0 = 1)$ using the parameterization by the formula $\Phi(x,y) = (x, Y_+)$, for $(x,y) \in (f_0 = 1)$ with $x > 0$ and $y > 0$; $\Phi(x,y) = (x, Y_-)$, for $(x,y) \in (f_0 = 1)$ with $x > 0$ and $y < 0$.
  - Case $x \leqslant -2$. Φ is defined by the same formulas $\Phi(x,y) = (x, Y_+)$ (for $y > 0$) or $\Phi(x,y) = (x, Y_-)$ (for $y < 0$).
  - Case $-2 \leqslant x \leqslant -1$. (Note: we do not use the above formulas in the neighborhood of the point $(-1,0)$ because the map $y_+ \mapsto Y_+$ is not bilipschitz near this point.) Let A, B be the two points of $(x = -2) \cap (f_0 = 1)$. Let $\tilde{A}, \tilde{B}$ be their images by Φ (i.e. A, B belong $(x = -2) \cap (f_1 = 1)$). Let γ be the compact part of $(f_0 = 1)$ between A and B and $\tilde{\gamma}$ be the compact part of $(f_1 = 1)$ between $\tilde{A}$ and $\tilde{B}$. We extend Φ in a bilipschitz way from γ to $\tilde{\gamma}$. This is possible as γ and $\tilde{\gamma}$ are two compact connected components of a smooth algebraic curve. Φ is now defined everywhere on $(f_0 = 1)$.
  - We extend Φ on $\mathbb{R}^2$ to a bilipschitz map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$. For instance we may suppose Φ is the identity outside a tubular neighborhood or radius 1 of $(f_0 = 1)$.
- **Bilipschitz on** $(f_0 = 1)$**.** It remains to justify that Φ is actually a bilipschitz map from $(f_0 = 1)$ to $(f_1 = 1)$.
  - Case $x > 0$ and $x \to 0$. Hence $y \to \pm\infty$. Then $y_+ \sim \frac{1}{x^{3/2}}$ and $Y_+ \sim \frac{1}{x^{3/2}} \sim y_+$ so that the map $\Phi(x, y_+) = (x, Y_+)$ is bilipschitz. The same applies for y_- and Y_-.
  - Case $x \to +\infty$. Hence $y \to 0$. Then $y_+ \sim \frac{1}{x}$ and $Y_+ \sim \frac{\sqrt{5}+1}{2} \cdot \frac{1}{x} \sim \sigma y_+$. Then, as in the proof of proposition 2, $\Phi(x, y_+) = (x, Y_+)$ is bilipschitz. The same applies for y_- and $Y_- \sim \tau y_-$ with $\tau = \frac{\sqrt{5}-1}{2}$.
  - Case $x \to -\infty$. It is similar to the previous case: $Y_+ \sim \tau y_+$, $Y_- \sim \sigma y_-$.

$\square$

3. Moduli

The following theorem proves that under bilipschitz equivalence at infinity a family of polynomials can have moduli. It is a version at infinity of the example of Henry and Parusiński [5]. Two functions $f, g : \mathbb{K}^n \to \mathbb{K}$ are *right-bilipschitz equivalent at infinity* if there exist compact sets C, C' and a bilipschitz map $\Phi : \mathbb{K}^n \setminus C \to \mathbb{K}^n \setminus C'$ such that $g \circ \Phi = f$.

Using this notion, we will prove the moduli affirmation of theorem 1 with the following refinement.

Theorem 1'.

$$f_s(x,y) = x(x^2y^2 - sxy - 1) \in \mathbb{K}[x,y].$$

- $\mathbb{K} = \mathbb{R}$. *Any two polynomials* f_s *and* $f_{s'}$ *with* $s, s' \in \mathbb{R}$, $s \neq s'$ *are not right-bilipschitz equivalent at infinity (hence not globally right-bilipschitz equivalent). Moreover they are also not left-right-equivalent if we assume* Φ *analytic at infinity.*
- $\mathbb{K} = \mathbb{C}$. *Fix* $s \in \mathbb{C}$, *with* $s^2 + 3 \neq 0$. *For all but finitely many (explicit)* $s' \in \mathbb{C}$, f_s *and* $f_{s'}$ *are not right-bilipschitz equivalent at infinity (hence not globally right-bilipschitz equivalent).*

3.1. Preliminaries.

- Let $f_s(x,y) = x(x^2y^2 - sxy - 1) = x^3y^2 - sx^2y - x$.

- Then $\partial_x f_s(x,y) = 3x^2y^2 - 2sxy - 1$.
- The equation $3z^2 - 2sz - 1 = 0$ has discriminant $\Delta = 4(s^2+3)$ and two solutions:

$$\alpha_s = \frac{s + \sqrt{s^2+3}}{3} \quad \text{and} \quad \beta_s = \frac{s - \sqrt{s^2+3}}{3}.$$

- The polar curve $\Gamma_s : (\partial_x f_s = 0)$, associated to the projection on the y-axis, has two components:

$$(xy = \alpha_s) \quad \text{and} \quad (xy = \beta_s),$$

parameterized by:

$$\left(\alpha_s t, \frac{1}{t}\right) \quad \text{and} \quad \left(\beta_s t, \frac{1}{t}\right) \qquad t \in \mathbb{K} \setminus \{0\}.$$

- We compute the values of f_s on the polar components. Near the point at infinity $(0:1:0)$, that is to say for $t \to 0$, we compute the values of f_s on each branch of Γ_s:

$$f_s\left(\alpha_s t, \frac{1}{t}\right) = \alpha_s(\alpha_s^2 - s\alpha_s - 1)t,$$

and

$$f_s\left(\beta_s t, \frac{1}{t}\right) = \beta_s(\beta_s^2 - s\beta_s - 1)t.$$

- We compare theses values for two branches at a same y-value:

$$\frac{f_s\left(\alpha_s t, \frac{1}{t}\right)}{f_s\left(\beta_s t, \frac{1}{t}\right)} = \frac{\alpha_s(\alpha_s^2 - s\alpha_s - 1)}{\beta_s(\beta_s^2 - s\beta_s - 1)}.$$

- Our arguments will only focus on a neighborhood of a the point $(0:1:0)$ at infinity. More precisely we will say that an analytic curve $(x(t), y(t))$ *tends to the point at infinity* $(0:1:0)$ if $y(t) \to +\infty$ and $\frac{|x(t)|}{|y(t)|} \to 0$ as $t \to 0$.

3.2. **Proof in the real case.**

- Fix $t > 0$. Let A, B, C, D, E be the following points having all y-coordinates equal to $\frac{1}{t}$:
  - $A \in (f_s = 0)$ with $x_A > 0$,
  - $B \in \Gamma_s : (\partial_x f_s = 0)$ with $x_B > 0$,
  - $C = (0, \frac{1}{t}) \in (f_s = 0)$,
  - $D \in \Gamma_s : (\partial_x f_s = 0)$ with $x_D < 0$,
  - $E \in (f_s = 0)$ with $x_E < 0$.
- Let us fix $s, s' \in \mathbb{R}$. By contradiction let us assume that there exists a bilipschitz homeomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f_{s'} \circ \Phi = f_s$. Let K be its bilipschitz constant. Let $\tilde{A}, \tilde{B}, \ldots$ be the image by Φ of $A, B, \ldots$ Let γ be the segment $[AB]$ and $\tilde{\gamma} = \Phi(\gamma)$.
- Φ sends $(f_s = 0)$ to $(f_{s'} = 0)$ and, as it is a homeomorphism, it should send the component $(x = 0)$ of $(f_s = 0)$ to the component $(x = 0)$ of $(f_{s'} = 0)$. Hence $x_{\tilde{C}} = 0$.
- A, B, C, D, E and γ are all included in the disk of radius rt centered at C, where r is a constant that depends only on the fixed value s. Hence by the bilipschitz map Φ, $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}$ and $\tilde{\gamma}$ are all included in a disk of radius Krt centered at $\tilde{C}$.
- There is an issue: the point B is on the polar curve Γ_s but $\tilde{B}$ has no reason to be on $\Gamma_{s'}$. We will replace $\tilde{B}$ by a point B' satisfying this condition.

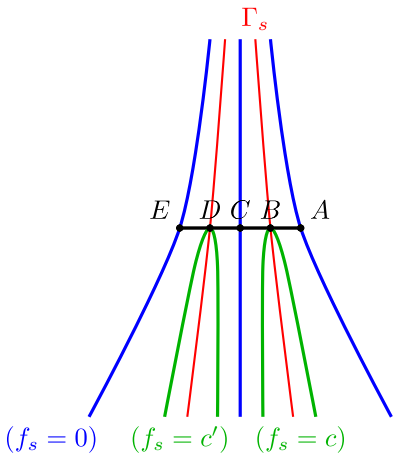

FIGURE 4. The situation for f_s.

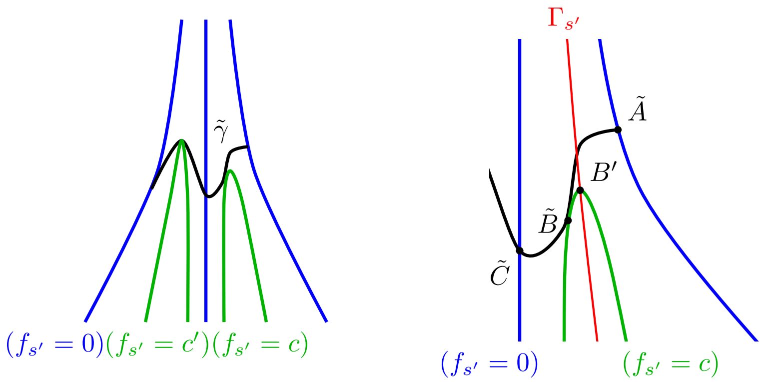

FIGURE 5. The situation for $f_{s'}$.

- Let $c = f_s(B)$. Let $\tilde{X}_c$ be the part of $(f_{s'} = c)$ in the ball of radius Krt centered at $\tilde{C}$. As $f_{s'}(\tilde{B}) = f_s(B) = c$, then $\tilde{B} \in \tilde{X}_c$ and $\tilde{X}_c$ is non empty. Moreover $\tilde{X}_c$ is contained between two components of $(f_{s'} = 0)$: $(x = 0)$ and one branch of $(x^2y^2 - s'xy - 1 = 0)$. Moreover $\tilde{X}_c$ is strictly below $\tilde{\gamma}$ except at $\tilde{B}$ (because $(f = c)$ is below $\gamma = [AB]$ and intersects it only at B).
- Let B' be the point of $\tilde{X}_c$ such that $y_{B'}$ is maximal among points of $\tilde{X}_c$. Then the tangent at B' is horizontal, that is to say $\partial_x f_{s'}(B') = 0$, hence $B' \in \Gamma_{s'}$. Remember also that $B' \in \tilde{X}_c$ so that $f_{s'}(B') = c$.
- Partial conclusion: we constructed a point $B' \in \Gamma_{s'} \cap (f_{s'} = c)$ such that $\|B' - \tilde{C}\| \leqslant Krt$ (with $x_{B'} > 0$).
- We carry on the same proof for the other side. Let $c' = f_s(D)$, we find a point $D' \in \Gamma_{s'} \cap (f_{s'} = c')$ such that $\|D' - \tilde{C}\| \leqslant Krt$ (with $x_{D'} < 0$).
- Now both these points B' and D' are in the same disk of radius Krt centered at $\tilde{C}$. In particular:

$$y_{D'} - 2Krt \leqslant y_{B'} \leqslant y_{D'} + 2Krt.$$

– Let $B' = (\alpha_{s'} t', \frac{1}{t'})$ be the coordinates of B' on the first branch of $\Gamma_{s'}$ and $D' = (\beta_{s'} t'', \frac{1}{t''})$ be the coordinates of D' on the second branch of $\Gamma_{s'}$. The former inequalities rewrite:
$$\frac{1}{t''} - 2Krt \leqslant \frac{1}{t'} \leqslant \frac{1}{t''} + 2Krt.$$
We consider $t \to 0$, so that $t' \to 0$, $t'' \to 0$ (a neighborhood of $(0:1:0)$ is send to a neighborhood of $(0:1:0)$). Hence $t'' = t' + O(tt't'') = t' + O(tt'^2)$.

– Now
$$\frac{f_{s'}\left(\alpha_{s'}t', \frac{1}{t'}\right)}{f_{s'}\left(\beta_{s'}t'', \frac{1}{t''}\right)} = \frac{\alpha_{s'}t'\left(\alpha_{s'}^2 - s'\alpha_{s'} - 1\right)}{\beta_{s'}t''\left(\beta_{s'}^2 - s'\beta_{s'} - 1\right)}$$
$$= \frac{\alpha_{s'}t'\left(\alpha_{s'}^2 - s'\alpha_{s'} - 1\right)}{\beta_{s'}(t' + O(tt'^2))\left(\beta_{s'}^2 - s'\beta_{s'} - 1\right)} \longrightarrow \frac{\alpha_{s'}(\alpha_{s'}^2 - s'\alpha_{s'} - 1)}{\beta_{s'}(\beta_{s'}^2 - s'\beta_{s'} - 1)}$$
as $t' \to 0$.

– On the other hand:
$$\frac{f_{s'}(B')}{f_{s'}(D')} = \frac{c}{c'} = \frac{f_{s'}(\tilde{B})}{f_{s'}(\tilde{D})} = \frac{f_s(B)}{f_s(D)} = \frac{\alpha_s(\alpha_s^2 - s\alpha_s - 1)}{\beta_s(\beta_s^2 - s\beta_s - 1)}.$$
Finally:
$$\frac{\alpha_s(\alpha_s^2 - s\alpha_s - 1)}{\beta_s(\beta_s^2 - s\beta_s - 1)} = \frac{\alpha_{s'}(\alpha_{s'}^2 - s'\alpha_{s'} - 1)}{\beta_{s'}(\beta_{s'}^2 - s'\beta_{s'} - 1)}.$$

– The map $s \mapsto \frac{\alpha_s(\alpha_s^2 - s\alpha_s - 1)}{\beta_s(\beta_s^2 - s\beta_s - 1)} = \frac{2(s^2+3)\alpha_s + s}{2(s^2+3)\beta_s + s}$ is strictly decreasing for $s \in \mathbb{R}$ so that $s = s'$.

– Conclusion: if $s, s' \in \mathbb{R}$, with $s \neq s'$, then there exists no bilipschitz homeomorphism sending f_s to $f_{s'}$. Since our arguments only care about situation near $(0:1:0)$ f_s and $f_{s'}$ are not right-bilipschitz equivalent at infinity.

3.3. **No left-right-equivalence.** We now prove that for $s \neq s'$ f_s and $f_{s'}$ are not left-right-equivalent, if we ask the homeomorphism Φ to be analytic near the point at infinity $(0:1:0)$. By contradiction we suppose that there exist bilipschitz homeomorphisms Φ and Ψ such that $f_{s'} \circ \Phi = \Psi \circ f_s$ and Φ is analytic near the point at infinity $(0:1:0)$. We continue with the same notation as above, but we cannot conclude as before because we no longer have $\frac{f_{s'}(B')}{f_{s'}(D')}$ equal to $\frac{f_s(B)}{f_s(D)}$.

– Let $C = (0, \frac{1}{t})$ and $\Phi(C) = \tilde{C} = (0, \frac{1}{\tilde{t}})$ $(t > 0)$. The map $\frac{1}{t} \mapsto \frac{1}{\tilde{t}}$ is a bilipschitz homeomorphism. We will assume $\Phi(0,0) = (0,0)$ so that $\frac{1}{K}\frac{1}{t} \leqslant \frac{1}{\tilde{t}} \leqslant K\frac{1}{t}$ hence $\frac{1}{K}t \leqslant \tilde{t} \leqslant Kt$. Define $\chi(t) = \tilde{t}$, for $t > 0$, and set $\chi(0) = 0$. In the following we will actually only need the relation $\frac{1}{K}t \leqslant \chi(t) \leqslant Kt$, but in fact the map $t \mapsto \chi(t)$ is a bilipschitz homeomorphism (with the constant K^3).

– We assumed that the map Φ is analytic at infinity around $(0:1:0)$. It implies that the map $t \mapsto \chi(t)$ is analytic for $t > 0$: $\chi(t) = a_0 t^{r_0} + a_1 t^{r_1} + \cdots$ The map χ being bilipschitz it implies $r_0 = 1$ so that $\chi(t) = a_0 t + a_1 t^{r_1} + \cdots$ with $r_1 > 1$.

– Notice that the relation $f_{s'} \circ \Phi = \Psi \circ f_s$ implies that the map Ψ is also an analytic map.

– Recall that $B = (\alpha_s t, \frac{1}{t})$ and $f_s(B) = c = \alpha_s(\alpha_s^2 - s\alpha_s - 1)t$, $D = (\beta_s t, \frac{1}{t})$ and $f_s(D) = c' = \beta_s(\beta_s^2 - s\beta_s - 1)t$. $\Phi(B) = \tilde{B}$ and $f_{s'}(\tilde{B}) = \tilde{c} = \Psi(c)$, $\Phi(D) = \tilde{D}$ and $f_{s'}(\tilde{D}) = \tilde{c}' = \Psi(c')$. We found $B' = (\alpha_{s'} t', \frac{1}{t'})$ close to $\tilde{B}$ such that $f_{s'}(B') = f_{s'}(\tilde{B}) = \tilde{c}$. Hence $\tilde{c} = \alpha_{s'}(\alpha_{s'}^2 - s'\alpha_{s'} - 1)t'$. Similarly $D' = (\beta_{s'} t'', \frac{1}{t''})$ is close to $\tilde{D}$ and $f_{s'}(D') = f_{s'}(\tilde{D}) = \tilde{c}'$. Hence $\tilde{c}' = \beta_{s'}(\beta_{s'}^2 - s'\beta_{s'} - 1)t''$.

B' is close to $\tilde{B}$ actually means $\left|\frac{1}{t'} - \frac{1}{\tilde{t}}\right| \leqslant Krt$, that implies $|t' - \tilde{t}| \leqslant Krtt'\tilde{t}$. That implies $t' = \chi(t) + O(t^3)$. Similarly $t'' = \chi(t) + O(t^3)$.

- The map Ψ is defined, for negative values, by $c \mapsto \tilde{c}$ that is to say $\alpha_s(\alpha_s^2 - s\alpha_s - 1)t \mapsto \alpha_{s'}(\alpha_{s'}^2 - s'\alpha_{s'} - 1)t'$. It implies that, for $u < 0$, the map Ψ is defined by
$$\Psi : u \mapsto \alpha_{s'}(\alpha_{s'}^2 - s'\alpha_{s'} - 1)\chi\left(\frac{u}{\alpha_s(\alpha_s^2 - s\alpha_s - 1)}\right) + O(u^3).$$
Hence, as $\chi(t) = a_0 t + o(t)$:
$$\Psi : u \mapsto \frac{\alpha_{s'}(\alpha_{s'}^2 - s'\alpha_{s'} - 1)}{\alpha_s(\alpha_s^2 - s\alpha_s - 1)} u + o(u).$$
Similarly $\Psi(d) = \tilde{d}$ so that for $u > 0$:
$$\Psi : u \mapsto \frac{\beta_{s'}(\beta_{s'}^2 - s'\beta_{s'} - 1)}{\beta_s(\beta_s^2 - s\beta_s - 1)} u + o(u).$$
- By analycity of Ψ, it implies that the coefficients of u are equal, whence
$$\frac{\alpha_s(\alpha_s^2 - s\alpha_s - 1)}{\beta_s(\beta_s^2 - s\beta_s - 1)} = \frac{\alpha_{s'}(\alpha_{s'}^2 - s'\alpha_{s'} - 1)}{\beta_{s'}(\beta_{s'}^2 - s'\beta_{s'} - 1)},$$
which is impossible for $s \neq s'$ as we have seen before in section 3.2.

3.4. **No left-right-equivalence (again).** It is not clear whether f_s and $f_{s'}$ $(s \neq s')$ are or not left-right bilipschitz equivalent when no restriction is made on Φ. However we can complicate our example in order to exclude left-right equivalence.

Lemma 4. *Let*
$$f_s(x, y) = x(x^4y^4 - 3sx^2y^2 + 1)$$
be a family of polynomials in $\mathbb{R}[x, y]$*. Then for* $s, s' > 1$*, with* $s \neq s'$*, the polynomials* f_s *and* $f_{s'}$ *are not left-right bilipschitz equivalent.*

Proof. The proof is similar to the proof of section 3.3.

- The equation $5z^4 - 9sz^2 + 1 = 0$ has 4 real solutions $-\alpha_s < -\beta_s < \beta_s < \alpha_s$ corresponding to 4 branches of the polar curve $(\partial_x f_s = 0)$.
- We use the same method as before in section 3.3 with $B = (-\alpha_s t, \frac{1}{t})$, $f_s(B) = -\alpha_s(\alpha_s^4 - 3s\alpha_s^2 + 1)t = c_s t > 0$ and $D = (\beta_s t, \frac{1}{t})$, $f_s(D) = \beta_s(\beta_s^4 - 3s\beta_s^2 + 1)t = d_s t > 0$ (with $t > 0$).
- This times for $u > 0$ we have two formulas for Ψ :
$$\Psi(u) = c_{s'}\chi\left(\frac{u}{c_s}\right) + O(u^3),$$
and
$$\Psi(u) = d_{s'}\chi\left(\frac{u}{d_s}\right) + O(u^3).$$
- It implies that the bilipschitz map χ verifies
$$\chi\left(\frac{c_s}{d_s}v\right) = \frac{c_{s'}}{d_{s'}}\chi(v) + O(v^3)$$
for all $v > 0$ near 0.
- Then by lemma 5 below, it implies $p = \frac{c_s}{d_s} > 1$ is equal to $q = \frac{c_{s'}}{d_{s'}} > 1$ which is impossible if $s \neq s'$.

□

Lemma 5. *Let $\chi : \mathbb{R} \to \mathbb{R}$ be a bilipschitz map such that*
$$\chi(pv) = q\chi(v) + O(v^3)$$
for some constant $p, q > 1$ and all v near 0. Then $p = q$.

Proof. We have $\chi(v) = q\chi(v/p) + v^3\eta(v)$, where $\eta(v)$ is a bounded function for v near 0. By induction it yields $\chi(v) = q^n\chi(v/p^n) + v^3\sum_{k=0}^{n-1}\eta(v/p^k)(q/p^3)^k$. Hence, except for the special case $p^3 = q$ that would be treated in a similar way, we have:
$$\left|\chi(v) - q^n\chi\left(\frac{v}{p^n}\right)\right| \leqslant Cv^3\frac{1-(q/p^3)^n}{1-q/p^3}. \tag{1}$$

Let $K > 0$ be a bilipschitz constant for χ. As $\chi(0) = 0$ we have $K^{-1} < \frac{|\chi(v)|}{|v|} < K$ for all $v \neq 0$. In particular $K^{-1} < p^n\frac{|\chi(v/p^n)|}{|v|} < K$.

Case $p > q$. Then we have $q^n\chi(v/p^n) \to 0$ as $n \to +\infty$. At the limit, when $n \to +\infty$, inequality (1) gives $|\chi(v)| \leqslant C'v^3$, which contradicts that χ is bilipschitz.

Case $p < q$. Inequality (1) gives
$$\left|\frac{p^n}{q^n}\chi(v) - p^n\chi\left(\frac{v}{p^n}\right)\right| \leqslant C'v^3\left(\frac{p^n}{q^n} - \frac{1}{p^{2n}}\right)$$
Fix $v \neq 0$. As $n \to +\infty$, the term $p^n\chi(\frac{v}{p^n})$ does not tend towards 0, it contradicts that all the other terms $\frac{p^n}{q^n}\chi(v)$, $\frac{p^n}{q^n}$ and $\frac{1}{p^{2n}}$ tends towards 0.

Conclusion: $p = q$.

□

We completed the proof of theorem 1 in the real setting.

4. Proof in the complex case

The proof in the complex case at infinity is an adaptation of the local proof of Henry and Parusiński [5].

4.1. **Notations.**

- Let $g : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial map and $p = (x, y)$ be a point near the point at infinity $(0 : 1 : 0)$, that is to say $|y| \gg 1$ and $|x| \ll |y|$.
- Fix p_0, let $c = g(p_0)$. Denote $B(p_0, \rho)$ the open ball centered at p_0 of radius ρ and $X(p_0, \rho) = (g = c) \cap B(p_0, \rho)$.
- Fix $K > 0$ and denote $\operatorname{dist}_{p_0,\rho,K}(p, q)$ the inner distance of p and q supposed to be in the same connected component of $X(p_0, K\rho)$.
- Let
$$\phi(p_0, K, \rho) = \sup\frac{\operatorname{dist}_{p_0,\rho,K}(p, q)}{\|p - q\|}$$
be the ratio between the inner and outer distances.
- Denote
$$\psi(p_0, K, \rho) = \sup_{\rho' \leqslant \rho}\phi(p_0, K, \rho').$$
- Finally let
$$Y(\rho, K, A) = \{p \mid \psi(p, K, \rho) \geqslant A\}.$$
be the set of points p where the curvature of the curve $(g = c)$ is large.
- Let $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ be a bilipschitz homeomorphism at infinity such that $\tilde{g} \circ \Phi = g$. Let L be a bilipschitz constant of Φ.

– Once Φ is fixed, we add a tilde to denote an object in then target space, for instance

$$\tilde{Y}(\rho, K, A) = \{\tilde{p} \mid \psi(\tilde{p}, K, \rho) \geqslant A\}$$

is the set of points $\tilde{p}$ in the target space where the curvature of the curve $(\tilde{g} = \tilde{c})$ is large.

We have the following lemma saying that points with large curvature are sent to points of large curvature by a bilipschitz map:

Lemma 6 ([5], Lemma 2.1). *For $K \geqslant L^2$:*

$$\tilde{Y}(L^{-1}\rho, K, AL^2) \subset \Phi(Y(\rho, K, A)) \subset \tilde{Y}(L\rho, K, AL^{-2}).$$

And a variant:

Lemma 7 ([5], Lemma 2.2). *Let $\delta > 0$ and*

$$Y(\delta, K, M, A) = \{p \mid \psi(p, M\|p\|^{-1+\delta}, K) \geqslant A\}.$$

If $K \geqslant L^2$ then:

$$\tilde{Y}(\delta, K, ML^{-\delta}, AL^2) \subset \Phi(Y(\delta, K, M, A)) \subset \tilde{Y}(\delta, K, ML^{+\delta}, AL^{-2}).$$

Remarks:

– There are two distinct uses of the norm:
  – $\|p - q\|$: distance between two “near” points: a “small” number.
  – $\|p\|$: distance to the origin: a “large” number. We will use it for $\frac{1}{\|p\|}$ in order to get a “small” number.
– If we denote $\tilde{p} = \Phi(p)$, then the bilipschitz property implies: $L^{-1}\|p\| \leqslant \|\tilde{p}\| \leqslant L\|p\|$ for some bilipschitz constant L, hence also:

$$L^{-1}\|p\|^{-1} \leqslant \|\tilde{p}\|^{-1} \leqslant L\|p\|^{-1}.$$

– Notice that in our definition of $Y(\delta, K, M, A)$ of lemma 7 there is a term in $\|p\|^{-1+\delta}$ while in [5] the term is $\|p\|^{1+\delta}$. After this modification, the proof is the same as in [5].
– We will restrict ourselves to a neighborhood of the point at infinity $(0 : 1 : 0)$, in particular we may suppose $|y| \gg |x|$ so that morally $\|p\| = \|(x, y)\| \simeq |y|$ (this is an equality in the case $\|\cdot\| = \|\cdot\|_\infty$).

Fix $s \in \mathbb{C}$ and denote $f_s(x, y) = x(x^2y^2 - sxy - 1)$. Let us denote $U = \{(x, y) \mid |\partial_x f_s| < |\partial_y f_s|\}$.

Lemma 8 (compare to [5], Lemma 3.2). *Let $(x(t), y(t)) \in U$ with $y(t) = \frac{1}{t}$. Then for $s^2 + 3 \neq 0$:*

$$x(t) = \gamma t + O(t^3) \quad \text{and} \quad f_s(x(t), y(t)) = \gamma(\gamma^2 - s\gamma - 1)t + O(t^3),$$

with

$$\gamma = \alpha_s \text{ or } \gamma = \beta_s \text{ a solution of } 3z^2 - 2sz - 1 = 0.$$

In this section we now suppose $s^2 + 3 \neq 0$.

Proof. Let $u = xy$. On U the inequality $|\partial_x f_s| < |\partial_y f_s|$ yields $|3u^2 - 2su - 1| < |x|^2|2u - s|$. In a neighborhood of the point at infinity $(0 : 1 : 0)$ we first prove that $|x(t)|$ is bounded as $t \to +\infty$. If this is not the case, then write $x(t) = a_0 t^{r_0} + a_1 t^{r_1} + \cdots$ with $r_i \in \mathbb{Q}$, $r_i < r_{i+1}$ and here $r_0 < 0$. As $y(t) = 1/t$, then $u(t) \sim a_0 t^{r_0 - 1} \to +\infty$. Then $|3u^2 - 2su - 1| < |x|^2|2u - s|$ implies $r_0 \leqslant -1$ in contradiction with $\frac{x(t)}{y(t)} \to 0$. Now, as $|x(t)|$ is bounded, inequality $|3u^2 - 2su - 1| < |x|^2|2u - s|$ implies that $|u(t)|$ is also bounded. Write again $x(t) = a_0 t^{r_0} + a_1 t^{r_1} + \cdots$ and using that $u(t)$ is bounded gives $r_0 \geqslant 1$: $x(t) = a_0 t + a_1 t^{r_1} + \cdots$ and $u(t) = a_0 + a_1 t^{r_1 - 1} + \cdots$ ($a_0 \in \mathbb{C}$). We plug $u(t)$ in the inequality $|3u^2 - 2su - 1| < |x|^2|2u - s|$:

$$\left|3(a_0 + a_1 t^{r_1-1} + \cdots)^2 - 2s(a_0 + a_1 t^{r_1-1} + \cdots) - 1\right| = O(t^2).$$

It implies:

$$3a_0^2 - 2sa_0 - 1 = 0$$

and

$$6a_0 a_1 t^{r_1-1} - 2sa_1 t^{r_1-1} = O(t^2).$$

We may suppose $a_1 \neq 0$ and we now prove $r_1 \geqslant 3$. Otherwise $6a_0 = 2s$, that is to say $s = 3a_0$, but a_0 is a solution of $3z^2 - 2sz - 1 = 0$. This is only possible if $s^2 + 3 = 0$. So that $x(t) = \gamma t + O(t^3)$ as required, where γ is a solution of $3z^2 - 2sz - 1 = 0$. Then $f_s(x(t), y(t)) = \gamma(\gamma^2 - s\gamma - 1)t + O(t^3)$. □

Lemma 9 (compare to [5], Lemma 3.3)**.** *Let $0 < \delta < 1$ and $C > 0$. On the set:*

$$\{p = (x, y) \mid \exists p_0 = (x_0, y_0) \in U, f_s(p) = f_s(p_0), |y - y_0| \leqslant C|y_0|^{-1+\delta}\},$$

if we denote $y(t) = \frac{1}{t}$, then

$$x(t) = O(t) \tag{2}$$

and

$$f_s(x(t), y(t)) = \gamma(\gamma^2 - s\gamma - 1)t + O(t^{2-\delta}). \tag{3}$$

Proof. We denote $y(t) = \frac{1}{t}$ and $y(t_0) = \frac{1}{t_0}$. As $|y - y_0| \leqslant C|y_0|^{-1+\delta}$, we have $|\frac{1}{t} - \frac{1}{t_0}| \leqslant C|t_0|^{1-\delta}$ hence $|t_0/t - 1| \leqslant C|t_0|^{2-\delta}$ hence $t_0/t \to 1$, i.e $t \sim t_0$. Then $|t_0/t - 1| \leqslant C'|t|^{2-\delta}$ so that $t_0 = t + O(t^{2-\delta})$.

Now by hypothesis and by lemma 8,

$$f_s(x(t), y(t)) = f_s(x(t_0), y(t_0)) = \gamma(\gamma^2 - s\gamma - 1)t_0 + O(t_0^3) = \gamma(\gamma^2 - s\gamma - 1)t + O(t^{2-\delta}).$$

So that

$$f_s(x(t), y(t)) = x(t)(x(t)^2 y(t)^2 - sx(t)y(t) - 1) = \gamma(\gamma^2 - s\gamma - 1)t + O(t^{2-\delta}).$$

We start over the computations of lemma 8. Set $x(t) = a_0 t^{r_0} + a_1 t^{r_1} + \cdots$ and $y(t) = 1/t$. Then

$$\frac{x(t)}{t}\left(\frac{x(t)^2}{t^2} - s\frac{x(t)}{t} - 1\right) = \gamma(\gamma^2 - s\gamma - 1) + O(t^{1-\delta}) \tag{4}$$

We cannot have $r_0 > 1$ since we would have $\frac{x(t)}{t} \to 0$ (as $t \to 0$) and the left-hand side of equation (4) would also tends to 0. We cannot either have $r_0 < 1$, since we would have $\left|\frac{x(t)}{t}\right| \to +\infty$ and the left-hand side of equation (4) would also tends to infinity. Then $r_0 = 1$ and $a_0(a_0^2 - sa_0 - 1) = \gamma(\gamma^2 - s\gamma - 1)$, so that $x(t) = O(t)$.

□

Lemma 10 (compare to [5], Corollary 3.4)**.** *Let $Y = Y(\delta, K, M, A) = \{p \mid \psi(p, M\|p\|^{-1+\delta}, K) \geqslant A\}$ where $0 < \delta < 1$, $M > 0$ and A, K are sufficiently large constants. Then the formulas (2) and (3) holds for $(x(t), y(t)) \in Y$ with $y(t) = \frac{1}{t}$.*

Proof. The proof is the same as in [5]: for $p_0 = (x_0, y_0) \in Y$ there exists $p = (x, y) \in U$ such that

$$\|p - p_0\| \leqslant KM\|p_0\|^{-1+\delta}.$$

As $\frac{1}{2}|y_0| \leqslant \|p_0\| \leqslant 2|y_0|$ (since $x_0 \leqslant y_0$), it implies $|y - y_0| \leqslant C|y_0|^{-1+\delta}$ and lemma 9 applies. □

Lemma 11 (compare to [5], Proposition 3.5). *Let $Y = Y(\delta, K, M, A)$, where $0 < \delta < 1$, $M > 0$ and A, K are sufficiently large constants. Suppose that p_1 and p_2 are in Y and there exists a $0 < \delta_1 < 1$ such that $\|p_1 - p_2\| \leqslant \|p_1\|^{-1+\delta_1}$. Then for $\max\{\delta, \delta_1\} < \delta_2 < 1$ and in a sufficiently small neighborhood of the point at infinity $(0 : 1 : 0)$:*

$$\left|\frac{f_s(p_1)}{f_s(p_2)} - a\right| \leqslant \|p_1\|^{-1+\delta_2},$$

with

$$a \in \left\{1, \frac{\alpha_s(\alpha_s^2 - s\alpha_s - 1)}{\beta_s(\beta_s^2 - s\beta_s - 1)}, \frac{\beta_s(\beta_s^2 - s\beta_s - 1)}{\alpha_s(\alpha_s^2 - s\alpha_s - 1)}\right\}.$$

Proof. Let $p_1 = (x_1(t), y_1(t))$ and $p_2 = (x_2(t'), y_2(t'))$ be two points in Y. Then by lemma 10

$$f_s(x_1(t), y_1(t)) = \gamma(\gamma^2 - s\gamma - 1)t + O(t^{2-\delta}),$$
$$f_s(x_2(t'), y_2(t')) = \gamma'(\gamma'^2 - s\gamma' - 1)t' + O(t'^{2-\delta}),$$

where γ and γ' are in $\{\alpha_s, \beta_s\}$.

Now as $\|p_1 - p_2\| \leqslant \|p_1\|^{-1+\delta_1}$ it implies $|y_1 - y_2| \leqslant 2|y_1|^{-1+\delta_1}$, as in the proof of lemma 9 we get $t' = t + O(t^{2-\delta_1})$. Whence

$$f_s(x_2(t'), y_2(t')) = \gamma'(\gamma'^2 - s\gamma' - 1)t + O(t^{2-\delta_1}) + O(t^{2-\delta}).$$

Then

$$\frac{f_s(p_1)}{f_s(p_2)} = \frac{\gamma(\gamma^2 - s\gamma - 1)}{\gamma'(\gamma'^2 - s\gamma' - 1)} + O(t^{1-\delta_1}) + O(t^{1-\delta}).$$

Then for $\delta_2 > \max\{\delta, \delta_1\}$ with $\delta_2 < 1$ and in neighborhood of the point at infinity $(0 : 1 : 0)$ we get:

$$\left|\frac{f_s(p_1)}{f_s(p_2)} - a\right| \leqslant \frac{1}{2}|t|^{1-\delta_2} \leqslant \|p_1\|^{-1+\delta_2},$$

where $a = \gamma/\gamma'$.

□

Lemma 12 (compare to [5], Lemma 3.6). *Let K and A sufficiently large and $0 < \delta < 1$. Fix s with $s^2 + 3 \neq 0$. Then $Y = Y(\delta, K, M, A)$ is nonempty and contains the polar curve Γ_s. Moreover all the limits of $f_s(p_1)/f_s(p_2)$ given in lemma 11 can be obtained by taking p_1 and p_2 along the branches of Γ_s associated to the point at infinity $(0 : 1 : 0)$.*

Proof. Fix δ and K. Let $\pi_c : (f_s = c) \to \mathbb{C}$ be the projection $(x, y) \mapsto y$. It is a triple covering branched at the points $\Gamma_s \cap (f_s = c)$. These points are of coordinates

$$(\alpha_s t, \frac{1}{t}) \quad \text{and} \quad (\beta_s t', \frac{1}{t'}) \quad \text{with} \quad f_s(\alpha_s t, \frac{1}{t}) = f_s(\beta_s t', \frac{1}{t'}) = c.$$

As $f_s(\alpha_s t, \frac{1}{t}) = \alpha_s t(\alpha_s^2 - s\alpha_s - 1)$ it implies

$$t = \frac{c}{\alpha_s(\alpha_s^2 - s\alpha_s - 1)} \quad \text{and similarly} \quad t' = \frac{c}{\beta_s(\beta_s^2 - s\beta_s - 1)}.$$

For $s^2 + 3 \neq 0$, $\alpha_s \neq \beta_s$ and it also implies $t \neq t'$ hence $|y(t) - y(t')|$ is of order $y(t)$, that is to say two points of ramifications are far enough. Let $p_0 = (x_0, y_0)$ be a point of ramification of π_c. Let $\mathcal{V} = \{y \mid |y-y_0| \leqslant \epsilon |y_0|\}$, with ϵ sufficiently small such that no other ramification point projects in $\mathcal{V}$. For a sufficiently large p_0 (i.e. small c), $X(p_0, KM\|p_0\|^{-1+\delta}) \subset \pi_c^{-1}(\mathcal{V})$. Now let $p = (x, y)$ such that:

$$|y - y_0| \leqslant |y_0|^{-1+\delta},$$

then by lemma 9, $x = O(\frac{1}{y})$ whence

$$\|p - p_0\| \leqslant 2\|p_0\|^{-1+\delta}.$$

Let $\mathcal{V}_\delta = \{y \mid |y-y_0| \leqslant \epsilon |y_0|^{-1+\delta}\}$, by the above inequality we get $\pi_c^{-1}(\mathcal{V}_\delta) \subset X(p_0, KM\|p_0\|^{-1+\delta})$. We restrict the triple branched covering π_c to a map $\tilde{\pi}_c$ from $\pi_c^{-1}(\mathcal{V}_\delta)$ composed by only two components of the triple cover. Let $y \in \mathcal{V}_\delta$ such that $|y - y_0| = \frac{1}{2}|y_0|^{-1+\delta}$. Let $p_1 = (x_1, y)$, $p_2 = (x_2, y)$ be the two points of $\tilde{\pi}_c^{-1}(y)$. These two points are in $\tilde{\pi}_c^{-1}(\mathcal{V}_\delta)$ which is a connected set. Any curve γ in $\tilde{\pi}_c^{-1}(\mathcal{V}_\delta)$ from p_1 to p_2 passes through p_0, hence the projection of γ by $\tilde{\pi}_c$ passes through y_0. Hence the inner distance (in $(f_s = c)$) of p_1 and p_2 is greater or equal than $2|y - y_0|$, it yields:

$$\mathrm{dist}_{p_0, M\|p_0\|^{-1+\delta}, K}(p_1, p_2) \geqslant 2|y - y_0| = |y_0|^{-1+\delta} = |t|^{1-\delta},$$

where we denote $y_0 = \frac{1}{t}$. By lemma 9 we have $x_1 = O(t)$ and $x_2 = O(t)$, so that

$$\|p_1 - p_2\| \leqslant C|t|.$$

Then

$$\frac{\mathrm{dist}_{p_0, M\|p_0\|^{-1+\delta}, K}(p_1, p_2)}{\|p_1 - p_2\|} \geqslant \frac{1}{C|t|^\delta} \xrightarrow[t \to 0]{} +\infty.$$

Then $\psi(p_0, M\|p_0\|^{-1+\delta}, K) \to +\infty$, as p_0 tends to the point at infinity $(0 : 1 : 0)$. It means that the branch of Γ_s near this point at infinity is included in $Y(\delta, K, M, A)$.

Finally we have already proved in subsection 3.1 that the list of values $f_s(p_1)/f_s(p_2)$ on Γ_s is the required one. □

We conclude by the proof of the theorem in the complex case.

Proof of theorem 1'. Fix s. By lemma 7 the set Y for f_s is sent into a set $\tilde{Y}$ for $f_{s'}$. The polar curve Γ_s is included in Y (lemma 12) and on this polar curve $f_s(p_1)/f_s(p_2)$ tends to a $\frac{\alpha_s(\alpha_s^2 - s\alpha_s - 1)}{\beta_s(\beta_s^2 - s\beta_s - 1)}$ for instance (lemma 11). On the one hand $f_{s'}(\tilde{p}_1)/f_{s'}(\tilde{p}_2)$ tends to the same value, because the bilipschitz homeomorphism Φ sends the levels of f_s to the levels of $f_{s'}$. On the other hand $\tilde{p}_1$, $\tilde{p}_2$ are in $\tilde{Y}$ so that $f_{s'}(\tilde{p}_1)/f_{s'}(\tilde{p}_2)$ is in

$$\left\{1, \frac{\alpha_{s'}(\alpha_{s'}^2 - s'\alpha_{s'} - 1)}{\beta_{s'}(\beta_{s'}^2 - s'\beta_{s'} - 1)}, \frac{\beta_{s'}(\beta_{s'}^2 - s'\beta_{s'} - 1)}{\alpha_{s'}(\alpha_{s'}^2 - s'\alpha_{s'} - 1)}\right\}.$$

This is only possible for a finite set of values s'. □

5. Topological equivalence

To complete the complex part of theorem 1 we prove the topological equivalence of any two polynomials.

Lemma 13. *Consider the following family of polynomials in* $\mathbb{C}[x, y]$*:*

$$f_s(x, y) = x(x^2y^2 - sxy - 1)$$

with $s^2 + 4 \neq 0$*. For any* s *and* s' *the polynomials* f_s *and* $f_{s'}$ *are topologically equivalent.*

This family is similar to examples in [1] of polynomials that are topologically equivalent but not algebraically equivalent. Recall that two polynomials $f, g : \mathbb{K}^n \to \mathbb{K}$ are *topologically equivalent* if there exist a homeomorphism $\Phi : \mathbb{K}^n \to \mathbb{K}^n$ and a homeomorphism $\Psi : \mathbb{K} \to \mathbb{K}$ such that $g \circ \Phi = \Psi \circ f$. We will use the following result that is global version of Lê-Ramanujam μ-constant theorem. See [2] for the two variables case and [3] for any number of variables.

Theorem 14. *Let $\{f_s\}_{s\in[0,1]}$ be a continuous family of complex polynomials with isolated singularities (in the affine space and at infinity), with $n \neq 3$ variables. Suppose that the following integers are constant w.r.t. the value of s:*

- $\deg f_s$*, the degree,*
- $\#\mathcal{B}_s$*, the number of irregular values,*
- $\chi(f_s = c_{gen})$*, the Euler characteristic of a generic fiber.*

Then f_0 and f_1 are topologically equivalent.
Proof of lemma 13.

- **Degree.** It is clear that the degree of the f_s is independent of s.
- **Affine singularities.** We search for points (x, y) where both derivatives vanish. $\partial_x f_s(x, y) = 3x^2y^2 - 2sxy - 1$ and $\partial_y f_s(x, y) = x^2(2xy - s)$. If $x = 0$ then $\partial_x f_s(x, y) \neq 0$. So that $\partial_y f_s(x, y) = 0$ implies $2xy - s = 0$. We plug $xy = s/2$ in $\partial_x f_s(x, y) = 0$ and get $s^2 + 4 = 0$. Notice that $s^2 + 4 = 0$ gives also the values where f_s is not a reduced polynomial. Conclusion: for $s^2 + 4 \neq 0$, the polynomials f_s has no affine singularities (nor affine critical values), so that its global affine Milnor number is $\mu_s = 0$.
- **Singularities at infinity.** The two points at infinity for this family are $P_1 = (0 : 1 : 0)$ and $P_2 = (1 : 0 : 0)$. Let $F_s(x, y, z) = x(x^2y^2 - sxyz^2 - z^4) - cz^5$ be the homogenization of $f_s(x, y) - c$.
  - **Milnor number at P_1.** We localize F_s at $P_1 = (0 : 1 : 0)$ to get $g_s(x, z) = F_s(x, 1, z) = x(x^2 - sxz^2 - z^4) - cz^5$. We compute the local Milnor of g_s at $(0, 0)$. For instance we may use the Newton polygon of g_s and Kouchnirenko formula. We get, for any s (with $s^2 + 4 \neq 0$) and depending on c:

    $$\mu(g_s) = 8 \quad \text{if } c \neq 0 \qquad \text{and} \qquad \mu(g_s) = 10 \quad \text{if } c = 0.$$

    Hence the value 0 is an irregular value at infinity and the jump of Milnor number is $\lambda_{P_1} = 10 - 8 = 2$.
  - **Milnor number at P_2.** At $P_2 = (1 : 0 : 0)$ we get $h_s(y, z) = F_s(1, y, z) = y^2 - syz^2 - z^4 - cz^5$. The local Milnor number of h_s at $(0, 0)$ is independent of s and c:

    $$\mu(h_s) = 3.$$

    So that there is no irregular values at infinity for this point and $\lambda_{P_2} = 0$.
  - Then the Milnor number at infinity is $\lambda_s = \lambda_{P_1} + \lambda_{P_2} = 2$ and the only irregular value at infinity is 0.
- **Conclusion.** For all s the only irregular value is 0: $\mathcal{B}_s = \{0\}$, the Euler characteristic of a generic fiber given by $\chi_s = 1 - \mu_s - \lambda_s = -1$ is also constant. Then by theorem 14 any f_s and $f_{s'}$ are topologically equivalent.

□

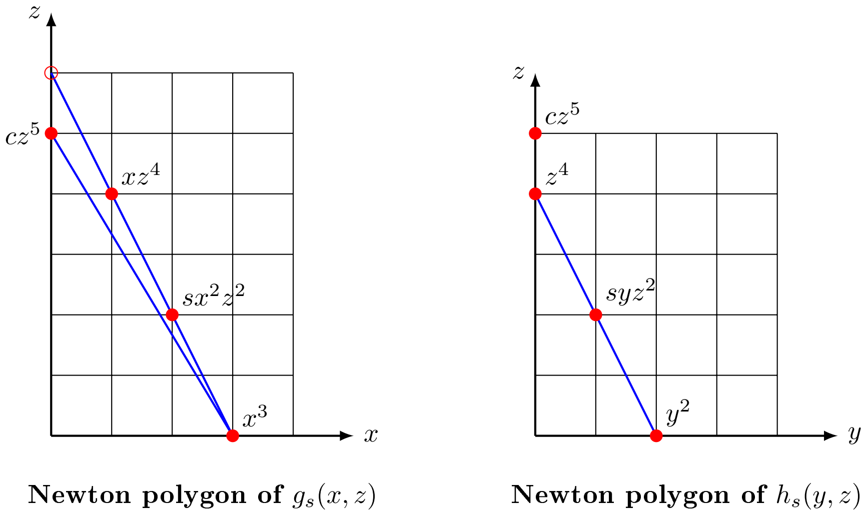

FIGURE 6. Computation of Milnor number at infinity.

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