

GENERATING SERIES FOR IRREDUCIBLE POLYNOMIALS OVER FINITE FIELDS

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ABSTRACT. We count the number of irreducible polynomials in several variables of a given degree over a finite field. The results are expressed in terms of a generating series, an exact formula and an asymptotic approximation. We also consider the case of the multi-degree and the case of indecomposable polynomials.

1. INTRODUCTION

A well-known formula that goes back to Gauss estimate the number I_n of monic irreducible polynomials among all the N_n monic polynomials of degree n in $\mathbb{F}_q[x]$:

$$\frac{I_n}{N_n} \sim \frac{1}{n}.$$

A proof of this fact from the combinatorial point of view has been formalized by Ph. Flajolet, X. Gourdon and D. Panario in [6]. They apply this formalism to count irreducible polynomials in one variable over \mathbb{F}_q with the help of generating series. These generating series are convergent. The goal of this paper is to count irreducible polynomials in several variables. While the formalism is the same, the series are now formal power series. At a first glance it seems a major obstruction, but despite that the series are non-convergent lots of manipulations are possible: computation of the terms of a generating series, approximation of this term, Möbius inversion formula,... For comments and examples in this vein we refer to the book of Ph. Flajolet and R. Sedgewick [7] (and especially Examples I.19, I.20, II.15, Note II.28 and Appendix A).

We start by extending the introduction and recall in Section 2 the formalism introduced in [6], we consider in Section 3 the case of irreducible polynomials in several variables of a given degree. Then in Section 4 the irreducible polynomials are counted with respect to the multi-degree (or vector-degree). Finally in Section 5 we deal with indecomposable polynomials. In each case we give an exact formula, a generating series, an asymptotic development at first order and end by

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an example. References to earlier works are given for each cases after the main statements.

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2. FORMALISM

Let us recall the formalism of [6]. Let \mathcal{I} be a family of primitive combinatorial objects and $\omega \in \mathcal{I}$, then $\frac{1}{1-\omega} = 1 + \omega + \omega^2 + \dots$ formally represents arbitrary sequences composed of ω . The product $\mathcal{N} = \prod_{\omega \in \mathcal{I}} \frac{1}{1-\omega}$ formally represents all multisets (sets with possible repetitions) composed of elements of \mathcal{I} .

In our situation \mathcal{I} will be the set of all monic irreducible polynomials and \mathcal{N} the set of all monic polynomials. The relation $\mathcal{N} = \prod_{\omega \in \mathcal{I}} \frac{1}{1-\omega}$ expresses that each monic polynomial admits a unique decomposition into monic irreducible polynomials.

In order to count polynomials over \mathbb{F}_q we make the following substitution $\omega \mapsto z^{\deg \omega}$, where z is a new formal variable and $\deg \omega$ is the degree of the polynomial ω . We get the generating series:

$$N(z) = \sum_{\omega \in \mathcal{N}} z^{\deg \omega} = \sum_{n \geq 1} N_n z^n, \quad I(z) = \sum_{\omega \in \mathcal{I}} z^{\deg \omega} = \sum_{n \geq 1} I_n z^n,$$

that count the total number of monic polynomials and monic irreducible polynomials. The decomposition of polynomials into irreducible factors gives the relation:

$$1 + N(z) = \prod_{\omega \in \mathcal{I}} \frac{1}{1 - z^{\deg \omega}} = \prod_{n \geq 1} \frac{1}{(1 - z^n)^{I_n}}.$$

By taking the logarithm we get:

$$\log(1 + N(z)) = \sum_{n \geq 1} -I_n \log(1 - z^n) = \sum_{n \geq 1} \frac{I(z^n)}{n}.$$

For one variable polynomials we know that the number of monic polynomials is $N_n = q^n$ so that $1 + N(z) = 1 + \sum_{n \geq 1} q^n z^n = \frac{1}{1 - qz}$. Hence

$$(1) \quad \log \frac{1}{1 - qz} = \sum_{n \geq 1} \frac{I(z^n)}{n},$$

by applying Möbius inversion formula to the (convergent) series we get

$$I(z) = \sum_{n \geq 1} \frac{\mu(n)}{n} \log \frac{1}{1 - qz}.$$

If we prefer, we can only consider the coefficients of the terms z^n in Equation (1) to find

$$\frac{q^n}{n} = \sum_{k|n} \frac{I_{n/k}}{k} \quad \text{and} \quad I_n = \frac{1}{n} \sum_{k|n} \mu(k) q^{n/k}$$

by applying the classical Möbius inversion formula (μ is Möbius function). From this last expression, we deduce that $I_n = \frac{q^n}{n} + O\left(\frac{q^{n/2}}{n}\right)$, that implies the first statement of the introduction.

3. IRREDUCIBLE POLYNOMIALS: DEGREE

Let

$$b_n = \binom{\nu + n}{\nu} \quad \text{and} \quad N_n = \frac{q^{b_n} - q^{b_{n-1}}}{q - 1}.$$

Then N_n is the number of irreducible, monic, polynomials of degree (exactly) n in $\mathbb{F}_q[x_1, \dots, x_\nu]$. The corresponding generating series is

$$N(z) = \sum_{n \geq 1} N_n z^n.$$

Let I_n be the number irreducible monic polynomials of degree n in $\mathbb{F}_q[x_1, \dots, x_\nu]$, we define the corresponding generating series:

$$I(z) = \sum_{n \geq 1} I_n z^n.$$

Theorem 1.

- (1) $I(z) = \sum_{n \geq 1} \frac{\mu(n)}{n} \log(1 + N(z^n))$,
- (2) $I_n = \sum_{k|n} \frac{\mu(k)}{k} [z^{\frac{n}{k}}] \log(1 + N(z))$,
- (3) $I_n = N_n - N_1 \cdot N_{n-1} + O(q^{b_{n-2} + b_2 - 2})$.

In item (3), ν and q are fixed while the error term O is for values of n that tend to infinity. The history for item (3) starts with L. Carlitz [3] who proved that almost all polynomials in several variables are irreducible. His work has been extended to the case of the bi-degree in [4] (as considered in Section 4) and by S. Cohen [5] for more variables. Here we get that most of polynomials are irreducible and an estimation of the error term. A similar formula has been obtained in [1]: $1 - \frac{I_n}{N_n} \sim \frac{N_1 \cdot N_{n-1}}{N_n}$ as n grows to infinity. A formula with explicit constants is given in [8] for polynomials in two variables $\left|1 - \frac{I_n}{N_n} - q^{-n+1}\right| \leq 2q^{-n}$, with $n \geq 6$. An asymptotic formula of higher order is proved in [11]: $I_n = N_n + \alpha_1 N_{n-1} + \alpha_2 N_{n-2} + \dots + \alpha_s N_{n-s} + O(N_{n-s-1})$, where α_i are constants and the order s is arbitrary large.

Our proof of (3) appears to be simple compare with these references. Finally, in the forthcoming [10], the formula $I(z)$ is also obtained and applied to get an approximation: $I_n = N_n - q^{b_n + \nu - 1} (1 + 2/q + O(1/q^2))$, where ν and $n \geq 5$ are fixed, while q grows to infinity.

The end of this section is devoted to the proof of Theorem 1.

Proof.

Generating series. We follow the formalism of the introduction:

$$1 + N(z) = \prod_{n \geq 1} \frac{1}{(1 - z^n)^{I_n}}.$$

Hence:

$$(2) \quad L(z) := \log(1 + N(z)) = \sum_{n \geq 1} -I_n \log(1 - z^n) = \sum_{n \geq 1} \frac{I(z^n)}{n}.$$

By Möbius inversion formula applied to the series (see comments after the proof) we get:

$$I(z) = \sum_{n \geq 1} \frac{\mu(n)}{n} L(z^n).$$

Another point of view is to make computations term by term: we take the coefficients of z^n in Equation (2) and get

$$[z^n]L(z) = \sum_{k|n} \frac{I_{n/k}}{k}.$$

Then we apply the usual Möbius inversion formula to $n[z^n]L(z)$ and find

$$(3) \quad I_n = \sum_{k|n} \frac{\mu(k)}{k} [z^{\frac{n}{k}}]L(z).$$

Asymptotic behaviour. We start with the computation of an approximation of $N(z)^2$.

$$\begin{aligned} [z^n]N(z)^2 &= [z^n] (N_1 z + N_2 z^2 + \dots)^2 \\ &= 2N_{n-1} \cdot N_1 + O(q^{b_{n-2}+b_2-2}). \end{aligned}$$

And for powers $k \geq 3$ we have:

$$[z^n]N(z)^k = O(q^{b_{n-2}+b_2-2}).$$

Then we compute the asymptotic behaviour of $[z^n]L(z)$.

$$\begin{aligned} [z^n]L(z) &= [z^n] \log(1 + N(z)) \\ &= [z^n]N(z) - \frac{1}{2}[z^n]N(z)^2 + \dots \\ &= N_n - \frac{1}{2} (2 \cdot N_1 \cdot N_{n-1} + O(q^{b_{n-2}+b_2-2})) + O(q^{b_{n-2}+b_2-2}) \\ &= N_n - N_1 \cdot N_{n-1} + O(q^{b_{n-2}+b_2-2}). \end{aligned}$$

Now we get:

$$\begin{aligned}
 I_n &= \sum_{k|n} \frac{\mu(k)}{k} [z^{\frac{n}{k}}] L(z) \\
 &= \frac{\mu(1)}{1} [z^n] L(z) + \sum_{k|n, k>1} \frac{\mu(k)}{k} [z^{\frac{n}{k}}] L(z) \\
 &= N_n - N_1 \cdot N_{n-1} + O(q^{b_{n-2}+b_2-2}) + O(q^{b_{n-2}}) \\
 &= N_n - N_1 \cdot N_{n-1} + O(q^{b_{n-2}+b_2-2}).
 \end{aligned}$$

□

Example. For example, in two variables ($\nu = 2$), we can compute the exact expression of I_{100} . We compute the first 100 coefficients of the expansion of $L(z) = \log(1 + N(z))$ into powers of z (by derivation an expansion of $L(z)$ can be obtained from the inverse of $1 + N(z)$). The exact value of I_{100} is then computed from formula (2) of Theorem 1:

$$I_{100} = \frac{1}{q-1} (q^{5151} - q^{5052} - q^{5051} - q^{5050} - q^{4955} + q^{4953} + 2q^{4952} + \dots)$$

which has a total of 4385 monomials! When we specialize the approximation of Theorem 1 to $n = 100$ we find that I_{100} is about $\frac{1}{q-1} (q^{5151} - q^{5052} - q^{5051} - q^{5050})$ with an error term of magnitude q^{4954} . Finally if we set $q = 2$ in the exact expression, we get

$$I_{100} = 4031880625288 \dots 8282220076800,$$

a number with 1551 digits.

Inversion formula. We shall now sketch the proof of inversion formula for formal power series. Suppose that $f, g \in K[[z]]$ with $f(0) = 0$ and

$$g(z) = \sum_{k \geq 1} f(z^k) = f(z) + f(z^2) + f(z^3) + \dots$$

then

$$f(z) = \sum_{n \geq 1} \mu(n) g(z^n) = \mu(1)g(z) + \mu(2)g(z^2) + \mu(3)g(z^3) + \dots$$

where $\mu(n)$ is the ordinary Möbius function. The proof is in fact the same as the classic one:

$$\sum_{n \geq 1} \mu(n) g(z^n) = \sum_{n \geq 1} \sum_{k \geq 1} \mu(n) f(z^{k \cdot n}).$$

In this expression the coefficient of $f(z^i)$ involves only a finite number of terms and is $\sum_{j|i} \mu(j) = 1$ so that the former sum becomes:

$$\sum_{i \geq 1} f(z^i) \sum_{j|i} \mu(j) = \sum_{i \geq 1} f(z^i).$$

Further readings for comments on computations with formal power series can be found in [7, Appendix A]. For convergent series, see [13, p. 98] and for a general version of inversion theorem, see [14].

4. IRREDUCIBLE POLYNOMIALS: MULTI-DEGREE

Let $N_{\underline{n}}$ be the number of irreducible, monic, polynomials of multi-degree $\underline{n} = (n_1, \dots, n_\nu)$ in $\mathbb{F}_q[x_1, \dots, x_\nu]$. By [5] or [11] we have

$$N_{\underline{n}} = \frac{1}{q-1} \sum_{(\delta_1, \dots, \delta_\nu) \in \{0,1\}^\nu} (-1)^{\nu + \delta_1 + \dots + \delta_\nu} q^{(n_1 + \delta_1) \dots (n_\nu + \delta_\nu)}.$$

The corresponding generating series is

$$N(z_1, \dots, z_\nu) = \sum_{(n_1, \dots, n_\nu) \neq (0, \dots, 0)} N_{n_1, \dots, n_\nu} \cdot z_1^{n_1} \cdots z_\nu^{n_\nu},$$

or for short

$$N(\underline{z}) = \sum_{\underline{n} \neq 0} N_{\underline{n}} \cdot \underline{z}^{\underline{n}}.$$

Let $I_{\underline{n}}$ be the number of irreducible monic polynomials of degree \underline{n} in $\mathbb{F}_q[x_1, \dots, x_\nu]$, we define the corresponding generating series:

$$I(\underline{z}) = \sum_{\underline{n} \neq 0} I_{\underline{n}} \cdot \underline{z}^{\underline{n}}.$$

Theorem 2.

- (1) $I(\underline{z}) = \sum_{k \geq 1} \frac{\mu(k)}{k} \log(1 + N(\underline{z}^k)),$
- (2) $I_{\underline{n}} = \sum_{k | \gcd(\underline{n})} \frac{\mu(k)}{k} [\underline{z}^{\frac{\underline{n}}{k}}] \log(1 + N(\underline{z})),$
- (3) $I_{\underline{n}} = N_{\underline{n}} - N_{1,0,\dots,0} \cdot N_{n_1-1,n_2,\dots,n_\nu} + O(N_{0,1,0,\dots,0} \cdot N_{n_1,n_2-1,n_3,\dots,n_\nu}) + O(N_{2,0,\dots,0} \cdot N_{n_1-2,n_2,\dots,n_\nu})$ if $n_1 \geq n_2 \geq \dots \geq n_\nu$.

For (3) X.-D. Hou and G. Mullen in [11] have already obtained a similar formula. The scheme of the proof is the same as in the previous section.

Proof.

Generating series. The proof is an application of the formalism of the introduction, (used in a similar way as in Section 3, see also [4] “heuristic proof” of (2.1)).

$$1 + N(\underline{z}) = \prod_{\underline{n} \neq 0} \frac{1}{(1 - \underline{z}^{\underline{n}})^{I_{\underline{n}}}}.$$

Hence:

$$(4) \quad L(\underline{z}) := \log(1 + N(\underline{z})) = \sum_{\underline{n} \neq 0} -I_{\underline{n}} \log(1 - \underline{z}^{\underline{n}}) = \sum_{k \geq 1} \frac{I(\underline{z}^k)}{k},$$

where $\underline{z}^k = (z_1^k, \dots, z_\nu^k)$. Now take the coefficient of \underline{z}^n in Equation (4), we get

$$[\underline{z}^n]L(\underline{z}) = \sum_{k|\gcd(\underline{n})} \frac{I_{\underline{n}/k}}{k}.$$

By Möbius formula applied to $\gcd(\underline{n})[\underline{z}^n]L(\underline{z})$ we get

$$I_{\underline{n}} = \sum_{k|\gcd(\underline{n})} \frac{\mu(k)}{k} [\underline{z}^{\frac{\underline{n}}{k}}]L(\underline{z}).$$

We can also directly apply Möbius inversion formula to $L(\underline{z})$:

$$I(\underline{z}) = \sum_{k \geq 1} \frac{\mu(k)}{k} L(\underline{z}^k).$$

Asymptotic behaviour. We suppose $n_1 \geq n_2 \geq \dots \geq n_\nu$. We have:

$$\begin{aligned} I_{\underline{n}} &= \sum_{k|\gcd(\underline{n})} \frac{\mu(k)}{k} [\underline{z}^{\frac{\underline{n}}{k}}]L(\underline{z}) \\ &= \frac{\mu(1)}{1} [\underline{z}^{\underline{n}}]L(\underline{z}) + \sum_{k|\gcd(\underline{n}), k>1} \frac{\mu(k)}{k} [\underline{z}^{\frac{\underline{n}}{k}}]L(\underline{z}) \\ &= N_{\underline{n}} - \frac{1}{2} (2N_{1,0,\dots,0} \cdot N_{n_1-1,n_2,\dots,n_\nu} + 2N_{0,1,0,\dots,0} \cdot N_{n_1,n_2-1,n_3,\dots,n_\nu} + \dots \\ &\quad + 2N_{0,\dots,0,1} \cdot N_{n_1,n_2,\dots,n_{\nu-1},n_\nu-1} + O(N_{1,1,,0,\dots,0} \cdot N_{n_1-1,n_2-1,n_3,\dots,n_\nu})) \\ &\quad + O(N_{2,0,\dots,0} \cdot N_{n_1-2,n_2,\dots,n_\nu}) \\ &= N_{\underline{n}} - N_{1,0,\dots,0} \cdot N_{n_1-1,n_2,\dots,n_\nu} + O(N_{0,1,0,\dots,0} \cdot N_{n_1,n_2-1,n_3,\dots,n_\nu}) + \\ &\quad + O(N_{2,0,\dots,0} \cdot N_{n_1-2,n_2,\dots,n_\nu}). \end{aligned}$$

Intuitively these formulas express that most of reducible polynomials are of type $P(x_1, \dots, x_\nu) = (x_1 + \alpha) \cdot Q(x_1, \dots, x_\nu)$, where $\deg_{x_1} Q = (\deg_{x_1} P) - 1$, α being a constant. \square

Example. For example, we get $I_{11,5} = \frac{1}{q-1} (q^{72} - q^{67} - q^{66} - q^{60} + \dots)$, while $N_{11,5} = \frac{1}{q-1} (q^{72} - q^{66} - q^{60} + q^{55})$. The approximation of Theorem 2 specialized to (11, 5) gives that $I_{11,5}$ is about $\frac{1}{q-1} (q^{72} - q^{67} - q^{66})$ with an error term of magnitude q^{61} . Finally if we fix $q = 2$, we get $I_{11,5} = 4499945769704095481856$ and $N_{11,5} = 4647462613867219124224$.

5. INDECOMPOSABLE POLYNOMIALS

Definition and known facts. A polynomial P in $\mathbb{F}_q[x_1, \dots, x_\nu]$ is *decomposable* if there exist $h \in \mathbb{F}_q[t]$ with $\deg h \geq 2$ and $Q \in \mathbb{F}_q[x_1, \dots, x_\nu]$ such that

$$P(x_1, \dots, x_\nu) = h \circ Q(x_1, \dots, x_\nu).$$

Otherwise P is said to be *indecomposable*. For example $P(x, y) = x^3y^3 + 2xy + 1$ is decomposable (with $h(t) = t^3 + 2t + 1$ and $Q(x, y) = xy$). Notice that if $P - c$ is irreducible in $\mathbb{F}_q[x_1, \dots, x_\nu]$ for some value $c \in \mathbb{F}_q$ then P is indecomposable.

Indecomposable polynomials have a special interest in regards of Stein's theorem and provide families of irreducible polynomials (see [15], [12], [2]):

Theorem (Stein's theorem). *If $P \in \mathbb{F}_q[x_1, \dots, x_\nu]$ is indecomposable then for all but finitely many values $c \in \overline{\mathbb{F}_q}$ (the algebraic closure of \mathbb{F}_q), $P - c$ is irreducible in $\overline{\mathbb{F}_q}[x_1, \dots, x_\nu]$. Moreover the number of values c such that $P - c$ is reducible is strictly less than $\deg P$.*

Let J_n be the number of indecomposable polynomials of degree (exactly) n in $\mathbb{F}_q[x_1, \dots, x_\nu]$, $\nu \geq 2$. Let \bar{N}_n be the number of polynomials of degree n (not necessarily monic) in $\mathbb{F}_q[x_1, \dots, x_\nu]$:

$$\bar{N}_n = q^{b_n} - q^{b_{n-1}}.$$

We define the following Dirichlet series:

$$N(s) = \sum_{n \geq 1} \frac{\bar{N}_n}{n^s}, \quad J(s) = \sum_{n \geq 1} \frac{J_n}{n^s}, \quad F(s) = \sum_{n \geq 1} \frac{q^{n-1}}{n^s}.$$

We also define a kind of Möbius function $\mu(d, n)$:

$$\mu(d, n) = \sum_{\substack{d=d_1 d_2 \dots d_k \\ d_1 < d_2 < \dots < d_k < n}} (-1)^k q^{\frac{d_2}{d_1} + \dots + \frac{d_{k+1}}{d_k} - k},$$

where the sum is over all the iterated divisors of n (k not being fixed). We define $\mu(n, n) = 1$. For a fixed n , we define $1 < \ell < \ell'$ to be the first two divisors of n .

Theorem 3.

- (1) $J(s) = \frac{N(s)}{F(s)}$,
- (2) $J_n = \sum_{d|n, 1 \leq d \leq n} \mu(d, n) \bar{N}_d$,
- (3) *If n is the product of at least three prime numbers:*

$$J_n = \bar{N}_n - q^{\ell-1} \bar{N}_{\frac{n}{\ell}} + O\left(q^{\ell+\ell'-2} \bar{N}_{\frac{n}{\ell'}}\right).$$

The item (3) has already been obtained in [2] and [9], where some explicit constants are computed. For example in two variables, and when n is the product of at least three prime numbers, Theorem 5.1 of [2] yields the same first order estimate of J_n : $\left| \bar{N}_n - J_n - q^{\ell-1} \bar{N}_{\frac{n}{\ell}} \frac{1-q^{-n-1}}{1-q^{-n/\ell-1}} \right| \leq n \frac{q^{\ell-1}}{q^{n/\ell}} \bar{N}_{\frac{n}{\ell}} \frac{1-q^{-n-1}}{1-q^{-n/\ell-1}}$.

Proof.

Generating series and inversion formula. The decomposition of a decomposable polynomial is unique once it has been normalized, see [2, Section 5]:

Lemma 4. *Let $P = h \circ Q$ be a decomposition with $\deg h \geq 2$, Q indecomposable, monic and its constant term equals zero. Then such a decomposition is unique.*

By induction it implies the following lemma (see [2]):

Lemma 5.

$$J_n = \bar{N}_n - \sum_{d|n, 1 \leq d < n} q^{\frac{n}{d}-1} \times J_d.$$

From Lemma 5 we get:

$$\bar{N}_n = \sum_{d|n, 1 \leq d \leq n} q^{\frac{n}{d}-1} \times J_d.$$

From the point of view of Dirichlet series it yields:

$$N(s) = J(s) \cdot F(s).$$

Then by induction we also deduce from Lemma 5 that

$$J_n = \sum_{d|n, 1 \leq d \leq n} \mu(d, n) \bar{N}_d.$$

Asymptotic behaviour. From this formula we deduce:

$$\begin{aligned} J_n &= \mu(n, n) \bar{N}_n + \mu(n/\ell, n) \bar{N}_{\frac{n}{\ell}} + O\left(\mu(n/\ell', n) \bar{N}_{\frac{n}{\ell'}}\right) \\ &= \bar{N}_n - q^{\ell-1} \bar{N}_{\frac{n}{\ell}} + O\left((q^{\ell+\ell'-2} - q^{\ell-1}) \bar{N}_{\frac{n}{\ell'}}\right) \\ &= N_n - q^{\ell-1} \bar{N}_{\frac{n}{\ell}} + O\left(q^{\ell+\ell'-2} \bar{N}_{\frac{n}{\ell'}}\right). \end{aligned}$$

□

Example. For example, in two variables, we have $\bar{N}_{100} = q^{5151} - q^{5050}$ and we compute J_{100} by formula (2) of Theorem 3. We get $J_{100} = q^{5151} - q^{5050} - q^{1327} + q^{1276} - q^{354} + \dots$. While the approximation given in Theorem 3 is that J_{100} is about $q^{5151} - q^{5050} - q^{1327} + q^{1276}$ with an error term of magnitude q^{355} .

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