

REALIZATION OF INTERMEDIATE LINKS OF LINE ARRANGEMENTS

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ABSTRACT. We investigate several topological and combinatorial properties of line arrangements. We associate to a line arrangement a link $\mathcal{A} \cap S_r^3(0)$ obtained by intersecting the arrangement with some sphere. Several topics are discussed: (a) some link configurations can be realized by complex line arrangements but not by real line arrangements; (b) if we intersect the arrangements with a vertical band instead of a sphere, what link configurations can be obtained? (c) relations between link configurations obtained by bands and spheres.

INTRODUCTION

The topic of this paper is the study of intermediate links. To an algebraic curve $(f(x, y) = 0)$ passing through the origin we classically associate a link $(f = 0) \cap S_\epsilon^3(0)$, which is independent of ϵ for all sufficiently small $\epsilon > 0$ (see Milnor [6]). Another well-known and studied situation are *links at infinity* when we consider the intersection with a sphere $S_R^3(0)$ of radius $R \gg 1$ sufficiently large. An *intermediate link* is the intersection $(f = 0) \cap S_r^3(0)$, with an arbitrary $r > 0$. There is no much literature on that subject initiated by L. Rudolph (see the surveys [9], [1] and also [8], [2], [3]).

We will extend and compare several concepts of intermediate links in the case of line arrangements.

- We compare the configurations obtained by intersecting a complex line arrangement with a ball of \mathbb{C}^2 and the configurations obtained by intersecting a real line arrangement with a ball of \mathbb{R}^2 .
- We compare the configurations obtained by intersecting a real line arrangement with a ball of \mathbb{R}^2 and the configurations obtained by intersecting with a band of type $[-r, +r] \times \mathbb{R}$ in \mathbb{R}^2 .

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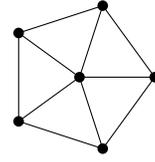
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To be more precise we define the intersection graph of an arrangement in a set. Let \mathcal{A} be a real or complex line arrangement and let B be a set (which will either be a ball $B_r^4(0)$ in \mathbb{C}^2 ; a disk $D_r^2(0)$ in \mathbb{R}^2 ; or a band $[-r, +r] \times \mathbb{R}$ in \mathbb{R}^2). The *intersection graph* of \mathcal{A} in B is the graph defined by:

- one vertex associated to one line;
- one edge connect two vertices if the corresponding lines have their intersection inside B .

Theorem 1.

This graph can be realized as the intersection graph of some complex line arrangement with a ball, but cannot be realized as the intersection graph of a real line arrangement with a disk.



Fix a realizable graph G : we can realize it by a real line arrangement \mathcal{A} inside a real disk $D_1^2(0)$ of radius fixed to 1. We define the *maximal radius* R_{\max} to be the maximal $r \geq 1$ such that there is no intersection point in $D_r^2(0) \setminus D_1^2(0)$. In other words the intersection points not in $D_1^2(0)$ are as far as possible.

We also defined a maximal radius for bands (instead of disks). The second and third parts are devoted to a numerical algorithm to compute this maximal number and –among other things– prove the following results:

Theorem 2.

- *The maximal radius for the band problem is an algebraic number;*
- *The maximal radius for the band problem is less or equal than the maximal radius for the disk problem.*

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Part 1. Real and complex intermediate links of arrangements

Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . An *arrangement* in \mathbb{K}^n is a finite collection of lines $\mathcal{A} = \{L_i\}$. In this part we intersect a real or complex line arrangement with a sphere $S_r^3(0)$ of arbitrary radius r . We will detail a configuration feasible with complex lines but not feasible with real lines.

1. THE PROBLEMS

To a line arrangement \mathcal{A} and a radius r we associate its link $\mathcal{A} \cap S_r^3(0)$. We can also associate an *intersection graph* $G_{\mathcal{A} \cap \mathbb{B}_r}$ as follows: a vertex is associated to each line, and two vertices are connected by one edge if and only if the corresponding lines have intersection inside $\mathbb{B}_r = \{(x, y) \in \mathbb{C}^2 \mid |x|^2 + |y|^2 \leq r^2\}$. In other words $G_{\mathcal{A} \cap \mathbb{B}_r}$ is build as follows: a vertex for each knot, an edge between two linked knots. For a given graph G is such a configuration of lines exists? In other words:

Question 1. Can any graph G (such that between any two vertices there is no or one edge) can be realized as the link of an arrangement?

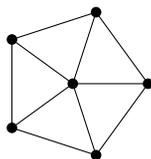
One clue that it could be true is the following.

- Lemma 3.** (1) *Any graph G (such that between any two vertices there is no or one edge) can be realized as a quasipositive link.*
 (2) *We may moreover suppose that each component of the link is a trivial knot and any pair of knots make a trivial link or a positive Hopf link.*
 (3) *This link is the intersection of $S_r^3(0)$ with some complex curve $f(x, y) = 0$.*

Proof. We use Rudolph’s theory of quasipositive link. By induction on the number of components: when adding a trivial link either it is unlinked with the other components so the braid word is unchanged, either it is linked with one (or more) component: it corresponds to the addition of a word (or several) of type: $w\sigma_i^2w^{-1}$. Then, by a deep result of [7], any quasipositive link is a transversal \mathbb{C} -link. \square

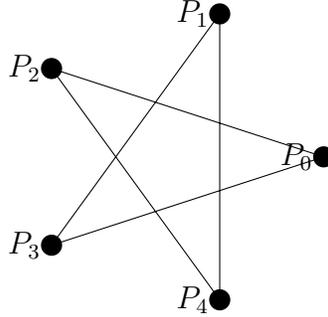
2. AN EXAMPLE AND A COUNTER-EXAMPLE

The following example is quite interesting. Let G_6 be the following graph:

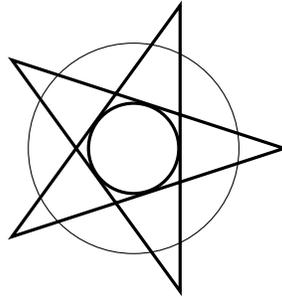


Proposition 4. *The graph G_6 can be realized as the intersection graph of some complex line arrangement but cannot be realized as the intersection graph of a real line arrangement.*

Proof. Realization as a complex line arrangement. Let $\omega = \exp(\frac{2i\pi}{5})$. Set $P_0 = (1, 1)$ and $P_i = (\omega^i, \omega^{5-i})$. Consider the 5-lines arrangement \mathcal{A} composed of (P_0P_2) , (P_2P_4) , (P_4P_1) , (P_1P_3) , (P_3P_0) . Finally define the sixth line L of equation $(6x - 4y = 1)$.



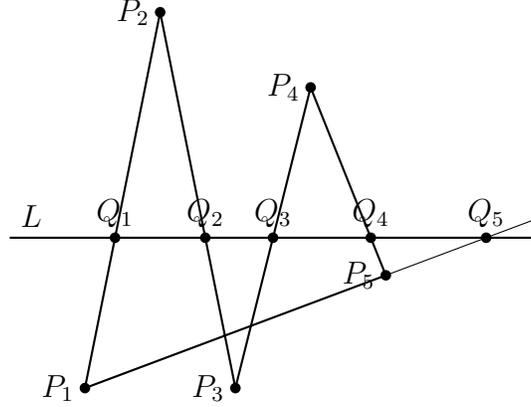
Let the 6-lines arrangement $\mathcal{A}' = \mathcal{A} \cup L$. It has the following picture: all bold curves are lines, including the bold circle! The thin circle is the sphere.



Fact: \mathcal{A}' has intersection graph G_6 . The proof is just a computation of the intersection points.

Menelaus theorem for polygons.

One of the key-point, due to Patrick Popescu-Pampu, is a Menelaus theorem for polygons in the real plane. The statement is given here for a pentagon, the proof for all polygons is the same as the one for triangles. Let a line L that intersects the edge lines of a pentagon P_1, \dots, P_5 at points Q_1, \dots, Q_5 : Q_i is the intersection of L with the line (P_iP_{i+1}) .



Theorem 5 (Menelaus theorem for polygons).

$$\frac{\overline{Q_1P_1}}{\overline{Q_1P_2}} \times \frac{\overline{Q_2P_2}}{\overline{Q_2P_3}} \times \cdots \times \frac{\overline{Q_5P_5}}{\overline{Q_5P_1}} = 1.$$

The overline \overline{AB} means the algebraic measure of AB with respect to an orientation of the line (AB) . The ratio $\frac{\overline{Q_iP_i}}{\overline{Q_iP_{i+1}}}$ is negative if and only if Q_i is in the *segment* $[P_i, P_{i+1}]$ (this is independent of the chosen orientation of the line).

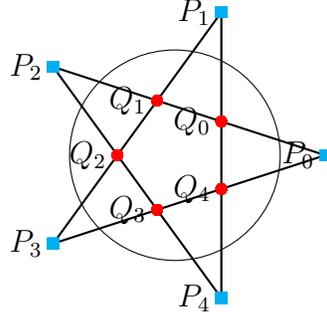
As a corollary we get:

Corollary 6. *A line cannot intersect an odd number of segments of a pentagon.*

Otherwise the product of the five ratios would be negative, that contradicts the fact that this product equals 1.

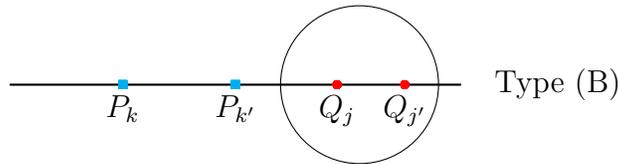
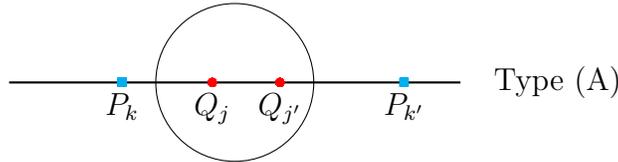
Non-realization as a real line arrangement.

We will apply this to our configurations. Suppose that 5 lines with *real equations* are disposed as follows: there are 5 intersection points P_1, \dots, P_5 outside the ball B_r and 5 intersection points inside the ball. We denote the lines as follows $\ell_i = (P_{i-1}P_{i+1})$ and $Q_i = \ell_i \cap \ell_{i+1}$ for i from $0, 1, \dots, 4$ (counting modulo 5).

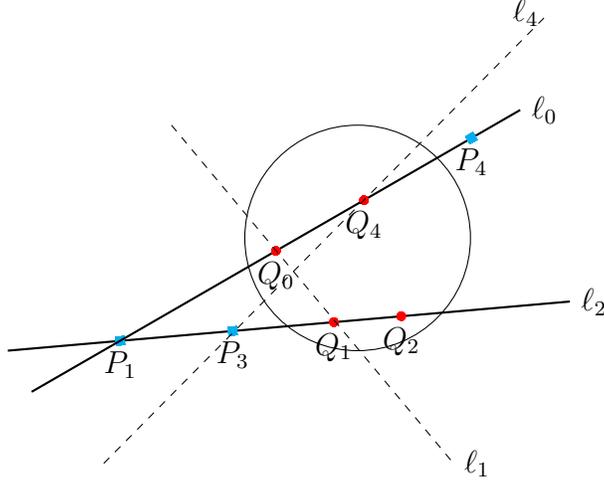


The first remark is that the polygon $Q_0Q_1 \cdots Q_4$ is convex otherwise two lines ℓ_j, ℓ_k would have an intersection point, distinct from the Q_i , inside the convex hull of $Q_0Q_1 \cdots Q_4$ and hence inside the ball B_r .

Each line ℓ_i contains two points Q_j and two points P_k . Two kinds of configurations for points on lines are possible. Type (A): $P_k - Q_j - Q_{j'} - P_{k'}$ (the ball separates the two P) or type (B): $P_k - P_{k'} - Q_j - Q_{j'}$ (the two P are on the same side of the ball).



We now prove that configurations of type (B) are associated by pairs. Suppose for instance that we have the following configuration of type (B) for the line ℓ_2 where P_1 is farthest point of the ball. Suppose now that the other line ℓ_0 that contains P_1 is of type (A).



As $P_3 \in [P_1Q_1]$, the line $l_4 = (P_3Q_4)$ intersects $l_1 = (Q_0Q_1)$ in $[Q_0Q_1]$. Then $P_0 = l_1 \cap l_4$ is inside the ball, that gives a contradiction. (The same phenomenon arise if we exchange the role of Q_1 and Q_2 on the line l_2 .) As a conclusion: there is an even number of type (B) line configurations.

Suppose now that there exists an additional line L that intersects our five lines l_i inside the ball (in order to realize the graph G_6). Consider the pentagon $P_0P_1 \dots P_4$ and its 5 segments:

- For a segment $[P_iP_{i+2}]$ supported by a line of type (A), L intersects (P_iP_{i+2}) in the ball, hence in the segment $[P_iP_{i+2}]$.
- For a segment $[P_iP_{i+2}]$ supported by a line of type (B), L intersects (P_iP_{i+2}) in the ball, hence outside the segment $[P_iP_{i+2}]$.

As there is an even number of segment supported by lines of type (B) among the 5 segments, there is odd number of segments supported by a line of type (A), hence the line L intersects an odd number of segments of the pentagon. By the Menelaus theorem it provides a contradiction. Then any line L with real equation cannot intersect the 5 lines inside the ball. It is quite surprising that in the realm of complex number this is possible. □

Part 2. Arrangements and numerical experiments

We study in details a variation of a problem of realisability of intermediate links of real line arrangements. We get two problems: the

realisability and maximization of the radius. After replacing spheres by bands, we transcribe the first problem into linear inequalities. For some examples we deduce (exact) lower bounds and (numerical) upper bounds for the maximum radius. We end by proving that this maximum radius is an algebraic number.

3. STATEMENT OF THE PROBLEM

In this part we focus on the following geometric problem, dealing with lines in the real plane. Let $\mathbb{D}_r = \{(x, y) \in \mathbb{R}^2 \mid |x| \leq r\}$ be the vertical band of radius r . Fix some $R \geq 1$. Given two lines, we will consider two conditions: the two lines have their intersection in \mathbb{D}_1 (the band of radius 1); they do not have their intersection in \mathbb{D}_R (the band of radius R).

More precisely: fix n and fix a graph G with n vertices. The problem is to find a set of n distinct lines $\{\ell_i\}$ such that for each pair (i, j) (with $i < j$): if an edge of G connect the vertex i to j then $\ell_i \cap \ell_j \in \mathbb{D}_1$ and if no edge connect the vertex i to j then $\ell_i \cap \ell_j \notin \mathbb{D}_R$.

For a given graph G and a given R the first question is: is such a configuration of lines exists? If it exists for some R , what is the maximal R that we can choose?

4. LINEAR PROGRAMS

We denote by $(y = a_i x + c_i)$ an equation of ℓ_i . The abscissa of the intersection $\ell_i \cap \ell_j$ is $x_{ij} = -\frac{c_i - c_j}{a_i - a_j}$. The condition $\ell_i \cap \ell_j \in \mathbb{D}_1$ becomes

$$(E_{i,j}^{\in}) \quad |a_i - a_j| \geq |c_i - c_j|.$$

while the condition $\ell_i \cap \ell_j \notin \mathbb{D}_R$ becomes

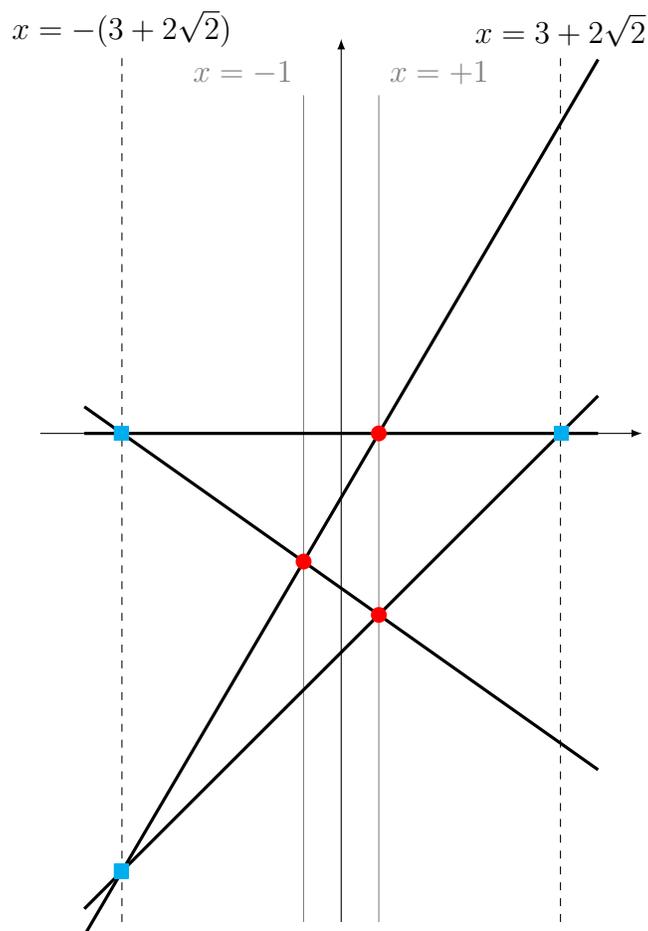
$$(E_{i,j}^{\notin}) \quad R|a_i - a_j| < |c_i - c_j|.$$

These conditions can be seen as linear inequalities, after discussion on cases depending on the sign of $a_i - a_j$ and $c_i - c_j$ (see below, paragraph 6).

5. NUMERICAL RESULTS

We will give some examples and results for several graphs. For the graph $G = A_4$ we conjecture numerically that $R_{\max}(A_4) = 3 + 2\sqrt{2}$. More precisely: we found a configuration of lines realizable for $R = 3 + 2\sqrt{2}$ and we numerically compute that no such configuration exists for $R = 3 + 2\sqrt{2} + \epsilon$ with $\epsilon = 10^{-6}$.





Here are the graph, lines and equations. The red dots are the intersections whose abscissa verify $|x| \leq 1$ (here all $|x| = 1$), the blue squares are the intersections whose abscissa have maximal $|x| \geq R$, for this example $|x| \geq R_{\max} = 3 + 2\sqrt{2}$ (here all $|x| = 3 + 2\sqrt{2}$).

$$(\ell_1) \quad y = 0$$

$$(\ell_2) \quad y = \left(1 + \frac{\sqrt{2}}{2}\right)x - 1 - \frac{\sqrt{2}}{2}$$

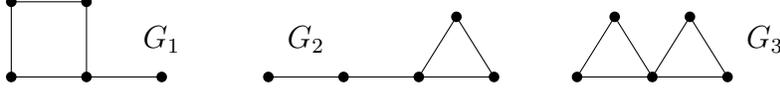
$$(\ell_3) \quad y = -\frac{\sqrt{2}}{2}x - 2 - 3\frac{\sqrt{2}}{2}$$

$$(\ell_4) \quad y = x - 3 - 2\sqrt{2}$$

For the graph $G = A_5$ we conjecture numerically that $R_{\max}(A_5) = 2 + \sqrt{3}$.

$$\begin{array}{l}
 (\ell_1) \quad y = 0 \\
 (\ell_2) \quad y = 3 + \sqrt{5} \\
 (\ell_3) \quad y = x \\
 (\ell_4) \quad y = x + 2 + \sqrt{5} \\
 (\ell_5) \quad y = -\frac{\sqrt{5}-1}{2}x + \frac{3+\sqrt{5}}{2}
 \end{array}$$


We also find for $G = G_1$, $R_{\max}(G_1) = 3$ and for $G = G_2$, $R_{\max}(G_2) = 3 + 2\sqrt{2}$. But for both these graphs the bound is obtained by a sequence of configuration that tends to a “degenerate” configuration with two lines that are equal.



For $G = G_3$, $R_{\max}(G_3) = \alpha = 2.60\dots$, where α is an algebraic number of degree 3, that is a root of $x^3 + x^2 - 9x - 1 = 0$.

Question 2. It would be interesting to know the value of $R_{\max}(A_n)$ (where A_n is the line-graph with n vertices). Conjecturally $R_{\max}(A_n) \rightarrow 3$ as $n \rightarrow +\infty$.

6. IMPLEMENTATION

An algorithm has been implemented in MATLAB to decide whether for a given graph G and a given R a corresponding configuration of lines exists. Moreover –if it exists– it gives a numeric solution.

The first step is to separate the situation in several linear problems. To each pair (i, j) with $i < j$ we have 4 possibilities for the two signs of $a_i - a_j$ and $c_i - c_j$. The number of pairs being $\frac{n(n-1)}{2}$. After reduction of the case by symmetry it yields $4^{n(n-1)/2-1}$ cases.

The second step is to study each case: for a fixed condition of sign for $a_i - a_j$ and $c_i - c_j$, the condition $(E_{i,j}^{\in})$ or the condition $(E_{i,j}^{\neq})$ yields a linear problem that can be solved numerically by standard tools.

This algorithm enables to find numerically $R_{\max}(G)$, by testing several R . Rigorously: it first gives a value R_0 such that $R_0 - \epsilon \leq R_{\max}(G) < R_0 + \epsilon$ where ϵ is a numerical value (say $\epsilon = 10^{-6}$ in practise).

Then it is possible to conjecture a value R_1 and the coefficients of the limit configuration and then check that this configuration works. We then rigorously have proved $R_1 \leq R_{\max}(G) < R_1 + \epsilon$.

Due to the exponential growth of the number of cases, we were only able to deal examples with 4 or 5 lines.

Question 3. Find an algorithm for the feasibility of band and ball problems over \mathbb{R} and \mathbb{C} that is efficient up to $n = 10$ lines. If a graph G is feasible then compute a configuration.

Question 4. Have a rigorous proof (other than numerical) for the upper bounds of R_{\max} .

7. THE MAXIMUM RADIUS IS AN ALGEBRAIC NUMBER

Consider the coefficients of the lines $(a_1, c_1, a_2, c_2, \dots, a_n, c_n) \in \mathbb{R}^{2n}$ as parameters. For a given graph G , the condition $(E_{i,j}^{\in})$ and the condition $(E_{i,j}^{\neq})$ for $R = 1$ define a semi-algebraic set $\mathcal{S} \subset \mathbb{R}^{2n}$. First define a function $F_1 : \mathcal{S} \rightarrow \mathbb{R}^{\frac{n(n-1)}{2}}$ by $(a_1, c_1, \dots) \mapsto (x_{ij})_{i < j}$ where $x_{ij} = -\frac{c_i - c_j}{a_i - a_j}$. Secondly define $F_2 : \mathbb{R}^{\frac{n(n-1)}{2}} \rightarrow \mathbb{R}$ by $(x_{ij}) \mapsto \min_{|x_{ij}| > 1} |x_{ij}|$ (equivalently the minimum runs over the pairs (i, j) such that no edge of G goes from i to j).

Let $F = F_2 \circ F_1 : \mathcal{S} \rightarrow \mathbb{R}$. Then by definition $R_{\max}(G) = \sup_{\mathcal{S}} F$. By general results in semi-algebraic geometry it implies:

Proposition 7. *Fix a graph G . If $R_{\max}(G)$ exists and is finite then it is an algebraic number.*

Part 3. Combinatorics

In this last part we will compare two problems of realisability and end with questions.

8. TWO REAL PROBLEMS

We consider two real problems. Firstly the problem already considered in section 3, dealing with the realisability of a graph as the configuration of lines within two bands. We define a similar problem for balls, by replacing a band \mathbb{D}_r by the ball $\mathbb{B}_r = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq r^2\}$.

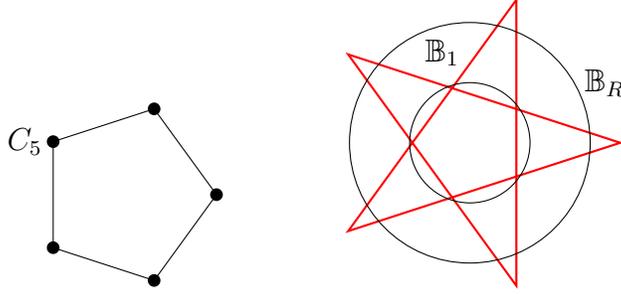
The questions are the same. Given a graph G with n vertices and a real number $R \geq 1$, find a set of n distinct lines $\{\ell_i\}$ such that for each pair (i, j) (with $i < j$): if an edge of G connect the vertex i to j then $\ell_i \cap \ell_j \in \mathbb{B}_1$ and if no edge connect the vertex i to j then $\ell_i \cap \ell_j \notin \mathbb{B}_R$. For a given graph G and a given R the questions are: is such a configuration of lines exists? If it exists for some R , what is the maximal

R that we can choose? We will compare the two problems from the combinatorial point of view.

9. FROM BANDS TO SPHERES

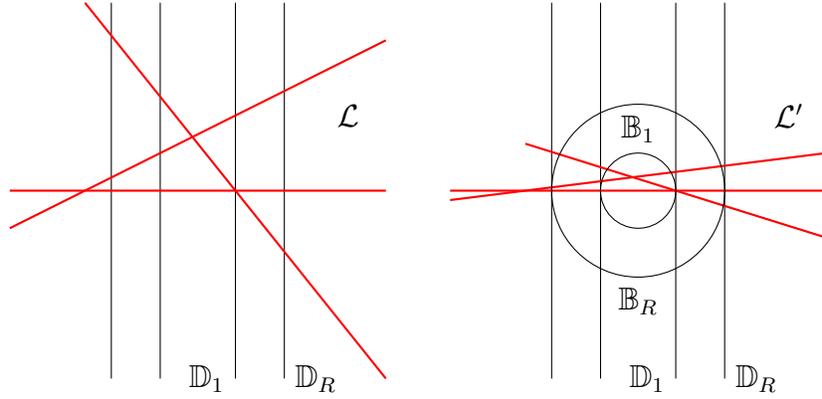
Lemma 8. *If G is feasible for the bands $(\mathbb{D}_1, \mathbb{D}_R)$ (in \mathbb{R}^2) then G is feasible for the balls $(\mathbb{B}_1, \mathbb{B}_{R'})$ (in \mathbb{R}^2) with $R' = R(1 - \epsilon)$ (for all $\epsilon > 0$). In particular the maximal radius R_{max} for the band problem is less or equal than the maximal radius R'_{max} for the ball problem.*

Remark 9. In general the reciprocal is false. For example let $G = C_5$. This graph is feasible for the ball problem $(\mathbb{B}_1, \mathbb{B}_R)$ for some $R \geq 1$ but not feasible for the band problem $(\mathbb{D}_1, \mathbb{D}_R)$ for any $R \geq 1$.



Drawing a 5-star proves the feasibility for balls. To prove that G is not feasible for bands, a first step is to remark that the five points of intersection in \mathbb{D}_1 draw a convex pentagon (otherwise there would be a sixth point of intersection inside \mathbb{D}_1). The second step is to notice that for the 5 points of intersection not in \mathbb{D}_1 , at least 3 of them are on the same side. So that, among this 3 points, you can choose Q_1 and Q_2 that are on a same line of the configuration. On this line two intersection points of the configuration are in \mathbb{D}_1 but by convexity of the pentagon they should also be in the segment $[Q_1, Q_2]$ which is entirely out of \mathbb{D}_1 . It yields a contradiction.

Proof of lemma 8. Fix $0 < \epsilon \ll 1$. Suppose that a configuration of lines \mathcal{L} realizes a graph G for the band problem $(\mathbb{D}_1, \mathbb{D}_R)$. The transformation $(x, y) \mapsto (x, \lambda y)$ preserves equations $(E_{i,j}^\epsilon)$ and $(E_{i,j}^\delta)$. So that by choosing a sufficiently small $0 < \lambda \ll 1$ we get a “flat” configuration of lines \mathcal{L}' . On the picture below the original configuration is on the left, the flattened one on the right.

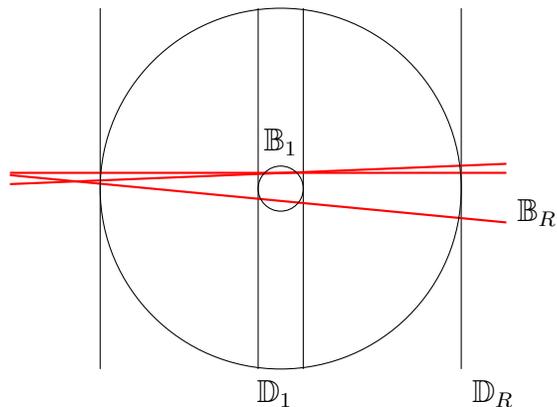


Now let $h : (x, y) \mapsto (1 - \epsilon) \cdot (x, y)$ be the homothety centred at the origin of ratio $1 - \epsilon$. Let P be a point of intersection of two lines of \mathcal{L}' . Due to the flatness if $P \in \mathbb{D}_1$ then $h(P) \in \mathbb{B}_1$ and if $P \notin \mathbb{D}_R$ then $P \notin \mathbb{B}_{R(1-\epsilon)}$. So that the configuration \mathcal{L}' proves the feasibility for the problem $(\mathbb{B}_1, \mathbb{B}_{R(1-\epsilon)})$. \square

10. FROM SPHERES TO BANDS

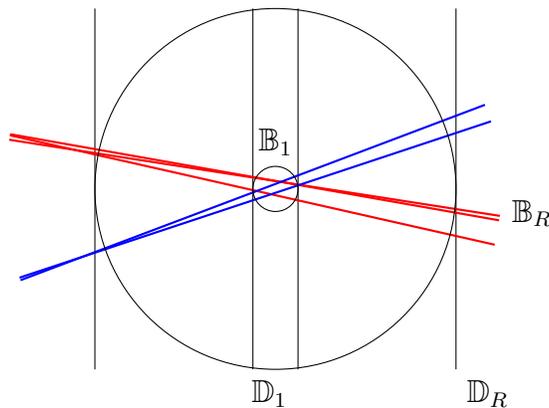
Lemma 10. *Let G be a graph. If G is feasible for the balls $(\mathbb{B}_1, \mathbb{B}_R)$ for all $R \gg 1$, then G is feasible for the bands $(\mathbb{D}_1, \mathbb{D}_R)$ for all $R \gg 1$.*

Proof. We give a heuristic proof, and start with the case where $\mathcal{C}G$ is a connected graph. Then for a big R , all lines of the configurations are nearly equal (see picture below): firstly any line of the configuration should pass through the ball \mathbb{B}_1 . Pick a line L_0 ; any other line L connected to L_0 in $\mathcal{C}G$ should pass through \mathbb{B}_1 that is very small compare to \mathbb{B}_R , so that we think of \mathbb{B}_1 as (nearly) a point. L should also intersect L_0 outside \mathbb{B}_R at Q (because $\mathcal{C}G$ is a connected). So the two “points” of intersection \mathbb{B}_1 and Q define (nearly) the same line L and L_0 . Because $\mathcal{C}G$ is supposed to be a connected set, it proves that all lines are (nearly) equal.



We may have supposed that L_0 was an horizontal line, then replacing \mathbb{B}_1 by \mathbb{D}_1 and \mathbb{B}_R by \mathbb{D}_R proves the feasibility.

If \mathcal{CG} is no longer connected, then each connected component of \mathcal{CG} yields a bundle of lines with (nearly) the same direction, any two bundles intersecting each other only in \mathbb{B}_1 . After choosing all directions sufficiently horizontal and replacing balls by bands, it gives the conclusion.



□

Question 5. For each class of problem (over \mathbb{R} or \mathbb{C}) characterize feasible graphs.

Question 6. More specifically for the complex problem with spheres, each component of a link of arrangement is in fact a true circle (a circle in the Euclidean meaning). For instance it is known that a Borromean ring cannot be obtained with true circles. See [4, Lemma 3.2] and [5]. The following questions seem to be open:

- Let two links of arrangements L_1 and L_2 with the same dual graph $G_1 = G_2$. Does it imply L_1 isotopic to L_2 ?

- Given a graph G , is it possible to construct a link L in S^3 whose components are true circles and whose dual graph is G ?
- Given a link L in S^3 whose components are true circles, is it possible to realize L as the link of a line arrangement?

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