STATIONARY PSEUDOHOLOMORPHIC DISCS

LÉA BLANC-CENTI

UNITÉ DE MATHÉMATIQUES PURES ET APPLIQUÉES (É.N.S. LYON)

Web: www.umpa.ens-lyon.fr/∼lblancce

Introduction

Stationary discs are natural invariants of manifolds with boundary with respect to biholomorphisms. They were originally introduced by L. Lempert in [4] as the geodesics for the Kobayashi metric on strictly pseudoconvex domains.

Holomorphic discs have proved to be a very important tool in the understanding of the local structure of complex manifolds. In order to study the local structure of almost complex manifolds and of pseudoholomorphic maps, we want to describe the set of pseudoholomorphic discs.

Almost complex geometry

Definition 1 *An almost complex manifold M is a (real) manifold equipped with an almost complex structure $J : M \to \mathrm{End}(TM)$, that is:*

$$\forall p,\ J_p^2 = -\mathrm{id}_{T_pM}.$$

⟶ dim$M$ is *even*.

⟶ example: $(\mathbb{R}^{2n}, J = i) \simeq \mathbb{C}^n$.

Definition 2 *A map $f : (M_1, J_1) \to (M, J)$ of class $\mathcal{C}^1$ is pseudoholomorphic if*

$$df \circ J_1 = J \circ df. \qquad (1)$$

A fundamental particular case

If *the source space is the unit disc $\Delta \subset \mathbb{C}$*, f is called a *pseudoholomorphic disc*. Condition (1) can then be written *locally* under one of the following equivalent forms:

- an elliptic PDE:

$$\overline{\partial} f + q(J_f)\partial f = 0;$$

- an almost complex version of the Cauchy-Riemann equations:

$$\frac{\partial f}{\partial y} = J_f \frac{\partial f}{\partial x}. \qquad (2)$$

Using (2), A. Nijenhuis and W. Woolf [5] proved that **there exist pseudoholomorphic discs**. In fact one can even prescribe the center $f(0)$, and $f'(0)$ (provided it is sufficiently small).

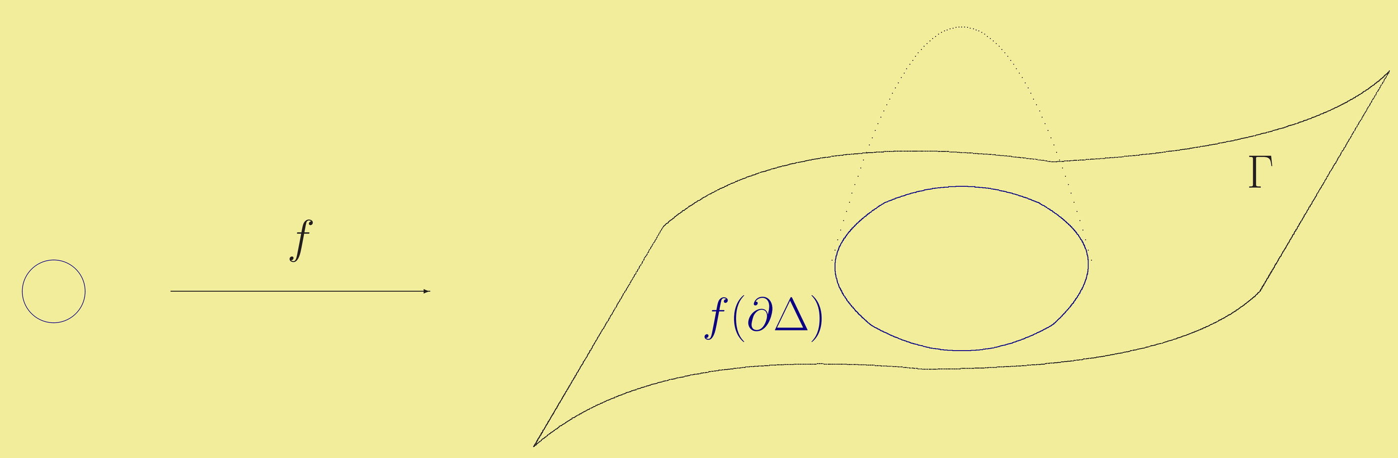
The set of stationary discs glued to a Levi-non-degenerate hypersurface

Since the statement will be local, we assume from now that the ambient manifold is $\mathbb{R}^{2n}$ equipped with an almost complex structure J.

Stationary discs

Let Γ be a real hypersurface in $\mathbb{R}^{2n}$.

Definition 3 *A pseudoholomorphic disc f is glued to Γ if f is continuous on $\overline{\Delta}$ and $f(\partial\Delta) \subset \Gamma$.*



Interest: stationary discs are globally invariant under the action of any pseudo-biholomorphism F.

But: this does not give a sufficiently precise information on F, since there are in some way "too many" discs glued to a hypersurface.

⟹ we introduce an additional condition on discs, *still preserved by biholomorphisms*.

We recall that

- the cotangent bundle $T^*\mathbb{R}^{2n}$ can be equipped with an almost complex structure J^*, obtained as a lift of J [6, 2];
- the *conormal bundle* $N^*\Gamma$ of Γ is a *subbundle of the cotangent bundle*:

$$N^*\Gamma = \{\phi \in T^*_{(1,0)}\Gamma /\ \Re\phi_{|T\Gamma} = 0\}.$$

Definition 4 *A pseudoholomorphic disc f glued to Γ is stationary if it admits a lift $f^* = (f, g)$ to the cotangent space $T^*\mathbb{R}^{2n}$, such that:*

$$i.\ (f, \zeta g) \text{ is } J^*\text{−holomorphic;}$$
$$ii.\ \forall\zeta \in \partial\Delta,\ g(\zeta) \in N^*_{f(\zeta)}\Gamma \setminus \{0\}.$$

Two examples

Γ= the unit sphere

Stationary discs glued to $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ and centered at 0 are exactly the linear discs $\zeta \mapsto \lambda\zeta$.

More generally:

Theorem 5 ([2]) *Γ=almost complex perturbation of $\mathbb{S}^{2n-1}$*

⟹ stationary discs centered at 0 form a manifold of dimension $2n - 1$.

Γ= a non-degenerate hyperquadric

In this case, stationary discs are explicitly determined [1]. We get:

Proposition 6 *Let $Q \subset \mathbb{C}^n$ be the hyperquadric*

$$Q :\ \Re z_n = \pm|z_1|^2 + \ldots + \pm|z_{n-1}|^2.$$

Then stationary discs centered at the same point $p \notin Q$ form a manifold of dimension $2n - 1$.

Remark 7 In Definition 4, we need to allow the fibered part to have a pole of order one at 0. For instance, no disc glued to $\mathbb{S}^{2n-1}$ admits a J^*-holomorphic lift glued to $N^*\mathbb{S}^{2n-1}$ without pole.

Main result

Here we study the case when Γ is only Levi-non-degenerate. Up to a linear change of coordinates, we can assume:

$$\begin{cases} \Gamma :\ \Re z_n = \pm|z_1|^2 + \ldots + \pm|z_{n-1}|^2 + O(|z|^3) \\ J \text{ is a small perturbation of } i \end{cases}.$$

Let us fix some stationary disc f_0 glued to Q and centered at $p = (0, \ldots, 0, p_n) \notin Q$.

Theorem 8

1. Γ=small almost complex perturbation of the Levi-non-degenerate hyperquadric Q

⟹ stationary discs centered at p form a manifold of dimension $2n - 1$ near f_0.

2. More precisely, the maps

$$f \mapsto f(1) \quad \text{and} \quad f \mapsto f'(0)$$

are local diffeomorphisms on their images.

In particular, this gives a local foliation of Γ by the boundaries of stationary discs.

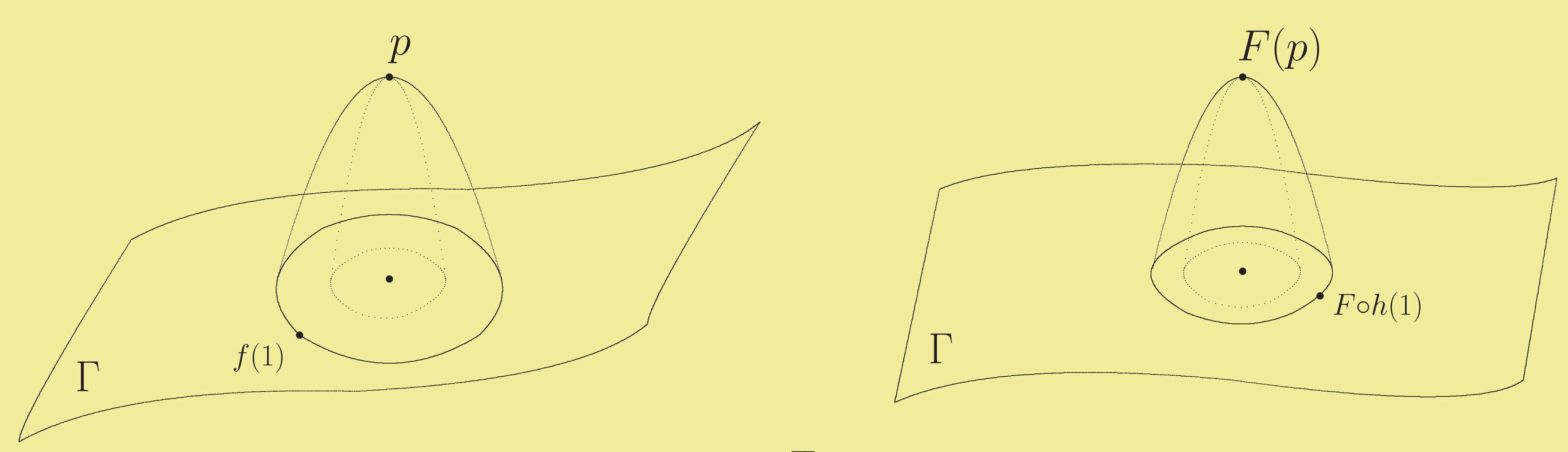
Ideas of proof

- Γ is *non-degenerate* $\Longleftrightarrow$ $N^*\Gamma \setminus \{zero\ section\}$ is *totally real* [3]. Hence stationary discs glued to Γ lift into *holomorphic discs glued to a totally real submanifold* $S(\Gamma)$.
- By Proposition 6, holomorphic discs glued to $S(Q)$ form a manifold of the right dimension (namely, the Maslov index along a lift of f_0); this property is *stable under a small perturbation of* (Q, J) since the partial indices along a lift of f_0 are positive.
- Projections of holomorphic discs glued to $S(\Gamma)$ are exactly the stationary discs glued to Γ.

Corollary: a uniqueness property

Theorem 8 also shows that the diffeomorphism $\phi : f(1) \mapsto f'(0)$ is well-defined, and *commutes with pseudo-biholomorphisms* in the following sense:

$$\boldsymbol{F = \phi^{-1} \circ dF_p \circ \phi.}$$



$$\begin{array}{ccc} f(1) & \xrightarrow{F} & F \circ f(1) \\ \phi \downarrow & & \downarrow \phi \\ f'(0) & \xrightarrow{dF_p} & (F \circ f)'(0) \end{array}$$

Corollary 9 *The pseudo-biholomorphism F is uniquely determined by its differential at p.*

References

[1] L. Blanc-Centi, Stationary holomorphic discs glued to a Levi non-degenerate hypersurface, *Trans. Amer. Math. Soc.* (to appear).

[2] B. Coupet, H. Gaussier, A. Sukhov, Riemann maps in almost complex manifolds, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **2** (2003), no. 4, 761-785.

[3] H. Gaussier, A. Sukhov, On the geometry of model almost complex manifolds with boundary, *Math. Z.* **254** (2006) no. 3, 567-589.

[4] L. Lempert, La métrique de Kobayashi et la représentation des domaines sur la boule, *Bull. Math. Soc. France* **109** (1981), 427-474.

[5] A. Nijenhuis, W. Woolf, Some integration problems in almost complex and complex manifolds, *Ann. Math.* **77** (1963), 424-489.

[6] K. Yano, Sh. Ishihara, Tangent and cotangent bundles: differential geometry, *Marcel Dekker, Inc., New York* (1973).