The Maynard–Tao method (2013 Oct–Nov)

Let $\mathcal{H} = \mathcal{H}_k$ admissible, $\mathcal{H} = \{h_i\}_{i=1}^{k}$

Main idea: the applied weights a_n should be

instead of
$$a_n = \Big(\sum_{\substack{d \mid P_{\mathcal{H}}(n) \\ d \le R}} \lambda_d\Big)^2 \qquad P_{\mathcal{H}}(n) = \prod_{i=1}^{k} (n+h_i)$$

$$a_n := \Big(\sum_{\substack{d_i \mid n+h_i \\ \prod d_i \le R}} \lambda_{d_1, d_2, \dots d_r}\Big)^2$$

Originally for $k=2$ this was mentioned in the Coll. Notes of Selberg (which appeared in 1992) in the proof of

$$2n = P_2 + P_3 \quad (n > n_0) \quad \text{or} \quad 2h = P_2 - P_3 \text{ or } P_3 - P_2$$

GPY: $\lambda_d = F\left(\frac{\log d}{\log R}\right) \mu(d)$

M–T $\lambda_{d_1, \dots d_k} \approx F\left(\frac{\log d_1}{\log R}, \quad \frac{\log d_k}{\log R}\right) \prod_{i=1}^{k} \mu(d_i)$

Remark
$$\sum_{\substack{d_i \mid (n+h_i) \\ \prod_{i=1}^{k} d_i = d}} F\left(\frac{\log d_1}{\log R}, \dots, \frac{\log d_k}{\log R}\right)$$

is not a continuous function of d (or $\frac{\log d}{\log R}$)

The GPY method is a special case of the M-T method

Since then $F(t_1, \dots, t_k)$ depends just on

$T_1 = t_1 + \dots + t_k$

Results: If $T_2 = t_1^2 + \dots + t_k^2$ and F depends just on T_1 and T_2 it is already possible to obtain more than one prime on average if n runs between N and $2N$ among $n + h_i$, namely the choice

$$F = \sum_{i=1}^{d} a_i (1-T_1)^{b_i} T_2^{c_i} \quad (b_i, c_i \in \mathbb{N}),\ a_i \in \mathbb{R}$$

is good with suitable small integers b_i, c_i and with $k = 105$ by using just $\vartheta = 1/2$ (the original BV theorem)

Remark: if b_i and c_i are fixed then the choice of a_i can be optimized using the following Lemma

Lemma: Let M_1, M_2 be real, symmetric positive definite matrices. Then

$$\frac{\underline{a}^T M_2 \underline{a}}{\underline{a}^T M_1 \underline{a}}$$

is maximal if $\underline{a}$ is an eigenvector of $M_1^{-1} M_2$ corresponding to the largest eigenvalue, which is the maximum value of the ratio.

This implies $\liminf_{n\to\infty} (p_{n+1} - p_n) \leq 600$.

How can we find for a given k the admissible set $\mathcal{H}_k^0$ with the minimal diameter.

(i) for "small" values of k by a computer

(ii) for "large" values of k the set

$\mathcal{H}_k^1 = \{p_t, \dots p_{t+k-1}\}$ where p_t is the smallest prime $> k$. This has diameter $(1+o(1))\, k\log k$, as $k\to\infty$

which is asymptotically optimal (Hardy-Littlewood 1923)

If EH Conj. is true, i.e. $\vartheta = 1$ then $k=5$ and

$\liminf_{n\to\infty} (p_{n+1} - p_n) \leq 12$ (GPY $k=6$ $\underline{\liminf} \leq 16$)

<u>Maynard-Tao</u>: If $k\to\infty$ then we have at least $(1-\varepsilon)\frac{\log k}{4}$ primes among $n+\mathcal{H}_k$ inf. often

(and in general $(1-\varepsilon)\frac{\log k\cdot\vartheta}{2}$ primes)

<u>Remark</u>: Even the BV theorem $\frac{1}{2}$ is not needed; one gets bounded gaps infinitely often with any $\vartheta > 0$. First such result was proved in 1947/48 by A. Rényi using Linnik's large sieve $\Rightarrow 2n = p + P_K$

Maynard - Tao thm. Dickson's theorem is true for a positive proportion of all admissible m-tuples. Precisely, let $m \in \mathbb{N}$, $r > r_0(m)$ $\mathcal{A} = \{a_1, \dots a_r\}$ a set of r distinct integers.

Then
$$\frac{\#\{(h_1, \dots h_m) \subseteq \mathcal{A} ; \{n+h_i\}_{i=1}^{m} \in \mathcal{P}^m \text{ i.o.}\}}{\#\{(h_1, \dots h_m) \subseteq \mathcal{A}\}} \gg_m 1$$

Some ideas of the Maynard - Tao method

(i) Let $\mathcal{H} = \{h_i\}_{i=1}^{k}$ be admissible, $n \sim N \Leftrightarrow n \in (N, 2N]$

$$D_0(N) = \log\log\log N \qquad W = \prod_{p \le D_0} p = e^{(1+o(1))D_0} \ll (\log\log N)^2$$

Let us choose ν_0 mod W so that for

$$\forall p \le D_0, \quad \forall h_i \in \mathcal{H} \quad -h_i \not\equiv \nu_0 \pmod p.$$

We will only consider numbers n with $n \equiv \nu_0 \pmod W$ *

Consequently $\forall p \le D_0$ $\forall i < k$ $n + h_i \equiv \nu_0 + h_i \not\equiv 0 \pmod p$

In this way $p \mid n+h_i \longrightarrow p > D_0$. We avoid complications with small primes

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* $a_n = 0$ unless $n \equiv \nu_0 \pmod W$

Let

$$S_1 := \sum_{\substack{n \sim N \\ n \equiv \nu_0 \,(\mathrm{mod}\, W)}} \Big(\sum_{\forall i\; d_i | n+h_i} \lambda_{d_1 \ldots d_k} \Big)^2$$

$$S_2 := \sum_{\substack{n \sim N \\ n \equiv \nu_0 \,(\mathrm{mod}\, W)}} \Big(\sum_{i=1}^{k} \chi_{\mathcal{P}}(n+h_i) \Big(\sum_{\forall i\; d_i | n+h_i} \lambda_{d_1 \ldots d_k} \Big)^2 \Big)$$

Let $\lambda_{d_1 \ldots d_k} := \Big(\prod_{i=1}^{k} \mu(d_i) d_i \Big) \sum_{\substack{r_i \\ \forall i\, d_i | r_i \\ \forall i\, (r_i, W) = 1}} \frac{\mu^2\big(\prod_i r_i\big)}{\prod_i \varphi(r_i)} F\Big(\frac{\log r_1}{\log R}, \ldots \frac{\log r_k}{\log R} \Big)$

whenever $\big(\prod_i d_i, W \big) = 1$. Otherwise let $\lambda_{d_1 \ldots d_k} = 1$.

Let F be supported on $R_k = \{ (x_1 \ldots x_k) \in [0,1]^k, \sum x_i \leq 1 \}$.

Main Proposition. We have

$$S_1 = \frac{(1+o(1))\, \varphi^k(W)\, N (\log R)^k}{W^{k+1}} I_k(F) \quad \text{if } I_k(F) \neq 0$$

$$S_2 = \frac{(1+o(1))\, \varphi^k(W)\, N (\log R)^k}{W^{k+1}} \sum_{m=1}^{k} J_k^{(m)}(F) \quad \text{if } J_k^{(m)}(F) \neq 0$$

where $I_k(F) = \int_0^1 \ldots \int_0^1 F^2(t_1 \ldots t_k)\, dt_1 \ldots dt_k$

$$J_k^{(m)}(F) = \int_0^1 \ldots \int_0^1 \Big(\int_0^1 F(t_1 \ldots t_k)\, dt_m \Big)^2 dt_1 \ldots dt_{m-1}\, dt_{m+1} \ldots dt_k$$

Let S_k denote the set of piecewise differentiable functions $F : [0,1]^k \longrightarrow R$ supported on R_k with $I_k(F) \neq 0$ $J_k^{(m)}(F) \neq 0$ $(m \in [1,k])$.

<u>Corollary</u>

If $M_k = \sup_{F \in S} \dfrac{\sum_{m=1}^{k} J_k^{(m)}(F)}{I_k(F)}$ $r_k = \left\lceil \dfrac{\vartheta M_k}{2} \right\rceil$

(ϑ = level of distribution of primes). If $\mathcal{H} = \{h_i\}_{i=1}^{k}$ admissible then at least r_k of the $n + h_i$ $(i = 1, \dots, k)$ are primes for infinitely many n values $\Longrightarrow$

$$\liminf_{n \to \infty} (p_{n+r_k-1} - p_n) \leq \operatorname{diam} \mathcal{H} = h_k - h_1 \quad (\text{if } h_1 < \dots < h_k)$$

REMARK: If we find any F_k with $F_k \in S_k$, any $\delta > 0$

$$G(F_k) = \frac{\sum_{m=1}^{k} J_k^{(m)}(F_k)}{I_k(F_k)} \quad \text{then at least} \left\lceil \frac{\vartheta G(F_k) - \delta}{2} \right\rceil$$

of the $n + h_i$ are primes i.o. (for infinitely many n)

A good choice for large k is the function

$$F_k(t_1 \dots t_k) = \begin{cases} \prod_{i=1}^{k} \dfrac{1}{1 + k \log k \cdot t_i} & \text{if } \sum_i t_i \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

This gives $G(F_k) \geqslant (1 + o(1)) \log k \Longrightarrow \dfrac{\vartheta \log k}{2}(1 + o(1))$ primes

unconditionally $\dfrac{\log k}{4}(1 + o(1))$ primes.

Results about M_k implying the mentioned theorems about primes

(i) $M_5 > 2 \Longrightarrow$ under EH each admissible 5-tuple $\mathcal{H}_5$ satisfies: $\{n+h_i\}_{i=1}^{5}$ contains at least 2 primes i.o. $\Longrightarrow \liminf (p_{n+1}-p_n) \leq 12$ under EH

Choice of F_5 with $G(F_5) > 2$ is:

$$F(t_1, \dots t_5) = (1-T_1)T_2 + \frac{7}{10}T_2^2 + \frac{T_2}{14} - \frac{3}{14}(1-T_2)$$

(ii) $M_{105} > 4 \Longrightarrow$ (unconditionally) each admissible 105-tuple $\mathcal{H}_{105}$ satisfies $\{n+h_i\}_{i=1}^{105}$ contains at least 2 primes i.o. $\Longrightarrow$ (unconditionally) $\liminf (p_{n+1}-p_n) \leq 600$

Choice of F_{105} with $G(F_{105}) > 4$ = a suitable linear combination of all functions of type $(1-T_1)^{b_i} T_2^{c_i}$ with $b_i, c_i \geq 0$, $b_i + 2c_i \leq 11$

If $c_i = 0, 1, 2, 3, 4, 5$ we have $12, 10, 8, 6, 4, 2$ choices for $b_i \Longrightarrow$ altogether 42 such terms. One needs to calculate the largest eigenvalue of $M_1^{-1}M_2$ where M_i are 42×42 symmetric positive definite matrices

The proof is based on a k-dimensional variant of Selberg's Λ^2 sieve where the two types of weights are connected by the two equivalent relations

$$\lambda_{d_1,\dots,d_k} = \prod_{i=1}^{k} \mu(d_i) d_i \sum_{\substack{r_i \\ \forall i\ d_i \mid r_i}} \frac{\prod_i \mu^2(r_i)}{\prod_i \varphi(r_i)} y_{r_1,\dots,r_k}$$

(mentioned already in the definition) and

$$y_{r_1,\dots,r_k} = \prod_i \mu(r_i)\varphi(r_i) \sum_{\substack{d_1,\dots,d_k \\ \forall i\ r_i \mid d_i}} \lambda_{d_1,\dots,d_k}$$

$$S_1 = \frac{N}{W} \sum_{\substack{d_1,\dots,d_k \\ e_1,\dots,e_k}}{}' \frac{\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}}{\prod_i [d_i, e_i]} + \text{error}$$

(where $\sum'$ means that $W, [d_1,e_1], \dots, [d_k,e_k]$ are pairwise coprime) is transformed after several steps into

$$S_1 = \frac{N}{W} \sum_{u_i} \frac{y^2_{u_1,\dots,u_k}}{\prod_i \varphi(u_i)} + \text{error}$$

$y_{r_1,\dots r_k}$ is defined as to depend in a piecewise differentiable way on $r_1, \dots r_k$, i.e.

$$y_{r_1,\dots r_k} = F\left(\frac{\log r_1}{\log R}, \dots \frac{\log r_k}{\log R}\right) \quad \text{where } F=0 \text{ unless}$$

$$\sum t_i \leq 1 \Leftrightarrow \prod r_i \leq R$$

When the sum twisted by primes is considered then one needs a statistically uniform distribution of primes in arithmetic progression, i.e. a Bombieri-Vinogradov type theorem, but as $M_k \to \infty$ with $k \to \infty$ it is sufficient to have any fixed positive level ϑ for the distribution of primes (Rényi 1947-48)

The original GPY method could be described with a situation of $M_k^* = 4 - \frac{C}{k^\alpha} \Rightarrow$

$$\frac{\vartheta}{2} M_k^* = \frac{\vartheta}{2}\left(4 - \frac{C}{k^\alpha}\right) \leq 1 - \frac{C}{4k^\alpha} \text{ unless we know}$$

that $\vartheta > 1/2$ can be chosen (at least for smooth moduli as shown in Motohashi-J.P.) which was proved by Zhang.

R.1. Important strategy: all calculations are carried out for a general (piecewise differentiable but in practice analytical) function $F: R^k \to R$ and optimization of the obtained average number of primes among $n+h_i$ i.e. optimization (more precisely just quasi-optimization) of the function F is carried out at the end

R.2. The proof of the sieve procedure follows by a k-fold iteration of a linear Selberg sieve since for each i: in step i we just sieve out the single residue class $-h_i \pmod p$ for any prime p (in GPY this happens in one step for all residue classes $\{-h_i\}_{i=1}^{k}$)

R 3. One can avoid complication caused by small primes, i.e. the use of singular series $\mathfrak{S}(\mathcal{H})$ by the W-trick (GT) in GPY too, but this trick is more important here due to the k-fold iteration.