ANALYTIC CONTINUATION OF RANDOM DIRICHLET SERIES

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Dedicated to the memory of Professor A. A. Karatsuba

1. Introduction

While studying analytic properties of Dirichlet series it is natural to ask whether and how far they can be meromorphically continued beyond their region of convergence. In general this is a difficult question and when an answer can be found it usually corresponds to the intuitive natural boundary. The consideration of random Dirichlet series that can be shown to have natural boundaries almost surely comfort us in our belief that Dirichlet series should be continuable to their expected domains.

Random Dirichlet series and their convergence are differently treated depending on the context. In this note we first give an overview of existing results on natural boundaries. For the most part the series considered are in one variable. We suggest, in the second part of our note, a multiple analogue of random Dirichlet series. We indicate only some initial tools whose applications enable us to compare the abscissa of convergence of the Goldbach generating function with its random version. A more detailed study of these multiple analogues might turn out to be interesting.

2. One variable

We let (Ω, A, P) or simply Ω be a probability space and we say that an event E in A happens almost surely if $P(E) = 1$. Random Dirichlet series are of the form $\sum_{n=1}^{\infty} X_n(\omega)e^{-\lambda_n s}$, where the λ_i form an increasing sequence of positive numbers, $X_n(\omega)$ are independent complex random variables in (Ω, A, P) and $s = \sigma + it$ a complex number. A sequence of independent random variables ε_n of Ω each of which takes only the values ± 1 with the same probability $1/2$ is called a Rademacher sequence.

Relations between the different abscissae of convergence of random Dirichlet series have been studied since long. For example Hartman in 1931 considered Dirichlet series of the form $\sum_{n=1}^{\infty}(\pm a_n)n^{-s}$. There exist numbers $\sigma_c \le \sigma_u \le \sigma_a$ which are almost surely the abscissae of convergence, uniform convergence and absolute convergence respectively of such Dirichlet series. Hartman showed that $\sigma_a - \sigma_u$ is atmost $1/2$ and can be exactly $1/2$ [10]. The Bohr-Toeplitz theorem says that the same difference exists between the abscissae of absolute and uniform convergence for the ordinary ordinary Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$.

Dvoretzky in 1945 showed that for a given ordinary Dirichlet series $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$, and for any σ squeezed between its abscissae of convergence and absolute convergence one can always construct a random Dirichlet series $\sum_{n=1}^{\infty} \varepsilon_n a_n e^{-\lambda_n s}$ with $\Re s = \sigma$ as its natural boundary by an appropriate choice of $\varepsilon_n = \pm 1$ [5]. This is the analogue of the Fatou-Pólya theorem for power series.

New developments in the theory of random Dirichlet series began in 1970s. Yu (1978) studied particular cases where λ_n satisfy the conditions $\lim_{n\to\infty} \frac{\log n}{\lambda_n} = 0$ to obtain analogues of results on the radius of almost sure convergence for random power series [18].

Where the random variables are symmetric (as for example the case of Rademacher variables) the abscissa of convergence coincides almost surely with the natural boundary. Kahane described a dichotomy for the general case. The random series $\sum_{n=1}^{\infty} X_n e^{-\lambda_n s}$ either has the abscissa of convergence as its natural boundary or there exists an ordinary Dirichlet series $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ with the same abscissa of convergence σ_c and a number $\sigma < \sigma_c$ such that the random series $\sum_{n=1}^{\infty}(X_n - a_n)e^{-\lambda_n s}$ has σ as its natural boundary almost surely [11]. For the special series $\sum_{n=1}^{\infty} \pm n^{-s}$ he showed that almost surely the abscissa of convergence $\sigma_c = 1/2$ would be its natural boundary [12]. This is the analogue of the Ryll-Nardzewski theorem for natural boundaries of random Taylor series. Some generalisations can be found in [7], [17].

Ding and Xiao studied explicit natural boundaries for uniformly nondegenerate variables X_n, i.e. $\sup_{n\ge 0}\sup_{a\in\mathbb{C}} P(X_n = a) < 1$, which is equivalent to the condition that there exists a sequence of positive numbers R_n such that $\sup_{n\ge 0}\sup_{a\in\mathbb{C}} P(|X_n - a| \le R_n) < 1$. If the abscissa of convergence of $\sum_{n=1}^{\infty} X_n e^{-\lambda_n s}$ is the same as that for $\sum_{n=1}^{\infty} R_n^2 e^{-2\lambda_n s}$ then this abscissa would almost surely be the natural boundary of the first series [4].

Queffélec enlarged the study by considering Euler products. Let p_n be the nth prime. He showed that $\prod(1 - \varepsilon_n p_n^{-s})^{-1}$, for (ε_n) a sequence of independent Rademacher variables, has $\sigma = 1/2$ almost surely as the natural boundary [14].

In the study of Euler products in both p and p^{-s} (for prime p), random variables were used in the exponents as an adaptation of the classical case. It was proved that the series $\prod_{n=1}^{\infty} \zeta(a_n s + b_n)^{c_n+\varepsilon_n}$, where a_n, b_n, c_n are increasing real sequences and ε_n a suitable sequence of independent real random variables, admits almost surely $\sigma = \limsup_{n\to\infty} \left(-\frac{b_n}{a_n}\right)$ as its natural boundary [2].

Convergence of Random Dirichlet polynomials, finite versions of the Dirichlet series, have been studied, for example in [13].

3. Several variables

Now we suggest a multiple analogue of Rademacher sequences and associated Dirichlet series. Let $m_1, \ldots, m_r \in \mathbb{N}$, and let $\varepsilon_{m_1,\ldots,m_r}$ be independent random variables which take only the values ± 1 and are defined on a certain probability space (Ω, A, P). Let

(3.1)
$$P(\omega \in \Omega \mid \varepsilon_{m_1,\ldots,m_r}(\omega) = 1) = P(\omega \in \Omega \mid \varepsilon_{m_1,\ldots,m_r}(\omega) = -1) = \frac{1}{2}.$$

The sequence $\{\varepsilon_{m_1,\ldots,m_r} \mid m_1, \ldots, m_r \in \mathbb{N}\}$ may be called a Rademacher sequence of r-tuple indices. The associated multiple Dirichlet series can be defined as

$$F_r(s,\omega) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{\varepsilon_{m_1,\ldots,m_r}(\omega) X(m_1,\ldots,m_r)}{(m_1+\cdots+m_r)^s}, \tag{3.2}$$

where $s \in \mathbb{C}$ and $X(m_1, \ldots, m_r) \in \mathbb{R}$.

We note that other definitions of multiple random Dirichlet series exist (see, for example, [15] or [16]).

First we write $F_r(s,\omega)$ in a form of a single Dirichlet series:

$$F_r(s,\omega) = \sum_{n=1}^{\infty} Y_r(n,\omega) n^{-s}, \tag{3.3}$$

where

$$Y_r(n,\omega) = \sum_{m_1+\cdots+m_r=n} \varepsilon_{m_1,\ldots,m_r}(\omega) X(m_1,\ldots,m_r). \tag{3.4}$$

Since $\epsilon_{m_1,\ldots,m_r}$ are independent variables, we see that $Y_r(n,\omega)$ are also independent of each other.

For each fixed $\omega \in \Omega$, the series (3.3) is a Dirichlet series, so we can find its abscissa of convergence $\sigma_c(F_r,\omega)$. Then, according to the zero-one law, we can find a constant $\sigma_c(F_r)$ such that $\sigma_c(F_r,\omega) = \sigma_c(F_r)$ almost surely (see [11, Section 6 of Chapter 4]). In general $-\infty \leq \sigma_c(F_r) \leq \infty$, but here we assume $-\infty < \sigma_c(F_r) < \infty$. The following result is obtained simply.

Proposition 1. *The line $\Re s = \sigma_c(F_r)$ is the natural boundary of $F_r(s,\omega)$ almost surely.*

Proof. We say that $Y_r(n,\omega)$ is symmetric if $Y_r(n,\omega)$ and $-Y_r(n,\omega)$ have the same distribution. We prove that $Y_r(n,\omega)$ is symmetric; and the conclusion then follows immediately from the expression (3.3) and [11, Theorem 4 in Section 6 of Chapter 4].

Let $L(n)$ be the number of tuples $(m_1,\ldots,m_r)$ satisfying $m_1+\cdots+m_r=n$. Let $\mathbf{b}$ be any tuple of $L(n)$ elements, each element being 1 or -1. Write $\mathbf{b}=(b_{m_1,\ldots,m_r})_{m_1+\cdots+m_r=n}$, where $b_{m_1,\ldots,m_r}\in\{\pm1\}$. Define

$$Z_r(n,\mathbf{b}) = \sum_{m_1+\cdots+m_r=n} b_{m_1,\ldots,m_r} X(m_1,\ldots,m_r).$$

Let A be any Borel subset of $\mathbb{R}$, and let $\mathcal{B}(A)$ be the set of all $\mathbf{b}$ such that $Z_r(n,\mathbf{b})\in A$. Then

$$\begin{aligned} &P(\omega\in\Omega \mid Y_r(n,\omega)\in A) \\ &= \sum_{\mathbf{b}\in\mathcal{B}(A)} P\left(\omega\in\Omega \mid (\varepsilon_{m_1,\ldots,m_r}(\omega))_{m_1+\cdots+m_r=n}=\mathbf{b}\right). \end{aligned} \tag{3.5}$$

Since $\varepsilon_{m_1,\ldots,m_r}$ are independent, we have

$$P\left(\omega\in\Omega \mid (\varepsilon_{m_1,\ldots,m_r}(\omega))_{m_1+\cdots+m_r=n}=\mathbf{b}\right) = 2^{-L(n)}, \tag{3.6}$$

and the same equality holds if we replace $\mathbf{b}$ by $-\mathbf{b}$. Therefore the right-hand side of (3.5) is equal to

$$\begin{aligned} &\sum_{\mathbf{b}\in\mathcal{B}(A)} P\left(\omega\in\Omega \mid (\varepsilon_{m_1,\ldots,m_r}(\omega))_{m_1+\cdots+m_r=n}=-\mathbf{b}\right) \\ &= P(\omega\in\Omega \mid Y_r(n,\omega)\in -A). \end{aligned}$$

This implies that two random variables $Y_r(n,\omega)$ and $-Y_r(n,\omega)$ have the same distribution. □

4. An example : The Goldbach generating functions

As an example, we will treat the case where $X(m_1,\ldots,m_r)$ is a product of the von Mangoldt functions, that is

$$\Phi_r(s,\omega) = \sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty} \frac{\varepsilon_{m_1,\ldots,m_r}(\omega)\Lambda(m_1)\cdots\Lambda(m_r)}{(m_1+\cdots+m_r)^s}. \tag{4.1}$$

The ordinary multiple Dirichlet series

$$\Phi_r(s) = \sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty} \frac{\Lambda(m_1)\cdots\Lambda(m_r)}{(m_1+\cdots+m_r)^s} \tag{4.2}$$

was introduced in [6] as the generating function of

$$G_r(n) = \sum_{m_1+\cdots+m_r=n} \Lambda(m_1)\cdots\Lambda(m_r), \tag{4.3}$$

the latter being connected to the problem of expressing positive integers as a sum of r prime numbers. Properties of Φ_r and G_r above have been studied actively in the recent past (see [1] for references). It may be interesting to note that under the assumption of the Riemann Hypothesis (RH) for the Riemann zeta-function $\zeta(s)$ whose non-trivial zeros are denoted by ρ,

$$\sum_{n\le X} G_k(n) = \frac{1}{k!}X^k + H_k(X) + \mathcal{O}_\epsilon(X^{r-1+\epsilon}) \tag{4.4}$$

with

$$H_k(X) = \sum_\rho \frac{X^{k-1+\rho}}{\rho(1+\rho)\cdots(k-1+\rho)}.$$

Concerning the question of convergence of $\Phi_r(s)$ it was conjectured in [6] that:

(C-r) The line $\Re s = r-1$ would be the natural boundary of $\Phi_r(s)$.

The conjecture is out of reach for the moment. However one can say more under the RH and other reasonable ones on the zeros of the Riemann zeta function like the following due to Fujii [8] [9] :

(Z) Let $\mathcal{I}$ be the set of all imaginary parts of non-trivial zeros of $\zeta(s)$. If $\gamma_j \in \mathcal{I}$ $(1 \le j \le 4)$ and $\gamma_1+\gamma_2 = \gamma_3+\gamma_4 \ne 0$, then $\{\gamma_1,\gamma_2\} = \{\gamma_3,\gamma_4\}$.

In [6] and [3], it was shown that (C-2) is indeed true under the RH and a certain quantitative version of (Z) or (Z) itself. The case of (C-r) is shown to be true if and only if (C-2) is.

The reason of introducing the random series (4.1) is to observe the situation from a stochastic viewpoint. Let

$$G_r(n,\omega) = \sum_{m_1+\cdots+m_r=n} \varepsilon_{m_1,\dots,m_r}(\omega)\Lambda(m_1)\cdots\Lambda(m_r). \tag{4.5}$$

Then

$$|G_r(n,\omega)| \le n^{r-1}(\log n)^r, \tag{4.6}$$

and hence (4.1), which can be written as

$$\Phi_r(s,\omega) = \sum_{n=1}^{\infty} G_r(n,\omega) n^{-s}, \tag{4.7}$$

is absolutely convergent for $\Re s > r$. Computations of expectation and variance give us :

Proposition 2. *The random series* (4.7) *converges for* $\Re s > r/2$ *almost surely.*

Proof. Let $E(\cdot)$ denote the expected value, and $V(\cdot)$ the variance. Write $X_n = G_r(n,\omega)n^{-s}$. From (4.6) we see that

$$\int_\Omega |X_n|^2 d\omega \le \left(\frac{n^{r-1}(\log n)^r}{n^\sigma}\right)^2 \int_\Omega d\omega < +\infty, \tag{4.8}$$

that is $X_n \in L^2(\Omega)$.

We use Theorem 2 in [11, Section 2 of Chapter 3], which asserts that if $X_n \in L^2(\Omega)$ are independent random variables satisfying $E(X_n) = 0$ and $\sum_{n=1}^\infty V(X_n) < +\infty$, then $\sum_{n=1}^\infty X_n$ converges almost surely.

Since

$$E(X_n) = \frac{1}{n^s} \sum_{m_1+\cdots+m_r=n} \Lambda(m_1)\cdots\Lambda(m_r) \int_\Omega \varepsilon_{m_1,\ldots,m_r}(\omega) d\omega, \tag{4.9}$$

and $\varepsilon_{m_1,\ldots,m_r}$ is an element of a Rademacher sequence, it is obvious that $E(X_n) = 0$ for any n. Next consider $V(X_n) = E(|X_n - E(X_n)|^2) = E(|X_n|^2)$. We see that

$$\begin{aligned} V(X_n) = \frac{1}{n^{2\sigma}} &\sum_{\substack{m_1+\cdots+m_r=n \\ m_1'+\cdots+m_r'=n}} \Lambda(m_1)\cdots\Lambda(m_r)\Lambda(m_1')\cdots\Lambda(m_r') \\ &\times \int_\Omega \varepsilon_{m_1,\ldots,m_r}(\omega)\varepsilon_{m_1',\ldots,m_r'}(\omega) d\omega. \end{aligned} \tag{4.10}$$

If $(m_1,\ldots,m_r) \ne (m_1',\ldots,m_r')$, then

$$P(\omega \in \Omega \mid \varepsilon_{m_1,\ldots,m_r}(\omega) = \pm 1,\ \varepsilon_{m_1',\ldots,m_r'}(\omega) = \pm 1) = \frac{1}{4}$$

for any choice of double signs, hence

$$\int_\Omega \varepsilon_{m_1,\ldots,m_r}(\omega)\varepsilon_{m_1',\ldots,m_r'}(\omega) d\omega = 0.$$

Therefore

$$\begin{aligned}V(X_n) &= \frac{1}{n^{2\sigma}} \sum_{m_1+\cdots+m_r=n} \Lambda(m_1)^2 \cdots \Lambda(m_r)^2 \int_\Omega \varepsilon_{m_1,\ldots,m_r}(\omega)^2 d\omega \\ &= \frac{1}{n^{2\sigma}} \sum_{m_1+\cdots+m_r=n} \Lambda(m_1)^2 \cdots \Lambda(m_r)^2,\end{aligned} \tag{4.11}$$

and the sum on the right-hand side can be estimated as $\leq n^{r-1}(\log n)^{2r}$. Therefore

$$\sum_{n=1}^{\infty} V(X_n) \leq \sum_{n=1}^{\infty} n^{r-1-2\sigma}(\log n)^{2r}, \tag{4.12}$$

which is convergent if $\sigma > r/2$. □

Proposition 2 thus gives the upper-bound $\sigma_c(\Phi_r) \leq r/2$. We also comment on its lower-bound. Since G_r is symmetric (as we have seen in the proof of Proposition 1), we have the criterion that $\sum_{n=1}^\infty X_n$ converges almost surely if and only if $\sum_{n=1}^\infty V(X_n')$ converges, where

$$X_n'(\omega) = \begin{cases} X_n(\omega) & \text{if} \quad |X_n(\omega)| \leq 1 \\ X_n(\omega)/|X_n(\omega)| & \text{if} \quad |X_n(\omega)| > 1 \end{cases} \tag{4.13}$$

([11, Theorem 7 in Section 5 of Chapter 3]).

In the case $r = 1$, we have $X_n(\omega) = \varepsilon_n(\omega)\Lambda(n)n^{-s}$. Hence if $\sigma > 0$ then $|X_n(\omega)| \leq 1$ for sufficiently large n, so we may assume $X_n' = X_n$ if $\sigma > 0$. From (4.11) we have

$$\sum_{n=1}^{\infty} V(X_n) = \sum_{n=1}^{\infty} \frac{\Lambda(n)^2}{n^{2\sigma}}, \tag{4.14}$$

which is convergent if and only if $\sigma > 1/2$. This implies that $\sigma_c(\Phi_1) = 1/2$, and hence, by Proposition 1, the line $\Re s = 1/2$ is the natural boundary of $\Phi_1(s,\omega)$ almost surely.

Next consider the case $r = 2$. Then

$$X_n(\omega) = \frac{1}{n^s} \sum_{m_1+m_2=n} \varepsilon_{m_1,m_2}\Lambda(m_1)\Lambda(m_2).$$

If $\sigma > 1$ then we may assume $X_n' = X_n$ for sufficiently large n. From (4.11) we have

$$\sum_{n=1}^{\infty} V(X_n) = \sum_{m_1+m_2=n} \frac{\Lambda(m_1)^2\Lambda(m_2)^2}{n^{2\sigma}}, \tag{4.15}$$

whose right-hand side is

$$\geq \frac{1}{n^{2\sigma}} \sum_{\substack{p_i,p_2:\text{prime} \\ p_1+p_2=n}} (\log p_1)^2(\log p_2)^2 \gg \frac{(\log n)^2}{n^{2\sigma}} \sum_{\substack{p_i,p_2:\text{prime} \\ p_1+p_2=n}} 1, \tag{4.16}$$

because one of p_1 or p_2 is $\geq n/2$. If the Hardy-Littlewood conjectural asymptotic formula for the Goldbach conjecture is true, then the last sum of (4.16) is $\gg n(\log n)^{-2}$. This implies that (4.15) diverges for $\sigma = 1$. Though $\sigma = 1$ is out of the range where $X'_n = X_n$ is valid, this argument suggests that $\sigma_c(\Phi_2) = 1$, and hence $\Re s = 1$ would be the natural boundary of $\Phi_2(s, \omega)$ almost surely. This is consistent with the existent conditional natural boundary of $\Phi_2(s)$.

The above observation in the cases $r = 1$ and $r = 2$ further suggests that in the case $r \geq 3$, perhaps $\sigma_c(\Phi_r) = r/2$ would hold.

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