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Tauberian Oscillation Theorems and the Distribution of Goldbach numbers

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Résumé. Soit $D(s) = \sum a_n n^{-s}$ une série de Dirichlet qui, en approchant par la droite un point non-réel sur la frontière de son domaine de convergence, tend vers ∞. Nous présentons un théorème d'oscillation localisé pour $\sum_{n\leq x} a_n$, et une application à $\sum_{n_1+\cdots+n_k\leq x} \Lambda(n_1)\cdots\Lambda(n_k)$.

Abstract. Let $D(s) = \sum a_n n^{-s}$ be a Dirichlet-series which tends to ∞ when approaching some non-real point on the boundary of its domain of convergence from the right. We give a localized oscillation theorem for $\sum_{n\leq x} a_n$, and apply this result to $\sum_{n_1+\cdots+n_k\leq x} \Lambda(n_1)\cdots\Lambda(n_k)$.

1. Introduction and Results

Let $D(s) = \sum a_n n^{-s}$ be a function represented by a Dirichlet series, holomorphic in a half-plane $\Re s > \sigma_0$. A large class of Tauberian theorems deal with the question of recovering the behaviour of $\sum_{n\leq N} a_n$ from the analytic behaviour of D. Under suitable assumptions one finds that the limit $s \searrow \sigma_0$ gives a main term, while the behaviour of D on the line $\Re s = \sigma_0$ determines the error term. If one is looking for more precise information, then the behaviour of D on the above line can be turned into oscillation type results. The first theorem of this type is Landau's theorem stating that if a_n is always positive, then D has a singularity at its abscissa of convergence. This result translates into oscillation type results for arithmetical functions, e.g. one immediately obtains $\Psi(x) = x + \Omega_\pm(x^{1/2-\epsilon})$ by exploiting the fact that $\frac{\zeta'}{\zeta}(s)$ is regular at $1/2$. Landau's original theorem may be used to prove the existence of infinitely many sign changes of a function. However it is impossible to deduce from it further information on the number of sign changes, the least sign change, localisation of sign changes and so forth. To do so would require more information on D and the use of less elementary methods. For a more detailed analysis of this problem we refer the reader

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to the introduction of [7], which could be seen as the first example of the use of non-elementary methods in this area.

To prove an effective version of Landau's theorem one requires more explicit data for D. For the rest of this article we fix the following notation. Put $F(s) = \int_1^\infty B(x)\,dx/x^{s+1}$, where B is a real function satisfying

$$\int_U^{2U} B(x)^2\,dx \ll U^{2\sigma_0+1}\eta(U), \tag{1.1}$$

for some positive, increasing, continuous function η with $\eta(U) \ll U^\epsilon$. Clearly, F is holomorphic for $\Re\, s > \sigma_0$. Assume that there exists a real number $t_0 \neq 0$ such that

$$c = \limsup_{\sigma \searrow \sigma_0} (\sigma - \sigma_0)|F(\sigma + it_0)| > 0, \tag{1.2}$$

and

$$a = \limsup_{\sigma \searrow \sigma_0} (\sigma - \sigma_0)F(\sigma) < \infty, \quad b = \liminf_{\sigma \searrow \sigma_0} (\sigma - \sigma_0)F(\sigma) > -\infty. \tag{1.3}$$

We denote the Lebesgue measure of a set of real numbers by μ. Then Kaczorowski & Szydło [6] proved the following :

Theorem 1.1. *Let B be a real function satisfying* (1.1)*. Define F as above, a and b by* (1.3)*, and assume that* (1.2) *holds true. Then for every $\epsilon > 0$ we have*

$$\mu\left(\{x \in [1,T] : B(x) > (b + c - \epsilon)x^{\sigma_0}\}\right) = \Omega\left(\frac{T}{\eta(T)}\right)$$

and

$$\mu\left(\{x \in [1,T] : B(x) < (a - c + \epsilon)x^{\sigma_0}\}\right) = \Omega\left(\frac{T}{\eta(T)}\right).$$

This theorem has the disadvantage that the points at which B is large may come in large blocks, so we get no explicit information on the location or even the number of sign changes. Here we prove the following.

Theorem 1.2. *Let B be a real function satisfying* (1.1)*. Define F as above, a and b by* (1.3)*, and assume that* (1.2) *holds true. Then we have, for every fixed $\epsilon > 0$ and as δ goes to 0,*

$$\int_1^\infty \mu(\{x \in [u,2u] : B(x) > (b + c - \epsilon)x^{\sigma_0}\}\frac{\eta(4u)du}{u^{2+\delta}} = \Omega(\delta^{-1}) \tag{1.4}$$

and

$$\int_1^\infty \mu(\{x \in [u,2u] : B(x) < (a - c + \epsilon)x^{\sigma_0}\}\frac{\eta(4u)du}{u^{2+\delta}} = \Omega(\delta^{-1}). \tag{1.5}$$

As an application we consider the error term for the summatory function of the number of representations of an integer as the sum of k primes. Put $G_k(n) = \sum_{n_1+\cdots+n_k=n} \Lambda(n_1)\cdot\dots\cdot\Lambda(n_k)$, $H_k(x) = -k\sum_\rho \frac{x^{k-1+\rho}}{\rho(1+\rho)\dots(k-1+\rho)}$, where the summation runs over all non-trivial zeros of the Riemann ζ function and define the error term $\Delta_k(x)$ by means of the formula

$$\sum_{n\leq x} G_k(n) = \frac{x^k}{k!} + H_k(x) + \Delta_k(x).$$

The function Δ_k was first considered by Fujii [5] who showed that under RH we have $\Delta_2(x) = \mathcal{O}((x\log x)^{4/3})$, which was later improved [3] to $\mathcal{O}(x\log^5 x)$ and finally by Languasco & Zaccagnini [8] to $\mathcal{O}(x\log^3 x)$ and for $k\geq 3$ to $\mathcal{O}(x^{k-1}\log^k x)$. On the other hand we have $\Delta_2(x) = \Omega(x\log\log x)$ (see [2]). The proof of the Ω-result uses the fact that G_2 gets surprisingly large, though $\Delta_2(x)$ is expected to be much more regular for most x. In fact, we have the following.

Proposition 1.3. *Assume RH. Then we have*

$$\lim_{\sigma\searrow k}(\sigma-k)\int_1^\infty \frac{\Delta_k(x)}{x^\sigma}\,dx = c_k$$

for some constant c_k. When $k=2$, we have

$$c_2 = \sum_\rho \frac{\pi}{\sin(\pi\rho)}$$

where the sum runs over all non-trivial zeros of ζ.

On the other hand we can apply Theorem 1.2 to show that the relation $\Delta_k(x) \asymp c_k x^{r-1}$ holds only in a very weak sense.

Proposition 1.4. *There exists a constant $c_k' > 0$, such that, as δ goes to 0, we have*

$$\int_1^\infty \mu(\{x\in[u,2u] : \Delta_k(x) > (c_k+c_k')x^{k-1}\})\frac{(\log u)^6 du}{u^{2+\delta}} = \Omega(\delta^{-1})$$

and

$$\int_1^\infty \mu(\{x\in[u,2u] : \Delta_k(x) < (c_k-c_k')x^{k-1}\})\frac{(\log u)^6 du}{u^{2+\delta}} = \Omega(\delta^{-1}).$$

Corollary 1.5. *We have $\Delta_2(x) = \Omega_\pm(x)$ and $\Delta_k(x) - c_k x^{k-1} = \Omega_\pm(x^{k-1})$ for $k\geq 3$.*

We mention that the same result was obtained in [3] with Ω in place of $\Omega_\pm$.

2. Proof of Theorem 1.2

We first note that (1.5) corresponds to (1.4) applied to $-B$ instead of B. We set $a' = a - \epsilon$ and consider the auxiliary functions

$$g(x) = B(x) - a'x^\theta, \quad G(s) = \int_1^\infty \frac{g(x)dx}{x^{s+1}} = F(s) - \frac{a'}{s-\theta}.$$

We introduce the positive and negative parts of g:

$$g_+(x) = \max(g(x), 0), \quad g_-(x) = \max(-g(x), 0), \quad (x \geq 1),$$

and the corresponding Mellin transforms

$$G_+(s) = \int_1^\infty \frac{g_+(x)dx}{x^{s+1}}, \quad G_-(s) = \int_1^\infty \frac{g_-(x)dx}{x^{s+1}}.$$

The reader will readily check that the three transforms G, G_+ and G_- and absolutely convergent when $\Re s > \theta$. We clearly have $G(s) = G_+(s) - G_-(s)$. The set we want to investigate is

$$A = \{x \geq 1 \,:\, g(x) > 0\}.$$

We define the following function:

$$m^\#(T) = \eta(T)\mu\{T \leq x \leq 2T \,:\, g(x) > 0\}/T. \tag{2.1}$$

Turning to the core of the proof, we assume that

$$\sum_{\ell \geq 0} \frac{m^\#(2^\ell)}{2^{\delta\ell}} = o(\delta^{-1}) \tag{2.2}$$

when $\delta > 0$. Our aim is to obtain a contradiction. We will then convert the negation of (2.2) into a "continuous" statement.

Let t be a real number and select $\sigma = \sigma_0 + \delta$ for some $\delta > 0$. We want to show that $(\sigma - \theta)|G_+(\sigma + it)|$ is small. In order to do so, we write

$$\begin{aligned}(\sigma - \sigma_0)|G_+(\sigma + it)| &\leq \delta\, G_+(\sigma) \\ &= \delta \int_A \frac{g(x)dx}{x^{\theta+\delta+1}} \\ &\leq \delta \sum_{k \geq 0} \int_{A \cap [2^k, 2^{k+1}]} \frac{g(x)dx}{x^{\theta+\delta+1}} \\ &\leq \delta \sum_{k \geq 0} \left(\eta(2^k) \int_{A \cap [2^k, 2^{k+1}]} \frac{dx}{x^{2\delta+1}} \right)^{1/2}.\end{aligned}$$

As a consequence, on simply using $\int_{A\cap[2^k,2^{k+1}]}\frac{dx}{x^{2\delta+1}} \le \int_{A\cap[2^k,2^{k+1}]}\frac{dx}{2^{(2\delta+1)k}}$, we have obtained

$$(2.3)\qquad \begin{aligned}(\sigma-\sigma_0)|G_+(\sigma+it)| &\le \delta\sum_{k\ge 0}\frac{\sqrt{m^\#(2^k)}}{2^{k\delta}}\\ &\le \delta\Big(\delta^{-1}sum_{k\ge 0}\frac{m^\#(2^k)}{2^{k\delta}}\Big)^{1/2} = o(1).\end{aligned}$$

We have thus established that

$$\lim_{\sigma\to\sigma^+}(\sigma-\sigma_0)|G_+(\sigma+it)| = 0.$$

To treat $G_-(\sigma+it)$ we simply use

$$\forall\sigma>\sigma_0,\quad |G_+(\sigma+it)|\le G_-(\sigma).$$

It remains for us to use our hypothesis concerning the existence of a singularity. There exists a sequence of σ tending to σ_0 from above for which

$$\begin{aligned}c &\le (\sigma-\sigma_0)|F(\sigma+it_0)| + o(1)\\ &\le (\sigma-\sigma_0)\Big|G(\sigma+it_0)+\frac{a'}{\sigma+it_0-\theta}\Big| + o(1)\\ &\le (\sigma-\sigma_0)|G(\sigma+it_0)| + o(1)\end{aligned}$$

since $t_0\neq 0$. We bound above $|G(\sigma+it_0)|$ by $|G(\sigma)|$, which is also $-G(\sigma)$, up to $o(1/(\sigma-\delta))$. We then employ $-G(\sigma) = -F(\sigma)+a'/(\sigma-\sigma_0)$. We finally reach

$$c\le a' - \liminf_{\sigma\to\theta^+}(\sigma-\sigma_0)F(\sigma)+o(1)$$

which contradicts our hypotheses.

To sum up, we have established that

$$(2.4)\qquad \sum_{\ell\ge 0}\frac{m^\#(2^\ell)}{2^{\delta\ell}} = \Omega(\delta^{-1})$$

and now we convert this to a 'continuous' form.

$$(2.5)\qquad \int_0^\infty m^\#(e^t)e^{-\delta t}dt = \Omega(\delta^{-1}).$$

The proof of (2.5) follows from (2.4). We notice that

$$\begin{aligned}
&\int_{\ell-1}^{\ell} m^{\#}(2^u)\frac{\eta(2^{u+2})du}{\eta(2^u)} \\
&= \int_{\ell-1}^{\ell}\int_{\substack{2^u\le t\le 2^{u+1},\\ g(t)>0}} \frac{\eta(2^{u+2})dt}{2^u}du \\
&= \eta(2^\ell)\int_{\substack{2^{\ell-1}\le t\le 2^{\ell},\\ g(t)>0}}\int_{t/2\le 2^u\le t}\frac{du}{2^u}dt + \eta(2^{\ell+1})\int_{\substack{2^{\ell}\le t\le 2^{\ell+1},\\ g(t)>0}}\int_{t/2\le 2^u\le t}\frac{du}{2^u}dt \\
&\ge m^{\#}(2^\ell) + m^{\#}(2^{\ell-1}).
\end{aligned}$$

The conclusion is now easy.

3. Application to Goldbach numbers

Let us put $\Phi_k(s) = \sum_{n\ge 1}\frac{G_k(n)}{n^s}$.

Lemma 3.1. *Suppose the RH holds. Then for any $k\ge 3$ there exist rational functions $f_{1,k}(s),\dots,f_{4,k}(s)$, such that*

$$\begin{aligned}
\Phi_k(s) &= f_{1,k}(s)\zeta(s-k+1) + f_{2,k}(s)\zeta(s-k+2) \\
&\qquad + f_{3,k}(s)\frac{\zeta'}{\zeta}(s-k+1) + f_{4,k}(s)\Phi_2(s-k+2) + R(s),
\end{aligned} \tag{3.1}$$

where $R(s)$ is holomorphic in the half-plane $\Re s > k-1-1/10$, and all the poles and zeros of the rational functions are real. Moreover, if we define $A_k^2(x) = \sum_{n\le x} G_k(n)(x-n)$, then there exist polynomials $g_{1,k},\dots g_{4,k}$, such that

$$\begin{aligned}
\sum_{n\ge 1} A_k^2(n)n^{-s} &= g_{1,k}(s)\zeta(s-k-1) + g_{2,k}(s)\zeta(s-k) \\
&\qquad + g_{3,k}(s)\frac{\zeta'}{\zeta}(s-k-1) + g_{4,k}(s)\Phi_2(s-k) + \tilde{R}(s),
\end{aligned} \tag{3.2}$$

where $\tilde{R}$ is holomorphic in the half-plane $\Re s > k+1-1/10$.

Proof. This is essentially [3, Theorem 3] together with the observation that the polynomials in [3, Lemma 2] have only real roots. □

The following is due to Egami-Matsumoto [4, Equation 4.3].

Lemma 3.2. *Assume the RH. Then we have*

$$\Phi_2(s) = -\frac{1}{\Gamma(s)}\sum_{\rho,\rho'}\frac{\Gamma(s+1-\rho)\Gamma(\rho)}{(s-\rho-\rho')\rho'} + R(s),$$

where R is a function holomorphic in the entire complex plane.

We now prove the first half of Proposition 1.3 and Corollary 1.5.

Let $\gamma_0 = 14.134\dots$ be the imaginary part of the first non-trivial zero of ζ. We claim that taking $t_0 = 2\gamma_0$ in Theorem 1.2 yields a non-trivial result. To do so we have to check that the conditions (1.2) and (1.3) are satisfied. It follows from Stirling's formula that $|\Gamma(\frac{1}{2}+it)| < e^{-\pi|t|/4}$. Hence if ρ, ρ' are non-trivial zeros of ζ, and s is a complex number with real part > 1 such that $|s-\rho-\rho'| < |\rho'|/2$ we have

$$\frac{\Gamma(s+1-\rho)\Gamma(\rho)}{(s-\rho-\rho')\rho'} \leq \frac{1}{\Re\, s - 1} e^{-\pi|\rho-1/2|/4},$$

whereas for $|s-\rho-\rho'| \geq |\rho'|/2$ we have

$$\frac{\Gamma(s+1-\rho)\Gamma(\rho)}{(s-\rho-\rho')\rho'} \leq \frac{2e^{-\pi|\rho-1/2|/4}}{|\rho'|^2}.$$

Clearly the sum over all ρ, ρ' of the latter form converges absolutely. For each fixed ρ there are $\mathcal{O}(|\rho|\log|\rho|)$ pairs of the former type, hence after multiplying with $\Re s - 1$ the sum over these pairs converges absolutely and uniformly as well, and we obtain

$$\lim_{\sigma\searrow 1}(\sigma-1)\sum_{\rho,\rho'} \frac{\Gamma(\sigma+it+1-\rho)\Gamma(\rho)}{(\sigma+it-\rho-\rho')\rho'} = \sum_{\rho,\rho'} \lim_{\sigma\searrow 1} \frac{(\sigma-1)\Gamma(\sigma+it+1-\rho)\Gamma(\rho)}{(\sigma+it-\rho-\rho')\rho'}$$

If $\rho+\rho' \neq 1+it$, then the denominator is bounded away from 0, while the numerator tends to 0, and we see that the limit is 0. If $\rho+\rho' = 1+it$, then $\sigma+it-\rho-\rho' = \sigma-1$, and using the functional equation of the Γ-function we obtain

$$\lim_{\sigma\searrow 1}(\sigma-1)\sum_{\rho,\rho'} \frac{\Gamma(\sigma+it+1-\rho)\Gamma(\rho)}{(\sigma+it-\rho-\rho')\rho'} = \sum_{\rho+\rho'=1+it} \Gamma(\rho')\Gamma(\rho)$$

Putting $t=0$ we obtain

$$(\sigma-1)\Phi_2(\sigma+it) \sim \frac{1}{\Gamma(1)}\sum_{\rho} |\Gamma(\rho)|^2 = 3.259\ldots\cdot 10^{-19},$$

which implies the "$k=2$"-part of Proposition 1.3.

For $t = 2\gamma_0$ the sum contains no pair ρ, ρ' with $|\Im\rho'| < 1000$ different from $\rho = \rho' = \frac{1}{2} + i\gamma_0$. To estimate the sum over zeros we use the estimates $N(T+1) - N(T) \leq \log T$, where $N(T)$ denotes the number of zeros of ζ with imaginary part in $(0,T]$, and $|\Gamma(1/2+it)| \leq e^{-\frac{\pi}{4}t}$. This bound follows from the work of Backlund [1], details have been given in [9, Lemma 3].

Hence

$$\begin{aligned}\sum_{\substack{\rho+\rho'=1+2i\gamma_0\\ \rho\neq\rho'}}\Gamma(\rho)\Gamma(1+2\gamma_1-\rho) &\leq \sum_{\Im\rho>100}\Gamma(\rho)\Gamma(1+2\gamma_1-\rho)\\ &\leq \sum_{\gamma>100} e^{-\frac{\pi}{2}(\gamma-\gamma_1)}\\ &\leq \sum_{n\geq 100} e^{-\frac{\pi(n-15)}{2}}\log n\\ &\leq e^{34}\sum_{10^3\leq n\leq 10^4} e^{-\frac{\pi n}{2}} + e^{24}\sum_{n>10^4} e^{-n}\\ &\leq \frac{e^{34-50\pi}}{1-e^{-\pi/2}}+\frac{e^{24-10^4}}{1-e^{-1}}\\ &\leq e^{-122}.\end{aligned}$$

Hence

$$\begin{aligned}\lim_{\sigma\searrow 1}(\sigma-1)\Phi_2(\sigma+2i\gamma_0) &= \frac{\Gamma^2(\frac{1}{2}+i\gamma_0)+\theta e^{-122}}{\Gamma(1+2i\gamma_0)}\\ &= 0.023049\ldots - 0.47088\ldots i+\theta e^{-122}.\end{aligned} \tag{3.3}$$

where θ is a complex number satisfying $|\theta|\leq 1$. Hence (1.2) holds true with some $c\geq 0.4713$

We conclude that $\Delta_2(x)-3.259\cdot 10^{-19}$ is infinitely often larger than 0.4713 and smaller then -0.4713. Since $0.4713 > 3.259\cdot 10^{-19}$, the first part of Corollary 1.5 follows.

For the general case of Proposition 1.3 we use (3.1) to see that there exists some integer A, such that

$$(\sigma-k)\int_1^\infty \frac{\Delta_k(x)}{x^\sigma\log^A x}\,dx = c_k,$$

where $A+1$ equals the total order of the pole of the right hand side of (3.1). Strictly speaking, Φ_2 does not have a pole at 1, however, we have just seen that as $\sigma\searrow 1$, $\Phi_2(\sigma)$ behaves like $\frac{3.259\ldots\cdot 10^{-19}}{(\sigma-1)}$, which is sufficiently pole-like. However, it then follows from the fact that the functions $g_{i,k}$ are polynomials, that $A=0$, thus the general case of Proposition 1.3 follows.

To obtain Corollary 1.5 for $k\geq 3$, we first note that Proposition 1.3 implies that Condition (3) is satisfied, while (3.3) together with the fact that the zeros of the functions $f_{i,k}$ are all real implies that Condition (2) is satisfied for $\sigma_0=k-1$ and $t_0=2\gamma$. Hence we can apply Theorem 1.2, and our claim follows.

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