(06/11-Sa80)

Forum Math. [redacted] (2006), 1–29
DOI 10.1515/FORUM.2006.[redacted]

Forum Mathematicum

Meromorphic continuation of multivariable Euler products

Gautami Bhowmik, Driss Essouabri and Ben Lichtin

(Communicated by Peter Sarnak)

Abstract. This article extends classical one variable results about Euler products, defined by integral valued polynomial or analytic functions, to several variables. We show there exists a meromorphic continuation up to a presumed natural boundary, and give a criterion, à la Estermann-Dahlquist, for the existence of a meromorphic extension to $\mathbb{C}^n$. In addition, we precisely describe the boundaries of analyticity and meromorphy for a multivariable Euler product determined by any toric variety (split over $\mathbb{Q}$). Using our method, we are also able to calculate a precise asymptotic for the number of n-fold products of integers that equal the n^{th} power of an integer, for any $n \geq 3$.

2000 Mathematics Subject Classifications: 11M41, 11N37, 14G05, 32D15.

Introduction

There are two fundamental problems in the study of Dirichlet series that admit an Euler product expansion in a region of absolute convergence. The first problem is to prove the existence of a meromorphic continuation into a larger region. Assuming this is possible, the second problem is to describe precisely the boundary of the domain for this meromorphic function. For Dirichlet series in one variable, the first important results are due to Estermann [6] who proved that if $h(Y) = \sum_d F(d) Y^d$, where $F(d)$ is a "ganzwertige" polynomial and $F(0) = 1$, then $Z(s) = \prod_p h(p^{-s})$ is absolutely convergent for $\Re(s) > 1$ and can be meromorphically continued to the half plane $\Re(s) > 0$. Moreover, $Z(s)$ can be continued to the whole complex plane if and only if $h(Y)$ is a cyclotomic polynomial. Dahlquist [2] extended this result to h any analytic function with isolated singularities within the unit circle. More than 30 years later, Kurokawa's deep work [9] extended that of Estermann by allowing polynomials $h(Y)$ whose coefficients were integral linear combinations of complex numbers that depended upon the traces of a certain class of representations of a topological group.

This paper extends these two basic properties to a general class of multivariate Dirichlet series that have an absolutely convergent Euler product expansion in some open domain of $\mathbb{C}^n$, $n \geq 2$. Thus, the object of our study is an Euler product

$$Z(h;\boldsymbol{s}) = \prod_p h(p^{-s_1}, \dots, p^{-s_n}, p)$$

when $h(X) = 1 + \sum_k h_k(X_1, \dots, X_n)X_{n+1}^k$ is either a polynomial or analytic function with integral coefficients. On pg. 28 [ibid.], Kurokawa asserted that he had proved certain multivariate analogues of his one variable results. To our knowledge, these have not yet been published. As a result, a multivariate extension such as that done here appears to be new. An essential role in our analysis is played by the polyhedra in $\mathbb{R}^n$, determined by the exponents of monomials appearing in the expression for each summand $h_k(X_1, \dots, X_n)X_{n+1}^k$. A variant of this polyhedron is a standard tool for studying hypersurface singularities, so it is, perhaps, not too surprising to see it appear here as well.

We first show in Section 1 that there is a meromorphic continuation up to a presumed natural boundary, whose geometry is that of a tube over a convex set with piecewise linear boundary. Using the polyhedra, this is not difficult. Our second main result applies to the case in which h depends only upon $X_1, \dots, X_n$. In this event, we prove a very precise result that is the multivariate extension of the work of Estermann-Dahlquist. This shows that the presumed natural boundary is *the* natural boundary (in the sense given to this expression in §1.2), unless h is a "cyclotomic" polynomial. A natural problem, to which we hope to return in the future, is to extend this result to the much larger class of multivariable Euler products of interest to Kurokawa.

An application of these results is given in Section 2 to a general problem in multiplicative number theory. For *any* $n \geq 3$, we give the *explicit* asymptotic for the number of n-fold products of relatively prime positive integers that equal the n^{th} power of an integer. Although earlier work had found the asymptotic when $n = 3$, nothing comparable for arbitrary $n > 3$ seems to have been published. As noted by Batyrev-Tschinkel, see [10, pg. 253], this problem is equivalent to the asymptotic description of a "height density function" on the maximal torus of the singular projective hypersurface $x_1 \cdots x_n = x_{n+1}^n$ in $\mathbb{P}^n(\mathbb{Q})$.

In general terms, any ample line bundle $\mathscr{L}$ on a projective toric variety $X(\mathbb{Q})$ determines a projective embedding of a maximal split torus $\imath_{\mathscr{L}} : U(\mathbb{Q}) \hookrightarrow \mathbb{P}^{n-1}(\mathbb{Q})$, for some n, and therefore a parametrization of the points of $\imath_{\mathscr{L}} U(\mathbb{Q})$. Using the standard height function $H(\boldsymbol{x}) = \prod_p \max_i\{|x_i|_p\}$ on $\mathbb{P}^{n-1}(\mathbb{Q})$, a natural problem, posed first by Manin, is to give a precise asymptotic for the height density function $\#\{\boldsymbol{x} \in \imath_{\mathscr{L}} U(\mathbb{Q}) : H(\boldsymbol{x}) \leq t\}$ as $t \to \infty$. This problem reduces to the asymptotic in t for the number of primitive lattice points $(m_1, \dots, m_n)$ such that $(m_1 : \dots : m_n) \in \imath_{\mathscr{L}} U(\mathbb{Q})$ and $\max_i\{|m_i|\} \leq t$. Descriptions of the asymptotic have been given with increasing levels of precision by [1], [10], and [4].

The starting point of each of these articles is with a desingularized model of the toric variety, constructed by means of a "fan decomposition" of some $\mathbb{R}^N$ into finitely many simplicial integral cones (i.e. each cone is generated by N 1-simplex integral vectors that generate $\mathbb{Z}^N$). Typically, one chooses for $\mathscr{L}$ the anticanonical bundle on the desingularized model, and assumes it is ample. This was the approach taken by Batyrev-Tschinkel for the particular cubic hypersurface $x_1x_2x_3 = x_4^3$.

The point of view adopted in this paper is rather different. We do not work with a desingularized model of the toric. Rather, our starting point consists of a finite set of simple defining equations for a toric variety $X(\mathbb{Q})$ that determines implicitly a projective embedding $U(\mathbb{Q}) \hookrightarrow \mathbb{P}^{n-1}(\mathbb{Q})$. As a result, we avoid having to construct and use a fan decomposition of Euclidean space N-space into simplical integral cones, which can become rather cumbersome when N is allowed to be arbitrarily large. In addition, it follows that we have no need for the hypothesis of ampleness of any line bundle to begin our analysis. An advantage of our method is that we can then work with some explicit examples in any number of variables, such as the hypersurface $\{x_1 \cdots x_n = x_{n+1}^n\}$, $n \geq 3$, with a certain facility and reasonable precision.

We then adapt an idea of La Bretèche [4] by introducing a multivariable Dirichlet series that encodes membership of each rational point on the embedded torus. The multiplicative nature of the defining equations implies that this series equals an Euler product in its domain of analyticity. Our Dirichlet series is rather different from that used in [ibid.] since we use a different embedding of the variety. We can also say a good deal more about this series than is done in [ibid.] (see the Remark at the end of §2.2 for further precision on this point).

We prove three basic analytical properties of our Dirichlet series in §2.2. The first two are given in Theorem 5, whose proof follows immediately from the discussion in Section 1. Here we show the existence of a meromorphic extension outside the domain of absolute convergence. In addition, we give a precise criterion for the existence of a natural boundary. The third result, Theorem 6, requires considerably more work to prove. This gives an intrinsic characterization of the *entire* boundary of the domain of analyticity of the Dirichlet series. Combining these two theorems results in a fairly complete description of the analytic behavior of this class of multivariable Dirichlet series. In §2.3, Theorem 7 gives the precise asymptotic for the general problem, described above, in multiplicative number theory. For this, an important ingredient is the tauberian theorem, Théorème 2, of [5].

Notations. For the reader's convenience, notations that will be used throughout the article are assembled here.

1. $\mathbb{N} = \{1, 2, \ldots\}$ denotes the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and p always denotes a prime.

2. The expression $f(\lambda, \boldsymbol{y}, \boldsymbol{x}) \ll_{\boldsymbol{y}} g(\boldsymbol{x})$ uniformly in $\boldsymbol{x} \in X$ and $\lambda \in \Lambda$ means there exists $A = A(\boldsymbol{y}) > 0$, which depends neither on $\boldsymbol{x}$ nor λ, but could eventually depend on the parameter vector $\boldsymbol{y}$, such that:

$$\forall \boldsymbol{x} \in X \textit{ and } \forall \lambda \in \Lambda \quad |f(\lambda, \boldsymbol{y}, \boldsymbol{x})| \leq A g(\boldsymbol{x}).$$

3. For every $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we set $\|\boldsymbol{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$ resp. $|\boldsymbol{x}| = |x_1| + \cdots + |x_n|$ to denote the length resp. weight of $\boldsymbol{x}$. We denote the canonical basis of $\mathbb{R}^n$ by $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$. For every $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, we also set $\alpha! = \alpha_1! \ldots \alpha_n!$. The standard inner product on $\mathbb{R}^n$ is denoted $\langle \, , \rangle$.

(AutoPDF V7 2/6/06 14:27) WDG (170×240mm) Tmath J-1468 Forum, : PMU: S(C) 19/05/2006 pp. 1–29 1468_06-11 (p. 3)

4. For every $s \in \mathbb{C}$, and for every non negative k, we define $\binom{s}{k} = \frac{s(s-1)\dots(s-k+1)}{k!}$. For two complex numbers w and z, we define $w^z = e^{z \log w}$, using the principal branch of the logarithm. We denote a vector in $\mathbb{C}^n$ by $\boldsymbol{s} = (s_1, \dots, s_n)$, and write $\boldsymbol{s} = \sigma + i\tau$, where $\sigma = (\sigma_1, \dots, \sigma_n)$ and $\tau = (\tau_1, \dots, \tau_n)$ are the real resp. imaginary components of $\boldsymbol{s}$ (i.e. $\sigma_i = \Re(s_i)$ and $\tau_i = \Im(s_i)$ for each i). We also write $\langle \boldsymbol{x}, \boldsymbol{s} \rangle$ for $\sum_i x_i s_i$ if $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{s} \in \mathbb{C}^n$. The unit polydisc in $\mathbb{C}^n$, that is, the domain $\{\boldsymbol{z} \in \mathbb{C}^n : \sup_i |z_i| < 1\}$, is denoted $P(1)$.

5. Given $\alpha \in \mathbb{N}_0^n$, we write X^α for the monomial $X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. For a polynomial $h(X_1, \dots, X_n) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha X^\alpha$, the set $S(h) := \{\alpha : a_\alpha \neq 0\}$ is called the *support of* h. We set $S^*(h) := S(h) \setminus \{0\}$ and denote the boundary of the convex hull of $\bigcup \{\alpha + \mathbb{R}^n : \alpha \in S^*(h)\}$ by $\mathscr{E}(h)$. This is called the *Newton polyhedron* of h. We denote by $Ext(h)$ the finite set of extremal points of $\mathscr{E}(h)$ (a point of $\mathscr{E}(h)$ is extremal if it does not belong to the interior of any closed segment of $\mathscr{E}(h)$).

Similarly, if $A \subset \mathbb{N}_0^n \setminus \{0\}$, we denote by $\mathscr{E}(A)$ the boundary of the convex hull of $\bigcup \{v + \mathbb{R}_+^n \mid v \in A\}$ and call it the *Newton polyhedron* of A. Its set of extremal points is denoted by $Ext(A)$.

6. Let A be a finite subset of $\mathbb{R}^n$. We set $A^o := \{\boldsymbol{x} \in \mathbb{R}_+^n : \forall v \in A, \langle \boldsymbol{x}, v \rangle \geq 1\}$ to denote the dual of A. Let $\imath(A)$ be the smallest weight of the elements of A^o. We call $\imath(A)$ the index of A. We define

$$R(A) := \{\alpha \in A^o : |\alpha| = \imath(A)\}.$$

For every $\alpha \in R(A)$, set $E(A, \alpha) := \{v \in A : \langle \alpha, v \rangle = 1\}$.

1 Analytic properties of multivariate Euler products

This section studies the analytic properties of an Euler product whose p^{th} factor is determined by a multivariate polynomial. We first show in §1.1 the existence of a meromorphic continuation from a region of absolute convergence into a product of halfplanes. The second result in §1.2 extends the classical Estermann criterion. This gives a criterion that insures the existence of a meromorphic continuation to $\mathbb{C}^n$.

1.1 Meromorphic continuation

We will first introduce some needed notations. Let Λ be an open subset of $\mathbb{C}^n$, $\boldsymbol{l} = (l_1, \dots, l_r) : \Lambda \to \mathbb{C}^r$ a vector of analytic functions, and $a_1, \dots, a_r$ integers. Define the Euler product

$$Z_{\boldsymbol{l}}(\boldsymbol{s}) = Z_{\boldsymbol{l}}(s_1, \dots, s_n) = \prod_p \left(1 + \sum_{k=1}^r \frac{a_k}{p^{l_k(\boldsymbol{s})}}\right),$$

and for any $\delta \in \mathbb{R}$, set

(AutoPDF V7 2/6/06 14:27) WDG (170×240mm) Tmath J-1468 Forum, : PMU: S(C) 19/05/2006 pp. 1–29 1468_06-11 (p. 4)

$$W(\boldsymbol{l};\delta) := \{\boldsymbol{s} \in \Lambda : \forall i = 1,\ldots,r\ \Re(l_i(\boldsymbol{s})) > \delta\}.$$

It is clear that $s \mapsto Z_{\boldsymbol{l}}(\boldsymbol{s})$ converges absolutely and defines a holomorphic function in the domain $W(\boldsymbol{l};1)$.

Lemma 1. *The function $Z_{\boldsymbol{l}}(\boldsymbol{s})$ can be continued into the domain $W(\boldsymbol{l};0)$ as a meromorphic function as follows:*

there exists a set $\{\gamma(\boldsymbol{n}) : \boldsymbol{n} \in \mathbb{N}_0^r\} \subset \mathbb{Z}$ and for each $\delta > 0$ a holomorphic function $G_\delta(\boldsymbol{s})$ that is expressible as an absolutely convergent and bounded Euler product on $W(\boldsymbol{l};\delta)$ such that

$$(1)\qquad Z_{\boldsymbol{l}}(\boldsymbol{s}) = \prod_{\substack{\boldsymbol{n}=(n_1,\ldots,n_r)\in\mathbb{N}_0^r\\ 1\le|\boldsymbol{n}|\le[\delta^{-1}]}} \zeta\left(\sum_{j=1}^r n_j l_j(\boldsymbol{s})\right)^{\gamma(\boldsymbol{n})} G_\delta(\boldsymbol{s}).$$

Proof of Lemma 1. Let $\delta \in (0,1)$ be arbitrary. To describe the continuation of $Z_{\boldsymbol{l}}(\boldsymbol{s})$ into $W(\boldsymbol{l};\delta)$, it will be convenient to work with a somewhat larger class of Euler products defined as follows:

$$(2)\qquad Z_{\boldsymbol{l}}(R_\delta;\boldsymbol{s}) = \prod_p \left(1 + \sum_{k=1}^r \frac{a_k}{p^{l_k(\boldsymbol{s})}} + R_\delta(p;\boldsymbol{s})\right)$$

where for all $p, s \mapsto R_\delta(p;\boldsymbol{s})$ is a holomorphic function on $W(\boldsymbol{l};\delta)$ satisfying $R_\delta(p;\boldsymbol{s}) \ll_{\boldsymbol{l},\delta} p^{-2}$ *uniformly in p and* $\boldsymbol{s} \in W(\boldsymbol{l};\delta)$. Evidently, $Z_{\boldsymbol{l}}(\boldsymbol{s}) = Z_{\boldsymbol{l}}(R_\delta;\boldsymbol{s})$ when $R_\delta(p;\boldsymbol{s}) \equiv 0$.

We next fix these notations:

1. For each $m \in \mathbb{N}$, set

$$\mathscr{L}_m(\boldsymbol{l}) = \mathscr{L}_m(l_1,\ldots,l_r) := \{n_1 l_1 + \cdots + n_r l_r : n_1 + \cdots + n_r \ge m\};$$

2. $N = [2\delta^{-1}]$;

3. $L(\boldsymbol{s}) := \prod_{k=1}^r \zeta(l_k(\boldsymbol{s}))^{-a_k}$ for $\boldsymbol{s} \in W(\boldsymbol{l};1)$.

By elementary computations, we obtain that for any $\boldsymbol{s} \in W(\boldsymbol{l};1)$:

$$L(\boldsymbol{s}) = \prod_p \prod_{k=1}^r \left(1 + \sum_{v_k=1}^N \frac{\binom{a_k}{v_k}(-1)^{v_k}}{p^{v_k l_k(\boldsymbol{s})}} + H_N^k(p;\boldsymbol{s})\right)$$

where, $\forall k = 1,\ldots,r$, $\boldsymbol{s} \mapsto H_N^k(p;\boldsymbol{s})$ is a holomorphic function in $W(\boldsymbol{l};\delta)$ such that: $H_N^k(p;\boldsymbol{s}) \ll_N p^{-\delta(N+1)} \ll_N p^{-2}$ *uniformly in p and* $\boldsymbol{s} \in W(\boldsymbol{l};\delta)$.

It is also clear that if $a_k \in \mathbb{N}$, then $H_N^k = 0$ once $N > a_k$.

Thus, there exist $f_1, \dots, f_m \in \mathscr{L}_2(\boldsymbol{l})$ and $d_1, \dots, d_m \in \mathbb{Z}$ such that:

$$L(\boldsymbol{s}) = \prod_p \left(1 - \sum_{k=1}^{r} \frac{a_k}{p^{l_k(\boldsymbol{s})}} + \sum_{i=1}^{m} \frac{d_i}{p^{f_i(\boldsymbol{s})}} + K_N(p;\boldsymbol{s})\right)$$

where $\boldsymbol{s} \mapsto K_N(p;\boldsymbol{s})$ is a holomorphic function in $W(\boldsymbol{l};\delta)$ that satisfies: $K_N(p;\boldsymbol{s}) \ll_N p^{-2}$ *uniformly in* p *and* $\boldsymbol{s} \in W(\boldsymbol{l},\delta)$.

Now an easy computation shows that for every $\boldsymbol{s} \in W(\boldsymbol{l},1)$:

$$\begin{aligned} &Z_{\boldsymbol{l}}(R_\delta;\boldsymbol{s})L(\boldsymbol{s}) \\ &= \prod_p \left(1 + \sum_{k=1}^{r} \frac{a_k}{p^{l_k(\boldsymbol{s})}} + R_\delta(p;\boldsymbol{s})\right)\left(1 - \sum_{k=1}^{r} \frac{a_k}{p^{l_k(\boldsymbol{s})}} + \sum_{i=1}^{m} \frac{d_i}{p^{f_i(\boldsymbol{s})}} + K_N(p;\boldsymbol{s})\right) \\ &= \prod_p \left(1 + \sum_{i=1}^{m} \frac{d_i}{p^{f_i(\boldsymbol{s})}} - \sum_{k_1=1}^{r}\sum_{k_2=1}^{r} \frac{a_{k_1}a_{k_2}}{p^{l_{k_1}(\boldsymbol{s})+l_{k_2}(\boldsymbol{s})}} + \sum_{k=1}^{r}\sum_{i=1}^{m} \frac{a_k d_i}{p^{l_k(\boldsymbol{s})+f_i(\boldsymbol{s})}} + V_N(p;\boldsymbol{s})\right) \end{aligned}$$

where $\boldsymbol{s} \mapsto V_N(p;\boldsymbol{s})$ is a holomorphic function in $W(\boldsymbol{l};\delta)$ that satisfies the bound: $V_N(p;\boldsymbol{s}) \ll_N p^{-2}$ *uniformly in* p *and* $\boldsymbol{s} \in W(\boldsymbol{l};\delta)$.

We have thus proved that there exist:

1. $g_1, \dots, g_\mu \in \mathscr{L}_2(\boldsymbol{l})$ and integers $c_1, \dots, c_\mu$;

2. for each p a holomorphic function $\boldsymbol{s} \mapsto R_{\delta,2}(p;\boldsymbol{s})$ on $W(\boldsymbol{l};\delta)$, that satisfies $R_{\delta,2}(p;\boldsymbol{s}) \ll_\delta p^{-2}$ *uniformly in* p *and* $\boldsymbol{s} \in W(\boldsymbol{l};\delta)$,

such that for every $\boldsymbol{s} \in W(\boldsymbol{l};1)$:

$$(3) \qquad Z_{\boldsymbol{l}}(R_\delta;\boldsymbol{s}) \prod_{k=1}^{r} \zeta(l_k(\boldsymbol{s}))^{-a_k} = \prod_p \left(1 + \sum_{k=1}^{\mu} \frac{c_k}{p^{g_k(\boldsymbol{s})}} + R_{\delta,2}(p;\boldsymbol{s})\right).$$

Since each $g_k \in \mathscr{L}_2(\boldsymbol{l})$, it is clear that $\Re(g_k(\boldsymbol{s})) > 1$ for any $\boldsymbol{s} \in W\left(\boldsymbol{l};\frac{1}{2}\right)$ and $k = 1, \dots, \mu$. This implies that for any $\delta' > \max\left(\frac{1}{2},\delta\right)$:

$$\boldsymbol{s} \mapsto \prod_p \left(1 + \sum_{k=1}^{\mu} \frac{c_k}{p^{g_k(\boldsymbol{s})}} + R_{\delta,2}(p;\boldsymbol{s})\right)$$

is an absolutely convergent and bounded Euler product that is holomorphic in the domain $W(\boldsymbol{l};\delta')$.

It is now evident how to proceed by induction. Let $M = [\log_2(N+1)] + 1 \in \mathbb{N}$. Repeating the above process M times, we conclude that there exist:

1. functions $h_1, \dots, h_q \in \mathscr{L}_1(\boldsymbol{l})$ and integers $\gamma_1, \dots, \gamma_q$;

2. functions $u_1, \dots, u_v \in \mathscr{L}_{2^M}(\boldsymbol{l})$ and integers $b_1, \dots, b_v$;

3. for each p, a holomorphic function $\boldsymbol{s} \mapsto R_{\delta,M}(p;\boldsymbol{s})$ on $W(\boldsymbol{l};\delta)$, satisfying $R_{\delta,M}(p;\boldsymbol{s}) \ll_\delta p^{-2}$ uniformly in p and $\boldsymbol{s} \in W(\boldsymbol{l};\delta)$

such that for every $\boldsymbol{s} \in W(\boldsymbol{l};1)$:

$$(4) \qquad Z_{\boldsymbol{l}}(R_\delta;\boldsymbol{s}) \prod_{k=1}^{q} \zeta(h_k(\boldsymbol{s}))^{-\gamma_k} = \prod_p \left(1 + \sum_{k=1}^{v} \frac{b_k}{p^{u_k(\boldsymbol{s})}} + R_{\delta,M}(p;\boldsymbol{s})\right),$$

where the right side is absolutely convergent and bounded on $W(\boldsymbol{l};\delta)$ since $2^{-M} < \delta/2$. We now define

$$G_\delta(\boldsymbol{s}) := Z_{\boldsymbol{l}}(R_\delta;\boldsymbol{s}) \cdot \left(\prod_{h_k \notin \mathscr{L}_{N+1}} \zeta(h_k(\boldsymbol{s}))^{-\gamma_k}\right).$$

In $W(\boldsymbol{l};1)$, it follows that

$$G_\delta(\boldsymbol{s}) = \prod_{\{k: h_k \in \mathscr{L}_{N+1}(\boldsymbol{l})\}} \zeta(h_k(\boldsymbol{s}))^{\gamma_k} \cdot \prod_p \left(1 + \sum_{k=1}^{v} \frac{b_k}{p^{u_k(\boldsymbol{s})}} + R_{\delta,M}(p;\boldsymbol{s})\right).$$

The preceding shows that the Euler product on the right is absolutely convergent and is bounded in $W(\boldsymbol{l};\delta)$. In addition, since $h_k \in \mathscr{L}_{N+1}(\boldsymbol{l})$ implies $\Re(h_k(\boldsymbol{s})) > (N+1)\delta > 2$, the product over k also admits an analytic continuation into $W(\boldsymbol{l};\delta)$ as an absolutely convergent Euler product. Thus, $G_\delta(\boldsymbol{s})$ admits an analytic continuation from $W(\boldsymbol{l};1)$ into $W(\boldsymbol{l};\delta)$ as an absolutely convergent and bounded Euler product. This completes the proof of Lemma 1. □

Let $h_0, \dots, h_d$ be polynomials in $\mathbb{Z}[X_1, \dots, X_n]$. Define

$$h(X_1, \dots, X_n, X_{n+1}) = 1 + \sum_{k=0}^{d} h_k(X_1, \dots, X_n) X_{n+1}^k,$$

$$Z(h;\boldsymbol{s}) = \prod_p h(p^{-s_1}, \dots, p^{-s_n}, p).$$

Given $h, h_0, \dots, h_d$, and $\delta \in \mathbb{R}$, we set:

$$V(h;\delta) := \bigcap_{k=0}^{d} \{\boldsymbol{s} \in \mathbb{C}^n : \langle \alpha, \sigma \rangle > k + \delta \ \forall \alpha \in Ext(h_k)\}.$$

Theorem 1. *$\boldsymbol{s} \mapsto Z(h;\boldsymbol{s})$ can be continued meromorphically from $V(h;1)$ (where $Z(h;\boldsymbol{s})$ converges absolutely), into $V(h;0)$.*

Proof. Apply the proof of Lemma 1 using the map $\boldsymbol{l}$, defined as follows. Writing $h_k = \sum_{\alpha \neq 0} a_{\alpha,k} X_1^{\alpha_1} \ldots X_n^{\alpha_n}$, set

$$\boldsymbol{l} = (l_{\alpha,k})_{(\alpha,k)}, \quad \text{where } l_{\alpha,k}(\boldsymbol{s}) = \langle \alpha, \boldsymbol{s} \rangle - k \quad \text{iff } \alpha \in S(h_k).$$

It is clear that for any δ, $\boldsymbol{s} \in W(\boldsymbol{l};\delta)$ if and only if $\boldsymbol{s} \in V(h;\delta)$. The proof then follows from the expression (1) for the continuation of $Z(h;\boldsymbol{s})$ into each $V(h;\delta)$, $\delta > 0$.

1.2 The natural boundary

This subsection studies the natural boundary of an Euler product

$$Z(h;\boldsymbol{s}) = \prod_p h(p^{-s_1}, \ldots, p^{-s_n}) \quad \text{where } h = 1 + \sum_{\alpha \neq 0} a_\alpha X^\alpha \in \mathbb{Z}[X_1, \ldots, X_n].$$

Theorem 1 has shown that $Z(h;\boldsymbol{s})$ can be meromorphically continued to $V(h;0)$. Of interest here are conditions satisfied by h that imply $Z(h;\boldsymbol{s})$ can or cannot be extended still further. We use the expression "$\partial V(h;0)$ is the natural boundary of $Z(h;\boldsymbol{s})$" to mean that $Z(h;\boldsymbol{s})$ can not be continued meromorphically into $V(h;\delta)$ for any $\delta < 0$.

In addition, we say that h is "cyclotomic" if there exists a finite set $(\boldsymbol{m}_j)_{j=1}^q$ of elements of $\mathbb{N}_0^n \backslash \{0\}$, and a finite set of integers $\{\gamma_j\}_{j=1}^q$ such that:

$$h(X) = \prod_{j=1}^q (1 - X^{\boldsymbol{m}_j})^{\gamma_j} = \prod_{j=1}^q (1 - X_1^{m_{1,j}} \ldots X_n^{m_{n,j}})^{\gamma_j}.$$

The following result extends Estermann's well known criterion [6] to several variables.

Theorem 2. *$Z(h;\boldsymbol{s})$ can be continued to $\mathbb{C}^n$ as a meromorphic function if and only if h is cyclotomic. In all other cases $\partial V(h;0)$ is the natural boundary.*

Proof. It is clear that if h is cyclotomic then $Z(h;\boldsymbol{s})$ has a meromorphic extension to $\mathbb{C}^n$. So, it suffices to prove the converse. To do so, it suffices to assume only that $Z(h;\boldsymbol{s})$ admits a meromorphic extension to $V(h;\delta_0)$ for some $\delta_0 < 0$. The argument to follow will then show that h must be cyclotomic, from which it follows immediately that $Z(h;\boldsymbol{s})$ is meromorphically extendible to $\mathbb{C}^n$.

It will first be convenient to reduce to the case in which $V(h;1) \cap \mathbb{R}^n \subset (0,\infty)^n$. By a permutation of coordinates, one can suppose that: $\{k \in \{1,\ldots,n\} : \exists a \in \mathbb{N}$ s.t. $a\boldsymbol{e}_k \in$

$S^*(h)\} = \{1, \ldots, r\}$. If the set is empty, then $r = 0$. It is clear that if $r = n$, then $V(h;1) \cap \mathbb{R}^n \subset (0, \infty)^n$.

Let us then suppose that $r < n$. We set

$$h^*(X_1, \ldots, X_n) = h(X_1, \ldots, X_n) \prod_{k=r+1}^{n} (1 - X_k).$$

A straightforward calculation, left to the reader, now shows that for each $k = 1, \ldots, n$, there exists a smallest positive integer c_k such that $c_k \boldsymbol{e}_k \in S^*(h^*)$. In particular, $c_k = 1$ if $k \geq r+1$. Moreover, it follows immediately that $V(h^*;1) \cap \mathbb{R}^n \subset (0, \infty)^n$, and $\sigma_k > \frac{1}{c_k}$ for each $k \geq 1$ implies:

$$(5) \qquad Z(h; \boldsymbol{s}) \prod_{k=r+1}^{n} \zeta(s_k)^{-1} = Z(h^*; \boldsymbol{s}).$$

Suppose that the theorem has been proved for h^*, *and* that there exists $\delta_0 < 0$ such that $\boldsymbol{s} \mapsto Z(h; \boldsymbol{s})$ can be meromorphically continued to $V(h; \delta_0)$. We set $\delta_1 = \delta_0/2 \cdot (\sup_{\alpha \in S^*(h)} \{\sum_{k=1}^{n} \alpha_k / c_k\})$. It is easy to check (exercise left to reader) that $V(h^*; \delta_1) \subset V(h; \delta_0)$. This, together with (5), then implies that $\boldsymbol{s} \mapsto Z(h^*; \boldsymbol{s})$ can be meromorphically continued to $V(h^*; \delta_1)$. It then follows that h^* is cyclotomic, from which it is clear that h must also be cyclotomic.

Thus, we may assume that a vector $c_k \boldsymbol{e}_k$ appears in $S^*(h)$ for each $k \geq 1$. We also denote the elements of $Ext(h)$ by setting $Ext(h) = \{\alpha_1, \ldots, \alpha_q\}$.

By Theorem 1, the expression for (the continuation of) $Z(h; \boldsymbol{s})$ into each $V\left(h; \frac{1}{r}\right)$, $r = 1, 2, \ldots$ is given by an equation (a priori, valid in $V(h;1)$)

$$(6) \qquad Z(h; \boldsymbol{s}) = \left(\prod_{\substack{m \in \mathbb{N}_0^n \\ 1 \leq |\boldsymbol{m}| \leq N_r}} \zeta(\langle \boldsymbol{m}, \boldsymbol{s} \rangle)^{\gamma(\boldsymbol{m})} \right) \times G_{1/r}(\boldsymbol{s}),$$

where $\{\gamma(\boldsymbol{m})\}_{\boldsymbol{m} \in \mathbb{N}_0^n} \subset \mathbb{Z}$, $\{N_r\}$ is an increasing sequence of positive integers, and $G_{1/r}(\boldsymbol{s})$ is holomorphic in $V\left(h; \frac{1}{r}\right)$, on which it equals an absolutely convergent Euler product.

Set $Ex := \{\boldsymbol{m} \in \mathbb{N}_0^n \backslash \{0\} : \gamma(\boldsymbol{m}) \neq 0\}$ and $Ex_- := \{\boldsymbol{m} \in \mathbb{N}_0^n \backslash \{0\} : \gamma(\boldsymbol{m}) < 0\}$.

There are two cases that will be treated separately.

Case 1: *Ex* is infinite

As above, let $\delta_0 < 0$ be such that $Z(h; \boldsymbol{s})$ has a meromorphic continuation to $V(h; \delta_0)$. Let ρ_0 be any fixed (and necessarily nonreal) zero of the Riemann zeta function satisfying $\Re(\rho_0) = \frac{1}{2}$.

Fix $\beta = (\beta_1, \ldots, \beta_n) \in (0, \infty)^n$ such that $\beta_1, \ldots, \beta_n$ are $\mathbb{Q}$-linearly independent, and set $Z_\beta(t) := Z(h; t\beta)$.

For all $\boldsymbol{m} \in Ex$ we set $t_{\boldsymbol{m}} = \frac{1}{\langle \boldsymbol{m}, \beta \rangle}$ if $\gamma(\boldsymbol{m}) < 0$, and $t_{\boldsymbol{m}} = \frac{\rho_0}{\langle \boldsymbol{m}, \beta \rangle}$ if $\gamma(\boldsymbol{m}) > 0$.

In addition, choose for each $\boldsymbol{m} \in K$, $r(\boldsymbol{m}) \in \mathbb{N}$ satisfying:

$$r(\boldsymbol{m}) > \frac{2 \cdot |\boldsymbol{m}| \cdot \sup_i \beta_i}{\inf_j \langle \alpha_j, \beta \rangle} \quad \text{and} \quad r(\boldsymbol{m}) \geq |\boldsymbol{m}|.$$

It follows that $N_{r(\boldsymbol{m})} \geq r(\boldsymbol{m}) \geq |\boldsymbol{m}|$. By (6), we have for each $\boldsymbol{m} \in Ex$ and $t\beta \in V\left(h; \frac{1}{r(\boldsymbol{m})}\right)$:

$$(7) \qquad Z_\beta(t) = Z(h; t\beta) = \zeta(t\langle \boldsymbol{m}, \beta \rangle)^{\gamma(\boldsymbol{m})} \left(\prod_{\substack{\boldsymbol{m}' \in \mathbb{N}_0^n \setminus \{\boldsymbol{m}\} \\ 1 \leq |\boldsymbol{m}'| \leq N_{r(\boldsymbol{m})}}} \zeta(t\langle \boldsymbol{m}', \beta \rangle)^{\gamma(\boldsymbol{m}')} \right) G_{1/r(\boldsymbol{m})}(t\beta).$$

From the definition of $r(\boldsymbol{m})$, it follows that for each $\alpha_j \in Ext(h)$:

$$\Re(\langle \alpha_j, t_{\boldsymbol{m}} \beta \rangle) \geq \frac{\langle \alpha_j, \beta \rangle}{2 \cdot \langle \boldsymbol{m}, \beta \rangle} \geq \frac{\langle \alpha_j, \beta \rangle}{2 \cdot |\boldsymbol{m}| \cdot \sup_i \beta_i} > \frac{1}{r(\boldsymbol{m})}.$$

Thus, $t \mapsto G_{1/r(\boldsymbol{m})}(t\beta)$ is holomorphic in a neighbourhood of $t = t_{\boldsymbol{m}}$.

We now distinguish two subcases:

First subcase: Ex_- is infinite

Let $\boldsymbol{m} \in Ex_-$, so that $t_{\boldsymbol{m}} = \frac{1}{\langle \boldsymbol{m}, \beta \rangle} > 0$. It follows that $t_{\boldsymbol{m}}$ is not a pole of $\zeta(t\langle \boldsymbol{m}', \beta \rangle)^{\gamma(\boldsymbol{m}')}$ for every $\boldsymbol{m}' \neq \boldsymbol{m} \in \mathbb{N}_0^n$. This is clear if $\gamma(\boldsymbol{m}') > 0$ since the only possible pole of this function occurs when $t = \frac{1}{\langle \boldsymbol{m}', \beta \rangle}$, which cannot equal $t_{\boldsymbol{m}}$ because $t_{\boldsymbol{m}} = \frac{1}{\langle \boldsymbol{m}, \beta \rangle} \neq \frac{1}{\langle \boldsymbol{m}', \beta \rangle}$. If $\gamma(\boldsymbol{m}') < 0$, then poles of $\zeta(t\langle \boldsymbol{m}', \beta \rangle)^{\gamma(\boldsymbol{m}')}$ must be zeroes of $\zeta(t\langle \boldsymbol{m}', \beta \rangle)$. A classical fact ([11], pg. 30) tells us that there are no positive zeroes of $\zeta(s)$. Thus, $t_{\boldsymbol{m}}$ cannot be a pole of $\zeta(t\langle \boldsymbol{m}', \beta \rangle)^{\gamma(\boldsymbol{m}')}$. On the other hand, $\gamma(\boldsymbol{m}) < 0$ implies that $t_{\boldsymbol{m}}$ *is* a zero of $Z_\beta(t)$ since $|\boldsymbol{m}| \leq N_{r(\boldsymbol{m})}$.

Furthermore, it is clear that the sequence $\{t_{\boldsymbol{m}}\}_{\boldsymbol{m} \in Ex_-}$ of zeroes of $Z_\beta(t)$ converges to 0 when $|\boldsymbol{m}| \to +\infty$.

Now, if $Z(h; \boldsymbol{s})$ had a meromorphic continuation to $V(h; \delta_0)$, then $Z_\beta(t)$ would have to have a meromorphic continuation to $U(\delta_1) := \{t \in \mathbb{C} : \Re(t) > \delta_1\}$, where $\delta_1 = \sup_{1 \leq j \leq q} \frac{\delta_0}{\langle \alpha_j, \beta \rangle} < 0$. Thus, $Z_\beta(t)$ would have to be identically zero, which is impossible because each $G_{1/r}(\boldsymbol{s})$ is an absolutely convergent Euler product in $V(h; 1/r)$, and cannot therefore be identically zero. We conclude that in this subcase, $Z(h; \boldsymbol{s})$ cannot be meromorphically extended to any $V(h; \delta)$ when $\delta < 0$.

Second subcase: Ex_- is finite

Choose $a > 0$ such that $\zeta(z) \neq 0$ for $|z| \leq a$.

Set

$$B := 2 \cdot \frac{(\sup_i \beta_i) \cdot |\rho_0| \cdot (\sup_{\boldsymbol{m} \in Ex_-} |\boldsymbol{m}|)}{a \cdot (\inf_i \beta_i)} > 0.$$

Define $Ex_+ := Ex \backslash Ex_-$, and fix $\boldsymbol{m} \in Ex_+$ such that $|\boldsymbol{m}| \geq B$. Then $\gamma(\boldsymbol{m}) > 0$ and $t_{\boldsymbol{m}} = \frac{\rho_0}{\langle \boldsymbol{m}, \beta \rangle} \in \mathbb{C} \backslash \mathbb{R}$.

We then observe the following:

1. for all $\boldsymbol{m}' \in Ex_+$ satisfying $\boldsymbol{m}' \neq \boldsymbol{m}$, $t_{\boldsymbol{m}}$ is not a pole of $\zeta(t\langle \boldsymbol{m}', \beta \rangle)^{\gamma(\boldsymbol{m}')}$ (since the only possible pole of this function is $\frac{1}{\langle \boldsymbol{m}', \beta \rangle} \in \mathbb{R}$ and $t_{\boldsymbol{m}} \notin \mathbb{R}$);

2. for all $\boldsymbol{m}' \in Ex_-$, $t_{\boldsymbol{m}}$ is not a pole of $\zeta(t\langle \boldsymbol{m}', \beta \rangle)^{\gamma(\boldsymbol{m}')}$. (if this were false, then $\rho := t_{\boldsymbol{m}} \langle \boldsymbol{m}', \beta \rangle$ would be a zero of $\zeta(s)$ satisfying:

$$|\rho| = |t_{\boldsymbol{m}}| \cdot \langle \boldsymbol{m}', \beta \rangle = \frac{|\rho_0| \cdot \langle \boldsymbol{m}', \beta \rangle}{\langle \boldsymbol{m}, \beta \rangle} \leq \frac{|\rho_0| \cdot |\boldsymbol{m}'| \cdot (\sup_i \beta_i)}{|\boldsymbol{m}| \cdot (\inf_i \beta_i)} \leq \frac{a \cdot B}{2 \cdot |\boldsymbol{m}|} \leq \frac{a}{2},$$

which is impossible).

By (7) and the fact that $|\boldsymbol{m}| \leq N_{r(\boldsymbol{m})}$, we conclude that for each $\boldsymbol{m} \in Ex_+$ satisfying $|\boldsymbol{m}| \geq B$, $t_{\boldsymbol{m}}$ is a zero of $Z_\beta(t)$. Since $t_{\boldsymbol{m}} \to 0$ when $|\boldsymbol{m}| \to +\infty$, it follows that $\{t_{\boldsymbol{m}}\}_{\{|\boldsymbol{m}| \geq B\}}$ contains a sequence of zeroes of $Z_\beta(t)$ with accumulation point in $U(\delta_1)$ *if* $Z(h; \boldsymbol{s})$ could be meromorphically extended to $V(h; \delta_0)$. As in the first subcase, this is not possible.

Case 2: Ex is finite

Set $G(\boldsymbol{s}) := (\prod_{m \in Ex} \zeta(\langle \boldsymbol{m}, \boldsymbol{s} \rangle)^{-\gamma(\boldsymbol{m})}) Z(h; \boldsymbol{s})$. We will prove that $G(\boldsymbol{s}) \equiv 1$.

By choosing r sufficiently large in the equation (6), we deduce that:

1. $G(\boldsymbol{s})$ is an Euler product of the form $G(\boldsymbol{s}) = \prod_p \left(\sum_{\alpha \in \mathbb{N}_0^n} \frac{m_\alpha}{p^{\langle \alpha, \boldsymbol{s} \rangle}} \right)$, where $m_0 = 1$, and there exist $C, D > 0$ such that $m_\alpha \leq C(1 + |\alpha|^D)$ for all α.

2. $G(\boldsymbol{s})$ converges absolutely in $V(h; 0) = \bigcup_r V\left(h; \frac{1}{r}\right)$.

Suppose that $G(\boldsymbol{s}) \not\equiv 1$. Then there exists $\alpha \neq 0$ such that $m_\alpha \neq 0$. Now fix $\beta = (\beta_1, \ldots, \beta_n) \in (0, \infty)^n$ as in Case 1. It follows that the Euler product

$$t \mapsto R_\beta(t) := G(t\beta) = \prod_p \left(\sum_{\alpha \in \mathbb{N}_0^n} \frac{m_\alpha}{p^{t\langle \alpha, \beta \rangle}} \right)$$

converges absolutely in the halfplane $\{t \in \mathbb{C} : \Re(t) > 0\}$.

Set $\mathscr{S} := \{\alpha \in \mathbb{N}_0^n : m_\alpha \neq 0\}$. Since $\langle \alpha, \beta \rangle \to +\infty$ as $|\alpha| \to +\infty$, it is clear that there exists $v \neq 0 \in \mathscr{S}$ such that $\langle v, \beta \rangle = \inf_{\alpha \neq 0 \in \mathscr{S}} \langle \alpha, \beta \rangle > 0$. We fix this v in the sequel.

Let $N = \left[\frac{8\langle v, \beta \rangle}{\inf_i \beta_i}\right] + |v| + 1 \in \mathbb{N}$. Then we have for $\Re(t) > \frac{1}{2\langle v, \beta \rangle}$ and uniformly in p:

$$\sum_{|\alpha|\geq N+1}\left|\frac{m_\alpha}{p^{t\langle\alpha,\beta\rangle}}\right| \ll \sum_{|\alpha|\geq N+1}\frac{|\alpha|^D}{p^{\Re(t)\cdot|\alpha|\cdot(\inf_i\beta_i)}}$$

$$\ll \sum_{|\alpha|\geq N+1}\frac{|\alpha|^D}{p^{\Re(t)\cdot|\alpha|/2\cdot(\inf_i\beta_i)}}\cdot\frac{1}{p^{\Re(t)\cdot(N+1)/2\cdot(\inf_i\beta_i)}}$$

$$\ll \frac{1}{p^{\Re(t)\cdot(N+1)/2\cdot(\inf_i\beta_i)}}\sum_{|\alpha|\geq N+1}\frac{|\alpha|^D}{2^{|\alpha|\inf_i\beta_i/4\langle v,\beta\rangle}}$$

$$\ll \frac{1}{p^{\Re(t)\cdot(N+1)/2\cdot(\inf_i\beta_i)}} \ll \frac{1}{p^2}.$$

From this we deduce that

$$R_\beta(t) = G(t\beta) = \prod_p\Bigg(\sum_{\substack{\alpha\in\mathbb{N}_0^n\\|\alpha|\leq N}}\frac{m_\alpha}{p^{t\langle\alpha,\beta\rangle}} + V_N(p;t)\Bigg),$$

where $t\mapsto V_N(p;t)$ is a holomorphic function that satisfies the bound $V_N(p;t)\ll_N p^{-2}$ uniformly in p and all $t\in\mathbb{C}$ such that $\Re(t)>\frac{1}{2\langle v,\beta\rangle}$. Since this Euler product converges absolutely for $t=\frac{1}{\langle v,\beta\rangle}>0$, it follows that

$$\prod_p\Bigg(1+\sum_{0<|\alpha|\leq N}\frac{m_\alpha}{p^{t\langle\alpha,\beta\rangle}}\Bigg)$$

also converges absolutely for $t=\frac{1}{\langle v,\beta\rangle}$. However, since $|v|\leq N$ it follows that $\sum_p \frac{m_v}{p^{t\langle v,\beta\rangle}}\Big|_{t=1/\langle v,\beta\rangle}$ must also converge, which is *not* possible. Thus, we conclude that $G(\boldsymbol{s})\equiv 1$.

As a result, we must have the following equation for all $\boldsymbol{s}\in V(h;A)$:

$$Z(h;\boldsymbol{s}) = \prod_{m\in Ex}\zeta(\langle\boldsymbol{m},\boldsymbol{s}\rangle)^{\gamma(\boldsymbol{m})} = \prod_{m\in Ex}\prod_p(1-p^{-\langle\boldsymbol{m},\boldsymbol{s}\rangle})^{-\gamma(\boldsymbol{m})}$$

$$= \prod_p\prod_{m\in Ex}(1-p^{-\langle\boldsymbol{m},\boldsymbol{s}\rangle})^{-\gamma(\boldsymbol{m})} = \prod_p h^*(p^{-s_1},\dots,p^{-s_n}),$$

where

$$h^*(X) = h^*(X_1,\dots,X_n) = \prod_{m\in Ex}(1-X^{\boldsymbol{m}})^{-\gamma(\boldsymbol{m})} = \prod_{m\in Ex}(1-X_1^{m_1}\dots X_n^{m_n})^{-\gamma(\boldsymbol{m})}.$$

Since the Euler product factorization is unique, we conclude that $h(X)=h^*(X)$. This completes the proof of Theorem 2. □

Remark. It is not difficult to extend the preceding discussion to analytic functions. Since these results will not be used in the article, we will give their statements and leave details of the straightforward proofs to the reader. Let $h_0, \dots, h_d$ be analytic functions on the unit polydisc $P(1)$ in $\mathbb{C}^n$, satisfying the property $h_k(0) = 0$ for each k. We also assume that each coefficient in the power series expansion of each h_k on $P(1)$ is an integer. Define

$$h(X_1, \dots, X_n, X_{n+1}) = 1 + \sum_{k=0}^{d} h_k(X_1, \dots, X_n) X_{n+1}^k,$$

$$Z(h; \boldsymbol{s}) = \prod_p h(p^{-s_1}, \dots, p^{-s_n}, p).$$

For each $\delta \in \mathbb{R}$, define

$$V^{\#}(h;\delta) := \bigcap_{k=0}^{d} \{\boldsymbol{s} \in \mathbb{C}^n : \langle \alpha, \sigma \rangle > k + \delta \ \forall \alpha \in Ext(h_k), \text{ and } \sigma_i > \delta \ \forall i\},$$

and for $\delta > 0$ define:

1. $N = \left[\frac{2(d+2)}{\delta}\right] + 1$;
2. $\mathscr{Y}_N := \{(\alpha, k) \in \mathbb{N}_0^n \times [0, d] : \alpha \in S(h_k) \text{ and } 1 \leq |\alpha| \leq N\}$, $r_N := \#\mathscr{Y}_N$, and $\mathscr{N}(\delta) := \{\boldsymbol{n} = (n_{\alpha,k}) \in \mathbb{N}_0^{r_N} : 1 \leq |\boldsymbol{n}| \leq \delta^{-1}\}$.

Theorem 3. *There exists $A > 0$ such that $Z(h; \boldsymbol{s})$ converges absolutely in $V^{\#}(h; A)$. In addition, $Z(h; \boldsymbol{s})$ can be continued into the domain $V^{\#}(h; 0)$ as a meromorphic function as follows. For any $\delta > 0$, there exists $\{\gamma(\boldsymbol{n}) : \boldsymbol{n} \in \mathscr{N}(\delta)\} \subset \mathbb{Z}$ and $G_\delta(\boldsymbol{s})$, a bounded holomorphic function on $V^{\#}(h;\delta)$, such that the equation*

$$(8) \qquad Z(h; \boldsymbol{s}) = \prod_{\boldsymbol{n}=(n_{\alpha,k}) \in \mathscr{N}(\delta)} \zeta\left(\sum_{(\alpha,k) \in \mathscr{Y}_N} n_{\alpha,k}(\langle \alpha, \boldsymbol{s} \rangle - k)\right)^{\gamma(\boldsymbol{n})} \cdot G_\delta(\boldsymbol{s}),$$

a priori valid in $V^{\#}(h; A)$, extends to $V^{\#}(h;\delta)$ outside the polar divisor of the product over $\boldsymbol{n} \in \mathscr{N}(\delta)$. Moreover G_δ can be expressed as an absolutely convergent Euler product in $V^{\#}(h;\delta)$.

Now assume $d = 0$ and let $h = 1 + \sum_{\alpha \neq 0} a_\alpha X^\alpha$ denote the power series expansion for h in $P(1)$.

Theorem 4. *If there exist $C, D > 0$ such that for all $\alpha \in \mathbb{N}_0^n$, $|a_\alpha| \leq C(1 + |\alpha|)^D$, then $Z(h; \boldsymbol{s})$ can be continued to $\mathbb{C}^n$ as a meromorphic function if and only if h is the quotient of cyclotomic polynomials. In all other cases the boundary $\partial V(h; 0)$ is the natural boundary.*

(AutoPDF V7 2/6/06 14:27) WDG (170×240mm) Tmath J-1468 Forum, : PMU: S(C) 19/05/2006 pp. 1–29 1468_06-11 (p. 13)

2 An application in diophantine geometry

We study in the first two subsections the analytic properties of a multivariable Dirichlet series whose coefficients encode membership in the maximal torus of a toric variety X. In §2.3, we apply our discussion to the toric defined by the equation $x_1 \cdots x_n = x_{n+1}^n$. We start with a given projective embedding, determined by a set of d monomial defining equations in n variables. This is not an unreasonable starting point since problems in multiplicative number theory, as one example, can sometimes be formulated in terms of such equations.

The set of exponents of the pertinent monomials therefore determines a $d \times n$ matrix $\boldsymbol{A}$ with entries in $\mathbb{Z}$, whose rows $\mathbf{a}_j = (a_{j,1}, \ldots, a_{j,n})$ each satisfy the property that $\sum_i a_{j,i} = 0$. The rational points of the variety resp. its maximal torus are defined as follows:

$$X(\boldsymbol{A}) := \left\{ (x_1 : \ldots : x_n) \in \mathbb{P}^{n-1}(\mathbb{Q}) : \prod_{\{i : a_{j,i} \geq 0\}} x_i^{a_{j,i}} = \prod_{\{i : a_{j,i} < 0\}} x_i^{-a_{j,i}} \ \forall j \right\};$$

$$U(\boldsymbol{A}) := \{ (x_1 : \ldots : x_n) \in X(\boldsymbol{A}) : x_1 \ldots x_n \neq 0 \}.$$

To each point $\boldsymbol{x}$ of $U(\boldsymbol{A})$ there corresponds a unique primitive lattice point $\boldsymbol{m} = \boldsymbol{m}(\boldsymbol{x}) = (m_1, \ldots, m_n) \in \mathbb{N}^n$, that is, $\gcd(m_1, \ldots, m_n) = 1$, and $(m_1 : \ldots : m_n) \in U(\boldsymbol{A})$.

Adapting an idea of Batyrev-Tschinkel [1], which was subsequently modified by La Bretèche [4] to exploit the formalism of universal torsors, see [10], we define a multivariable Dirichlet series with Euler product in the open set $\Omega := \{ \boldsymbol{s} : \sigma_i > 1, i = 1, \ldots, n \}$ by first introducing the function $F_{\boldsymbol{A}} : \mathbb{N}^n \to \mathbb{Z}$:

1. $F_{\boldsymbol{A}}(m_1, \ldots, m_n) = 1$ if $\gcd(m_1, \ldots, m_n) = 1$ and $\prod_i m_i^{a_{j,i}} = 1 \ \forall j \leq d$,
2. $F_{\boldsymbol{A}}(m_1, \ldots, m_n) = 0$ if not.

It is clear that $F_{\boldsymbol{A}}$ is multiplicative (see [5] for the definition), $F_{\boldsymbol{A}}(m_1, \ldots, m_n) = 1$ iff $(m_1 : \ldots : m_n) \in U(\boldsymbol{A})$, and that for each p and all $\nu \in \mathbb{N}_0^n$,

$$F_{\boldsymbol{A}}(p^{\nu_1}, \ldots, p^{\nu_n}) = 1 \quad \text{iff } \nu \in T(\boldsymbol{A}) := \{ \nu \in \mathbb{N}_0^n : \boldsymbol{A}(\nu) = 0 \}.$$

Our Dirichlet series is initially defined, if $\boldsymbol{s} \in \Omega$, to equal

$$Z_{\boldsymbol{A}}(\boldsymbol{s}) := \sum_{(m_1, \ldots, m_n) \in \mathbb{N}^n} \frac{F_{\boldsymbol{A}}(m_1, \ldots, m_n)}{m_1^{s_1} \ldots m_n^{s_n}} = \prod_p h_{\boldsymbol{A}}(p^{-s_1}, \ldots, p^{-s_n}),$$

where $h_{\boldsymbol{A}}(X) := \sum_{\nu \in T(\boldsymbol{A})} X^\nu$ is analytic on the polydisc $P(1)$. The three properties of $Z_{\boldsymbol{A}}(\boldsymbol{s})$, described in the Introduction, are proved in §2.2. The remark at the end of §2.2 compares in more detail this work with that in [4]. The reader may find this to be of value.

(AutoPDF V7 2/6/06 14:27) WDG (170×240mm) Tmath J-1468 Forum, : PMU: S(C) 19/05/2006 pp. 1–29 1468_06-11 (p. 14)

As shown in §2.3, a tauberian theorem, combined with knowledge of the analytic properties of $Z_{\boldsymbol{A}}(\boldsymbol{s})$, can be used to deduce the asymptotic behavior of a height density function on $U(\boldsymbol{A})$. The reason for this is as follows. Using the preceding notation, the function $\boldsymbol{x} \in X(\boldsymbol{A}) \to \max_i |m_i|$, where $\boldsymbol{m} = \boldsymbol{m}(\boldsymbol{x})$, equals a height function on $X(\boldsymbol{A})$ that is induced from the standard height on $\mathbb{P}^{n-1}(\mathbb{Q})$ (see Introduction). Defining the constant

$$C(\boldsymbol{A}) := \frac{1}{2} \cdot \# \left\{ \epsilon \in \{\pm 1\}^n : \prod_{i=1}^{n} \epsilon_i^{a_{j,i}} = 1 \text{ for all } j = 1, \ldots, d \right\},$$

the equation

$$\#\{\boldsymbol{x} \in U(\boldsymbol{A}) : H(\boldsymbol{x}) \leq t\} = C(\boldsymbol{A}) \cdot \sum_{\substack{(m_1, \ldots, m_n) \in \mathbb{N}^n \\ 1 \leq m_i \leq t \ \forall i}} F_{\boldsymbol{A}}(m_1, \ldots, m_n) \tag{9}$$

therefore interprets the height density function on $U(\boldsymbol{A})$ at t in terms of the sum of those coefficients of $Z_{\boldsymbol{A}}(\boldsymbol{s})$ contained in a box, each of whose sides has length t.

2.1 A basic property of $h_{\boldsymbol{A}}(X)$

As noted above, the only thing that we know of for sure about the function $h_{\boldsymbol{A}}(X)$ is that it is analytic on the polydisc $P(1)$. However, we must be more precise. The crucial property is the following.

Definition 1. An analytic function h on $P(1)$ is unitary if there exist a finite set $K \subset \mathbb{N}_0^n \setminus \{0\}$, positive integers $\{c(v)\}_{v \in K}$, and a polynomial $W \in \mathbb{Z}[X_1, \ldots, X_n]$, such that for all $X \in P(1)$:

$$h(X) = \left(\prod_{v \in K} (1 - X^v)^{-c(v)} \right) W(X).$$

The data $(K; \langle c(v) \rangle_{v \in K}; W)$ determines a presentation of h when $1 - X^v$ does not divide $W(X)$ for each $v \in K$.

The result we will need in §2.2 is the following.

Lemma 2. *The function $h_{\boldsymbol{A}}(X)$ is unitary.*

Lemma 2 is a simple consequence of a more general result which analyzes the behavior of an analytic function, all of whose monomial exponents belong to an affine plane

$$T(\boldsymbol{A}, \boldsymbol{b}) := \{v \in \mathbb{N}_0^n : \boldsymbol{A}(v) = \boldsymbol{b}\}.$$

(AutoPDF V7 2/6/06 14:27) WDG (170×240mm) Tmath J-1468 Forum, : PMU: S(C) 19/05/2006 pp. 1–29 1468_06-11 (p. 15)

Lemma 3. *For any integral $d \times n$ matrix $\boldsymbol{A}$ (the rows of which need not sum to 0!), and any $\boldsymbol{b} \in \mathbb{Z}^d$, the function*

$$h_{\boldsymbol{A};\boldsymbol{b}}(X) := \sum_{\nu \in T(\boldsymbol{A},\boldsymbol{b})} X^{\nu}$$

is unitary.

Proof that Lemma 3 implies Lemma 2. For all $X = (X_1, \ldots, X_n) \in P(1)$ we have:

$$h_{\boldsymbol{A}}(X) = \sum_{\substack{\nu \in T(\boldsymbol{A},0) \\ \nu_1 \ldots \nu_n = 0}} X^{\nu} = \sum_{\nu \in T(\boldsymbol{A},0)} X^{\nu} - \sum_{\substack{\nu \in T(\boldsymbol{A},0) \\ \nu_1 \geq 1, \ldots, \nu_n \geq 1}} X^{\nu}$$

$$= (1 - X_1 \ldots X_n) h_{\boldsymbol{A},0}(X).$$

Since Lemma 3 says that $h_{\boldsymbol{A},0}$ is unitary it follows that $h_{\boldsymbol{A}}$ is also unitary. □

Proof of Lemma 3. We shall prove the lemma by induction on n.

For $n = 1$ the result is trivially true.

Let $n \geq 2$. The induction hypothesis allows us to assume that for any $m < n$, any $d \times m$ integral matrix $\boldsymbol{A}'$, and any $\boldsymbol{b}' \in \mathbb{Z}^d$, we have that $h_{\boldsymbol{A}',\boldsymbol{b}'}(X_1, \ldots, X_m)$ is unitary.

Now, let $\boldsymbol{A}$ be a $d \times n$ integral matrix, and $\boldsymbol{b} = (b_1, \ldots, b_d) \in \mathbb{Z}^d$. It suffices to assume that $T(\boldsymbol{A},\boldsymbol{b}) \neq \emptyset$ since the proof of Lemma 2 is trivial when $T(\boldsymbol{A},\boldsymbol{b}) = \emptyset$.

It will be convenient to distinguish two cases:

Case 1: $\{0\} \subsetneq T(\boldsymbol{A},0)$
We choose and fix $\alpha \neq 0 \in T(\boldsymbol{A},0)$ in the following. For any $I \subset \{1, \ldots, n\}$, we define

$$L(I,\alpha) := \{\nu \in T(\boldsymbol{A},\boldsymbol{b}) : \nu_i \geq \alpha_i \textit{ iff } i \in I\},$$

and

$$h_{\boldsymbol{A},\boldsymbol{b}}(I;\alpha;X) := \sum_{\nu \in L(I,\alpha)} X^{\nu}. \tag{10}$$

If $L(I,\alpha) = \emptyset$, the value is defined to be 0. A straightforward calculation then shows:

$$(1 - X^{\alpha}) h_{\boldsymbol{A},\boldsymbol{b}}(X) = \sum_{\substack{I \subset \{1,\ldots,n\} \\ I \neq \{1,\ldots,n\}}} h_{\boldsymbol{A},\boldsymbol{b}}(I;\alpha;X) \quad \forall X \in P(1). \tag{11}$$

So, we need to show that each $h_{\boldsymbol{A},\boldsymbol{b}}(I;\alpha;X)$ is unitary. By permuting coordinates, it suffices to prove this for any $I_q := \{1,2,\dots,q\}$ with $q \le n-1$.

To express the necessary equation in a concise manner, we first introduce the following notations:

1. $X = (Y,Z)$ with $Y = (X_1,\dots,X_q)$ and $Z = (X_{q+1},\dots,X_n)$;

2. $\boldsymbol{x}' = (x_1,\dots,x_q)$ and $\boldsymbol{x}'' = (\boldsymbol{x}_{q+1},\dots,\boldsymbol{x}_n)$, for any n-vector $\boldsymbol{x}$, and $\boldsymbol{A}'$ is the $d \times q$ matrix with rows $\mathbf{a}_j' = (a_{j,1},\dots,a_{j,q})$ for each $j \le d$;

3. $\mathscr{D}(=\mathscr{D}(\alpha)) := \{v'' = (v_{q+1},\dots,v_n) \in \prod_{i=q+1}^{n}\{0,1,2,\dots,\alpha_i - 1\}\}$;

4. $\forall v'' \in \mathscr{D}$,

$$\boldsymbol{l}(v'') := (b_1 - \langle \mathbf{a}_1'', v''\rangle - \langle \mathbf{a}_1', \alpha'\rangle, \dots, b_d - \langle \mathbf{a}_d'', v''\rangle - \langle \mathbf{a}_d', \alpha'\rangle).$$

We then observe that for all $X = (Y,Z) \in P(1)$,

$$\begin{aligned}
h_{\boldsymbol{A},\boldsymbol{b}}(I_q;\alpha;X) &= \sum_{\substack{v \in T(\boldsymbol{A},\boldsymbol{b}) \\ \forall i \le q\ v_i \ge \alpha_i \text{ and } \forall i > q\ v_i < \alpha_i}} X^{v} \\
&= \sum_{v''=(v_{q+1},\dots,v_n)\in\mathscr{D}} \ \sum_{\substack{\mu=(\mu_1,\dots,\mu_q)\in\mathbb{N}_0^q \\ (\alpha'+\mu,v'')\in T(\boldsymbol{A},\boldsymbol{b})}} Y^{\alpha'+\mu} Z^{v''} \\
&= \sum_{v''=(v_{q+1},\dots,v_n)\in\mathscr{D}} Y^{\alpha'} Z^{v''} \sum_{\mu=(\mu_1,\dots,\mu_q)\in T(\boldsymbol{A}',\boldsymbol{l}(v''))} Y^{\mu}.
\end{aligned}$$

So the following equation is true:

$$h_{\boldsymbol{A},\boldsymbol{b}}(I_q;\alpha;X) = \sum_{v''\in\mathscr{D}} Y^{\alpha'} Z^{v''} h_{\boldsymbol{A}',\boldsymbol{l}(v'')}(Y). \tag{12}$$

We conclude by induction.

Case 2: $T(\boldsymbol{A},0) = \{0\}$
Since $T(\boldsymbol{A},\boldsymbol{b}) \ne \emptyset$, there exists $\gamma \in T(\boldsymbol{A},\boldsymbol{b})$. We begin by observing that: $v \in T(\boldsymbol{A},\boldsymbol{b})$ is equivalent to one of the two following conditions:

1. $v = \gamma$ (i.e. $v_i \ge \gamma_i\ \forall i$ implies $v - \gamma \in T(\boldsymbol{A},0) = \{0\}$);

2. $v \in T(\boldsymbol{A},\boldsymbol{b})$ and $\exists i \in \{1,\dots,n\}$ such that $v_i < \gamma_i$.

This observation implies that for all $X \in P(1)$:

$$h_{\boldsymbol{A},\boldsymbol{b}}(X) = X^{\gamma} + \sum_{\substack{I \subset \{1,\dots,n\} \\ I \ne \{1,\dots,n\}}} h_{\boldsymbol{A},\boldsymbol{b}}(I;\gamma;X)$$

where each $h_{\boldsymbol{A},\boldsymbol{b}}(I;\gamma;X)$ is defined as in (10), replacing α by γ. We now conclude by induction as in Case 1. This completes the proof of Lemma 3. $\square$

Remark. The proof of Lemma 3 actually gives an explicit procedure to find a presentation of $h_{\boldsymbol{A}}$. This is used in §2.3.

2.2 Analytic properties of $Z_{\boldsymbol{A}}(\boldsymbol{s})$

The essential first step needed to deduce the analytic properties of $Z_{\boldsymbol{A}}(\boldsymbol{s})$ is given by Lemma 2, which gives a presentation of $h_{\boldsymbol{A}}(X)$ as a rational function:

$$(13)\qquad h_{\boldsymbol{A}}(X) = \prod_{v\in K}(1-X^v)^{-c(v)}\cdot W(X).$$

Note. Although K and W certainly depend upon $\boldsymbol{A}$, the notation will not indicate this for the sake of simplicity. The reader should not find this confusing. $\square$

Since both $h_{\boldsymbol{A}}(X)$ and each $(1-X^v)^{-c(v)}$ equal 1 when $X=0$, it is clear that W is a polynomial with integer coefficients that satisfies $W(0)=1$. Define the Euler product $Z(W;\boldsymbol{s}) = \prod_p W(p^{-s_1},\ldots,p^{-s_n})$.

We set $I = K\cup S^*(W)$, and define for every $\delta\in\mathbb{R}$, $V(I;\delta) := \{\boldsymbol{s}\in\mathbb{C}^n : \langle v,\sigma\rangle > \delta\ \forall v\in I\}$. It is then clear that $Z_{\boldsymbol{A}}(\boldsymbol{s})$ converges absolutely in $V(I;1)$ and for any $\boldsymbol{s}\in V(I;1)$:

$$(14)\qquad Z_{\boldsymbol{A}}(\boldsymbol{s}) = \left(\prod_{\boldsymbol{m}\in K}\zeta(\langle v,\boldsymbol{s}\rangle)^{c(v)}\right)\cdot Z(W;\boldsymbol{s}).$$

Theorems 1, 2 (whose notations are used below) can now be immediately applied to tell us the following.

Theorem 5. 1. $\boldsymbol{s}\mapsto Z_{\boldsymbol{A}}(\boldsymbol{s})$ *can be meromorphically continued to* $V(W;0)$*;*

2. $\boldsymbol{s}\mapsto Z_{\boldsymbol{A}}(\boldsymbol{s})$ *can be meromorphically continued to* $\mathbb{C}^n$ *if and only if* W *is cyclotomic;*

3. *if* W *is not cyclotomic, then* $\partial V(W;0)$ *is the natural boundary of meromorphic continuation.*

To proceed, we will need to introduce some additional notations, and prove a preliminary result. First, we fix the expression for W by setting $W(X_1,\ldots,X_n) = 1+\sum_{v\in S^*(W)} u(v)X^v$.

We note that $\partial V(I;1) = \partial I^o$ (see Notations). For any $\alpha\in\partial V(I;1)$, we set $E(I,\alpha) = \{v\in I : \langle\alpha,v\rangle = 1\}$.

Finally, for all $v\in I$, we define $c'(v)$ as follows:

1. $c'(v) = c(v)$ if $v \in K \backslash S^*(W)$;

2. $c'(v) = u(v)$ if $v \in S^*(W) \backslash K$;

3. $c'(v) = c(v) + u(v)$ if $v \in K \cap S^*(W)$.

The following lemma plays an important role in the proof of Theorem 6 below.

Lemma 4. *For each $\alpha \in \partial V(I;1)$, and each $v \in E(I,\alpha)$, $c'(v) = 1$.*

Proof. We start with the presentation (13), and choose $\eta < \frac{1}{4}\min_{v \in I \backslash E(I,\alpha)}(\langle \alpha, v\rangle - 1)$ *if $E(I,\alpha) \neq I$.* Otherwise, we choose $\eta \in (0, 1/6)$.

We set $\mathscr{F} = \{\varepsilon \in (0,1)^{2n} : 1, \varepsilon_1, \ldots, \varepsilon_{2n}$ are linearly independant over $\mathbb{Q}\}$.

For each $\varepsilon \in \mathscr{F}$ we define:

1. $\alpha(\varepsilon) = (\alpha_1(\varepsilon), \ldots, \alpha_n(\varepsilon))$, where $\alpha_i(\varepsilon) = (1 - \varepsilon_i)\alpha_i + \varepsilon_{n+i}$ for all $i = 1, \ldots, n$;

2. $g_\varepsilon(t) = h_{\boldsymbol{A}}(t^{\alpha_1(\varepsilon)}, \ldots, t^{\alpha_n(\varepsilon)})$ for all $t \in (0,1)$.

By using the bound for η, as above, and the fact that $\langle \alpha(\varepsilon), v\rangle = \langle \alpha, v\rangle + O(|\varepsilon|)$ as $|\varepsilon| \to 0$ (since I is finite), it is clear that one can choose $\varepsilon \in \mathscr{F}$ with $|\varepsilon|$ so small that the following property is satisfied:

(15) $\quad v \in E(I,\alpha)$ implies

$$\langle \alpha(\varepsilon), v\rangle < 1 + \eta \quad \text{and} \quad g_\varepsilon(t) = 1 + \sum_{v \in E(I,\alpha)} c'(v) t^{\langle \alpha(\varepsilon), v\rangle} + O_\varepsilon(t^{1+\eta}) \quad (t \to 0).$$

We fix any such ε in the following.

On the other hand, it is also clear that there exist $N = N(\eta, \varepsilon)$ such that

$$(16) \qquad g_\varepsilon(t) = \sum_{\substack{v \in T(\boldsymbol{A}) \\ |v| \leq N}} t^{\langle \alpha(\varepsilon), v\rangle} + O_\varepsilon(t^{1+\eta}) \quad (t \to 0).$$

Since $\epsilon \in \mathscr{F}$, it follows that if $v \neq v' \in \mathbb{N}_0^n$, then $\langle \alpha(\varepsilon), v\rangle \neq \langle \alpha(\varepsilon), v'\rangle$. In particular, this insures that for any $v \in E(I,\alpha)$, the coefficient of $t^{\langle \alpha(\epsilon), v\rangle}$ in (15) equals $c'(v)$, and in (16) equals 1. Since the two partial asymptotic expansions must be equal up to terms of order $t^{1+\eta}$, this shows that $c'(v) = 1$ if $v \in E(I,\alpha)$. □

We know that $Z_{\boldsymbol{A}}(\boldsymbol{s})$ converges absolutely in $V(I;1)$. Our second basic observation identifies the boundary of this domain as *the* boundary of the domain of analyticity of $Z_{\boldsymbol{A}}(\boldsymbol{s})$.

Theorem 6. *For each point $\alpha \in \partial V(I;1)$, the meromorphic continuation of $Z_{\boldsymbol{A}}(\boldsymbol{s})$ is not analytic at α.*

Proof. Let $\alpha \in \partial V(I;1)$ be arbitrary and fixed, and assume $\boldsymbol{s}$ is such that $\sigma_i > \alpha_i$ for each i. It is then clear that $Z_{\boldsymbol{A}}(\boldsymbol{s})$ converges absolutely since $\langle \sigma, v\rangle > \langle \alpha, v\rangle \geq 1$ for any $v \in I$.

We next introduce the product of linear forms $\mathscr{L}_\alpha(\boldsymbol{s}) := \prod_{v\in E(I,\alpha)} \langle v, \boldsymbol{s}\rangle$, and use Lemma 4 to write it as follows:

$$\mathscr{L}_\alpha(\boldsymbol{s}) = \prod_{v\in K\cap E(I,\alpha)} \langle v, \boldsymbol{s}\rangle^{c(v)} \cdot \prod_{v\in S^*(W)\cap E(I,\alpha)} \langle v, \boldsymbol{s}\rangle^{u(v)}.$$

The function $\mathscr{H}_\alpha(\boldsymbol{s}) := Z(F_{\boldsymbol{A}}; \alpha + \boldsymbol{s}) \cdot \mathscr{L}_\alpha(\boldsymbol{s})$ is evidently analytic in $V(I;0)$. We first show that it is analytic in some larger domain $V(I;-\delta_1)$ for some positive δ_1, by grouping each factor in $\mathscr{L}_\alpha(\boldsymbol{s})$ with an appropriate factor of $Z(F_{\boldsymbol{A}}; \alpha + \boldsymbol{s})$ obtained from (14).

For the leftmost factor on the rightside of (14), we have:

$$\prod_{v\in K} \zeta(\langle v, \alpha\rangle + \langle v, \boldsymbol{s}\rangle)^{c(v)} \cdot \prod_{v\in K\cap E(I,\alpha)} \langle v, \boldsymbol{s}\rangle^{c(v)}$$

$$= \prod_{v\in K\cap E(I,\alpha)} [\langle v, \boldsymbol{s}\rangle \cdot \zeta(1 + \langle v, \boldsymbol{s}\rangle)]^{c(v)} \cdot \prod_{v\in K\setminus E(I,\alpha)} \zeta(\langle v, \alpha\rangle + \langle v, \boldsymbol{s}\rangle)^{c(v)}.$$

For δ_0 chosen small enough, it is clear that each of the two products on the last line, one over $v \in K \cap E(I,\alpha)$, the other over $v \in K - E(I,\alpha)$, is analytic in $V(I;-\delta_0)$.

For the rightmost factor on the right side of (14), observe first that (14) and the proof of Lemma 1 imply that there exists $\delta \in (0,1)$ such that

$$(17) \qquad G_\delta(\boldsymbol{s}) := Z(W; \boldsymbol{s}) \cdot \prod_{v\in S^*(W)\cap E(I,\alpha)} \zeta(\langle v, \boldsymbol{s}\rangle)^{-u(v)}$$

is analytic and bounded in $V(W; 1-\delta)$.

Thus,

$$Z(W; \alpha + \boldsymbol{s}) \prod_{v\in S^*(W)\cap E(I,\alpha)} \langle v, \boldsymbol{s}\rangle^{u(v)}$$

$$= \prod_{v\in S^*(W)\cap E(I,\alpha)} [\langle v, \boldsymbol{s}\rangle \zeta(1 + \langle v, \boldsymbol{s}\rangle)]^{u(v)} \cdot G_\delta(\alpha + \boldsymbol{s}),$$

and $G_\delta(\alpha + \boldsymbol{s})$ is analytic for $\boldsymbol{s} \in V(I; -\delta_0')$, for some $\delta_0' > 0$.

We conclude that $\mathscr{H}_\alpha(\boldsymbol{s})$ can be written in $V(I;0)$ as follows:

$$\begin{aligned}\mathscr{H}_\alpha(\boldsymbol{s}) = &\prod_{v\in K\cap E(I,\alpha)} [\langle v,\boldsymbol{s}\rangle\cdot\zeta(1+\langle v,\boldsymbol{s}\rangle)]^{c(v)}\cdot\prod_{v\in K\setminus E(I,\alpha)}\zeta(\langle v,\alpha\rangle+\langle v,\boldsymbol{s}\rangle)^{c(v)}\\ &\cdot\prod_{v\in S^*(W)\cap E(I,\alpha)}[\langle v,\boldsymbol{s}\rangle\cdot\zeta(1+\langle v,\boldsymbol{s}\rangle)]^{u(v)}\cdot G_\delta(\alpha+\boldsymbol{s})\\ = &\prod_{v\in K\cap E(I,\alpha)}[\langle v,\boldsymbol{s}\rangle\cdot\zeta(1+\langle v,\boldsymbol{s}\rangle)]^{c(v)}\\ &\cdot\prod_{v\in S^*(W)\cap E(I,\alpha)}[\langle v,\boldsymbol{s}\rangle\cdot\zeta(1+\langle v,\boldsymbol{s}\rangle)]^{u(v)}\\ &\cdot\prod_{v\in K\setminus E(I,\alpha)}\zeta(\langle v,\alpha\rangle+\langle v,\boldsymbol{s}\rangle)^{c(v)}\cdot G_\delta(\alpha+\boldsymbol{s}).\end{aligned}$$

Moreover, we know that there exists $\delta_1'>0$ such that the product of the two functions on the last line is analytic in $V(I;-\delta_1')$.

Applying Lemma 4 a second time now shows that for any $\boldsymbol{s}\in V(I;0)$:

$$\begin{aligned}(18)\qquad \mathscr{H}_\alpha(\boldsymbol{s}) = &\prod_{v\in E(I,\alpha)}[\langle v,\boldsymbol{s}\rangle\cdot\zeta(1+\langle v,\boldsymbol{s}\rangle)]\\ &\cdot\prod_{v\in K\setminus E(I,\alpha)}\zeta(\langle v,\alpha\rangle+\langle v,\boldsymbol{s}\rangle)^{c(v)}\cdot G_\delta(\alpha+\boldsymbol{s}).\end{aligned}$$

We then deduce the existence of $\delta_1>0$, such that the product over $v\in E(I,\alpha)$ in the first line of (18) is analytic in $V(I;-\delta_1)$. Since the product of functions on the second line is analytic if δ_1 is chosen sufficiently small, we have verified what we needed to show, that is, $\mathscr{H}_\alpha(\boldsymbol{s})$ is analytic in some neighborhood $V(I;-\delta_1)$ containing $\boldsymbol{s}=0$.

The second part of the argument is an immediate consequence of the following essential property:

$$(19)\qquad \mathscr{H}_\alpha(0)\neq 0.$$

To prove this, we start with (18) and rewrite the product by writing

$$1=\prod_{v\in K\cap E(I,\alpha)}\zeta(1+\langle v,\boldsymbol{s}\rangle)^{c(v)}\cdot\prod_{v\in K\cap E(I,\alpha)}\zeta(1+\langle v,\boldsymbol{s}\rangle)^{-c(v)}.$$

Multiplying together all the terms with exponent $-c(v)$ with $\prod_{v\in S^*(W)\cap E(I,\alpha)}\zeta(1+\langle v,\boldsymbol{s}\rangle)^{-u(v)}$ (a term that equals a factor in (17) when evaluated at $\alpha+\boldsymbol{s}$), and applying Lemma 4 again, gives a factor of $\mathscr{H}_\alpha(\boldsymbol{s})$ that equals $\prod_{v\in E(I,\alpha)}\zeta(1+\langle v,\boldsymbol{s}\rangle)^{-1}$. Multiplying together all the terms with exponent $c(v)$ with the product over $v\in K-E(I,\alpha)$ in (18) gives a factor equal to

$\prod_{m\in K}\zeta(\langle v,\alpha+\boldsymbol{s}\rangle)^{c(v)}$. Thus, we find a different expression for $\mathscr{H}_\alpha(\boldsymbol{s})$ as a product of functions, each of which is analytic, at least, in $V(I;0)$:

$$\mathscr{H}_\alpha(\boldsymbol{s}) = \prod_{m\in K}\zeta(\langle v,\alpha+\boldsymbol{s}\rangle)^{c(v)}\cdot\prod_{v\in E(I,\alpha)}\zeta(1+\langle v,\boldsymbol{s}\rangle)^{-1}\cdot Z(W,\alpha+\boldsymbol{s}) \cdot\prod_{v\in E(I,\alpha)}[\langle v,\boldsymbol{s}\rangle\cdot\zeta(1+\langle v,\boldsymbol{s}\rangle)]. \tag{20}$$

Since there exists a neighborhood of $\boldsymbol{s}=0$ in which the function $\prod_{v\in E(I,\alpha)}[\langle v,\boldsymbol{s}\rangle\cdot\zeta(1+\langle v,\boldsymbol{s}\rangle)]$ is both analytic and *never* 0, it follows that the product in (20) is actually analytic in a neighborhood of $\boldsymbol{s}=0$. In such a neighborhood, we therefore have:

$$\mathscr{H}_\alpha(\boldsymbol{s}) = \prod_p H(p;\boldsymbol{s})\cdot\prod_{v\in E(I,\alpha)}[\langle v,\boldsymbol{s}\rangle\cdot\zeta(1+\langle v,\boldsymbol{s}\rangle)], \tag{21}$$

where

$$H(p;\boldsymbol{s}) = \prod_{v\in K}(1-p^{-\langle v,\alpha\rangle-\langle v,\boldsymbol{s}\rangle})^{-c(v)}\cdot\prod_{v\in E(I,\alpha)}(1-p^{-1-\langle v,\boldsymbol{s}\rangle})\cdot W(p^{-\alpha_1-s_1},\ldots,p^{\alpha_n-s_n}).$$

The function $\boldsymbol{s}\to\prod_p H(p;\boldsymbol{s})$ is analytic at $\boldsymbol{s}=0$, but we still need to understand its value at this point. For $r\in(0,1)$ we define the open neighborhood $\mathscr{B}(r)=V(I;0)\cup\{\boldsymbol{s}\in\mathbb{C}^n\mid|s_i|<r\}$ of 0, and write out $H(p;\boldsymbol{s})|_{\mathscr{B}(r)}$. For our purposes, it now suffices to observe the existence of $u>1$ such that the following holds, to which we apply Lemma 4 for the last equation:

$$\begin{aligned}
H(p;\boldsymbol{s}) &= \left(1+\sum_{v\in K\cap E(I,\alpha)}\frac{c(v)}{p^{1+\langle v,\boldsymbol{s}\rangle}}+O(p^{-u+r})\right)\\
&\quad\cdot\left(1-\sum_{v\in E(I,\alpha)}\frac{1}{p^{1+\langle v,\boldsymbol{s}\rangle}}+O(p^{-u+r})\right)\\
&\quad\cdot\left(1+\sum_{v\in S^*(W)\cap E(I,\alpha)}\frac{u(v)}{p^{1+\langle v,\boldsymbol{s}\rangle}}+O(p^{-u+r})\right)\\
&= 1-\sum_{v\in E(I,\alpha)}\frac{1-c(v)-u(v)}{p^{1+\langle v,\boldsymbol{s}\rangle}}+O(p^{-u+r})\\
&= 1-\sum_{v\in E(I,\alpha)}\frac{1-c'(v)}{p^{1+\langle v,\boldsymbol{s}\rangle}}+O(p^{-u+r})\\
&= 1+\mathcal{O}(p^{-u+r})\quad\text{uniformly in } \boldsymbol{s}\in\mathscr{B}(r).
\end{aligned}$$

Thus, by choosing r so small that $-u + r < -1$ for all $\boldsymbol{s} \in \mathscr{B}(r)$, we conclude that $\boldsymbol{s} \mapsto \prod_p H(p; \boldsymbol{s})$ also converges absolutely in $\mathscr{B}(r)$. We can therefore evaluate both sides of (21) at $\boldsymbol{s} = 0$. In this way, we find the following Euler product expansion that *converges* to $\mathscr{H}_\alpha(0)$:

$$\mathscr{H}_\alpha(0) = \prod_p \left((1 - p^{-1})^{\# E(I,\alpha)} \cdot W(p^{-\alpha_1}, \ldots, p^{-\alpha_n}) \cdot \prod_{v \in K} (1 - p^{-\langle v, \alpha \rangle})^{-c(v)} \right). \tag{22}$$

The distinct advantage of (22) is that it easily is seen to imply that $\mathscr{H}_\alpha(0) > 0$. Indeed, we know that

$$W(p^{-\alpha_1}, \ldots, p^{-\alpha_n}) \cdot \prod_{v \in K} (1 - p^{-\langle v, \alpha \rangle})^{-c(v)} = h_{\boldsymbol{A}}(p^{-\alpha_1}, \ldots, p^{-\alpha_n}) > 0 \quad \text{for each } p.$$

Thus, each factor of the Euler product in (22) is *positive*. This implies $\mathscr{H}_\alpha(0)$ is also positive. As a result, the equation that gives the meromorphic continuation of $Z_{\boldsymbol{A}}(\boldsymbol{s})$ in a neighborhood of α,

$$Z_{\boldsymbol{A}}(\alpha + \boldsymbol{s}) = \frac{\mathscr{H}_\alpha(\boldsymbol{s})}{\mathscr{L}_\alpha(\boldsymbol{s})},$$

now implies that the right side *cannot* be analytic at $\boldsymbol{s} = 0$. This completes the proof of Theorem 6. □

Remark. It may be instructive for the reader to compare the preceding discussion with that in [4]. As noted in the Introduction, the starting point of [ibid.] is an explicit projective embedding (defined by choosing the anticanonical line bundle and assuming it is ample) of a desingularized model of the variety into $\mathbb{P}^{d-1}(\mathbb{Q})$ for an appropriate d. The model is constructed by patching together a set of affine charts that is bijective with a set of simplicial cones with integral 1-skeletal vectors that forms a "fan decomposition" of $\mathbb{R}^d$. Following the discussion in [10, §11], the 1-skeletal vectors then determine the entries of an $n \times m$ integral matrix $\boldsymbol{B}$ (where n denotes the number of 1-skeletal vectors and m the number of cones of maximal dimension). To each rational point on the maximal torus of height t, there corresponds (see [10, (11.4), (11.5)]), a unique point $(\boldsymbol{x}^{\boldsymbol{b}_1}, \ldots, \boldsymbol{x}^{\boldsymbol{b}_m})$, where $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{N}^n$, the $\boldsymbol{b}_j$ denote the column vectors of $\boldsymbol{B}$, and $\max_j\{\boldsymbol{x}^{\boldsymbol{b}_j}\} = t$. The point is also subject to a certain *gcd* condition that is important but need not be defined here. Up to an additional scalar factor, this gives the analogue of (9).

The analogue of $Z_{\boldsymbol{A}}(\boldsymbol{s})$ in [op cit., §4.2] is the series denoted $F(s_1, \ldots, s_m)$, which also has an Euler product expression. Note however that its coefficients are no longer restricted to 0, 1 in value. The analogue of (14) for this product is as follows:

$$F(s_1, \ldots, s_m) = \prod_i \zeta(\langle \boldsymbol{c}_i, (s_1, \ldots, s_m) \rangle) \cdot G, \quad \text{where } G = \prod_p g(p^{-s_1}, \ldots, p^{-s_m})$$

is such that $g(Y) := \sum_{v \in \mathbb{N}_0^\ell} \mu(p^{v_1}, \ldots, p^{v_\ell}) \prod_{j=1}^m Y_j^{\langle v, \boldsymbol{b}_j \rangle}$ is a polynomial, and $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n$ denote the rows of $\boldsymbol{B}$. The details of this were worked out in [10]. An approximation to the support of g is also given in [4], see §4.2. The preceding equation actually gives a meromorphic extension of F because it is shown in [ibid., Lemme 4.2 (iv)] that G is absolutely convergent in the region $\bigcap_i \{\Re(\boldsymbol{c}_i \cdot (s_1, \ldots, s_m)) > \kappa\}$ for some $\kappa < 1$.

The analogue of Theorem 6 is, however, only proved at exactly one real point $\alpha = \frac{1}{m}(1, \ldots, 1)$. Moreover, the proof that $G(\alpha) \neq 0$, that is, the analogue of the nonvanishing property (19), is indirect, and actually is a consequence of the version of Manin's conjecture proved in [1]. As a result, this argument would not seem to extend to find other points on the boundary of analyticity of F. It would therefore be interesting to know if the proof of Theorem 6 can be adapted to give more complete information about the polar locus of $F(s_1, \ldots, s_m)$.

2.3 How often is the product of n integers an n^{th} power?

A natural problem in multiplicative number theory is to describe the asymptotic density of n-fold products of positive integers that equals the n^{th} power of an integer. When $n = 3$, several authors have given a precise asymptotic for the density [7], [8], [3]. Our starting point was an observation of Batyrev-Tschinkel ([10], 11.50) who noted that the problem is equivalent to finding the asymptotic of the height density function on a certain singular cubic toric variety. This interpretation naturally extends to any $n \geq 3$. However, until now, no extension of these results to arbitrary n seems to have been published in the literature, although we have learned from the referee that Salberger lectured about the case $n = 4$ in 1998.

This subsection solves the problem for arbitrary $n \geq 3$ by combining the method in §2.2 with a tauberian theorem of La Bretèche [5].

In the following discussion, we use the notations from the preceding subsections and the introduction to §2, with the role of the matrix $\boldsymbol{A}$ played here by the $1 \times (n+1)$ matrix $\boldsymbol{A}_n = (1, \ldots, 1, -n)$. Note that the torus $U(\boldsymbol{A}_n)$ of the toric $X(\boldsymbol{A}_n)$ is now defined to equal

$$U(\boldsymbol{A}_n) = \{\boldsymbol{x} = (x_1 : \ldots : x_{n+1}) \in \mathbb{P}^n(\mathbb{Q}) : x_1 \cdots x_n = x_{n+1}^n \text{ and } x_1 \cdots x_n \neq 0\},$$

and the Dirichlet series $Z_{\boldsymbol{A}_n}(\boldsymbol{s})$ of interest becomes a function of $\boldsymbol{s} = (s_1, \ldots, s_{n+1})$. Setting $\boldsymbol{r} = (r_1, \ldots, r_n)$ and $|\boldsymbol{r}| = r_1 + \cdots + r_n$, we also define

$$J_n = \{\boldsymbol{r} + \boldsymbol{e}_{n+1} : \boldsymbol{r} \in \{0, \ldots, n\}^n \text{ and } |\boldsymbol{r}| = n\} \setminus \{(1, \ldots, 1)\},$$

$$D_n = \{\boldsymbol{r} \in \{0, \ldots, n-1\}^n : n \mid |\boldsymbol{r}|\},$$

$$\ell(\boldsymbol{r}) = (r_1, \ldots, r_n, |\boldsymbol{r}|/n) \quad \text{for any } \boldsymbol{r} \in D_n,$$

and for every $\delta \in \mathbb{R}$,

$$V(\delta) = \{\boldsymbol{s} \in \mathbb{C}^{n+1} : \langle \ell(\boldsymbol{r}), \sigma \rangle > \delta \ \forall \boldsymbol{r} \in D_n\}.$$

Theorem 7. *For any $n \geq 3$ the following three assertions are satisfied.*

1. $\boldsymbol{s} \mapsto Z_{\boldsymbol{A}_n}(\boldsymbol{s})$ *converges absolutely in $V(1)$ and satisfies:*

$$Z_{\boldsymbol{A}_n}(\boldsymbol{s}) = \frac{\prod_{i=1}^{n} \zeta(ns_i + s_{n+1})}{\zeta(s_1 + \cdots + s_{n+1})} \cdot \prod_p \left(\sum_{\boldsymbol{r} \in D_n} \frac{1}{p^{\langle \ell(\boldsymbol{r}), \boldsymbol{s} \rangle}} \right);$$

2. $\boldsymbol{s} \mapsto Z_{\boldsymbol{A}_n}(\boldsymbol{s})$ *can be meromorphically continued to $V(0)$ and $\partial V(0)$ is the natural boundary of $Z_{\boldsymbol{A}_n}(\boldsymbol{s})$;*

3. *there exists $\theta > 0$ such that:*

$$\#\left\{ \boldsymbol{x} \in U(\boldsymbol{A}_n) : H(\boldsymbol{x}) = \max_i |m_i(\boldsymbol{x})| \leq t \right\} = tQ_n(\log t) + O(t^{1-\theta}) \quad \text{as } t \to \infty,$$

where Q_n is a non-vanishing polynomial of degree $d_n = \binom{2n-1}{n} - n - 1$ satisfying

$$Q_n(\log t) = C_0(n) t^{-1}\, Vol(A_n(t)) + O(\log^{d_n - 1}(t)) \quad \text{as } t \to \infty,$$

$A_n(t)$ is defined with the help of the vector $\beta := \left(1, \ldots, 1, 1 + \frac{1}{d_n + 1}\right)$ to equal

$$A_n(t) = \left\{ \boldsymbol{x} = (x_v)_{v \in J_n} \in [1, +\infty[^{d_n + n} : \prod_{v \in J_n} x_v^{v_j} \leq t^{\beta_j} \ \forall j = 1, \ldots, n+1 \right\},$$

and

$$C_0(n) = 2^{n-1} \cdot \prod_p \left((1 - p^{-1})^{d_n + 1} \cdot \sum_{\boldsymbol{r} \in D_n} p^{-|\boldsymbol{r}|/n} \right) > 0$$

Proof. Defining

$$T(\boldsymbol{A}_n) = \{\alpha \in \mathbb{N}_0^{n+1} : \alpha_1 + \cdots + \alpha_n = n\alpha_{n+1} \text{ and } \alpha_1 \ldots \alpha_{n+1} = 0\},$$

we first need to construct an explicit presentation of

$$h_{\boldsymbol{A}_n}(X) = \sum_{\alpha \in T(\boldsymbol{A}_n)} X_1^{\alpha_1} \ldots X_{n+1}^{\alpha_{n+1}}.$$

To do so, we observe that for every $X \in P(1)$:

$$h_{\boldsymbol{A}_n}(X) = \sum_{\substack{\alpha_1 + \cdots + \alpha_n = n\alpha_{n+1} \\ \alpha_1 \ldots \alpha_{n+1} = 0}} X^\alpha = (1 - X_1 \ldots X_{n+1}) \cdot \sum_{\alpha_1 + \cdots + \alpha_n = n\alpha_{n+1}} X^\alpha$$

$$= (1 - X_1 \ldots X_{n+1}) \cdot \sum_{n | \alpha_1 + \cdots + \alpha_n} X_1^{\alpha_1} \ldots X_n^{\alpha_n} X_{n+1}^{(\alpha_1 + \cdots + \alpha_n)/n}$$

$$= (1 - X_1 \dots X_{n+1}) \cdot \sum_{\boldsymbol{r} \in D_n} X_1^{r_1} \dots X_n^{r_n} X_{n+1}^{|\boldsymbol{r}|/n} \cdot \sum_{\alpha \in \mathbb{N}_0^n} X_1^{n\alpha_1} \dots X_n^{n\alpha_n} X_{n+1}^{|\alpha|}$$

$$= \left(\prod_{i=1}^{n} (1 - X_i^n X_{n+1})^{-1} \right) \cdot W_n(X_1, \dots, X_{n+1}).$$

We conclude that $(K, \langle c(v) \rangle_{v \in K}, W_n)$ is a presentation of $h_{\boldsymbol{A}_n}(X)$ where:

$$W_n(X_1, \dots, X_{n+1}) = (1 - X_1 \dots X_{n+1}) \cdot \sum_{\boldsymbol{r} \in D_n} X_1^{r_1} \dots X_n^{r_n} X_{n+1}^{|\boldsymbol{r}|/n}$$

$$= \sum_{\substack{\boldsymbol{r} \in D_n \\ \boldsymbol{r} \neq (1,\dots,1)}} X_1^{r_1} \dots X_n^{r_n} X_{n+1}^{|\boldsymbol{r}|/n}$$

$$- \sum_{\substack{\boldsymbol{r} \in D_n \\ \boldsymbol{r} \neq (0,\dots,0)}} X_1^{1+r_1} \dots X_n^{1+r_n} X_{n+1}^{1+|\boldsymbol{r}|/n}$$

$$K = \{n\boldsymbol{e}_i + e_{n+1} : i = 1, \dots, n\}$$

$$c(v) = 1 \quad \forall v \in K.$$

Assertion 1 and the first part of Assertion 2 of the Theorem now follow immediately from Theorem 5.

To prove that $\partial V(0)$ is the natural boundary of $Z_{\boldsymbol{A}_n}(\boldsymbol{s})$, it suffices to show that the polynomial W_n is not cyclotomic when $n \geq 3$. We show this by contradiction.

Thus, suppose that W_n is cyclotomic. It is then clear that the polynomial

$$W_n^*(X_1, \dots, X_{n+1}) := \sum_{\boldsymbol{r} \in D_n} X_1^{r_1} \dots X_n^{r_n} X_{n+1}^{|\boldsymbol{r}|/n}$$

is also cyclotomic. From this it follows that the polynomial in one variable $R(t) := W_n^*(t, t, 0, \dots, 0, 1) = 1 + (n-1)t^n$ is cyclotomic. But this is impossible since $R(t)$ has roots of modulus different from 1. This completes the proof of Assertion 2.

Proof of Assertion 3. We first note that the constant $C(\boldsymbol{A}_n)$, defined at the end of the Introduction to §2, satisfies $C(\boldsymbol{A}_n) = 2^{n-1}$. Thus,

$$\#\left\{ \boldsymbol{x} \in U(\boldsymbol{A}_n) : H(\boldsymbol{x}) = \max_i |m_i(\boldsymbol{x})| \leq t \right\} = 2^{n-1} \sum_{1 \leq m_i \leq t \ \forall i} F_{\boldsymbol{A}_n}(m_1, \dots, m_{n+1}).$$

Setting $I_n = K \cup S^*(W_n)$, it is elementary to check that $\alpha := \left(\frac{1}{n}, \dots, \frac{1}{n}, 0\right) \in \partial V(I_n; 1)$, and $\iota(I_n) = 1$. We then define $\mathscr{H}_\alpha(\boldsymbol{s}) := (\prod_{v \in E(I_n; \alpha)} \langle v, \boldsymbol{s} \rangle) Z_{\boldsymbol{A}_n}(\alpha + \boldsymbol{s})$, and apply both

Theorem 6 (see (17) and (19) in particular) and a standard growth estimate for the Riemann zeta function to conclude the following:

(23.i) there exists $\delta > 0$ such that $\mathscr{H}_\alpha(\boldsymbol{s})$ is analytic in $V(I_n; -\delta)$;

(23.ii) $\mathscr{H}_\alpha(0) \neq 0$;

(23.iii) there exists $u > 0$ such that for all $\boldsymbol{s} \in V(I_n; -\delta)$,

$$\mathscr{H}_\alpha(\boldsymbol{s}) \ll_\varepsilon \prod_{v \in E(I_n;\alpha)} (1 + |\langle v, \tau \rangle|)^{(1 - u\min\{0, \langle v, \sigma \rangle\})} \cdot \left(1 + \left(\sum_{i=1}^{n} |\tau_i|\right)^{\epsilon}\right).$$

We now try to apply the multivariable tauberian theorem, Théorème 2 of [5], with pole α and exponent vector $(1, \ldots, 1)$, whose components determine the exponents of t (this vector is denoted β in [ibid.]). The first point is to identify the sets of vectors $J(\alpha)$ and $E(I_n, \alpha)$ as well as the rank of their union. To this end, it is elementary to check the following:

- $J(\alpha) := \{\boldsymbol{e}_i \mid \alpha_i = 0\} = \{\boldsymbol{e}_{n+1}\}$ and $E(I_n, \alpha) = J_n$;
- $Rank(E(I_n, \alpha) \cup J(\alpha)) = n + 1$ and $\#E(I_n, \alpha) = \binom{2n-1}{n} - 1 = d_n + n$.

The second point is more delicate since $(1, \ldots, 1)$ need not satisfy the criterion in part (iv) of Théorème 2 [ibid.]. That is, there may not exist $\{\gamma_v\}_{v \in J_n \cup \{\boldsymbol{e}_{n+1}\}} \subset (0, \infty)$ such that $(1, \ldots, 1) = \sum_{v \in J_n \cup \{\boldsymbol{e}_{n+1}\}} \gamma_v v$. To circumvent this difficulty, the idea is to find an equivalent vector as follows. Setting $\beta = \left(1, \ldots, 1, 1 + \frac{1}{d_n+1}\right) := (\beta_1, \ldots, \beta_{n+1})$, it is clear that $\forall (m_1, \ldots, m_{n+1}) \in \mathbb{N}^{n+1}$ satisfying $(m_1 : \ldots : m_{n+1}) \in U(\boldsymbol{A}_n)$ and $\gcd(m_1, \ldots, m_{n+1}) = 1$, we have

$$\max_i m_i \leq t \Leftrightarrow m_j \leq t^{\beta_j} \quad \forall j = 1, \ldots, n+1, \ \forall t \geq 1.$$

To finish the proof, it suffices to show that β *does* belong to the interior of the cone generated by $J_n \cup \{\boldsymbol{e}_{n+1}\}$. We first define $t(n) = \#\{\boldsymbol{r} \in \{0, \ldots, n-1\}^n : |\boldsymbol{r}| = n\}$. It is well known that $t(n) = d_n + 1$. Next, we set

$$\gamma_v = t(n)^{-1} \quad \forall v \in (J_n \cup \{\boldsymbol{e}_{n+1}\}) \backslash \{n\boldsymbol{e}_i + \boldsymbol{e}_{n+1}\}_{i=1}^n,$$

$$\gamma_{n\boldsymbol{e}_i + \boldsymbol{e}_{n+1}} = 1/nt(n) \quad \forall i = 1, \ldots, n.$$

We then note that the value of $\sum_{\substack{\boldsymbol{r} \in \{0,\ldots,n-1\}^n \\ |\boldsymbol{r}|=n}} r_j$ is independent of j and satisfies:

$$\sum_{\substack{\boldsymbol{r} \in \{0,\ldots,n-1\}^n \\ |\boldsymbol{r}|=n}} r_j = \frac{1}{n} \sum_{i=1}^{n} \sum_{\substack{\boldsymbol{r} \in \{0,\ldots,n-1\}^n \\ |\boldsymbol{r}|=n}} r_i = \frac{1}{n} \sum_{\substack{\boldsymbol{r} \in \{0,\ldots,n-1\}^n \\ |\boldsymbol{r}|=n}} |\boldsymbol{r}| = t(n).$$

A straightforward computation then shows:

(AutoPDF V7 2/6/06 14:27) WDG (170×240mm) Tmath J-1468 Forum, : PMU: S(C) 19/05/2006 pp. 1–29 1468_06-11 (p. 27)

$$\sum_{v \in J_n \cup \{\boldsymbol{e}_{n+1}\}} \gamma_v v = (1 + t(n)^{-1})\boldsymbol{e}_{n+1} + \sum_{j=1}^{n} t(n)^{-1} \Bigg(\sum_{\substack{\boldsymbol{r} \in \{0,\ldots,n-1\}^n \\ |\boldsymbol{r}|=n}} r_j \Bigg) \boldsymbol{e}_j$$

$$= (1 + t(n)^{-1})\boldsymbol{e}_{n+1} + \sum_{j=1}^{n} \boldsymbol{e}_j = \beta.$$

We can now apply Théorème 2 of [5] with pole α and exponent vector β for t to finish the proof of Assertion 3, and complete the proof of Theorem 7.

Concluding Remark. Additional information about the distribution of primitive integral solutions to the equation $x_1 \cdots x_n = x_{n+1}^n$ can also be deduced. This uses two facts. The first is that Theorem 6 characterizes *all* the boundary points of analyticity as poles of $Z_{\boldsymbol{A}_n}(\boldsymbol{s})$ at which (23.i)–(23.iii) are satisfied. The second is that the tauberian theorem of La Bretèche gives an explicit asymptotic for the sum of coefficients $F_{\boldsymbol{A}_n}(m_1, \ldots, m_{n+1})$ when each m_i is allowed to grow at a different rate in t. Precisely, given a vector $\gamma = (\gamma_1, \ldots, \gamma_{n+1}) \in (0, \infty)^{n+1}$, one can also calculate a precise (and nonzero!) asymptotic for the counts

$$\sum_{\{1 \le m_i \le t^{\gamma_i} \ \forall i\}} F_{\boldsymbol{A}_n}(m_1, \ldots, m_{n+1}),$$

provided that γ is a "generic" vector, that is, γ belongs to an open dense subset of $(0, \infty)^{n+1}$. The expression for the dominant term is similar to that in part 3 of Theorem 7. Working out the details in general for this multiplicative equation, as well as any other, that is, $x_1^{k_1} \cdots x_n^{k_n} = x_{n+1}^{l_1} \cdots x_{n+q}^{l_q}$ with $\sum k_i = \sum l_j$, would seem to be an interesting problem.

References

[1] Batyrev V. V. and Tschinkel Yu.: Manin's conjecture for toric varieties. J. Algebraic Geometry **7** (1998), 15–53

[2] Dahlquist G.: On the analytic continuation of eulerian products. Ark. Mat. **1** (1952), 533–554

[3] de la Bretèche R.: Sur le nombre de points de hauteur bornée d'une certaine surface cubique singulière. Peyre, Emmanuel (ed.), Nombre et répartition de points de hauteur bornée. Astérisque **251** (1998), 51–57

[4] de la Bretèche R.: Compter des points d'une variété torique. J. Number Theory **87** (2001), 315–331

[5] de la Bretèche R.: Estimation de sommes multiples de fonctions arithmétiques. Compositio Math. **128** (2001), 261–298

[6] Estermann T.: On certain functions represented by Dirichlet series. Proc. London Math. Soc. **27** (1928), 435–448

[7] Fouvry E.: Sur la hauteur des points d'une certaine surface cubique singulière. Peyre, Emmanuel (ed.), Nombre et répartition de points de hauteur bornée. Astérisque **251** (1998), 31–49

(AutoPDF V7 2/6/06 14:27) WDG (170×240mm) Tmath J-1468 Forum, : PMU: S(C) 19/05/2006 pp. 1–29 1468_06-11 (p. 28)

[8] Heath-Brown D. R. and Moroz B. Z.: The density of rational points on the cubic surface $x_0^3 = x_1x_2x_3$. Math. Proc. Camb. Philos. Soc. **125** (1999), 385–395

[9] Kurokawa N.: On the meromorphy of Euler products I, II. Proc. London Math. Soc. **53** (1986), 1–47 and 209–236

[10] Salberger P.: Tamagawa measures on universal torsors and points of bounded height on Fano varieties. Peyre, Emmanuel (ed.), Nombre et répartition de points de hauteur bornée. Astérisque **251** (1998), 91–258

[11] Titchmarsh E. C.: The theory of the Riemann Zeta-function. Clarendon Press. Oxford, 2 revised by d.r. heath-brown edition, 1986

Received March 2005; revised February 2006

Université de Lille 1, UFR de Math., Laboratoire de Math. Paul Painlevé U.M.R. CNRS 8524, 59655 Villeneuve d'Ascq Cedex, France
bhowmik@math.univ-lille1.fr

Université de Caen, UFR des Sciences, Campus 2, Laboratoire de Math. Nicolas Oresme, CNRS UMR 6139, Bd. Mal Juin, B.P. 5186, 14032 Caen, France
essoua@math.unicaen.fr

Rochester N.Y., USA
lichtin@math.rochester.edu