Journal of Number Theory **73**, 59–71 (1998)
Article No. NT982281

A Turán–Kubilius Inequality for Integer Matrices

Gautami Bhowmik and Olivier Ramaré

Laboratoire de Mathématiques, URA CNRS 751, Université de Lille 1, 59655 Villeneuve d'Asq Cedex, France
E-mail: bhowmik@gat.univ-lille1.fr and ramare@gat.univ-lille1.fr

Communicated by A. Granville

Received May 27, 1997; revised March 5, 1998

We prove a general Turán–Kubilius inequality and use it to derive that the number $\tau(S)$ of divisors of an integer $r\times r$ matrix S verifies $\tau(S)=(\mathrm{Log}\,|S|)^{\mathrm{Log}\,2+o(1)}$ for all but $o(X)$ matrices of determinant $\leqslant X$. This is in sharp contrast with the average order which is $\asymp |S|^{\beta_r-1}(\mathrm{Log}\,|S|)^{\gamma_r}$ for β_r that are >1 as soon as $r\geqslant 4$ and some non-negative γ_r. We further extract a fairly large set of matrices over which the normal order is much closer to the average order. © 1998 Academic Press

I. INTRODUCTION

In 1934–1936, Turán devised a simple but powerful process of obtaining normal orders for some additive functions of integers, a process which was later (1956–1964) extended by Kubilius to every additive function of integers and which is now called the Turán–Kubilius inequality (A classical introduction on this subject may be found in [El].) The simplicity of this proof enables one to extend it to different situations: Horadam [Ho] for instance extended it to some additive functions over a set of regular Beurling generalized integers, while Hinz [Hi] recently extended it to additive functions of integer ideals of a number field. In Section III we describe a general setting which covers these three applications as well as the two new ones that have in fact been the starting point of this paper. The hypotheses are expected to be wide enough to cover most results of this kind. Note, however, that our proof resembles Turán's.

Our first aim is to study the distribution of the number of divisors of non-singular integer $r\times r$ matrices, where $r\geqslant 2$. We denote the set of such matrices by $\mathfrak{Inv}_r$. Since the number of divisors of S, which we shall denote by $\tau(S)$, depends only on the Smith Normal Form (SNF) of S we study the distribution of the values of $\tau(S)$ when S ranges over all SNF matrices with determinant atmost X, where X is a large enough real parameter. We recall that the cardinality of such a set is asymptotic to $C(r)\,X$ (cf. [B1]) where

0022-314X/98 $25.00

$C(r)$ is defined in (1.3) (cf. [BN, N2, Ne] for the general theory of arithmetical functions over integer matrices). We use the notation $|M|$ for the determinant of a matrix M and $(n, m) = 1$ to mean that the two integers m and n are coprime. We have shown in [BR1] that

$$X^{\beta_r - 1} \ll \frac{1}{X} \sum_{\substack{S\ SNF \\ |S| \leqslant X}} \tau(S) \ll X^{\beta_r - 1} \operatorname{Log}^{\gamma_r} X, \tag{1.1}$$

where $\gamma_r \geqslant 0$ and β_r is given by

$$\beta_r = ([r^2/4] + 1)/r, \tag{1.2}$$

and $[n]$ denotes the integer part of n. In fact, in [BW2] it is shown that $1/X \sum \tau(S) = C'(r)\, X^{\beta_r - 1} (\operatorname{Log} X)^{\gamma_r} (1 + o(1))$ where $C'(r)$ is an effective constant; $\gamma_2 = 2$, $\gamma_3 = 4$, $\gamma_{2k} = 1$, and $\gamma_{2k+1} = 2$ if $k \geqslant 2$.

In particular, we have $\beta_2 = \beta_3 = 1$, $\beta_4 = 5/4$, and $\beta_5 = 7/5$. The question then naturally arises as to whether this average value properly reflects the behavior of most S or whether it is influenced by a minority of summands in (1.1). It turns out that the second case is the correct one. We first show here that

THEOREM 1.

$$\sum_{\substack{S\ SNF \\ |S| \leqslant X}} |\operatorname{Log} \tau(S) - (\operatorname{Log} 2) \operatorname{Log} \operatorname{Log} X|^2 \ll X \operatorname{Log} \operatorname{Log} X.$$

In particular, we have $\tau(S) = (\operatorname{Log} |S|)^{(1 + o(1)) \operatorname{Log} 2}$ for all but $o(X)$ SNF matrices of determinant $\leqslant X$. This result is similar to the classical result of Hardy and Ramanujan which is of course the case $r = 1$ of the above theorem. We first have to express the value of $\operatorname{Log} \tau$ in terms of its value over SNF matrices whose determinant is a power of a prime. A generic matrix satisfying these properties will be denoted by P. Using the interpretation in terms of abelian groups developed in [BR2], we show that

$$\operatorname{Log} \tau(M) = \sum_{P \to S} \operatorname{Log} \tau(P),$$

where we use $M \to S$ to say that there exists $N \in \mathfrak{I}nv_r$ such that $SNF(N) = SNF(M)$ and $N \parallel S$ (i.e., $N \mid S$ and $(|N|, |S|/|N|) = 1$). The next step consists in counting properly the number of M in SNF such that $P \to M$ which we do in Lemma 2.2. This approach extends to any additive arithmetical function over matrices. Note finally that in [B3], the first named author has given an exact expression of $\tau(S)$ but that this expression is too complicated to understand the normal value of Log τ. To explain further we need some definitions.

Put

$$\rho_r(m) = \prod_{k=1}^{r} \prod_{p \mid m} (1 - p^{-k}), \qquad C(r) = \prod_{k=2}^{r} \prod_{p \geqslant 2} (1 - p^{-k})^{-1}. \quad (1.3)$$

Define further

$$t(b) = \sum_{c_c^3 c_3^3 \cdots c_r^r = b} 1, \quad (1.4)$$

so that the number of SNF matrices of determinant n is given by $1 \star t(n)$. A function f of non-singular integer $r \times r$ matrices is said to be *additive* if $f(A)$ depends only on the SNF of A and if $f(AB) = f(A) + f(B)$ whenever $(|A|, |B|) = 1$. We then have

THEOREM 2. *Let f be an additive function. We have*

$$\sum_{\substack{S \text{ SNF} \\ |S| \leqslant X}} |f(S) - M(f, X)|^2 \ll XD(f, X),$$

with

$$M(f, X) = \sum_{|P| \leqslant X} \frac{f(P)}{|P|} \rho_r(|P|), \qquad \text{and}$$

$$D(f, X) = \sum_{|P| \leqslant X} \frac{|f(P)|^2}{|P|} \rho_r(|P|),$$

where P denotes an $r \times r$ matrix in SNF whose determinant is a power of an integer prime. Moreover the constant implied in the Vinogradoff symbol depends at most on r and not on f.

In particular f may depend on X.

In the case of integers, the study of the normal order of Log τ is usually replaced by the study of the omega functions by using the bounds $2^{\omega(n)} \leqslant \tau(n) \leqslant 2^{\Omega(n)}$. It is possible to get an analog for ω by defining $\omega(M)$ to be the number of matrices P of prime-power determinant such that $P \parallel M$. It is then also the number of factors in any decomposition of $M = P_1 \cdots P_k$ where the P_j's are determinant-wise coprime matrices of prime power determinant. A little reflection later we see that

$$\omega(M) = \sum_{P \to M} 1$$

and hence Theorem 2 readily gives its normal order to be $\sim \operatorname{Log}\operatorname{Log} X$ while its average order is also seen to be $\sim \operatorname{Log}\operatorname{Log} X$ (on application of Lemma 2.2). It is however not obvious to connect ω and τ and our primary aim being the investigation of τ we prefer to study $\operatorname{Log}\tau$ rather than ω. Note, moreover, that $\omega(M)=\omega(|M|)$ so that this function carries no further information than that already in the determinant.

Since the average order of $\tau(S)$ does not reflect its normal behavior we seek subsets of $\mathfrak{I}nv_r$ where its normal value is large (or more precisely where the normal value of $\operatorname{Log}\tau$ is large). Note that this phenomenon has been a classical field of investigation in the case $r=1$ and that the case $r=2$ is dealt with in [BW1].

Let $M \in \mathfrak{I}nv_r$. We say that M has *total rank* r if each of the matrices P with prime power determinant and such that $P \to M$ corresponds to an abelian group of rank r and not less ($M=\mathrm{Id}$ is accepted). Summations with a $*$ as a superscript are restricted to matrices of total rank r. We further denote by $g(n)$ the number of SNF matrices of determinant n and total rank r. We define

$$\sum_{n\geqslant 1}\frac{g(n)}{n^s}=\zeta(rs)\sum_{n\geqslant 1}\frac{t^*(n)}{n^s}, \tag{1.5}$$

and

$$C^*(r)=\sum_{n}\frac{t^*(n)}{n^{1/r}}, \qquad \rho_r^*(m)=\prod_{p\,|\,m}\Big(\sum_{u\geqslant 0}t^*(p^u)\,p^{-u/r}\Big)^{-1}. \tag{1.6}$$

(An explanation of these formulae as well as proofs of the convergence of the above series may be found in the course of Section II.) We show in Lemma 2.3 that the number of matrices of determinant less than X and total rank r is asymptotic to $C^*(r)\,X^{1/r}$. We can then make precise Theorem 2 in

THEOREM 3. *Let f be an additive function. We have*

$$\sum_{\substack{S\ SNF\\ |S|\leqslant X}}^{*} |f(S)-M^*(f,X)|^2 \ll X^{1/r}D^*(f,X),$$

with

$$M^*(F,X)=\sum_{|P|\leqslant X}^{*}\frac{f(P)}{|P|}\,\rho_r^*(|P|), \qquad \textit{and}$$

$$D^*(f,X)=\sum_{|P|\leqslant X}\frac{|f(P)|^2}{|P|}\,\rho_r^*(|P|),$$

where P denotes an $r \times r$ matrice in SNF whose determinant is a power of an integer prime. Moreover the constant implied in the Vinogradoff symbol depends at most on r and not on f.

Using this result for $\mathrm{Log}(\tau(S)/|S|^{[r^2/4]/r})$ we shall get

THEOREM 4. *With $\xi = \mathrm{Log}\, 2$ if r is even and $\xi = 0$ if r is odd, we have*

$$\sum_{\substack{S\ SNF \\ |S| \leqslant X}}^{*} \left| \mathrm{Log}\, \tau(S) - \frac{[r^2/4]}{r} \mathrm{Log}\, |S| - \xi\, \mathrm{Log}\, \mathrm{Log}\, X \right|^2 \ll X^{1/r}(\xi\, \mathrm{Log}\, \mathrm{Log}\, X + 1).$$

This result is sharp since it yields

$$X^{[r^2/4]/r + 1/r} \ll \sum_{|S| \leqslant X}^{*} \tau(S) \leqslant \sum_{|S| \leqslant X} \tau(S) \ll X^{([r^2/4]+1)/r}(\mathrm{Log}\, X)^{\gamma_r}. \tag{1.7}$$

II. AUXILIARY LEMMAS

We shall require the number $h_r(n)$ of HNF having determinant n. Its associated Dirichlet series is given by (cf. [B1, B2])

$$\sum_{n \geqslant 1} h_r(n)\, n^{-s} = \prod_{k=1}^{r} \zeta(s+k-1). \tag{2.1}$$

LEMMA 2.1. *For $X > 0$, we have $\sum_{c_2^2 c_3^3 \cdots c_r^r \leqslant X} 1 \leqslant r!\, X^{1/2}/2$.*

Proof. Put

$$\Delta_j(X) = \sum_{c_j^j c_{j+1}^{j+1} \cdots c_r^r \leqslant X} 1 \qquad (2 \leqslant j \leqslant r).$$

A simple recursion shows that $\Delta_j(X) \leqslant (r!/j!)\, X^{1/j}$. Taking $j = 2$ yields the result. See [B2] for an asymptotic evaluation of $\Delta_2(X)$. ▮

LEMMA 2.2. *For $M \in \mathfrak{Inv}_r$ and $X > 0$, we put*

$$\mathcal{N}(M, X) = \sum_{\substack{S\ SNF \\ |S| \leqslant X,\, M \to S}} 1.$$

We have $\mathcal{N}(M, X) = C(r)\, X \rho_r(m)/m + \mathcal{O}(X^{1/2} m^{-1/2} 2^{\omega(m)})$ with $m = |M|$.

Proof. We first prove that

$$\mathcal{N}(M, X) = \sum_{\substack{n \leqslant X/m \\ (n,m)=1}} \sum_{ab=n} t(b). \tag{2.2}$$

To do so note that the number of S in SNF with determinant $\leqslant X$ and such that $M \to S$ is equal to the number of finite abelian groups G of rank $\leqslant r$, cardinality $\leqslant X$, which admit a subgroup H isomorphic to the abelian group associated with M and such that $(|H|, |G|/|H|)=1$. Note that $|H|=m$. Such a subgroup is a direct summand of G and thus the number we seek is the number of finite abelian groups G' of rank $\leqslant r$, cardinality $\leqslant X/m$, and coprime with m, which we now convert into the number of SNF S of determinant $\leqslant X/m$ and coprime with m, thus proving the claimed formula.

We now evaluate $\mathcal{N}(M, X)$ by using

$$\sum_{\substack{n \leqslant Y \\ (n,m)=1}} 1 = \frac{\phi(m)}{m} Y + \mathcal{O}(2^{\omega(m)}) \tag{2.3}$$

and Lemma 2.1. ∎

Lemma 2.2 enables one to carry out usual evaluations like sieving processes. We shall first concentrate on a Turán–Kubilius inequality.

Denoting by $k(n)$ the squarefree kernel of n, we check that $g(n)=0$ if $k(n)^r \nmid n$ and that $g(n) = 1 \star t(n/k(n)^r)$ if $k(n)^r | n$, thus getting

$$\sum_{n \geqslant 1} \frac{g(n)}{n^s} = \prod_{p \geqslant 2} \left(1 + \frac{1}{p^{rs}} \prod_{i=1}^{r} (1 - 1/p^{is})^{-1}\right) = \zeta(rs) \sum_{n \geqslant 1} \frac{t^*(n)}{n^s}, \tag{2.4}$$

where $t^*(n)$ is the multiplicative function defined over the powers of the prime p by $t^*(p^u)=0$ if $u \leqslant r$, $t^*(p^u) = t(p^{u-r})$ if $r < u < 2r$, and $t^*(p^u) = t(p^{u-r}) - t(p^{u-2r})$ if $2r \leqslant u$. We verify that $t^* \geqslant 0$. Note further that $t(p^u) \leqslant u^{r-1}$, that $t^*(p^{r+1}) = 0$, and that $t^*(p^{r+2}) = 1$ if $r \geqslant 2$ and 0 if $r=2$, so that

$$\sum_{n \geqslant 1} t^*(n)\, n^{-1/(r+1)} \ll 1,$$

which by Rankin's method yields

$$\sum_{n \leqslant Y} t^*(n) \ll Y^{1/(r+1)} \qquad (Y > 0).$$

Define

$$C^*(r) = \sum_n \frac{t^*(n)}{n^{1/r}}, \qquad \rho_r^*(m) = \prod_{p|m} \left(\sum_{u \geqslant 0} t^*(p^u)\, p^{-u/r}\right)^{-1}. \tag{2.5}$$

LEMMA 2.3. *For $M \in \mathfrak{Inv}_r$ of total rank r and $X > 0$, we put*

$$\mathcal{N}^*(M, X) = \sum_{\substack{S \text{ SNF} \\ |S| \leqslant X,\, M \to S}}^{*} 1.$$

We have $\mathcal{N}^*(M, X) = C^*(r)\, \rho_r^*(m)(X/m)^{1/r} + \mathcal{O}(2^{\omega(m)}(X/m)^{1/(r+1)})$ *with* $m = |M|$.

Proof. Arguing as in Lemma 2.2, we find that

$$\mathcal{N}^*(M, X) = \sum_{\substack{|S| \leqslant X/m \\ (|S|, m) = 1}}^{*} 1,$$

We complete the proof easily. ∎

LEMMA 2.4. *Let P denote an SNF-matrix whose determinant is a power of a prime p. For $X \geqslant 3$, we have*

$$\sum_{|P| \leqslant X} 1 \ll X/\operatorname{Log} X.$$

Proof. We have

$$\sum_{|P| \leqslant X} 1 = \sum_{p^a \leqslant X} 1 * t(p^a) \leqslant \sum_{p \leqslant X} 1 + \sum_{\substack{p^a \leqslant X \\ a \geqslant 2}} a^{r-1} \ll X/\operatorname{Log} X. \quad ∎$$

LEMMA 2.5. *Let P denote an SNF-matrix whose determinant is a power of a prime p. We have*

$$\sum_{|P| \leqslant X}^{*} 1 \ll X^{1/r}/\operatorname{Log} X.$$

Proof. We have

$$\sum_{|P| \leqslant X}^{*} 1 \leqslant \sum_{p^a \leqslant X} g(p^a) \leqslant \sum_{\substack{p^a \leqslant X \\ a \geqslant r}} g(p^a) \ll X^{1/r}/\operatorname{Log} X. \quad ∎$$

We finally recall the following lemma from [BW2]:

LEMMA 2.6. *Let $P = \operatorname{diag}(p^{f_1}, ..., p^{f_1 + \cdots + f_r})$. Then*

$$c(f_1, ..., f_r)\, p^{\eta(f_1, ..., f_r)} \leqslant \tau(P) \leqslant p^{\eta(f_1, ..., f_r)} \left(c(f_1, ..., f_r) + \frac{1}{p} d(f_1, ..., f_r) \right)$$

with

$$\begin{cases} \eta(f_1, ..., f_r) = \sum_{j=1}^{r} \left[\dfrac{(r-j+1)^2}{4} \right] f_j, \\ c(f_1, ..., f_r) = (f_r+1)(f_{r-2}+1) \cdots (f_{r-2[r/2]}+1), \\ d(f_1, ..., f_r) = 1 + (rf_1 + (r-1)\, f_2 + \cdots + f_r)^{r-1}. \end{cases}$$

III. A GENERAL TURÁN–KUBILIUS INEQUALITY

We prove a Turán–Kubilius inequality in a general setting in order to derive Theorems 4.1 and 4.3 without having to do the same proof twice. Moreover it also yields the usual Turán–Kubilius inequality over integers or over algebraic number fields. It also works for Beurling numbers if some regularity assumptions are made to ensure (H_0) and (H_1) below.

Let $X \geqslant 3$ be a real number. Let $\mathscr{P}$ be a set which will be considered as a set of primes. Let $\mathscr{S}$ be a set which will be considered as a set of integers. Let $\hookrightarrow$ be a relation from $\mathscr{P}$ to $\mathscr{S}$ such that for any $S \in \mathscr{S}$, the number of $P \in \mathscr{P}$ such that $P \hookrightarrow S$ is finite. Let $\simeq$ be an equivalence relation over $\mathscr{P}$ such that $P_1 \hookrightarrow S$ and $P_2 \hookrightarrow S$ implies $P_1 \not\simeq P_2$, for $P_1, P_2 \in \mathscr{P}$ and $S \in \mathscr{S}$. Let $|\cdot|$ be a function from $\mathscr{P} \sqcup \mathscr{S}$ (disjoint union) to $]0, +\infty[$ such that the number of points P in $\mathscr{P}$ (resp. in $\mathscr{S}$) such that $|P| \leqslant X$ is finite and such that $P_i \hookrightarrow S$ for $i+1, ..., k$ implies $|P_1| \cdots |P_k| \leqslant |S|$ if the P_i are all distinct. We now make regularity assumptions on the system $(\mathscr{P}, \mathscr{S}, \hookrightarrow, |\cdot|, X)$.

There exist a non-negative function δ over $\mathscr{P}$ and non-negative real numbers σ, $c_1, ..., c_6$, α and β such that $1 \geqslant \alpha \geqslant \beta \geqslant 0$ and

$$N(X) = \sum_{|S| \leqslant X} 1 \leqslant \sigma X^{\alpha} + c_7 X^{\beta}, \tag{H_0}$$

$$N(P, X) = \sum_{\substack{|S| \leqslant X \\ P \hookrightarrow S}} 1 = \sigma\delta(P)(X/|P|)^{\alpha} + \mathscr{O}^*(c_5 \delta(P)(X/|P|)^{\beta}), \tag{H_1}$$

$$\begin{aligned} N(P_1, P_2, X) = \sum_{\substack{|S| \leqslant X \\ P_1 \hookrightarrow S \\ P_2 \hookrightarrow S}} 1 \leqslant \sigma\delta(P_1)\, \delta(P_2) \left(\frac{X}{|P_1|\,|P_2|} \right)^{\alpha} \\ + c_6 \delta(P_1)\, \delta(P_2) \left(\frac{X}{|P_1|\,|P_2|} \right)^{\beta}, \end{aligned} \tag{H_2}$$

$$\sum_{|P| \leqslant X} \delta(P)\, |P|^{-\alpha} \leqslant c_2 \operatorname{Log} \operatorname{Log} X, \tag{H_3}$$

$$\sum_{|P| \leqslant X} \delta(P)\,|P|^{-2\beta+\alpha} \leqslant c_3 X^{2(\alpha-\beta)}/\mathrm{Log}\,X, \tag{H$_4$}$$

$$\sum_{\substack{P_1 \neq P_2 \\ |P_1|\,|P_2| \leqslant X}} \delta(P_1)\,\delta(P_2)(|P_1|\,|P_2|)^{\alpha-2\beta} \leqslant c_4 X^{2(\alpha-\beta)}(\mathrm{Log}\,\mathrm{Log}\,X)/\mathrm{Log}\,X. \tag{H$_5$}$$

Now let f be a function over $\mathscr{P}$. We extend it into a function over $\mathscr{S}$ by putting

$$f(S) = \sum_{P \hookrightarrow S} f(P).$$

We define further

$$M(f, X) = \sum_{|P| \leqslant X} \frac{\delta(P)\,f(P)}{|P|^{\alpha}}, \qquad D(f, X) = \sum_{|P| \leqslant X} = \sum_{|P| \leqslant X} \frac{\delta(P)\,|f(P)|^2}{|P|^{\alpha}}.$$

Then we have

THEOREM 3.1. $\sum_{|S| \leqslant X} |f(S) - M(f, X)|^2 \leqslant 2(\sigma+\varepsilon)\, X^{\alpha} D(f, X)$, *where*

$$\varepsilon = c_5 + (c_6 c_4^{1/2} + 2c_5\sqrt{c_3 c_2})\sqrt{(\mathrm{Log}\,\mathrm{Log}\,X)/\mathrm{Log}\,X} + c_7 c_2 X^{\beta-\alpha}\,\mathrm{Log}\,\mathrm{Log}\,X.$$

Note that $M(f, X)^2 \leqslant D(f, X)\, c_2\,\mathrm{Log}\,\mathrm{Log}\,X$. In case $M(f, X)$ is much greater than $D(f, X)^{1/2}$ and $N(X) \asymp X^{\alpha}$, this theorem tells us that $f(S) \sim M(f, X)$ for almost all $S \leqslant X$.

Proof. Since X is fixed throughout the proof, we abbreviate $M(f, X)$ and $D(f, X)$ as $M(f)$ and $D(f)$. We further call $\Delta(f)$ the LHS in Theorem 3.1. We first assume f to be non-negative. Then

$$\Delta(f) = \sum_{|S| \leqslant X} f(S)^2 - 2M(f) \sum_{|S| \leqslant X} f(S) + M(f)^2\,N(X).$$

We study each of these three terms separately.

$$\sum_{|S| \leqslant X} f(S)^2 = \sum_{\substack{|P_1|,\,|P_2| \leqslant X \\ P_1 \neq P_2}} f(P_1)\,f(P_2)\,N(P_1, P_2, X) + \sum_{|P| \leqslant X} f(P)^2\,N(P, X),$$

and using (H_0) and (H_1), we find that the above expression is

$$\begin{aligned}
&\leqslant \sigma X^\alpha \sum_{\substack{|P_1|,\,|P_2|\leqslant X\\ P_1\neq P_2}} \frac{\delta(P_1)\,f(P_1)\,\delta(P_2)\,f(P_2)}{|P_1|^\alpha\,|P_2|^\alpha}\\
&\quad + c_6 X^\beta \sum_{\substack{|P_1|\,|P_2|\leqslant X\\ P_1\neq P_2}} \frac{\delta(P_1)\,f(P_1)\,\delta(P_2)\,f(P_2)}{|P_1|^\beta\,|P_2|^\beta}\\
&\quad + \sigma X^\alpha \sum_{|P|\leqslant X} \frac{\delta(P)\,f(P)^2}{|P|^\alpha} + c_5 X^\beta \sum_{|P|\leqslant X} \frac{\delta(P)\,f(P)^2}{|P|^\beta},\\
&\leqslant \sigma X^\alpha M(f)^2 + (\sigma + c_5)\,X^\alpha D(f)\\
&\quad + c_6 D(f)\,X^\beta \Bigg(\sum_{\substack{|P_1|\,|P_2|\leqslant X\\ P_1\neq P_2}} \frac{\delta(P_1)\,\delta(P_2)}{(|P_1|\,|P_2|)^{2\beta-\alpha}}\Bigg)^{1/2},
\end{aligned}$$

which is now less than

$$\sigma X^\alpha M(f)^2 + (\sigma + c_5 + c_6 c_4^{1/2}\sqrt{(\mathrm{Log\,Log}\,X)/\mathrm{Log}\,X})\,X^\alpha D(f).$$

We also have

$$\begin{aligned}
\sum_{|S|\leqslant X} f(S) &\geqslant \sigma X^\alpha \sum_{|P|\leqslant X} \frac{\delta(P)\,f(P)}{|P|^\alpha} - c_5 X^\beta \sum_{|P|\leqslant X} \frac{\delta(P)\,f(P)}{|P|^\beta},\\
&\geqslant \sigma M(f) - c_5 D(f)^{1/2} \Bigg(X^{2\beta} \sum_{|P|\leqslant X} \frac{\delta(P)}{|P|^{2\beta-\alpha}}\Bigg)^{1/2},\\
&\geqslant \sigma M(f) - c_5 D(f)^{1/2}\,X^\alpha \sqrt{c_3/\mathrm{Log}\,X}.
\end{aligned}$$

Furthermore $M(f)^2 \leqslant c_2 D(f)\mathrm{Log\,Log}\,X$, and thus

$$\begin{aligned}
\Delta(f) \leqslant{}& (\sigma + c_5 + c_6 c_4^{1/2}\sqrt{(\mathrm{Log\,Log}\,X)/\mathrm{Log}\,X})\,X^\alpha D(f)\\
&+ 2c_5 X^\alpha D(f)\sqrt{c_3 c_2 (\mathrm{Log\,Log}\,X)/\mathrm{Log}\,X}\\
&+ c_7 c_2 X^\alpha D(f)\,X^{\beta-\alpha}\,\mathrm{Log\,Log}\,X
\end{aligned}$$

which is what was to be proved. If f is real valued, we write it as a difference of two non-negative functions and use $(x+y)^2 \leqslant x^2 + y^2$. If f is complex-valued, we split it into real and imaginary parts. ∎

We note here that the first term in the definition of ε in Theorem 3.1 (this term is equal to c_5) does not appear when working with integers due to the inequality $\sum_{d\mid n\leqslant X} 1 \leqslant X/d$ which has no error term. We could have introduced such a refinement but since the constant ε is not optimal even in the case of integers it seems superfluous.

We can of course replace (H_1) and (H_2) by L^2-average bounds of the shape

$$\sum_{|P| \leqslant X} |N(P, X) - \sigma\delta(P)(X/|P|)^{\alpha}|^2 \ll X^{\alpha}$$

and similarly for $N(P_1, P_2, X)$.

We now comment on (H_3) and (H_4). We have made these three assumptions because we wanted to allow all the constants and f to depend on X. However, the hypothesis

$$\sum_{|P| \leqslant t} 1 \leqslant c(X)\, t^{\alpha}/\text{Log}\, t \qquad (2 \leqslant t \leqslant X) \tag{H$_6$}$$

is sufficient if $\beta < \alpha$ and easier to verify. We recall rapidly how to derive (H_3), (H_4), and (H_5) from it. Define $h(n)$ to be the number of $P \in \mathscr{P}$ such that $|P| = n$. By partial summation, (H_6) implies

$$\sum_{n \leqslant t} h(n)\, n^{-\alpha} \ll c(X)\, \text{Log Log}(2t) \qquad (2 \leqslant t \leqslant X),$$

where the implied constant depends only on α. Similarly we get

$$\sum_{n \leqslant t} h(n)\, n^{-2\beta + \alpha} \ll c(X)\, t^{2(\alpha - \beta)}/\text{Log}\, t \qquad (2 \leqslant t \leqslant X).$$

Moreover by using the Dirichlet hyperbola principle we get

$$\sum_{nm \leqslant t} h(n)\, h(m) \ll c(X)\, t^{\alpha}(\text{Log Log}(2t))/\text{Log}\, t \qquad (2 \leqslant t \leqslant X)$$

and using the previous inequality with h changed into $h \star h$ and $c(X)$ into $c(X)$ Log Log$(2X)$, we get (H_5).

IV. TURÁN–KUBILIUS INEQUALITIES FOR INTEGER MATRICES: PROOF OF THEOREMS 1–4

Throughout this part P denotes an integer matrix in SNF whose determinant is a power of a prime. Let f be a complex-valued additive function, i.e., a function which verifies $f(AB) = f(A) + f(B)$ whenever $(|A|, |B|) = 1$. We further assume that f is arithmetical, meaning that $f(A) = f(SNF\, A)$. For an additive arithmetical function, we have

$$f(A) = \sum_{P \to A} f(P).$$

To prove such a decomposition, the simplest way is to interpret it in terms of abelian groups, following [BR2]. The above equality says that $f(A)$ equals the sum of the values of f on each p-component. Each of these p-components corresponds uniquely to a subgroup of $G(A)$ and thus to a divisor of A in HNF, which in turn corresponds to a unique SNF matrix P with $P \to A$. We put

$$M(f, X) = \sum_{|P| \leqslant X} \frac{f(P)}{|P|} \rho_r(|P|), \qquad D(f, X) = \sum_{|P| \leqslant X} \frac{|f(P)|^2}{|P|} \rho_r(|P|).$$

Using Lemmas 2.2, 2.4, and Theorem 3.1 with $\alpha = 1$ and $\beta = 1/2$ we readily get Theorem 2.

Since $\operatorname{Log} \tau(S)$ is an additive arithmetical function, we deduce Theorem 1 from it provided we estimate the mean and the dispersion which we do below.

Proof of Theorem 1. We evaluate the mean and the dispersion by using Lemma 2.6 (Proposition 5 of [BR1] would be enough) which yields

$$\operatorname{Log} \tau(P) \ll \operatorname{Log} |P|.$$

We thus get

$$M(\operatorname{Log} \tau, X) = (\operatorname{Log} 2) \operatorname{Log} \operatorname{Log} X + \mathcal{O}(1) \tag{4.1}$$

and

$$D(\operatorname{Log} \tau, X) = (\operatorname{Log} 2)^2 \operatorname{Log} \operatorname{Log} X + \mathcal{O}(1). \tag{4.2}$$

We replace $M(\operatorname{Log} \tau, X)$ by $(\operatorname{Log} 2) \operatorname{Log} \operatorname{Log} X$ by using $(x+y)^2 \leqslant x^2 + y^2$. ∎

We then use Theorem 3.1 but with SNF matrices of total rank r to deduce Theorem 3. The hypotheses (H_1), (H_2), and (H_6) are dealt with in Lemmas 2.3 and 2.5.

It remains to prove Theorem 4. Since $f(S) = \operatorname{Log} \tau(S) - ([r^2/4]/r) \operatorname{Log} |S|$ is an additive arithmetical function, we only have to evaluate its mean and dispersion.

Proof of Theorem 4. We evaluate the mean and the dispersion by using Lemma 2.6. We first discard the contribution of the P's such that

$rf_1 + \cdots + f_r \geqslant r+1$ (notations of Lemma 2.6). Indeed their contribution to the mean is not more than

$$\sum_{\substack{p^{rf_1+\cdots+f_r} \leqslant X \\ rf_1+\cdots+f_r \geqslant r+1 \\ f_1 \geqslant 1}} \frac{\begin{bmatrix} (\eta(f_1, ..., f_r) + ([r^2/4]/r)(fr_1 + \cdots + f_r)) \operatorname{Log} p \\ + \operatorname{Log}(c(f_1, ..., f_r) + (1/p)\, d(f_1, ..., f_r)) \end{bmatrix}}{p^{rf_1 + \cdots + f_r)/r}}$$

where we sum over p and $f_1, ..., f_r \geqslant 0$. This sum is clearly $\mathcal{O}(1)$. The same holds for the dispersion. We are thus left with the contribution of matrices $\operatorname{diag}(p, ..., p)$ for which we have $\eta = [r^2/4]$, $c=1$ if r is odd, and $c=2$ if r is even and $d = 1 + r^{r-1}$. The contribution to the mean is thus

$$\begin{cases} \sum_{p \leqslant X^{1/r}} \dfrac{\operatorname{Log} 2}{p} + \mathcal{O}(1) & \text{if } r \text{ is even} \\ O(1) & \text{if } r \text{ is odd.} \end{cases} \tag{4.3}$$

The dispersion is computed in a similar way. ∎

REFERENCES

[Be] A. Beurling, Analyse de la loi asymptotique de la distribution des nombres premiers généralisés, *Acta Math.* **68** (1937), 255–291.

[B1] G. Bhowmik, Divisor functions of integer matrices: Evalutations, average orders and applications, *Astérisque* **209** (1992), 169–177.

[B2] G. Bhowmik, Average orders of certain functions connected with arithmetic of matrices, *J. Indian Math. Soc.* **59** (1993), 97–106.

[B3] G. Bhowmik, Evaluations of the divisor function, *Acta Math.* **74** (1996), 155–159.

[BN] G. Bhowmik and V. C. Nanda, Arithmetic of matrices, manuscript.

[BR1] G. Bhowmik and O. Ramaré, Average orders of multiplicative arithmetical functions of integer matrices, *Acta Math.* **66**, No. 1 (1994), 45–62.

[BR2] G. Bhowmik and O. Ramaré, Algebra of matrix arithmetic, *J. Algebra*, in press.

[BW1] G. Bhowmik and J. Wu, On the asymptotic behavior of the number of subgroups of a finite abelian group, *Arch. Math.* **69** (1997), 95–104.

[BW2] G. Bhowmik and J. Wu, On the zeta-function of subgroups of finite abelian groups, *J. London Math. Soc.* (1998).

[El] P. D. T. A. Elliott, "Probabilisitc Number Theory: Mean-Value Theorems," Grundlehren Math. Wiss., Vol. 239, Springer-Verlag, New York/Berlin/Heidelberg.

[Hi] J. G. Hinz, On the prime ideal theorem, *J. Indian Math. Soc.* **59** (1993), 243–260.

[Ho] E. M. Horadam, Normal order for divisor functions of generalized integers, *Portugal. Math.* **27** (1968), 201–207.

[N2] V. C. Nanda, On arithmetical functions of integral matrices, *J. Indian Math. Soc.* **55** (1990), 175–188.

[Ne] M. Newman, "Integral Matrices," Academic Press, New York/London, 1972.