GOLDBACH REPRESENTATIONS IN ARITHMETIC PROGRESSIONS AND ZEROS OF DIRICHLET L-FUNCTIONS

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Abstract. Assuming a conjecture on distinct zeros of Dirichlet L-functions we get asymptotic results on the average number of representations of an integer as the sum of two primes in arithmetic progression. On the other hand the existence of good error terms gives information on the location of zeros of L-functions. Similar results are obtained for an integer in a congruence class expressed as the sum of two primes.

1. Introduction and Results

The Goldbach problem of representing every even integer larger than 2 as the sum of two primes has several variants, one being that in which the summands are primes in given arithmetic progressions. Similar to the original problem it is known that almost all even integers satisfying some congruence condition can be written as the sum of two primes in congruence classes. Quantitatively, the exceptional set of integers less than X and satisfying the necessary congruence condition, which can not be written as the sum of primes congruent to a common modulus q may be estimated as $O(\varphi(q)^{-1}X^{1-\delta})$ for a computable positive constant δ and all $q \leq X^{\delta}$ (See , for example, [13], [1]) .

Though the complete solution of these binary Goldbach problems is out of sight, the related question of the average number of representations of integers as sums of primes seems more accessible. The study of the average order of the weighted function

$$G(n) = \sum_{\ell+m=n} \Lambda(\ell)\Lambda(m)$$

where Λ is the von Mangoldt function has begun with Fujii [8] and continues to be actively pursued. However the current state of knowledge on the zeros of the Riemann zeta function $\zeta(s)$ is not enough to obtain "good" error terms unconditionally and the Riemann Hypothesis is always assumed in such studies. In fact obtaining sufficiently sharp error terms for average orders of the mean value of $G(n)$ is expected to solve other conjectures like the Riemann Hypothesis, as elaborated by Granville [11] in the classical case of unrestricted primes. This paper is an analogous study with the two primes in arithmetic progressions with a common modulus.

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The function that we consider here, with a, b positive integers coprime to q, is

$$G(n;q,a,b) = \sum_{\substack{\ell+m=n \\ \ell\equiv a,\, m\equiv b \,(\mathrm{mod}\ q)}} \Lambda(\ell)\Lambda(m)$$

whose summatory function defined as

$$S(x;q,a,b) = \sum_{n\le x} G(n;q,a,b)$$

was introduced by Rüppel [16] and further studied by the fourth author [17].

On the lines of Granville we consider the relations between an explicit formula for $S(x;q,a,b)$ and zeros of L-functions. In [11, 1A] it is stated that there is an equivalence between the estimate

$$\sum_{n\le x} (G(n) - J(n)) \ll x^{3/2+o(1)} \tag{1.1}$$

and the Riemann Hypothesis (RH) for $\zeta(s)$ where $J(n) = 0$ for odd n and, with $C_2 = 2\prod_{p>2}(1 - \frac{1}{(p-1)^2})$ being the twin prime constant,

$$J(n) = n \cdot C_2 \prod_{\substack{p|n \\ p>2}} \frac{p-1}{p-2}$$

for even n. The function $J(n)$ is believed since Hardy and Littlewood to be a good approximation for $G(n)$ (cf. [12]).

We denote by χ a Dirichlet character (mod q), by $L(s,\chi)$ the associated Dirichlet L-function and by ρ_χ its non-trivial zeros. Let $B_\chi = \sup\{\Re\rho_\chi\}$ and $B_q = \sup\{B_\chi \mid \chi \,(\mathrm{mod}\ q)\}$. Hence $1/2 \le B_q \le 1$ for $q \ge 1$.

In the context of primes in congruence classes we first need to formulate the Distinct Zero Conjecture (DZC) on zeros of L-functions as:

> *For any $q \ge 1$, any two distinct Dirichlet L-functions associated with characters of modulus q do not have a common non-trivial zero, except for a possible multiple zero at $s = 1/2$.*

Though weaker than the non-coincidence conjecture found in literature that expects all zeros of all primitive L-functions to be linearly independent except for the possible multiple zero at $s = 1/2$ (cf. [5, p.353]), this suffices for our purpose.

In this paper, q denotes an arbitrary positive integer. If an implied constant for Landau's or Vinogradov's symbol depends on certain parameters then we indicate this dependence by attaching subscripts, e.g. $O_A(x\log x)$, or $\ll_{A,B} x\log x$. When there is no explicit mention an implied constant should be understood to be absolute.

Theorem 1. *Let a, b, q be integers with $(ab, q) = 1$, $q \ge 1$.*

(1) *For $x \ge 2$ and $\delta > 0$ we have*

$$S(x;q,a,b) = \frac{x^2}{2\varphi(q)^2} + O(x^{1+B_q^*}). \tag{1.2}$$

and

$$B_q^* = B_q^*(x) = \min(B_q, 1-\eta),$$

with

$$\eta = \eta_q(x) = \frac{c_1(\delta)}{\max(q^\delta, (\log x)^{2/3}(\log\log x)^{1/3})}$$

for a small positive constant $c_1(\delta)$ depending only on δ.

(2) *Let* DZC *be true, let $\chi(a) + \chi(b) \neq 0$ for all characters $\chi \pmod q$ and let $1/2 \leq d < 1$. If the asymptotic formula*

$$S(x; q, a, b) = \frac{x^2}{2\varphi(q)^2} + O_{q,\varepsilon}(x^{1+d+\varepsilon}). \tag{1.3}$$

holds for any $\varepsilon > 0$, then either $B_q \leq d$ or $B_q = 1$. Further if (1.3) *holds with $a = b$, then we obtain $B_q \leq d$.*

Remark 1. The parameter B_q^* introduced above enables us to obtain a non-trivial estimate in (1.2) even for the case $B_q = 1$, where B_q alone would have given no information on the distribution of non-trivial zeros of Dirichlet L-functions.

Remark 2. Our result thus falls short of an equivalence with the Generalized Riemann Hypothesis (GRH) for L-functions modulo q since we have an additional possibility of $B_q = 1$ where the meromorphic continuation is not available (see Proposition 3 below). Using the idea from [2] we were able to exclude the possibility of $B_q = 1$ for the case $a = b$ still assuming the DZC.

Though the proof of the equivalence between the RH and (1.1) is now complete (see also [2]) the equivalences for primes in arithmetic progressions are still partial.

Remark 3. We need the condition that $\chi(a) + \chi(b) \neq 0$ for all $\chi \pmod q$ in order to assure that the residue $r_1(\rho_q)$ of Proposition 3 does not vanish. It is easy to see that $\chi(a) + \chi(b) \neq 0$ for all $\chi \pmod q$ if and only if the residue $ab^{-1} \pmod q$ is of odd order in the multiplicative group of the reduced residue classes $\pmod q$. (See also Remarks 6 and 7 below.)

To prove Theorem 1 we need an explicit formula for $S(x; q, a, b)$, which can be stated as follows.

Theorem 2. *Let a, b, q be integers with $(ab, q) = 1$, $q \geq 1$. Then, for $x \geq 2$ and for any $\delta > 0$,*

$$\begin{aligned} S(x; q, a, b) = {} & \frac{x^2}{2\varphi(q)^2} \\ & - \frac{1}{\varphi(q)^2} \sum_{\chi \,(\mathrm{mod}\, q)} (\overline{\chi(a)} + \overline{\chi(b)}) \sum_{\rho_\chi} \frac{x^{\rho_\chi + 1}}{\rho_\chi(\rho_\chi + 1)} + O(x^{2B_q^*}(\log qx)^5). \end{aligned} \tag{1.4}$$

Remark 4. The constant $c_1(\varepsilon)$ can not be effectively computed due to Siegel's theorem.

In Section 3 we prove an explicit formula with a weaker error estimate (Proposition 1) using a generalized Landau–Gonek formula for L-functions (Proposition 2 in Section 4). This weaker form is an analogue of Granville's [11, Corrigendum, (2)], that states

$$\sum_{n \leq x} G(n) = \frac{x^2}{2} - 2 \sum_{\substack{\rho \\ |\Im \rho| \leq x}} \frac{x^{\rho+1}}{\rho(\rho+1)} + O\left(x^{(2+4B)/3}(\log x)^2\right), \tag{1.5}$$

where ρ runs through the non-trivial zeros of $\zeta(s)$ and $B = \sup\{\Re\rho\}$, and the proof of Proposition 1 essentially runs along the line suggested by [11]. Thus Sections 3 and 4 include a reconstruction of Granville's argument for the asymptotic order. However we can go further; we take this opportunity to prove (in Sections 5 and 6) the stronger error estimate (1.4), an analogue of that announced in [11, (5.1)] (cf. [11, Corrigendum, comments before (2)]), using a kind of circle method of the first author and Schlage-Puchta [3].

With the help of Theorem 2 above the analytic continuation of the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{G(n;q,a,b)}{n^s}$$

is examined in Proposition 3 (in Section 7) and this enables us to establish relations between the error terms in the average of Goldbach problems in arithmetic progressions and zeros of Dirichlet L-functions.

Further, we examine the case of n with congruence conditions as in [11, 1B] where it is stated that the GRH for Dirichlet L-functions $L(s,\chi)$, over all characters χ, the modulus of which are odd squarefree divisors of q, is equivalent to the estimate

$$\sum_{\substack{n\le x\\ n\equiv 2 \,(\mathrm{mod}\ q)}} (G(n)-J(n)) \ll x^{3/2+o(1)}. \tag{1.6}$$

Moreover in [11, Corrigendum, 1C] it is stated that if the estimate

$$\sum_{\substack{n\le x\\ q|n}} G(n) = \frac{1}{\varphi(q)}\sum_{n\le x} G(n) + O_q(x^{1+o(1)}) \tag{1.7}$$

is attained then the GRH for Dirichlet L-functions $L(s,\chi)$, $\chi\,(\mathrm{mod}\ q)$ holds; and under this hypothesis the last estimate would have the error term $O(x^{4/3}(\log x)^2)$.

Here we extend (1.6) with the general congruence condition $n\equiv c$ for an arbitrary positive integer c instead of the special case $n\equiv 2$. Assuming the GRH for L-functions (mod q) we can deduce the estimate

$$\sum_{\substack{n\le x\\ n\equiv c \,(\mathrm{mod}\ q)}} (G(n)-J(n)) \ll x^{3/2}. \tag{1.8}$$

However in the other direction, we could not deduce satisfactory conclusions on the size of B_q when $a\neq b$. In particular we were unable then to reconstruct the reverse implications for (1.6) and (1.7). In the following we elaborate conditions under which reverse implications on the distribution of zeros would be true.

Theorem 3. *Let q,c be integers such that $(2,q)\mid c$.*

(1) *For $x\ge 2$, we have*

$$\sum_{\substack{n\le x\\ n\equiv c \,(\mathrm{mod}\ q)}} (G(n)-J(n)) \ll x^{1+B_q^*}. \tag{1.9}$$

(2) *Assume that*

$$\sum_{\substack{n\le x\\ n\equiv c \,(\mathrm{mod}\ q)}} (G(n)-J(n)) \ll_{q,\varepsilon} x^{1+d+\varepsilon} \tag{1.10}$$

holds for some $1/2 \le d \le 1$ *and any* $\varepsilon > 0$*. If there exists a zero* ρ_0 *of* $\prod_{\chi \,(\mathrm{mod}\ q)} L(s,\chi)$ *such that*
(a) $B_q = \Re\rho_0$
(b) ρ_0 *belongs to a unique character* $\chi_1 \pmod q$
(c) *the conductor* q^* *of* $\chi_1 \pmod q$ *is squarefree and satisfies* $(c, q^*) = 1$*,*
then $B_q = \Re\rho_0 \le d$*.*

One possible candidate for ρ_0 in the above theorem is the Siegel zero β_1. However, we cannot exclude the existence of some other non-trivial zero $\rho_0 = \beta_0 + i\gamma_0$ such that $\beta_0 > \beta_1$ and $|\gamma_0|$ is large.

Remark 5. When $B_q = 1/2$, the bound in (1.9) is better than that of (1.6).

To obtain the above results we require an asymptotic formula with B_q^* as in Theorem 2. We denote the principal character mod q by $\chi_0 \pmod q$. For a given Dirichlet character $\chi \pmod q$, let $\chi^* \pmod{q^*}$ be the primitive character inducing $\chi \pmod q$.

Theorem 4. *For* $x \ge 2$*, and for any positive integer* c *we have*

$$\sum_{\substack{n \le x \\ n \equiv c \,(\mathrm{mod}\ q)}} G(n) = \frac{\mathfrak{S}_q(c)}{2} x^2 - 2 \sum_{\chi \,(\mathrm{mod}\ q)} \mathfrak{S}_q(c, \overline{\chi}) \sum_{\rho_\chi} \frac{x^{\rho_\chi + 1}}{\rho_\chi(\rho_\chi + 1)} + O(x^{2B_q^*} (\log qx)^5),$$

where

$$\mathfrak{S}_q(c,\chi) = \frac{\mu^2(q^*)\chi^*(q)}{\varphi(q)\varphi(q^*)} \prod_{\substack{p \mid q \\ p \nmid q^* c}} \frac{p-2}{p-1} \quad \text{and} \quad \mathfrak{S}_q(c) = \mathfrak{S}_q(c, \chi_0).$$

Note that $\mathfrak{S}_q(c) = 0$ if $(2,q) \nmid c$. The above theorem is proven in Section 6. The singular series $\mathfrak{S}_q(c,\chi)$ appears again in Lemma 12. To ensure the uniformity of q it is not enough to sum Theorem 2 over residues and we need other tools like Lemma 11 . Finally, using Theorem 4, we give the proof of (1.8) and Theorem 3 in Section 7. In Section 8 we give the proof for the supplement of Theorem 1 (2) in the case $a = b$.

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2. Some preliminaries

In this section we fix notations and give some basic lemmas on Dirichlet L-functions.

As mentioned in Section 1, we denote by $\chi \pmod q$ a Dirichlet character modulo q and by $\chi^* \pmod{q^*}$ the primitive character inducing $\chi \pmod q$. We denote the principal character modulo q by $\chi_0 \pmod q$. We refer to the constant function 1 as the trivial character and regard it as the primitive character (mod 1) . If there is no specific mention a statement with $\chi \pmod q$ holds true for any $q \ge 1$ and any character $\chi \pmod q$.

We denote the Dirichlet L-function associated to $\chi \pmod q$ by $L(s,\chi)$. We say that a zero of $L(s,\chi)$ is non-trivial if it is contained in the strip $0<\sigma<1$. We denote by ρ_χ non-trivial zeros of $L(s,\chi)$ with the real part β_χ and the imaginary part γ_χ. As a summation variable, the letter ρ_χ runs through all non-trivial zeros of $L(s,\chi)$ counted with multiplicity. We denote the Siegel zero by β_1.

We set $\delta_0(\chi)=1$ if $\chi=\chi_0$ is the principal character and $\delta_0(\chi)=0$ otherwise. Similarly we let $\delta_1(\chi)=1$ if χ is the exceptional character (that is, whose L-function has a Siegel zero) and $\delta_1(\chi)=0$ otherwise.

We now recall the Vinogradov–Korobov zero free region in a suitable form .

Lemma 1. *For any $\delta>0$, there is some constant $c_1=c_1(\delta)>0$ such that*

$$L(s,\chi)\neq 0 \quad \textit{for} \quad \sigma > 1-\frac{c_1(\delta)}{\max(q^\delta,(\log x)^{2/3}(\log\log x)^{1/3})}=1-\eta \quad \textit{and} \quad |t|\le x.$$

Proof. By the Korobov–Vinogradov estimate [14, p.176], we have, for c_2 an absolute positive constant, $L(s,\chi)\neq 0$ for

$$\sigma > 1-\frac{c_2}{\max(\log q,(\log x)^{2/3}(\log\log x)^{1/3})}=1-\eta \quad \text{and} \quad |t|\le x$$

except real zeros. For the real zeros, we can apply Siegel's theorem [15, Corollary 11.15].

□

We next evoke some lemmas for sums over non-trivial zeros.

Lemma 2 ([15, Theorem 10.17])**.** *For any $T\ge 0$, we have*

$$\sum_{\substack{\rho_\chi \\ T\le|\gamma_\chi|\le T+1}} 1 \ll \log q(|T|+2).$$

Lemma 3. *For any $T\ge 1$ and $\chi \pmod q$, we have*

$$\sum_{\substack{\rho_\chi\neq 1-\beta_1 \\ |\gamma_\chi|\le T}} \frac{1}{|\rho_\chi|} \ll (\log 2qT)^2, \qquad \sum_{\substack{\rho_\chi \\ |\gamma_\chi|\le T}} \frac{1}{|\rho_\chi|} \ll (\log 2qT)^2+\delta_1(\chi)q^{1/2}(\log q)^2.$$

Proof. For the first estimate we dissect the sum at $|\gamma_\chi|=1$ as

$$\sum_{\substack{\rho_\chi\neq 1-\beta_1 \\ |\gamma_\chi|\le T}} \frac{1}{|\rho_\chi|} = \sum_{\substack{\rho_\chi\neq 1-\beta_1 \\ |\gamma_\chi|\le 1}} \frac{1}{|\rho_\chi|} + \sum_{\substack{\rho_\chi \\ 1<|\gamma_\chi|\le T}} \frac{1}{|\rho_\chi|}.$$

For the first sum Lemma 2 gives

$$\begin{aligned}\sum_{\substack{\rho_\chi\neq 1-\beta_1 \\ |\gamma_\chi|\le 1}} \frac{1}{|\rho_\chi|} \le \sum_{\substack{\rho_\chi\neq 1-\beta_1 \\ |\gamma_\chi|\le 1}} \frac{1}{\beta_\chi} = \sum_{\substack{\rho_\chi\neq \beta_1 \\ |\gamma_\chi|\le 1}} \frac{1}{1-\beta_\chi} \\ \ll \sum_{\substack{\rho_\chi \\ |\gamma_\chi|\le 1 \\ \beta_\chi>1-c_0/\log 2q}} \frac{1}{1-\beta_\chi} \ll (\log 2q)\sum_{\substack{\rho_\chi \\ |\gamma_\chi|\le 1}} 1 \ll (\log 2q)^2 \ll (\log 2qT)^2,\end{aligned}$$

where $c_0 > 0$ is some small absolute constant while for the second sum Lemma 2 gives

$$\sum_{\substack{\rho_\chi \\ 1<|\gamma_\chi|\le T}} \frac{1}{|\rho_\chi|} \le \sum_{n\le T} \sum_{\substack{\rho_\chi \\ n<|\gamma_\chi|\le n+1}} \frac{1}{|\rho_\chi|} \ll \sum_{n\le T} \frac{\log q(n+2)}{n} \ll (\log 2qT)^2.$$

Thus the first estimate follows. The second estimate is obtained from the first combined with the well-known bound [15, Corollary 11.12] of the Siegel zero

$$\beta_1 > 1 - \frac{c_2}{q^{1/2}(\log q)^2}, \tag{2.1}$$

where $c_2 > 0$ is some absolute constant. □

Lemma 4. *For any $T \ge 1$ and $\chi \pmod q$, we have*

$$\sum_{\substack{\rho_\chi \\ |\gamma_\chi|>T}} \frac{1}{|\rho_\chi|^2} \ll \frac{\log 2qT}{T}, \quad \sum_{\rho_\chi} \frac{1}{|\rho_\chi(\rho_\chi+1)|} \ll (\log 2q)^2 + \delta_1(\chi) q^{1/2}(\log q)^2.$$

Proof. The first estimate is again obtained by using Lemma 2 ,i.e.

$$\sum_{\substack{\rho_\chi \\ |\gamma_\chi|>T}} \frac{1}{|\rho_\chi|^2} \le \sum_{n=[T]}^{\infty} \sum_{\substack{\rho_\chi \\ n<|\gamma_\chi|\le n+1}} \frac{1}{|\rho_\chi|^2} \ll \sum_{n=[T]}^{\infty} \frac{\log qn}{n^2} \ll \frac{\log qT}{T},$$

whereas the last estimate can be obtained by comparison with an integral. For the latter estimate, we combine the former with Lemma 3. This gives

$$\sum_{\rho_\chi} \frac{1}{|\rho_\chi(\rho_\chi+1)|} \le \sum_{\substack{\rho_\chi \\ |\gamma_\chi|\le 1}} \frac{1}{|\rho_\chi|} + \sum_{\substack{\rho_\chi \\ |\gamma_\chi|>1}} \frac{1}{|\rho_\chi|^2} \ll (\log 2q)^2 + \delta_1(\chi) q^{1/2}(\log q)^2.$$

□

We next consider some explicit formulæ for the sum

$$\psi(x,\chi) = \sum_{n\le x} \chi(n)\Lambda(n).$$

For a Dirichlet character $\chi \pmod q$, we introduce the constant

$$C(\chi) = \frac{L'}{L}(1,\overline{\chi}) + \log\frac{q}{2\pi} - \gamma, \tag{2.2}$$

where γ is the Euler–Mascheroni constant.

Lemma 5. *For any $u, T \ge 2$, the explicit formula*

$$\psi(u,\chi) = \delta_0(\chi)u - \sum_{\substack{\rho_\chi \\ |\gamma_\chi|\le T}} \frac{u^{\rho_\chi}}{\rho_\chi} + C(\chi^*) + E(u,T,\chi)$$

holds, where

$$E(u,T,\chi) \ll (\log 2q)(\log u) + \frac{u}{T}(\log quT)^2$$

and $C(\chi)$ is defined by (2.2).

Proof. For primitive χ, this follows immediately from Theorem 12.5 and 12.10 of [15] as below. If χ is trivial so that $q=1$, we use Theorem 12.5 of [15] with $x=u$. We can estimate the last three terms on the right-hand side of (12.3) of [15] as

$$-\log 2\pi - \frac{1}{2}\log(1-1/u^2) + R(u,T) \ll (\log 2q)(\log u) + \frac{u}{T}(\log quT)^2,$$

by using (12.4) of [15]. This gives the assertion for the case when χ is trivial. If χ is non-principal, we use Theorem 12.10 of [15] with $x=u$. The last four terms on the right-hand side of (12.6) of [15] can be rewritten as

$$\begin{aligned}&-\frac{1}{2}\log(u-1) - \frac{\chi(-1)}{2}\log(u+1) + C(\chi) + R(u,T;\chi)\\ &\quad = C(\chi) + O\left((\log 2q)(\log u) + \frac{u}{T}(\log quT)^2\right).\end{aligned}$$

This gives the assertion for the case of χ being primitive and non-principal.

If χ is imprimitive, then it suffices to note that the non-trivial zeros of $L(s,\chi)$ are those of $L(s,\chi^*)$ and that

$$\begin{aligned}\psi(u,\chi)-\psi(u,\chi^*) &\ll \sum_{\substack{n\le u\\ (n,q)>1}} \Lambda(n) = \sum_{p|q}(\log p)\left[\frac{\log u}{\log p}\right]\\ &\le \left(\frac{\log u}{\log 2}\right)\sum_{p|q}(\log p) \ll (\log 2q)(\log u),\end{aligned} \tag{2.3}$$

which is absorbed into $E(u,T,\chi)$. □

When we substitute the above explicit formula into some sum or integral, we need to use a uniform parameter T and a uniform bound of the error term. Also it is convenient to work with the case $0\le u<2$. Thus we modify the above explicit formula in the following form.

Lemma 6. *Let $x\ge 2$ be a real number. For any $x\ge T\ge 2$ and $x\ge u\ge 0$, the explicit formula*

$$\psi(u,\chi) = \delta_0(\chi)u - \sum_{\substack{\rho_\chi\\ |\gamma_\chi|\le T}} \frac{u^{\rho_\chi}}{\rho_\chi} + O\left(\frac{x}{T}(\log qx)^2 + \delta_1(\chi)q^{1/2}(\log q)^2\right) \tag{2.4}$$

holds.

Proof. We first consider the case $u\ge 2$. For this we use Lemma 5. Since $u,T\le x$, the error term of Lemma 5 is

$$E(u,T,\chi) \ll (\log 2q)(\log x) + \frac{x}{T}(\log qxT)^2 \ll \frac{x}{T}(\log qxT)^2.$$

Further, by Theorem 11.4 of [15], we know that $(L'/L)(1,\overline{\chi^*}) \ll \log 2q$ if χ is not exceptional, while for the exceptional χ and Siegel zero β_1, (2.1) gives

$$\frac{L'}{L}(1,\overline{\chi^*}) = \frac{1}{1-\beta_1} + O(\log 2q) \ll q^{1/2}(\log q)^2.$$

Therefore by (12.7) of [15], we obtain

$$C(\chi^*) \ll \log 2q + \delta_1(\chi)q^{1/2}(\log q)^2 \ll \frac{x}{T}(\log qx)^2 + \delta_1(\chi)q^{1/2}(\log q)^2 \tag{2.5}$$

for non-principal χ, from which the lemma follows for the case $u\ge 2$.

The remaining case is when $0 \le u < 2$. Now the sum on the right-hand side of (2.4) is estimated by using Lemma 3 as

$$\sum_{\substack{\rho_\chi \\ |\gamma_\chi| \le T}} \frac{u^{\rho_\chi}}{\rho_\chi} \ll u \sum_{\substack{\rho_\chi \\ |\gamma_\chi| \le T}} \frac{1}{|\rho_\chi|} \ll (\log qT)^2 + \delta_1(\chi) q^{1/2} (\log q)^2$$
$$\ll \frac{x}{T} (\log qx)^2 + \delta_1(\chi) q^{1/2} (\log q)^2$$

since $T \le x$. On the other hand, when $u < 2$, the left-hand side of (2.4) is zero. Therefore the assertion holds trivially. □

3. An asymptotic formula for $S(x; q, a, b)$

In this section we deduce a prototype of Theorem 2 along the line of [11]:

Proposition 1. *For integers a, b, q with $(ab, q) = 1$, we have*

$$S(x; q, a, b) = \frac{x^2}{2\varphi(q)^2} - \frac{1}{\varphi(q)^2} \sum_{\chi \,(\mathrm{mod}\ q)} (\overline{\chi(a)} + \overline{\chi(b)}) \sum_{\rho_\chi} \frac{x^{\rho_\chi + 1}}{\rho_\chi(\rho_\chi + 1)} + O\left(x^{(2+4B_q^*)/3} (\log qx)^4 \right).$$

Proof. The proof is divided into three steps.

Step 1 : The first substitution in the explicit formula :

With

$$\psi(x; q, a) = \sum_{\substack{m \le x \\ m \equiv a \,(\mathrm{mod}\ q)}} \Lambda(m),$$

we can write

$$S(x; q, a, b) = \sum_{\substack{\ell \le x \\ \ell \equiv a \,(\mathrm{mod}\ q)}} \Lambda(\ell) \psi(x - \ell; q, b). \tag{3.1}$$

Using the orthogonality relation of Dirichlet characters, we have

$$\psi(x; q, b) = \frac{1}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\ q)} \overline{\chi(b)} \psi(x, \chi).$$

We substitute the explicit formula given by Lemma 6 here. This gives

$$\psi(u; q, b) = \frac{1}{\varphi(q)} A(u, T; q, b) + B(u, T; q, b) \tag{3.2}$$

for $x \ge T \ge 2$ and $x \ge u \ge 0$, where

$$A(u, T; q, b) = u - \sum_{\chi \,(\mathrm{mod}\ q)} \overline{\chi(b)} \sum_{\substack{\rho_\chi \\ |\gamma_\chi| \le T}} \frac{u^{\rho_\chi}}{\rho_\chi}$$

and $B(u, T; q, b)$ is the error term satisfying

$$B(u, T; q, b) \ll \frac{x}{T} (\log qx)^2 + \frac{1}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\ q)} \delta_1(\chi) q^{1/2} (\log q)^2 \ll \frac{x}{T} (\log qx)^2,$$

where for estimating the term involving $\delta_1(\chi)q^{1/2}(\log q)^2$, we use the fact that there is at most one exceptional character (mod q). Substituting (3.2) into (3.1), we obtain

$$S(x;q,a,b)=\frac{1}{\varphi(q)}\sum_{\substack{\ell\le x\\ \ell\equiv a\ (\mathrm{mod}\ q)}}\Lambda(\ell)A(x-\ell,T;q,b)+O\left(\frac{x^2}{T}(\log qx)^2\right). \tag{3.3}$$

Step 2 : The second substitution in the explicit formula :

Now we evaluate the sum on the right-hand side of (3.3) by splitting it in two parts Σ_1 and Σ_2.

$$\begin{aligned}\frac{1}{\varphi(q)}&\sum_{\substack{\ell\le x\\ \ell\equiv a\ (\mathrm{mod}\ q)}}\Lambda(\ell)A(x-\ell,T;q,b)\\ =\frac{1}{\varphi(q)}&\sum_{\substack{\ell\le x\\ \ell\equiv a\ (\mathrm{mod}\ q)}}\Lambda(\ell)(x-\ell)\\ &-\frac{1}{\varphi(q)}\sum_{\substack{\ell\le x\\ \ell\equiv a\ (\mathrm{mod}\ q)}}\Lambda(\ell)\sum_{\chi\ (\mathrm{mod}\ q)}\overline{\chi(b)}\sum_{\substack{\rho_\chi\\ |\gamma_\chi|\le T}}\frac{(x-\ell)^{\rho_\chi}}{\rho_\chi}\\ =\Sigma_1-\Sigma_2,&\ \text{say.}\end{aligned} \tag{3.4}$$

Consider Σ_1. We have

$$\Sigma_1=\frac{1}{\varphi(q)}\sum_{\substack{\ell\le x\\ \ell\equiv a\ (\mathrm{mod}\ q)}}\Lambda(\ell)\int_\ell^x du=\frac{1}{\varphi(q)}\int_0^x\psi(u;q,a)du,$$

so, using (3.2), we obtain

$$\Sigma_1=\frac{1}{\varphi(q)^2}\int_0^x A(u,T;q,a)du+O\left(\frac{x^2}{T}(\log qx)^2\right). \tag{3.5}$$

Inserting the definition of $A(u,T;q,a)$ yields

$$\Sigma_1=\frac{1}{\varphi(q)^2}\left(\frac{x^2}{2}-\sum_{\chi\ (\mathrm{mod}\ q)}\overline{\chi(a)}\sum_{\substack{\rho_\chi\\ |\gamma_\chi|\le T}}\frac{x^{\rho_\chi+1}}{\rho_\chi(\rho_\chi+1)}\right)+O\left(\frac{x^2}{T}(\log qx)^2\right). \tag{3.6}$$

Next consider Σ_2. We have

$$\Sigma_2=\frac{1}{\varphi(q)}\sum_{\chi\ (\mathrm{mod}\ q)}\overline{\chi(b)}\sum_{\substack{\rho_\chi\\ |\gamma_\chi|\le T}}\Psi(\rho_\chi,x;q,a), \tag{3.7}$$

where

$$\Psi(\rho_\chi,x;q,a)=\frac{1}{\rho_\chi}\sum_{\substack{\ell\le x\\ \ell\equiv a\ (\mathrm{mod}\ q)}}\Lambda(\ell)(x-\ell)^{\rho_\chi}.$$

Again using (3.2),

$$\Psi(\rho_\chi,x;q,a)=\frac{1}{\rho_\chi}\int_0^x(x-u)^{\rho_\chi}d\psi(u;q,a)$$

$$
\begin{aligned}
&= \frac{1}{\rho_\chi}\int_0^x (x-u)^{\rho_\chi}\left(\frac{1}{\varphi(q)}dA(u,T;q,a)+dB(u,T;q,a)\right)\\
&= \frac{1}{\varphi(q)\rho_\chi}\int_0^x (x-u)^{\rho_\chi}du\\
&\quad -\frac{1}{\varphi(q)\rho_\chi}\sum_{\chi' \,(\mathrm{mod}\ q)}\overline{\chi'(a)}\sum_{\substack{\rho_{\chi'}\\ |\gamma_{\chi'}|\le T}}\int_0^x (x-u)^{\rho_\chi}u^{\rho_{\chi'}-1}du\\
&\qquad +\frac{1}{\rho_\chi}\int_0^x (x-u)^{\rho_\chi}dB(u,T;q,a)\\
&= J_1 - J_2 + J_3, \text{ say.}
\end{aligned}
$$

Obviously

$$J_1 = \frac{x^{\rho_\chi+1}}{\varphi(q)\rho_\chi(\rho_\chi+1)}.$$

Since

$$\int_0^x (x-u)^{\rho_\chi}u^{\rho_{\chi'}-1}du = x^{\rho_\chi+\rho_{\chi'}}\frac{\rho_\chi\Gamma(\rho_\chi)\Gamma(\rho_{\chi'})}{(\rho_\chi+\rho_{\chi'})\Gamma(\rho_\chi+\rho_{\chi'})},$$

we have

$$J_2 = \frac{1}{\varphi(q)}\sum_{\chi' \,(\mathrm{mod}\ q)}\overline{\chi'(a)}\sum_{\substack{\rho_{\chi'}\\ |\gamma_{\chi'}|\le T}}\mathcal{Z}(\rho_\chi,\rho_{\chi'})x^{\rho_\chi+\rho_{\chi'}},$$

where

$$\mathcal{Z}(\rho_\chi,\rho_{\chi'}) = \frac{\Gamma(\rho_\chi)\Gamma(\rho_{\chi'})}{(\rho_\chi+\rho_{\chi'})\Gamma(\rho_\chi+\rho_{\chi'})} = \frac{\Gamma(\rho_\chi)\Gamma(\rho_{\chi'})}{\Gamma(1+\rho_\chi+\rho_{\chi'})}. \tag{3.8}$$

Lastly,

$$
\begin{aligned}
J_3 &= \frac{1}{\rho_\chi}\Big[(x-u)^{\rho_\chi}B(u,T;q,a)\Big]_{u=0}^{x} + \int_0^x (x-u)^{\rho_\chi-1}B(u,T;q,a)du\\
&= O\left(\frac{1}{|\rho_\chi|}\frac{x^2}{T}(\log qx)^2\right) + \int_0^x (x-u)^{\rho_\chi-1}B(u,T;q,a)du.
\end{aligned}
$$

Therefore we now obtain

$$
\begin{aligned}
\Psi(\rho_\chi,x;q,a) &= \frac{x^{\rho_\chi+1}}{\varphi(q)\rho_\chi(\rho_\chi+1)} - \frac{1}{\varphi(q)}\sum_{\chi' \,(\mathrm{mod}\ q)}\overline{\chi'(a)}\sum_{\substack{\rho_{\chi'}\\ |\gamma_{\chi'}|\le T}}\mathcal{Z}(\rho_\chi,\rho_{\chi'})x^{\rho_\chi+\rho_{\chi'}}\\
&\quad + \int_0^x u^{\rho_\chi-1}B(x-u,T;q,a)du + O\left(\frac{1}{|\rho_\chi|}\frac{x^2}{T}(\log qx)^2\right).
\end{aligned}
$$

Substituting this into (3.7) and using Lemma 3, we obtain

$$\Sigma_2 = \frac{1}{\varphi(q)^2}\sum_{\chi \,(\mathrm{mod}\ q)}\overline{\chi(b)}\sum_{\substack{\rho_\chi\\ |\gamma_\chi|\le T}}\frac{x^{\rho_\chi+1}}{\rho_\chi(\rho_\chi+1)} - \Sigma_3 + \Sigma_4 + O\left(\frac{x^2}{T}(\log qx)^4\right). \tag{3.9}$$

where

$$\Sigma_3 = \frac{1}{\varphi(q)^2}\sum_{\chi \,(\mathrm{mod}\ q)}\overline{\chi(b)}\sum_{\chi' \,(\mathrm{mod}\ q)}\overline{\chi'(a)}\sum_{\substack{\rho_\chi\\ |\gamma_\chi|\le T}}\sum_{\substack{\rho_{\chi'}\\ |\gamma_{\chi'}|\le T}}\mathcal{Z}(\rho_\chi,\rho_{\chi'})x^{\rho_\chi+\rho_{\chi'}},$$

$$\Sigma_4 = \frac{1}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\ q)} \overline{\chi(b)} \sum_{\substack{\rho_\chi \\ |\gamma_\chi| \le T}} \int_0^x u^{\rho_\chi - 1} B(x-u, T; q, a) du.$$

Combining (3.3) with (3.4) yields

$$S(x;q,a,b) = \Sigma_1 - \Sigma_2 + O\left(\frac{x^2}{T}(\log qx)^2\right),$$

so with (3.6) and (3.9) we now arrive at

$$S(x;q,a,b) = \frac{x^2}{2\varphi(q)^2} - \frac{1}{\varphi(q)^2} \sum_{\chi \,(\mathrm{mod}\ q)} (\overline{\chi(a)} + \overline{\chi(b)}) \sum_{\substack{\rho_\chi \\ |\gamma_\chi| \le T}} \frac{x^{\rho_\chi + 1}}{\rho_\chi(\rho_\chi + 1)} + \Sigma_3 - \Sigma_4 + O\left(\frac{x^2}{T}(\log qx)^4\right).$$

We next extend the sum over zeros. By Lemma 4, we have

$$\sum_{\substack{\rho_\chi \\ |\gamma_\chi| > T}} \frac{x^{\rho_\chi + 1}}{\rho_\chi(\rho_\chi + 1)} \ll x^2 \sum_{\substack{\rho_\chi \\ |\gamma_\chi| > T}} \frac{1}{|\gamma_\chi|^2} \ll \frac{x^2}{T}(\log qx). \tag{3.10}$$

Therefore our sum becomes

$$S(x;q,a,b) = \frac{x^2}{2\varphi(q)^2} - \frac{1}{\varphi(q)^2} \sum_{\chi \,(\mathrm{mod}\ q)} (\overline{\chi(a)} + \overline{\chi(b)}) \sum_{\rho_\chi} \frac{x^{\rho_\chi + 1}}{\rho_\chi(\rho_\chi + 1)} + \Sigma_3 - \Sigma_4 + O\left(\frac{x^2}{T}(\log qx)^4\right). \tag{3.11}$$

Step 3 : The estimation of Σ_3 and Σ_4.

Lastly we estimate the remaining error terms Σ_3 and Σ_4.

First consider Σ_4. The contribution of the integral on the interval $0 \le u \le 3$ is

$$\begin{aligned}
&\ll \frac{1}{\varphi(q)} \frac{x}{T} (\log qx)^2 \sum_{\chi \,(\mathrm{mod}\ q)} \sum_{\substack{\rho_\chi \\ |\gamma_\chi| \le T}} \int_0^3 u^{\beta_\chi - 1} du \\
&\ll \frac{1}{\varphi(q)} \frac{x}{T} (\log qx)^2 \sum_{\chi \,(\mathrm{mod}\ q)} \sum_{\substack{\rho_\chi \\ |\gamma_\chi| \le T}} \frac{1}{\beta_\chi} \\
&\ll \frac{1}{\varphi(q)} \frac{x}{T} (\log qx)^2 \sum_{\chi \,(\mathrm{mod}\ q)} \sum_{\substack{\rho_\chi \ne 1 - \beta_1 \\ |\gamma_\chi| \le T}} \frac{1}{\beta_\chi} + \frac{q^{1/2}}{\varphi(q)} \frac{x}{T} (\log qx)^4 \\
&\ll \frac{1}{\varphi(q)} \frac{x}{T} (\log qx)^2 \sum_{\chi \,(\mathrm{mod}\ q)} \sum_{\substack{\rho_\chi \\ |\gamma_\chi| \le T \\ \beta_\chi > 1 - c_0/\log qT}} \frac{1}{\beta_\chi} + \frac{x}{T} (\log qx)^4 \\
&\ll \frac{1}{\varphi(q)} \frac{x}{T} (\log qx)^3 \sum_{\chi \,(\mathrm{mod}\ q)} \sum_{\substack{\rho_\chi \\ |\gamma_\chi| \le T}} 1 + \frac{x}{T} (\log qx)^4 \ll x (\log qx)^4
\end{aligned}$$

by (2.1) provided $T \le x$, so we have

$$\Sigma_4 = \frac{1}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\ q)} \overline{\chi(b)} \int_3^x \Bigg(\sum_{\substack{\rho_\chi \\ |\gamma_\chi| \le T}} u^{\rho_\chi - 1} \Bigg) B(x-u, T; q, a) du + O(x(\log qx)^4). \tag{3.12}$$

If $3 \le u \le x$ and $T \le x$, then Proposition 2, proven in the next section, yields

$$\sum_{\substack{\rho_\chi \\ |\gamma_\chi| \le T}} u^{\rho_\chi} \ll u \log(quT) \log\log u + T \log u \ll u(\log qx)^2 + x(\log qx).$$

Note that Proposition 2 is stated for primitive χ, but the above estimate is valid for any χ. Using this estimate from (3.12) we obtain

$$\Sigma_4 \ll \frac{x^2}{T} (\log qx)^4 \tag{3.13}$$

if $T \le x$.

Next consider Σ_3. The following argument is inspired by [11, Corrigendum], but in our case we have to treat the zeros near the real line more carefully since $\Gamma(s)$ has a simple pole at $s = 0$. We evaluate $\mathcal{Z}(\rho_\chi, \rho_{\chi'})$ defined in (3.8) for $|\gamma_\chi| \le |\gamma_{\chi'}|$ by using Stirling's formula

$$\Gamma(s) \ll (|t|+1)^{\sigma - 1/2} e^{-(\pi/2)|t|}, \quad s = \sigma + it,\ 0 \le \sigma \le 3,\ |t| \ge 1.$$

If $|\gamma_\chi| \le |\gamma_{\chi'}| \le 1$, then $|\Gamma(1+\rho_\chi+\rho_{\chi'})| \asymp 1$, and hence

$$\mathcal{Z}(\rho_\chi, \rho_{\chi'}) \ll |\rho_\chi|^{-1} |\rho_{\chi'}|^{-1} \ll T^{1/2} |\rho_\chi|^{-1} |\rho_{\chi'}|^{-1}.$$

If $|\gamma_\chi| \le 1 \le |\gamma_{\chi'}|$, then applying Stirling's formula to $\Gamma(\rho_{\chi'})$ and $\Gamma(1+\rho_\chi+\rho_{\chi'})$,

$$\begin{aligned} \mathcal{Z}(\rho_\chi, \rho_{\chi'}) &\ll |\rho_\chi|^{-1} \frac{|\gamma_{\chi'}|^{\beta_{\chi'} - 1/2} e^{-(\pi/2)|\gamma_{\chi'}|}}{(|\gamma_\chi + \gamma_{\chi'}| + 1)^{\beta_\chi + \beta_{\chi'} + 1/2} e^{-(\pi/2)|\gamma_\chi + \gamma_{\chi'}|}} \\ &\ll |\rho_\chi|^{-1} |\gamma_{\chi'}|^{-\beta_\chi - 1} \ll |\rho_\chi|^{-1} |\rho_{\chi'}|^{-1} \ll T^{1/2} |\rho_\chi|^{-1} |\rho_{\chi'}|^{-1} \end{aligned}$$

as in the case $|\gamma_\chi| \le |\gamma_{\chi'}| \le 1$. If $1 \le |\gamma_\chi| \le |\gamma_{\chi'}| \le T$, we have

$$\mathcal{Z}(\rho_\chi, \rho_{\chi'}) \ll \frac{|\gamma_\chi|^{\beta_\chi - 1/2} e^{-(\pi/2)|\gamma_\chi|} |\gamma_{\chi'}|^{\beta_{\chi'} - 1/2} e^{-(\pi/2)|\gamma_{\chi'}|}}{(|\gamma_\chi + \gamma_{\chi'}| + 1)^{\beta_\chi + \beta_{\chi'} + 1/2} e^{-(\pi/2)|\gamma_\chi + \gamma_{\chi'}|}}.$$

When γ_χ and $\gamma_{\chi'}$ have the same sign, then the exponential factors are cancelled and we obtain

$$\mathcal{Z}(\rho_\chi, \rho_{\chi'}) \ll |\gamma_\chi|^{\beta_\chi - 1/2} |\gamma_{\chi'}|^{-\beta_\chi - 1} \ll |\gamma_\chi|^{-1/2} |\gamma_{\chi'}|^{-1} \ll T^{1/2} |\gamma_\chi|^{-1} |\gamma_{\chi'}|^{-1} \tag{3.14}$$

since $|\gamma_\chi| \le |\gamma_{\chi'}| \le T$. When they have opposite signs the contribution of the exponential factors is $O(e^{-\pi|\gamma_\chi|})$, and

$$\begin{aligned} &(|\gamma_\chi + \gamma_{\chi'}| + 1)^{-(\beta_\chi + \beta_{\chi'} + 1/2)} \\ &= (1 + |\gamma_{\chi'}| - |\gamma_\chi|)^{-(\beta_\chi + \beta_{\chi'} + 1/2)} \\ &= (1 + |\gamma_{\chi'}|)^{-(\beta_\chi + \beta_{\chi'} + 1/2)} \left(1 - \frac{|\gamma_\chi|}{1 + |\gamma_{\chi'}|}\right)^{-(\beta_\chi + \beta_{\chi'} + 1/2)} \\ &\le (1 + |\gamma_{\chi'}|)^{-(\beta_\chi + \beta_{\chi'} + 1/2)} (1 + |\gamma_\chi|)^{(\beta_\chi + \beta_{\chi'} + 1/2)} \\ &\le (1 + |\gamma_{\chi'}|)^{-(\beta_\chi + \beta_{\chi'} + 1/2)} (1 + |\gamma_\chi|)^{\pi} \\ &\le (1 + |\gamma_{\chi'}|)^{-(\beta_\chi + \beta_{\chi'} + 1/2)} e^{\pi |\gamma_\chi|}, \end{aligned}$$

we again obtain $\mathcal{Z}(\rho_\chi, \rho_{\chi'}) \ll T^{1/2}|\gamma_\chi|^{-1}|\gamma_{\chi'}|^{-1}$. Therefore, the estimate

$$\mathcal{Z}(\rho_\chi, \rho_{\chi'}) \ll T^{1/2}|\rho_\chi|^{-1}|\rho_{\chi'}|^{-1} \tag{3.15}$$

holds for all $|\gamma_\chi|, |\gamma_{\chi'}| \le T$ by symmetry between ρ_χ and $\rho_{\chi'}$.

By using the estimate (3.15) and Lemma 1, we have

$$\sum_{\substack{\rho_\chi \\ |\gamma_\chi|\le T}} \sum_{\substack{\rho_{\chi'} \\ |\gamma_{\chi'}|\le T}} \mathcal{Z}(\rho_\chi, \rho_{\chi'}) x^{\rho_\chi+\rho_{\chi'}} \ll x^{2B_q^*} T^{1/2} \sum_{\substack{\rho_\chi \\ |\gamma_\chi|\le T}} \sum_{\substack{\rho_{\chi'} \\ |\gamma_{\chi'}|\le T}} |\rho_\chi|^{-1}|\rho_{\chi'}|^{-1}.$$

Therefore, by Lemma 3, we have

$$\begin{aligned} \Sigma_3 &\ll x^{2B_q^*} T^{1/2} \left(\frac{1}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\ q)} \sum_{\substack{\rho_\chi \\ |\gamma_\chi|\le T}} \frac{1}{|\rho_\chi|} \right)^2 \\ &\ll x^{2B_q^*} T^{1/2} \left((\log qx)^2 + \frac{q^{1/2}(\log q)^2}{\varphi(q)} \right)^2 \ll x^{2B_q^*} T^{1/2} (\log qx)^4 \end{aligned}$$

if $T \le x$.

All the error terms on the right-hand side of (3.11) have now been estimated. We choose the optimal T by requiring that $x^{2B_q^*}T^{1/2} = x^2/T$, hence $T = x^{4(1-B_q^*)/3}$. Since $B_q^* \ge 1/2$ this choice satisfies the condition $T \le x$. Substituting this choice of T into (3.11), we obtain the assertion of Proposition 1. □

4. The Landau–Gonek formula for L-functions

The Landau–Gonek result [10], originally a formula on the zeros of $\zeta(s)$, has been extended to other situations, for example Ford et al. [7] worked on a general setting of the Selberg class. We did not find any instance where the uniformity with respect to q was treated and we do so here for the sake of completeness.

Proposition 2. *Let $x, T, q > 1$ and χ be a primitive character* (mod q). *Then*

$$\begin{aligned} \sum_{\substack{\rho_\chi \\ |\gamma_\chi|\le T}} x^{\rho_\chi} &= -\frac{T}{\pi}\chi(x)\Lambda(x) + O(x(\log 2qxT)(\log\log 3x)) \\ &\quad + O\left((\log x) \min\left\{ T, \frac{x}{\langle x \rangle} \right\} \right) + O\left((\log 2qT) \min\left\{ T, \frac{1}{\log x} \right\} \right), \end{aligned}$$

where $\chi(x) = \Lambda(x) = 0$ if x is not an integer, and $\langle x \rangle$ denotes the distance from x to the nearest prime power other than x itself.

Proof. The proof essentially follows the original one of Gonek [10]. Let $c = 1 + 1/\log 3x$, and consider the integral

$$\begin{aligned} I &= \frac{1}{2\pi i} \left(\int_{c-iT}^{c+iT} + \int_{c+iT}^{1-c+iT} + \int_{1-c+iT}^{1-c-iT} + \int_{1-c-iT}^{c-iT} \right) \frac{L'}{L}(s,\chi) x^s ds \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned} \tag{4.1}$$

say. First suppose that the horizontal paths do not cross any zero of $L(s, \chi)$. The poles inside the rectangle are the non-trivial zeros of $L(s, \chi)$ and at most one trivial zero of $L(s,\chi)$ at $s=0$. Hence the residue theorem gives

$$I = \sum_{\substack{\rho_\chi \\ |\gamma_\chi|\le T}} x^{\rho_\chi} + O(1). \tag{4.2}$$

We evaluate I_1, I_2, I_3 and I_4. First consider I_2. We use the well-known formula

$$\frac{L'}{L}(s,\chi) = \sum_{\substack{\rho_\chi \\ |\gamma_\chi - t|\le 1}} \frac{1}{s-\rho_\chi} + O(\log q(|t|+2)) \tag{4.3}$$

uniformly in $-1 \le \sigma \le 2$ ([15, Lemma 12.6]). We have

$$I_2 = \sum_{\substack{\rho_\chi \\ |\gamma_\chi - T|\le 1}} \int_{c+iT}^{1-c+iT} \frac{x^s}{s-\rho_\chi} ds + O\left(\log 2qT \int_{1-c}^{c} x^\sigma \, d\sigma\right), \tag{4.4}$$

whose error term is

$$\ll x^c \log 2qT \ll x \log 2qT. \tag{4.5}$$

For each integral on the first sum of (4.4), we first observe that $T-1 \le \gamma_\chi \le T+1$. When $T-1 \le \gamma_\chi \le T$, we deform the path of integration as

$$\int_{c+iT}^{1-c+iT} = \int_{c+iT}^{c+i(T+1)} + \int_{c+i(T+1)}^{1-c+i(T+1)} + \int_{1-c+i(T+1)}^{1-c+iT}.$$

Noting that the denominator on the second term is $\gg 1$ and that $\log 3x \gg 1$, $\log\log 3x \gg 1$, we obtain in this case

$$\begin{aligned}
\int_{c+iT}^{1-c+iT} \frac{x^s}{s-\rho_\chi} ds &\ll x^c \int_T^{T+1} \frac{dt}{|(c-\beta_\chi) + i(t-\gamma_\chi)|} + \int_{1-c}^{c} x^\sigma \, d\sigma + \frac{x^{1-c}}{\beta_\chi - (1-c)} \\
&\ll x \int_{\gamma_\chi}^{\gamma_\chi + 2} \min\left\{\log 3x, \frac{1}{t-\gamma_\chi}\right\} dt + x + \log 3x \\
&\ll x \left(\int_{\gamma_\chi}^{\gamma_\chi + 1/\log 3x} \log 3x \, dt + \int_{\gamma_\chi + 1/\log 3x}^{\gamma_\chi + 2} \frac{dt}{t-\gamma_\chi} \right) + x \\
&\ll x \left(\frac{1}{\log 3x} \cdot \log 3x + \Big[\log(t-\gamma_\chi)\Big]_{t=\gamma_\chi + 1/\log 3x}^{\gamma_\chi + 2} \right) + x \\
&\ll x \log\log 3x.
\end{aligned}$$

If $T < \gamma_\chi \le T+1$, we deform the path to that including the segment with the imaginary part $T-1$, and argue similarly. We can conclude that

$$I_2 \ll x \log\log 3x \sum_{\substack{\rho_\chi \\ |\gamma_\chi - T|\le 1}} 1 + x \log 2qT$$

and hence, by using Lemma 2,

$$I_2 \ll x(\log 2qxT)(\log\log 3x). \tag{4.6}$$

The estimate of I_4 is similar.

Next consider I_3. We first quote

$$\frac{L'}{L}(s,\chi) = -\frac{L'}{L}(1-s,\overline{\chi}) - \log \frac{q}{2\pi} - \frac{\Gamma'}{\Gamma}(1-s) + \frac{\pi}{2} \cot \frac{\pi}{2}(s+\kappa) \tag{4.7}$$

([15, (10.35)]), where $\kappa = 0$ or 1 depending on whether χ is an even or an odd character, respectively. We see easily that

$$\frac{\pi}{2}\cot\frac{\pi}{2}(s+\kappa) = \pm i + O(e^{-\pi|t|})$$

for $|t| \geq 1$ and $\Re s = 1-c$. Therefore, putting $s = 1-c+it$ and applying Stirling's formula we have

$$\frac{L'}{L}(1-c+it,\chi) = -\frac{L'}{L}(c-it,\overline{\chi}) - \log qt + C + O(t^{-1}) \tag{4.8}$$

for $t \geq 1$, where C denotes a constant. Hence, the part $[1-c\pm i, 1-c\pm iT]$ of the integral I_3 is

$$= \frac{\pm 1}{2\pi}\int_1^T \left(\frac{L'}{L}(c\mp it,\overline{\chi}) + \log qt - C\right) x^{1-c\pm it}dt + O\left(\int_1^T \frac{x^{1-c}}{t}dt\right),$$

whose error term is $O(\log T)$. The integral is

$$= \mp\frac{x^{1-c}}{2\pi}\sum_{n=2}^{\infty}\frac{\overline{\chi(n)}\Lambda(n)}{n^c}\int_1^T (nx)^{\pm it}dt \pm \frac{x^{1-c}}{2\pi}\int_1^T (\log qt - C)x^{\pm it}dt,$$

whose first part is

$$\ll x^{1-c}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^c\log nx} \ll \sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^c} = -\frac{\zeta'}{\zeta}(c) \ll \frac{1}{c-1} \ll \log 3x.$$

The second part is trivially $O(T\log 2qT)$, while integration by parts gives

$$= \frac{x^{1-c}}{2\pi}\left(\left[(\log qt - C)\frac{x^{\pm it}}{i\log x}\right]_{t=1}^T - \int_1^T \frac{x^{it}}{it\log x}dt\right) \ll \frac{\log 2qT}{\log x}.$$

(Note that $\log x \gg 1$ does not hold.) The part $[1-c-i, 1-c+i]$ of the integral I_3 is $\ll \log 2qx$ by (4.7). Therefore we conclude

$$\begin{aligned} I_3 &\ll \log 2qx + (\log 2qT)\min\left\{T, \frac{1}{\log x}\right\} + \log T \\ &\ll x(\log 2qT)(\log\log 3x) + (\log 2qT)\min\left\{T, \frac{1}{\log x}\right\} \end{aligned} \tag{4.9}$$

since $T > 1$.

We next consider I_1. Substituting the Dirichlet series expansion, we have

$$\begin{aligned} I_1 &= -\sum_{n=2}^{\infty}\chi(n)\Lambda(n)\frac{1}{2\pi}\int_{-T}^T (x/n)^{c+it}dt \\ &= -\frac{T}{\pi}\chi(x)\Lambda(x) + O\left(\sum_{n\neq x}\Lambda(n)(x/n)^c\min\left\{T, \frac{1}{|\log x/n|}\right\}\right). \end{aligned} \tag{4.10}$$

The error term here can be estimated by [10, Lemma 2], and so

$$I_1 = -\frac{T}{\pi}\chi(x)\Lambda(x) + O(x(\log 2x)(\log\log 3x)) + O\left((\log x)\min\left\{T, \frac{x}{\langle x\rangle}\right\}\right). \tag{4.11}$$

The formula of the lemma follows by combining (4.6), (4.9) and (4.11).

Finally if the path of I_2 or I_4 crosses some zero we choose T' slightly larger than T, and define I', similar to I but now T is replaced by T'. Then instead of (4.2) we obtain

$$I' = \sum_{\substack{\rho_\chi \\ |\gamma_\chi| \le T}} x^{\rho_\chi} + O(x \log 2qT)$$

(by Lemma 2), while the evaluation of integrals on the edges of the rectangle can be done in the same way as for I, so the assertion of the lemma is also valid in this case. □

5. Lemmas for the proof of Theorem 2

In this section we present some preparatory material for the improvement of the error term of Proposition 1. We start with a lemma on an integral of the Selberg-type. In order to prove this first lemma, we need to calculate the following sum over the non-trivial zeros of $L(s,\chi)$.

Lemma 7. *For any $x \ge 2$ and real number y, we have*

$$\sum_{\substack{\rho_\chi \\ |\gamma_\chi| \le x}} \frac{1}{1+|\gamma_\chi - y|} \ll (\log qx)^2.$$

Proof. If $|y| > 2x$, then each term above is $\ll x^{-1}$ and the lemma holds trivially. So we consider the case $|y| \le 2x$. Then by the triangle inequality, we have

$$|\gamma_\chi - y| \le x + y \le 3x.$$

Thus we can extend the sum and dissect it as

$$\begin{aligned}\sum_{\substack{\rho_\chi \\ |\gamma_\chi| \le x}} \frac{1}{1+|\gamma_\chi - y|} &\le \sum_{\substack{\rho_\chi \\ |\gamma_\chi - y| \le 3x}} \frac{1}{1+|\gamma_\chi - y|} \\ &\le \sum_{\substack{\rho_\chi \\ |\gamma_\chi - y| \le 1}} \frac{1}{1+|\gamma_\chi - y|} + \sum_{1 \le n \le 3x} \sum_{\substack{\rho_\chi \\ n < |\gamma_\chi - y| \le n+1}} \frac{1}{1+|\gamma_\chi - y|} \\ &\le \sum_{\substack{\rho_\chi \\ |\gamma_\chi - y| \le 1}} 1 + \sum_{1 \le n \le 3x} \frac{1}{n} \sum_{\substack{\rho_\chi \\ n < |\gamma_\chi - y| \le n+1}} 1.\end{aligned}$$

By using Lemma 2, we can estimate the last sum to be

$$\ll (\log qx) \left(1 + \sum_{1 \le n \le 3x} \frac{1}{n} \right) \ll (\log qx)(\log x) \ll (\log qx)^2.$$

□

We now obtain an estimate for an integral of the Selberg-type.

Lemma 8. *For any $2 \le h \le x$ and any $\chi \pmod q$, we have*

$$\int_x^{2x} \Big| \sum_{t < n \le t+h} \chi(n)\Lambda(n) - \delta_0(\chi) h \Big|^2 dt \ll h x^{2B_q^*} (\log qx)^4.$$

Proof. By taking the difference between $u = t + h$ and $u = t$ in Lemma 5, we have

$$\sum_{t<n\le t+h} \chi(n)\Lambda(n) = \delta_0(\chi)h - \sum_{\substack{\rho_\chi \\ |\gamma_\chi|\le T}} \frac{(t+h)^{\rho_\chi} - t^{\rho_\chi}}{\rho_\chi} + E(t+h, T, \chi) - E(t, T, \chi),$$

where the term $C(\chi^*)$ is cancelled out since it is independent of the main variable u in Lemma 5. We take the parameter $T = x$. Then the error term is estimated by

$$E(t+h, T, \chi) - E(t, T, \chi) \ll (\log 2q)(\log x) + \frac{x}{T}(\log qxT)^2 \ll (\log qx)^2.$$

Therefore, we obtain

$$\begin{aligned} &\sum_{t<n\le t+h} \chi(n)\Lambda(n) - \delta_0(\chi)h \\ &= -\sum_{\substack{\rho_\chi \\ |\gamma_\chi|\le x}} \frac{(t+h)^{\rho_\chi} - t^{\rho_\chi}}{\rho_\chi} + O((\log qx)^2) \\ &= -\sum_{\substack{\rho_\chi \\ |\gamma_\chi|\le x/h}} \int_t^{t+h} u^{\rho_\chi - 1} du - \sum_{\substack{\rho_\chi \\ x/h<|\gamma_\chi|\le x}} \frac{(t+h)^{\rho_\chi} - t^{\rho_\chi}}{\rho_\chi} + O((\log qx)^2) \\ &= -\int_t^{t+h} \Bigg(\sum_{\substack{\rho_\chi \\ |\gamma_\chi|\le x/h}} u^{\rho_\chi - 1} \Bigg) du - \sum_{\substack{\rho_\chi \\ x/h<|\gamma_\chi|\le x}} \frac{(t+h)^{\rho_\chi} - t^{\rho_\chi}}{\rho_\chi} + O((\log qx)^2) \\ &= -\psi_1(t) - \psi_2(t) + O\left((\log qx)^2\right), \text{ say.} \end{aligned}$$

Substituting this explicit formula into the integral of the assertion, we have

$$\begin{aligned} &\int_x^{2x} \Big| \sum_{t<n\le t+h} \chi(n)\Lambda(n) - \delta_0(\chi)h \Big|^2 dt \\ &\ll \int_x^{2x} |\psi_1(t)|^2 dt + \int_x^{2x} |\psi_2(t)|^2 dt + x(\log qx)^4. \end{aligned}$$

Hence it suffices to prove the two estimates

$$\int_x^{2x} |\psi_1(t)|^2 dt, \quad \int_x^{2x} |\psi_2(t)|^2 dt \ll hx^{2B_q^*}(\log qx)^4,$$

since $B_q^* \ge 1/2$ implies $x(\log qx)^4 \le hx^{2B_q^*}(\log qx)^4$.

Applying the Cauchy–Schwarz inequality to the first integral we have

$$|\psi_1(t)|^2 \ll h \int_t^{t+h} \Big| \sum_{\substack{\rho_\chi \\ |\gamma_\chi|\le x/h}} u^{\rho_\chi - 1} \Big|^2 du$$

so that

$$\begin{aligned} \int_x^{2x} |\psi_1(t)|^2 dt &\ll h \int_x^{2x} \int_t^{t+h} \Big| \sum_{\substack{\rho_\chi \\ |\gamma_\chi|\le x/h}} u^{\rho_\chi - 1} \Big|^2 du dt \\ &= h \int_x^{2x+h} \Big| \sum_{\substack{\rho_\chi \\ |\gamma_\chi|\le x/h}} u^{\rho_\chi - 1} \Big|^2 \left(\int_{\max(u-h,x)}^{\min(u,2x)} dt \right) du \end{aligned}$$

$$\ll h^2 \int_x^{3x} \Big| \sum_{\substack{\rho_\chi \\ |\gamma_\chi| \le x/h}} u^{\rho_\chi - 1} \Big|^2 du \ll h^2 x^{-2} \int_x^{3x} \Big| \sum_{\substack{\rho_\chi \\ |\gamma_\chi| \le x/h}} u^{\rho_\chi} \Big|^2 du$$

since $h \le x$. We now expand the square and integrate over u. This gives

$$\int_x^{2x} |\psi_1(t)|^2 dt \ll h^2 x^{-2} \sum\sum_{\substack{\rho_\chi, \rho'_\chi \\ |\gamma_\chi|, |\gamma'_\chi| \le x/h}} \frac{x^{\beta_\chi + \beta'_\chi + 1}}{1 + |\gamma_\chi - \gamma'_\chi|}.$$

From Lemma 1 we can estimate β_χ and $\beta_{\chi'}$ by B_q^*. Therefore,

$$\begin{aligned}\int_x^{2x} |\psi_1(t)|^2 dt &\ll h^2 x^{2B_q^* - 1} \sum_{\substack{\rho_\chi \\ |\gamma_\chi| \le x/h}} \sum_{\substack{\rho'_\chi \\ |\gamma'_\chi| \le x}} \frac{1}{1 + |\gamma_\chi - \gamma'_\chi|} \\ &\ll h^2 x^{2B_q^* - 1} (\log qx)^2 \sum_{\substack{\rho_\chi \\ |\gamma_\chi| \le x/h}} 1 \ll h x^{2B_q^*} (\log qx)^3\end{aligned}$$

by Lemma 7.

For the latter integral, we ignore the difference as

$$\begin{aligned}\int_x^{2x} |\psi_2(t)|^2 dt &\ll \int_x^{2x} \Big| \sum_{\substack{\rho_\chi \\ x/h < |\gamma_\chi| \le x}} \frac{(t+h)^{\rho_\chi}}{\rho_\chi} \Big|^2 dt + \int_x^{2x} \Big| \sum_{\substack{\rho_\chi \\ x/h < |\gamma_\chi| \le x}} \frac{t^{\rho_\chi}}{\rho_\chi} \Big|^2 dt \\ &= \int_{x+h}^{2x+h} \Big| \sum_{\substack{\rho_\chi \\ x/h < |\gamma_\chi| \le x}} \frac{t^{\rho_\chi}}{\rho_\chi} \Big|^2 dt + \int_x^{2x} \Big| \sum_{\substack{\rho_\chi \\ x/h < |\gamma_\chi| \le x}} \frac{t^{\rho_\chi}}{\rho_\chi} \Big|^2 dt \\ &\ll \int_x^{3x} \Big| \sum_{\substack{\rho_\chi \\ x/h < |\gamma_\chi| \le x}} \frac{t^{\rho_\chi}}{\rho_\chi} \Big|^2 dt.\end{aligned}$$

Then we expand the square and integrate over t. This gives

$$\int_x^{2x} |\psi_2(t)|^2 dt \ll \sum\sum_{\substack{\rho_\chi, \rho'_\chi \\ x/h < |\gamma_\chi|, |\gamma'_\chi| \le x}} \frac{x^{\beta_\chi + \beta'_\chi + 1}}{|\gamma_\chi||\gamma'_\chi|(1 + |\gamma_\chi - \gamma'_\chi|)}.$$

Obviously,

$$\frac{x^{\beta_\chi + \beta'_\chi}}{|\gamma_\chi||\gamma'_\chi|} \ll \frac{x^{2\beta_\chi}}{|\gamma_\chi|^2} + \frac{x^{2\beta'_\chi}}{|\gamma'_\chi|^2}.$$

By using the symmetry of the terms in γ_χ and $\gamma_{\chi'}$) we have

$$\begin{aligned}\int_x^{2x} |\psi_2(t)|^2 dt &\ll x^{1+2B_q^*} \sum_{\substack{\rho_\chi \\ x/h < |\gamma_\chi| \le x}} \frac{1}{|\gamma_\chi|^2} \sum_{\substack{\rho'_\chi \\ |\gamma'_\chi| \le x}} \frac{1}{1 + |\gamma_\chi - \gamma'_\chi|} \\ &\ll x^{1+2B_q^*} (\log qx)^2 \sum_{\substack{\rho_\chi \\ x/h < |\gamma_\chi| \le x}} \frac{1}{|\gamma_\chi|^2} \ll h x^{2B_q^*} (\log qx)^3,\end{aligned}$$

where we used Lemma 4 for the last estimate. This completes the proof. □

Now let

$$T(\alpha)=\sum_{n\le x}e(n\alpha),\quad S(\alpha,\chi)=\sum_{n\le x}\chi(n)\Lambda(n)e(n\alpha),$$

and

$$W(\alpha,\chi)=S(\alpha,\chi)-\delta_0(\chi)T(\alpha),$$

where $e(\alpha)=\exp(2\pi i\alpha)$. Our next task is to translate the previous estimate of the integral into this exponential sum setting.

Lemma 9. *Let $x^{-1}\le\xi\le 1/2$. For any $\chi\pmod q$, we have*

$$\int_{-\xi}^{\xi}|W(\alpha,\chi)|^2d\alpha\ll\xi x^{2B_q^*}(\log qx)^4.$$

Proof. We first note that

$$W(\alpha,\chi)=\sum_{0<n\le x}\left(\chi(n)\Lambda(n)-\delta_0(\chi)\right)e(n\alpha).$$

Thus using Gallagher's lemma [9, Lemma 1], we have

$$\begin{aligned}\int_{-\xi}^{\xi}|W(\alpha,\chi)|^2d\alpha&=\int_{-\xi}^{\xi}\left|\sum_{0<n\le x}\left(\chi(n)\Lambda(n)-\delta_0(\chi)\right)e(n\alpha)\right|^2d\alpha.\\&\ll\xi^2\int_{-(2\xi)^{-1}}^{x}\left|\sum_{a(t)<n\le b(t)}\left(\chi(n)\Lambda(n)-\delta_0(\chi)\right)\right|^2dt,\end{aligned}$$

where

$$a(t)=\max(t,0),\quad b(t)=\min(t+(2\xi)^{-1},x).$$

We decompose this integral as

$$\ll\xi^2\int_{-(2\xi)^{-1}}^{(2\xi)^{-1}}+\xi^2\int_{(2\xi)^{-1}}^{x-(2\xi)^{-1}}+\xi^2\int_{x-(2\xi)^{-1}}^{x}=\xi^2I_-+\xi^2I+\xi^2I_+,\ \text{say.}$$

By Lemma 1 and Lemma 5, we obtain

$$\sum_{n\le t}\chi(n)\Lambda(n)-\delta_0(\chi)t\ll x^{B_q^*}\sum_{\substack{\rho_\chi\ne1-\beta_1\\|\gamma_\chi|\le x}}\frac{1}{|\rho_\chi|}+\left|\delta_1(\chi)\frac{t^{1-\beta_1}}{1-\beta_1}-C(\chi^*)\right|+(\log qx)^2$$

for $2\le t\le 2x$. The second term on the right-hand side is estimated by using Theorem 11.4 and formula (12.7) of [15] as

$$\begin{aligned}\delta_1(\chi)\frac{t^{1-\beta_1}}{1-\beta_1}-C(\chi^*&=\delta_1(\chi)\frac{t^{1-\beta_1}-1}{1-\beta_1}+O(\log 2q)\\&=\delta_1(\chi)\frac{\log t}{1-\beta_1}\int_0^{1-\beta_1}t^\sigma d\sigma+O(\log 2q)\\&\ll x^{1-\beta_1}(\log 2x)+\log 2q\ll x^{1/2}(\log qx),\end{aligned}$$

since $\beta_1>1/2$. Thus, by Lemma 3, we have

$$\sum_{n\le t}\chi(n)\Lambda(n)-\delta_0(\chi)t\ll x^{B_q^*}(\log qx)^2+x^{1/2}(\log qx)+(\log qx)^2\ll x^{B_q^*}(\log qx)^2,$$

which also holds trivially for $0 \le t < 2$. Thus for any $0 \le a \le b \le 2x$, we have

$$\sum_{a<n\le b} (\chi(n)\Lambda(n) - \delta_0(\chi)) = \sum_{a<n\le b} \chi(n)\Lambda(n) - \delta_0(\chi)(b-a) + O(1) \ll x^{B_q^*}(\log qx)^2.$$

By substituting this estimate into $I_\pm$, we obtain

$$\xi^2 I_\pm \ll \xi x^{2B_q^*}(\log qx)^4,$$

since $I_\pm$ are integrals taken over intervals of length $\le \xi^{-1}$. Finally,

$$\begin{aligned} \xi^2 I &= \xi^2 \int_{(2\xi)^{-1}}^{x-(2\xi)^{-1}} \left| \sum_{t<n\le t+(2\xi)^{-1}} (\chi(n)\Lambda(n) - \delta_0(\chi)) \right|^2 dt \\ &\ll \xi^2 \int_{(2\xi)^{-1}}^{x} \left| \sum_{t<n\le t+(2\xi)^{-1}} \chi(n)\Lambda(n) - \delta_0(\chi)(2\xi)^{-1} \right|^2 dt + \xi^2 x \\ &\ll \xi^2 \sum_{k=0}^{O(\log x)} \int_{x/2^{k+1}}^{x/2^k} \left| \sum_{t<n\le t+(2\xi)^{-1}} \chi(n)\Lambda(n) - \delta_0(\chi)(2\xi)^{-1} \right|^2 dt + \xi^2 x \\ &\ll \xi x^{2B_q^*}(\log qx)^4 \end{aligned}$$

by Lemma 8. Summing up the above calculations, we obtain the lemma. □

Let

$$J(\chi) = \int_{-1/2}^{1/2} |W(\alpha,\chi)|^2 |T(\alpha)| d\alpha.$$

Now by using the previous results we can obtain an estimate for this quantity.

Lemma 10. *We have*

$$J(\chi) \ll x^{2B_q^*}(\log qx)^5.$$

Proof. We dissect the integral dyadically as

$$J(\chi) \le \int_{|\alpha|\le 1/x} |W(\alpha,\chi)|^2 |T(\alpha)| d\alpha + \sum_{k=1}^{O(\log x)} \int_{1/2^{k+1}<|\alpha|\le 1/2^k} |W(\alpha,\chi)|^2 |T(\alpha)| d\alpha.$$

Then since $T(\alpha) \ll \min(x, |\alpha|^{-1})$ for $|\alpha| \le 1/2$, we have

$$J(\chi) \ll (\log x) \sup_{1/x<\xi\le 1/2} \xi^{-1} \int_{|\alpha|\le\xi} |W(\alpha,\chi)|^2 d\alpha \ll x^{2B_q^*}(\log qx)^5$$

by Lemma 9. □

6. Proofs of Theorems 2 and 4

We let

$$G(n;\chi_1,\chi_2) = \sum_{\ell+m=n} \chi_1(\ell)\Lambda(\ell)\chi_2(m)\Lambda(m), \quad S(x;\chi_1,\chi_2) = \sum_{n\le x} G(n;\chi_1,\chi_2)$$

and prove the following intermediate lemma.

Lemma 11. *For $x \geq 2$ and $\chi_1, \chi_2 \pmod q$, we have*

$$S(x;\chi_1,\chi_2) = \frac{\delta_0(\chi_1)\delta_0(\chi_2)}{2}x^2 - \delta_0(\chi_2)H(x,\chi_1) - \delta_0(\chi_1)H(x,\chi_2) + R(x;\chi_1,\chi_2)$$
$$+ O\Big(\delta_0(\chi_2)(1+\delta_1(\chi_1)q^{1/2})x(\log qx)^2 + \delta_0(\chi_1)(1+\delta_1(\chi_2)q^{1/2})x(\log qx)^2\Big),$$

where

$$H(x,\chi) = \sum_{\rho_\chi} \frac{x^{\rho_\chi+1}}{\rho_\chi(\rho_\chi+1)}, \quad R(x;\chi_1,\chi_2) = \int_0^1 W(\alpha,\chi_1)W(\alpha,\chi_2)T(-\alpha)d\alpha.$$

Proof. By the orthogonality of the exponential function we have

$$S(x;\chi_1,\chi_2) = \int_0^1 S(\alpha,\chi_1)S(\alpha,\chi_2)T(-\alpha)d\alpha. \tag{6.1}$$

From the definition we have an expansion

$$\begin{aligned} S(\alpha,\chi_1)S(\alpha,\chi_2) = \delta_0(\chi_2)S(\alpha,\chi_1)T(\alpha) + \delta_0(\chi_1)S(\alpha,\chi_2)T(\alpha) \\ - \delta_0(\chi_1)\delta_0(\chi_2)T(\alpha)^2 + W(\alpha,\chi_1)W(\alpha,\chi_2). \end{aligned} \tag{6.2}$$

Substituting this decomposition into the integral expression (6.1), we have

$$S(x;\chi_1,\chi_2) = \delta_0(\chi_2)I(\chi_1) + \delta_0(\chi_1)I(\chi_2) - \delta_0(\chi_1)\delta_0(\chi_2)I + R(x;\chi_1,\chi_2),$$

where

$$I = \int_0^1 T(\alpha)^2 T(-\alpha)d\alpha, \quad I(\chi) = \int_0^1 S(\alpha,\chi)T(\alpha)T(-\alpha)d\alpha.$$

Therefore it is sufficient to show that

$$I = \frac{x^2}{2} + O(x), \quad I(\chi) = \frac{\delta_0(\chi)x^2}{2} - H(x,\chi) + O((1+\delta_1(\chi)q^{1/2})x(\log qx)^2). \tag{6.3}$$

The first integral I is evaluated by using the orthogonality as

$$I = \sum_{\ell+m\leq x} 1 = \sum_{n\leq x}(n-1) = \frac{x^2}{2} + O(x).$$

The second integral $I(\chi)$ is

$$\begin{aligned} I(\chi) &= \sum_{\ell+m\leq x} \chi(\ell)\Lambda(\ell) = \sum_{n\leq x}(x-n)\chi(n)\Lambda(n) + O(x) \\ &= \sum_{n\leq x}\chi(n)\Lambda(n)\int_n^x du + O(x) = \int_0^x \psi(u,\chi)du + O(x) \end{aligned}$$

by partial summation. We substitute Lemma 6 with $T = x$. Then

$$I(\chi) = \frac{\delta_0(\chi)x^2}{2} - \sum_{\substack{\rho_\chi \\ |\gamma_\chi|\leq x}} \frac{x^{\rho_\chi+1}}{\rho_\chi(\rho_\chi+1)} + O(x(\log qx)^2 + \delta_1(\chi)q^{1/2}(\log q)^2 x). \tag{6.4}$$

We then extend the sum over zeros which this gives the error term of the size

$$\sum_{\substack{\rho_\chi \\ |\gamma_\chi|>x}} \frac{x^{\rho_\chi+1}}{\rho_\chi(\rho_\chi+1)} \ll x^2 \sum_{\substack{\rho_\chi \\ |\gamma_\chi|>x}} \frac{1}{|\rho_\chi|^2} \ll x(\log qx)^2,$$

by the use of Lemma 4 in the last estimate. Substituting this estimate into (6.4), we obtain (6.3) for $I(\chi)$.

□

Now Theorem 2 can be proven.

Proof of Theorem 2. By the orthogonality of characters, we have

$$S(x;q,a,b)=\frac{1}{\varphi(q)^2}\sum_{\chi_1,\chi_2 \ (\mathrm{mod}\ q)}\overline{\chi_1(a)\chi_2(b)}S(x;\chi_1,\chi_2).$$

Thus, by Lemma 11, it suffices to show that

$$\frac{1}{\varphi(q)^2}\sum_{\chi_1,\chi_2 \ (\mathrm{mod}\ q)}|R(x;\chi_1,\chi_2)|\ll x^{2B_q^*}(\log qx)^5.$$

By the Cauchy–Schwarz inequality and Lemma 10, the left-hand side above is

$$\ll\frac{1}{\varphi(q)^2}\sum_{\chi_1,\chi_2 \ (\mathrm{mod}\ q)}J(\chi_1)^{1/2}J(\chi_2)^{1/2}\ll x^{2B_q^*}(\log qx)^5.$$

This completes the proof. □

Remark 6. Though it is true that characters $\chi \pmod q$ with $\chi(a)+\chi(b)=0$ have no contribution to the second term on the right-hand side of Theorem 2, we see in the last proof that such characters may contribute to the last term on the right-hand side. This is also true of Theorem 4.

We move on to the proof of Theorem 4. At first, it might seem that we can obtain this asymptotic formula by summing up Theorem 2 over residues. However this procedure violates the uniformity over q and so instead we take advantage of the "bilinear nature" of the error term $R(x;\chi_1,\chi_2)$ in Lemma 11. With this in mind we prove the following which will be used in its full generality in the proof of Theorem 3.

Lemma 12. *For positive integers c,q and a character $\chi \pmod q$, we have*

$$\sum_{\substack{a=1\\(a(c-a),q)=1}}^{q}\chi(a)=\mu(q^*)\chi^*(c)\frac{\varphi(q)}{\varphi(q^*)}\prod_{\substack{p|q\\p\nmid q^*c}}\frac{p-2}{p-1}=\mathfrak{S}_q(c,\chi),$$

where $\chi^* \pmod{q^*}$ is the primitive character which induces χ.

Proof. By using the Chinese Remainder Theorem and decomposing the character into the product of characters of prime power moduli it is sufficient to prove the lemma in the case where q is a prime power, say $q=p^k$ and $q^*=p^\ell$. If $\ell=0$, then

$$\sum_{\substack{a=1\\(a(c-a),q)=1}}^{q}\chi(a)=\sum_{\substack{a=1\\(a(c-a),p)=1}}^{p^k}1=\sum_{\substack{a=1\\a\not\equiv0,c\ (\mathrm{mod}\ p)}}^{p^k}1=\begin{cases}p^{k-1}(p-1) & (\text{if } p\mid c),\\ p^{k-1}(p-2) & (\text{if } p\nmid c),\end{cases}$$

which coincides with the assertion. If $\ell\geq1$, then we have

$$\sum_{\substack{a=1\\(a(c-a),q)=1}}^{q}\chi(a)=\sum_{\substack{a=1\\(a(c-a),p)=1}}^{p^k}\chi^*(a)=p^{k-\ell}\sum_{\substack{a=1\\a\not\equiv0,c\ (\mathrm{mod}\ p)}}^{p^\ell}\chi^*(a).$$

If $\ell = 1$, then by the orthogonality, this is

$$= p^{k-1}\left(\sum_{a=1}^{p}\chi^*(a) - \sum_{\substack{a=1\\ a\equiv c \,(\mathrm{mod}\ p)}}^{p}\chi^*(a)\right) = \mu(p)\chi^*(c)\frac{\varphi(p^k)}{\varphi(p)},$$

which also coincides with the assertion. If $\ell \geq 2$, then by Theorem 9.4 of [15] we have

$$\sum_{\substack{a=1\\ (a(c-a),q)=1}}^{q}\chi(a) = p^{k-\ell}\left(\sum_{a=1}^{p^\ell}\chi^*(a) - \sum_{\substack{a=1\\ a\equiv c \,(\mathrm{mod}\ p)}}^{p^\ell}\chi^*(a)\right) = 0,$$

which again satisfies the claimed equality. □

Proof of Theorem 4. By using the symmetry between ℓ and m, we have

$$\sum_{\substack{n\leq x\\ n\equiv c \,(\mathrm{mod}\ q)}} G(n) = \sum_{\substack{\ell+m\leq x\\ \ell+m\equiv c \,(\mathrm{mod}\ q)\\ (\ell m,q)=1}}\Lambda(\ell)\Lambda(m) + O\Bigg(\sum_{\substack{\ell+m\leq x\\ \ell+m\equiv c \,(\mathrm{mod}\ q)\\ (m,q)>1}}\Lambda(\ell)\Lambda(m)\Bigg).$$

This error term can be estimated as

$$\sum_{\substack{\ell+m\leq x\\ \ell+m\equiv c \,(\mathrm{mod}\ q)\\ (m,q)>1}}\Lambda(\ell)\Lambda(m) \ll \sum_{\ell\leq x}\Lambda(\ell)\sum_{\substack{m\leq x\\ (m,q)>1}}\Lambda(m) \ll x(\log qx)^2,$$

using the same estimate as in (2.3). Thus it suffices to consider

$$\sum_{\substack{\ell+m\leq x\\ \ell+m\equiv c \,(\mathrm{mod}\ q)\\ (\ell m,q)=1}}\Lambda(\ell)\Lambda(m) = \frac{1}{\varphi(q)^2}\sum_{\substack{a=1\\ (a(c-a),q)=1}}^{q}\sum_{\chi_1,\chi_2 \,(\mathrm{mod}\ q)}\overline{\chi_1(a)\chi_2(c-a)}S(x;\chi_1,\chi_2).$$

We apply Lemma 11 to the right-hand side, and evaluate the resulting expression. Clearly,

$$\frac{1}{\varphi(q)^2}\sum_{\substack{a=1\\ (a(c-a),q)=1}}^{q}\sum_{\chi_1,\chi_2 \,(\mathrm{mod}\ q)}\delta_0(\chi_2)(1+\delta_1(\chi_1)q^{1/2})x(\log qx)^2 \ll x(\log qx)^2.$$

Also,

$$\sum_{\substack{a=1\\ (a(c-a),q)=1}}^{q} 1 = \varphi(q)^2\mathfrak{S}_q(c) \tag{6.5}$$

by Lemma 12 with the principal character. Therefore, it suffices to show that

$$R = \frac{1}{\varphi(q)^2}\sum_{\substack{a=1\\ (a(c-a),q)=1}}^{q}\sum_{\chi_1,\chi_2 \,(\mathrm{mod}\ q)}\overline{\chi_1(a)\chi_2(c-a)}R(x;\chi_1,\chi_2) \ll x^{2B_q^*}(\log qx)^5.$$

We have

$$\sum_{\chi_1,\chi_2 \,(\mathrm{mod}\ q)}\overline{\chi_1(a)\chi_2(c-a)}R(x;\chi_1,\chi_2)$$

$$= \int_0^1 \left(\sum_{\chi_1 \pmod q} \overline{\chi_1(a)} W(\alpha, \chi_1) \right) \left(\sum_{\chi_2 \pmod q} \overline{\chi_2(c-a)} W(\alpha, \chi_2) \right) T(-\alpha) d\alpha.$$

Next the Cauchy–Schwarz inequality gives

$$R \ll \frac{1}{\varphi(q)^2} \sum_{\substack{a=1 \\ (a(c-a),q)=1}}^{q} \int_{-1/2}^{1/2} \left| \sum_{\chi \pmod q} \overline{\chi(a)} W(\alpha, \chi) \right|^2 |T(\alpha)| d\alpha.$$

By the orthogonality of characters, we have

$$\sum_{\substack{a=1 \\ (a(c-a),q)=1}}^{q} \left| \sum_{\chi \pmod q} \overline{\chi(a)} W(\alpha, \chi) \right|^2 \ll \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| \sum_{\chi \pmod q} \overline{\chi(a)} W(\alpha, \chi) \right|^2$$

$$= \varphi(q) \sum_{\chi \pmod q} |W(\alpha, \chi)|^2.$$

Thus, by Lemma 10,

$$R \ll \frac{1}{\varphi(q)} \sum_{\chi \pmod q} J(\chi) \ll x^{2B_q^*} (\log qx)^5.$$

Summing up the above calculations, we complete the proof. □

7. The connection between $S(x; q, a, b)$ and GRH

Consider the Dirichlet series

$$F(s) = F(s; q, a, b) = \sum_{n=1}^{\infty} \frac{G(n; q, a, b)}{n^s},$$

which converges absolutely and is analytic for $\sigma > 2$. Analytic properties of $F(s)$ have been studied by Egami and the third author [6], the first author and Schlage-Puchta [4] (in the case $q = 1$), and by Rüppel [16] (general case). In particular, the connection between $S(x; q, a, b)$ and GRH can be understood through the analytic continuation of $F(s)$. We first find the meromorphic continuation of $F(s)$ via Theorem 2 in the following proposition. This type of result , under GRH was obtained in [6], [16].

Proposition 3. *Assume $B_q < 1$. The function $F(s)$ can be continued meromorphically to the half plane $\sigma > 2B_q$. Its poles in the half plane $\sigma > 2B_q$ are*

(i) a simple pole at $s = 2$ with residue $\varphi(q)^{-2}$,
(ii) a possible pole at $s = \rho_q + 1$ of at most order 1 with residue

$$r(\rho_q) = -\frac{1}{\varphi(q)^2} \frac{1}{\rho_q} \sum_{\substack{\chi \pmod q \\ L(\rho_q, \chi)=0}} (\overline{\chi(a)} + \overline{\chi(b)}) m_\chi(\rho_q),$$

where ρ_q is a zero of $\prod_{\chi \pmod q} L(s, \chi)$ with $0 < \Re\rho_q < 1$ and $m_\chi(\rho_q)$ is the multiplicity of ρ_q as a zero of $L(s, \chi)$.

In particular, assuming DZC *and* $\overline{\chi}(a)+\overline{\chi}(b)\neq 0$ *for all* $\chi \pmod q$ *we obtain*

$$1+B_q=\inf\left\{\sigma_0\geq \frac{3}{2}\;\middle|\; F(s)-\frac{1}{\varphi(q)^2}\frac{1}{s-2}\ \textit{is analytic on}\ \sigma>\sigma_0\right\}.$$

Proof. From Theorem 2, we have

$$S(x;q,a,b)=\frac{x^2}{2\varphi(q)^2}+\sum_{\rho_q} r(\rho_q)\frac{x^{\rho_q+1}}{\rho_q+1}+E(x;q,a,b) \tag{7.1}$$

for $x\geq 1$, where

$$E(x;q,a,b)\ll x^{2B_q}(\log 2qx)^5 \tag{7.2}$$

since $B_q^*\leq B_q$. For $\sigma>2$, we have

$$F(s)=\int_1^\infty u^{-s}dS(u;q,a,b)=s\int_1^\infty S(u;q,a,b)u^{-s-1}du$$

with integration by parts, because $S(x;q,a,b)=0$ for $x<4$. Substitute (7.1) in the right-hand side of the above. The swapping of summation and integration is justified due to absolute convergence. Therefore we have

(7.3)

$$\begin{aligned}F(s)&=\frac{s}{2\varphi(q)^2(s-2)}+\sum_{\rho_q}\frac{r(\rho_q)s}{(\rho_q+1)(s-\rho_q-1)}+s\int_1^\infty E(u;q,a,b)u^{-s-1}du\\&=\frac{1}{\varphi(q)^2(s-2)}+\sum_{\rho_q}\frac{r(\rho_q)}{s-\rho_q-1}+s\int_1^\infty E(u;q,a,b)u^{-s-1}du+C_1(q,a,b),\end{aligned}$$

where

$$C_1(q,a,b)=\frac{1}{2\varphi(q)^2}+\sum_{\rho_q}\frac{r(\rho_q)}{\rho_q+1}$$

and the sum converges due to Lemma 4 to yield a certain constant depending on a,b,q. The sum on the right-hand side of (7.3) converges uniformly on every compact subset of $\mathbb{C}\setminus\{\rho_q+1\}$ and determines a meromorphic function on $\mathbb{C}$. (Since for $|\Im\rho_q|=|\gamma_\chi|>2|\Im s|$, we have $|\rho_q+1-s|\geq|\gamma_\chi-\Im s|\geq|\gamma_\chi|/2$, then the uniform convergence on compact subsets can be justified by Lemma 4. Further note that in Lemma 4, each zero appears with multiplicity.) The first and second term on the right-hand side of (7.3) already give the announced residues of the proposition. Using the estimate (7.2) we see that the integral

$$\int_1^\infty E(u;q,a,b)u^{-s-1}du$$

converges uniformly on the half plane $\sigma>2B_q$ and so it defines an analytic function on $\sigma>2B_q$. This completes the proof of the meromorphic continuation.

For the last assertion, the inequality

$$1+B_q\geq\inf\left\{\sigma_0\geq\frac{3}{2}\;\middle|\;F(s)-\frac{1}{\varphi(q)^2}\frac{1}{s-2}\ \text{is analytic on}\ \sigma>\sigma_0\right\}$$

follows from the above meromorphic continuation, since $1+B_q\geq 2B_q$. We next prove the reverse inequality. If $B_q=1/2$, then the implication is trivial. Hence we can assume that $1/2<B_q<1$ and we have $\max(2B_q,3/2)<1+B_q$, so that we can take $\varepsilon>0$ such that $\max(2B_q,3/2)<1+B_q-\varepsilon$. By the definition of B_q, we can find a zero ρ_q such that $1/2 < B_q - \varepsilon < \Re\rho_q$. Then by the above meromorphic continuation, we have a possible pole of $F(s)$ of residue $r(\rho_q)$ at $\rho_q + 1$. (Note that we do not necessarily have meromorphic continuation on $\sigma > 1 + B_q - \varepsilon$ if $B_q = 1$, since then $B_q + 1 - \varepsilon = 2B_q - \varepsilon < 2B_q$.) By DZC and the assumption that $\Re\rho_q > 1/2$ we have

$$r(\rho_q) = -\frac{1}{\varphi(q)^2}\frac{1}{\rho_q}(\overline{\chi(a)} + \overline{\chi(b)})m,$$

where $m \geq 1$. Since we have assumed that $\overline{\chi}(a) + \overline{\chi}(b) \neq 0$ for all $\chi \pmod q$, this residue is non-zero so that $\rho_q + 1$ is a pole of $F(s)$ in the half plane $\sigma > 1 + B_q - \varepsilon > 3/2$. This implies

$$1 + B_q - \varepsilon \leq \inf\left\{ \sigma_0 \geq \frac{3}{2} \;\middle|\; F(s) - \frac{1}{\varphi(q)^2}\frac{1}{s-2} \text{ is analytic on } \sigma > \sigma_0 \right\}$$

so that on letting $\varepsilon \to 0$ we obtain the reverse inequality. □

Remark 7. We could replace the condition $\chi(a) + \chi(b) \neq 0$ for all $\chi \pmod q$ that occurs naturally in the proof of Proposition 3 by a weaker condition which would be rather difficult to state .

We can now prove Theorem 1.

Proof of Theorem 1. First we prove the assertion (1). If $q > x^{(1-B_q^*)/2}(\log x)$, then the left-hand side of (1.2) is

$$\leq \left(\sum_{\substack{\ell \leq x \\ \ell \equiv a \pmod q}} \Lambda(\ell)\right)\left(\sum_{\substack{m \leq x \\ m \equiv b \pmod q}} \Lambda(m)\right) \ll \frac{(x\log x)^2}{q^2} + (\log x)^2 \ll x^{1+B_q^*}.$$

Also the first term on the right-hand side of (1.2) is $\ll x^{1+B_q^*}$. Thus (1.2) holds trivially. Therefore, we may assume $q \leq x^{(1-B_q^*)/2}(\log x)$. By using Lemma 1 and the second estimate of Lemma 4,

$$\frac{1}{\varphi(q)^2}\sum_{\chi \pmod q}(\overline{\chi(a)} + \overline{\chi(b)})\sum_{\substack{\rho_\chi \\ |\gamma_\chi| \leq x}} \frac{x^{\rho_\chi+1}}{\rho_\chi(\rho_\chi+1)}$$
$$\ll \frac{x^{B_q^*+1}}{\varphi(q)^2}\sum_{\chi \pmod q}\left((\log 2q)^2 + \delta_1(\chi)q^{1/2}(\log q)^2\right) \ll x^{1+B_q^*}.$$

Also, by the second estimate of Lemma 4, we have

$$\frac{1}{\varphi(q)^2}\sum_{\chi \pmod q}(\overline{\chi(a)} + \overline{\chi(b)})\sum_{\substack{\rho_\chi \\ |\gamma_\chi| > x}} \frac{x^{\rho_\chi+1}}{\rho_\chi(\rho_\chi+1)} \ll \frac{x}{\varphi(q)^2}\sum_{\chi \pmod q}\log qx \ll x^{1+B_q^*}.$$

Therefore, by Theorem 2, we obtain

$$S(x;q,a,b) = \frac{x^2}{2\varphi(q)^2} + O(x^{1+B_q^*} + x^{2B_q^*}(\log x)^5).$$

If $1 - B_q^* > 5\log\log x/\log x$, then we see that

$$x^{2B_q^*}(\log x)^5 \leq x^{1+B_q^*} \cdot x^{-(1-B_q^*)}(\log x)^5 \leq x^{1+B_q^*}$$

so that (1.2) follows. Thus we may assume $1 - B_q^* \le 5 \log\log x/\log x$. Further using the assumption on q we find that $q \le x^{(1-B_q^*)/2}(\log x) \le (\log x)^{7/2}$. Thus, by recalling the definition of $\eta = \eta_q(x)$ and choosing $\delta = 1/7$, we obtain

$$\begin{aligned} B_q^* &\le 1 - \eta \le 1 - \frac{c_1}{\max((\log x)^{1/2}, (\log x)^{2/3}(\log\log x)^{1/3})} \\ &= 1 - \frac{c_1}{(\log x)^{2/3}(\log\log x)^{1/3}}, \end{aligned}$$

where $c_1 = c_1(1/7)$ is an absolute constant. This gives

$$x^{2B_q^*}(\log x)^5 \le x^{1+B_q^*}(\log x)^5 \exp\left(-c_1\left(\frac{\log x}{\log\log x}\right)^{\frac{1}{3}}\right) \ll x^{1+B_q^*}.$$

Therefore we always arrive at (1.2).

We next prove the assertion (2). Assume that the formula (1.3) holds, i.e.,

$$S(x;q,a,b) = \frac{x^2}{2\varphi(q)^2} + E_d(x), \quad E_d(x) \ll_{q,\varepsilon} x^{1+d+\varepsilon} \tag{7.4}$$

for arbitrary $\varepsilon > 0$. Now we use this formula to obtain the meromorphic continuation of $F(s)$. In the same manner as in the proof of Proposition 3 we have

$$F(s) - \frac{1}{\varphi(q)^2}\frac{1}{s-2} = s\int_1^\infty E_d(u)u^{-s-1}du + \frac{1}{2\varphi(q)^2}$$

for $\sigma > 2$. Then, by (7.4), the right-hand side gives an analytic function on $\sigma > 1+d$. Therefore under the last assertion of Proposition 3, we have $B_q \le d$ provided DZC, $B_q < 1$ and that $\overline{\chi(a)} + \overline{\chi(b)} \ne 0$ for any $\chi \pmod q$. The supplement for $a = b$ is proved in Section 8. This now completes the proof. □

We next move on to Theorem 3. The strategy is the same as in the proof of Theorem 1. We consider the Dirichlet series

$$F_1(s) = F_1(s;q;c) = \sum_{\substack{n=1 \\ n\equiv c \pmod q}}^\infty \frac{G(n)}{n^s}.$$

The meromorphic continuation of $F_1(s)$ is obtained via Theorem 4.

Proposition 4. *Assume $B_q < 1$. The function $F_1(s)$ can be continued meromorphically to the half plane $\sigma > 2B_q$. Its poles in the half plane $\sigma > 2B_q$ are*

(i) a possible pole at $s = 2$ of order at most 1 with residue $\mathfrak{S}_q(c)$,
(ii) a possible pole at $s = \rho_q + 1$ of at most order 1 with residue

$$r_1(\rho_q) = -\frac{2}{\varphi(q)^2}\frac{1}{\rho_q}\sum_{\substack{\chi \pmod q \\ L(\rho_q,\chi)=0}} m_\chi(\rho_q) \sum_{\substack{a=1 \\ (a(c-a),q)=1}}^q \overline{\chi(a)},$$

where ρ_q is a zero of $\prod_{\chi \pmod q} L(s,\chi)$ with $0 < \Re\rho < 1$.

Proof. This can be proven in the same manner as Proposition 3.

□

We also require the following lemma.

Lemma 13. *For $x \geq 2$ and positive integers c, q, we have*

$$\sum_{\substack{n \leq x \\ n \equiv c \,(\mathrm{mod}\ q)}} J(n) = \frac{\mathfrak{S}_q(c)}{2} x^2 + O(x \log x),$$

where the implied constant is absolute.

Proof. We first consider the case of $(2, q) \nmid c$, i.e. q is even while c is odd. Since there is no even number $n \equiv c \pmod q$ the sum on the left-hand side is $= 0$ since $J(n) = 0$ for odd n. Also, $\mathfrak{S}_q(c) = 0$ by definition in this case. Thus the assertion trivially holds for $(2, q) \nmid c$. We next consider the case $(2, q) \mid c$. We use an expression

$$J(2N) = 2C_2 N \sum_{\substack{d \mid N \\ d\colon \mathrm{odd}}} \frac{\mu(d)^2}{\varphi_2(d)}, \quad \varphi_2(n) = \prod_{p \mid n} (p - 2),$$

to obtain

$$\sum_{\substack{n \leq x \\ n \equiv c \,(\mathrm{mod}\ q)}} J(n) = \sum_{\substack{2N \leq x \\ 2N \equiv c \,(\mathrm{mod}\ q)}} J(2N) = 2C_2 \sum_{\substack{d \leq x \\ d\colon \mathrm{odd}}} \frac{\mu(d)^2 d}{\varphi_2(d)} \sum_{\substack{2dn \leq x \\ 2dn \equiv c \,(\mathrm{mod}\ q)}} n. \tag{7.5}$$

Let $q_1 = q/(2d, q)$. The congruence

$$2dn \equiv c \pmod q$$

has a solution, say $c_1 \pmod{q_1}$ if $(2d, q) \mid c$, and no solution if $(2d, q) \nmid c$. Moreover, the condition $(2d, q) \mid c$ is equivalent to $(d, q) \mid c$ since d is odd and $(2, q) \mid c$. Hence from (7.5), we have

$$\begin{aligned} \sum_{\substack{n \leq x \\ n \equiv c \,(\mathrm{mod}\ q)}} J(n) &= 2C_2 \sum_{\substack{d \leq x/2 \\ d\colon \mathrm{odd} \\ (d,q) \mid c}} \frac{\mu(d)^2 d}{\varphi_2(d)} \sum_{\substack{n \leq x/2d \\ n \equiv c_1 \,(\mathrm{mod}\ q_1)}} n \\ &= \frac{C_2 x^2}{4q} \sum_{\substack{d \leq x/2 \\ d\colon \mathrm{odd} \\ (d,q) \mid c}} \frac{\mu(d)^2 (2d, q)}{d \varphi_2(d)} + O\left(x \sum_{\substack{d \leq x/2 \\ d\colon \mathrm{odd}}} \frac{\mu(d)^2}{\varphi_2(d)} \right). \end{aligned} \tag{7.6}$$

As for the second term on the right-hand side of (7.6), we have

$$\sum_{\substack{d \leq x/2 \\ d\colon \mathrm{odd}}} \frac{\mu(d)^2}{\varphi_2(d)} \leq \prod_{2 < p \leq x} \left(1 + \frac{1}{p - 2}\right) \ll \log x. \tag{7.7}$$

For the first term on the right-hand side of (7.6), we have

$$\frac{C_2}{4q} \sum_{\substack{d \leq x/2 \\ d\colon \mathrm{odd} \\ (d,q) \mid c}} \frac{\mu(d)^2 (2d, q)}{d \varphi_2(d)} = \frac{C_2 \cdot (2, q)}{4q} \sum_{\substack{d\colon \mathrm{odd} \\ (d,q) \mid c}} \frac{\mu(d)^2 (d, q)}{d \varphi_2(d)} + O\left(\sum_{\substack{d > x/2 \\ d\colon \mathrm{odd}}} \frac{\mu(d)^2}{d \varphi_2(d)} \right). \tag{7.8}$$

This remainder term is estimated by using (7.7) as

$$\sum_{\substack{d>x/2\\ d:\,\mathrm{odd}}} \frac{\mu(d)^2}{d\varphi_2(d)} = \sum_{\substack{d>x/2\\ d:\,\mathrm{odd}}} \frac{\mu(d)^2}{\varphi_2(d)} \int_d^\infty \frac{du}{u^2} \le \int_{x/2}^\infty \left(\sum_{\substack{d\le u\\ d:\,\mathrm{odd}}} \frac{\mu(d)^2}{\varphi_2(d)} \right) \frac{du}{u^2} \ll \frac{\log x}{x}. \tag{7.9}$$

On the other hand we have

$$\frac{C_2\cdot(2,q)}{4q} \sum_{\substack{d:\,\mathrm{odd}\\ (d,q)\mid c}} \frac{\mu(d)^2(d,q)}{d\varphi_2(d)} = \frac{C_2\cdot(2,q)}{4q} \prod_{\substack{p>2\\ p\mid(q,c)}} \frac{p-1}{p-2} \prod_{\substack{p>2\\ p\nmid q}} \frac{(p-1)^2}{p(p-2)}.$$

Since the definition of C_2 is

$$C_2 = 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right) = 2\prod_{p>2}\frac{p(p-2)}{(p-1)^2},$$

the right-hand side of the above is equal to

$$= \frac{(2,q)}{2q} \prod_{\substack{p>2\\ p\mid(q,c)}} \frac{p}{p-1} \prod_{\substack{p>2\\ p\mid q\\ p\nmid c}} \frac{p(p-2)}{(p-1)^2} = \frac{\mathfrak{S}_q(c)}{2} \tag{7.10}$$

since $(2,q)\mid c$. Substituting (7.9) and (7.10) into (7.8), we have

$$\frac{C_2}{4q} \sum_{\substack{d\le x\\ d:\,\mathrm{odd}\\ (d,q)\mid c}} \frac{\mu(d)^2(2d,q)}{d\varphi_2(d)} = \frac{\mathfrak{S}_q(c)}{2} + O\left(\frac{\log x}{x}\right).$$

Combining this with (7.7) and (7.6), we obtain the lemma. □

We finally prove Theorem 3.

Proof of Theorem 3. We first prove (1). By using Lemma 1 and Lemma 12,

$$\sum_{\chi\,(\mathrm{mod}\ q)} \mathfrak{S}_q(c,\overline{\chi}) \sum_{\substack{\rho_\chi\\ |\gamma_\chi|\le x}} \frac{x^{\rho_\chi+1}}{\rho_\chi(\rho_\chi+1)} \ll \frac{x^{B_q^*+1}}{\varphi(q)} \sum_{\chi\,(\mathrm{mod}\ q)} \frac{1}{\varphi(q^*)} \sum_{\rho_\chi} \frac{1}{|\rho_\chi(\rho_\chi+1)|}.$$

By Lemma 4, this can be estimated as

$$\ll x^{1+B_q^*} \frac{q^{1/2}(\log 2q)^2}{\varphi(q)} \sum_{\chi\,(\mathrm{mod}\ q)} \frac{1}{\varphi(q^*)} \ll x^{1+B_q^*} \frac{q^{1/2}(\log 2q)^2}{\varphi(q)} \sum_{q^*\mid q} \frac{1}{\varphi(q^*)} \sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\ q^*)} 1$$
$$\le x^{1+B_q^*} \frac{q^{1/2}(\log 2q)^2\tau(q)}{\varphi(q)} \ll x^{1+B_q^*}$$

where the summation symbol with $*$ denotes the sum over primitive characters and $\tau(q)$ denotes the number of divisors of q. Similarly, by Lemma 4, we also have

$$\sum_{\chi\,(\mathrm{mod}\ q)} \mathfrak{S}_q(c,\overline{\chi}) \sum_{\substack{\rho_\chi\\ |\gamma_\chi|>x}} \frac{x^{\rho_\chi+1}}{\rho_\chi(\rho_\chi+1)} \ll \frac{x}{\varphi(q)} \sum_{\chi\,(\mathrm{mod}\ q)} \frac{\log qx}{\varphi(q^*)}$$
$$\ll x\frac{\tau(q)\log qx}{\varphi(q)} \ll x^{1+B_q^*}.$$

Thus Theorem 4 gives

$$\sum_{\substack{n\le x\\ n\equiv c\,(\mathrm{mod}\ q)}} G(n) = \frac{\mathfrak{S}_q(c)}{2}x^2 + O(x^{1+B_q^*} + x^{2B_q^*}(\log qx)^5).$$

Then an argument similar to that for (1) of Theorem 1 gives (1.9).

We next prove (2). Assume that formula (1.10) holds. Then by Lemma 13,

$$\sum_{\substack{n\le x\\ n\equiv c\,(\mathrm{mod}\ q)}} G(n) = \frac{\mathfrak{S}_q(c)}{2}x^2 + E_d(x), \quad E_d(x) \ll_{q,\varepsilon} x^{1+d+\varepsilon} \tag{7.11}$$

for arbitrary $\varepsilon > 0$. As in the case of $F(s)$ we can obtain the meromorphic continuation of $F_1(s)$ to the half plane $\sigma > 1 + d$, which has only one possible pole at $s = 2$. We compare this analytic continuation with Proposition 4. By assumption (a) we have $B_q = \Re\rho_0 < 1$ so that $2B_q < 1 + \Re\rho_0$. Thus by using assumption (b) $F_1(s)$ has a possible pole of order ≤ 1 with residue

$$-\frac{2}{\varphi(q)^2}\frac{m}{\rho_0}\sum_{\substack{a=1\\ (a(c-a),q)=1}}^{q}\overline{\chi(a)},$$

where $m \ge 1$. By Lemma 12, we find that this residue is non-zero provided under the assumption (c): q^* is squarefree, $(c, q^*) = 1$, and yet another assumption

$$2 \nmid q \quad \text{or} \quad 2|q^*c,$$

the last of which is assured by the condition $(2, q) \mid c$ of Theorem 3. Therefore $\rho_0 + 1$ is a pole of $F_1(s)$. By comparing the position of this pole and the analytic continuation we have $1 + B_q \le 1 + d$. This completes the proof. □

8. Exclusion of $B_q = 1$ for $a = b$

In this last section, we exclude the possibility of $B_q = 1$ for $a = b$ in Theorem 1 (2) following an idea of Ruzsa.

Let $G_{a,q}(n) = G(n; q, a, a)$, so that $S(x; q, a, a) = \sum_{n\le x} G_{a,q}(n)$.

Then our assumption (1.3) in Theorem 1 (2) reads

$$S(x; q, a, a) = \frac{x^2}{2\varphi(q)^2} + O_q(x^{1+d+\varepsilon}) \tag{8.1}$$

for some $1/2 \le d < 1$ and any $\varepsilon > 0$. We prove that (8.1) together with DZC implies that $B_q < 1$.

Proof. Step 1. For $|z| < 1$ let

$$F_{a,q}(z) = \sum_{\substack{n\ge 1\\ n\equiv a\ (q)}} \Lambda(n)z^n, \text{ so } F_{a,q}^2(z) = \sum_{n\ge 1} G_{a,q}(n)z^n.$$

Then, since $(1 - z)^{-1} = 1 + z + z^2 + \cdots$, we obtain an identity

$$\frac{1}{1-z}F_{a,q}^2(z) = \sum_{n\ge 1}\left(\sum_{\substack{i+j=n\\ i\ge 1,\, j\ge 0}} G_{a,q}(i)\right)z^n = \sum_{n\ge 1} S(n; q, a, a)z^n.$$

From (8.1) we deduce that

$$\frac{1}{1-z}F_{a,q}^2(z) = \frac{1}{2\varphi(q)^2}\sum_{n\geq 1} n^2 z^n + O_q\left(\sum_{n\geq 1} n^{1+d+\varepsilon}|z|^n\right). \tag{8.2}$$

By the derivative of the geometric series twice, we find that

$$\frac{2}{(1-z)^3} = \sum_{n\geq 1} n^2 z^n + O\left(\sum_{n=1}^{\infty} n|z|^n\right)$$

for $|z| < 1$ so we can evaluate the main term in (8.2). The above error terms are estimated with the help of the following Lemma.

Lemma 14. *For a sequence of positive real numbers $(a_n)_{n=0}^{\infty}$ satisfying*

$$A(x) := \sum_{n\leq x} a_n \leq Cx^\kappa \tag{8.3}$$

for all $x \geq 0$ with some constants $C, \kappa \geq 0$, we have

$$\sum_{n\geq 0} a_n e^{-n/X} \leq C\Gamma(\kappa+1)X^\kappa. \tag{8.4}$$

for any real number $X \geq 1$.

Proof. By partial summation and (8.3), the above series is expressed as

$$\sum_{n\geq 0} a_n e^{-n/X} = \frac{1}{X}\int_0^\infty e^{-u/X} A(u)du.$$

Also by (8.3), we estimate this integral as

$$\leq \frac{C}{X}\int_0^\infty e^{-u/X}u^\kappa du = CX^\kappa \int_0^\infty e^{-u}u^\kappa du = C\Gamma(\kappa+1)X^\kappa.$$

Thus the lemma follows. □

In what follows we work on the circle $|z| = R$ with $R = e^{-1/N}$ for a large positive integer N. Since $\sum_{n\leq x} n^{1+d+\varepsilon} \ll x^{2+d+\varepsilon}$, by Lemma 14, we continue (8.2) by

$$\frac{1}{1-z}F_{a,q}^2(z) = \frac{1}{\varphi(q)^2(1-z)^3} + O_q\left(N^C\right), \quad C := 2+d+\varepsilon < 3$$

on the circle $|z| = R$. Therefore, we obtain

$$F_{a,q}(z)^2 = \frac{1}{(1-z)^2\varphi(q)^2} + O_q\left(|1-z|N^C\right).$$

Note that the second term on the right-hand side is smaller than the first term if

$$|1-z| \leq cN^{-C/3} \tag{8.5}$$

with sufficiently small constant $c > 0$ depending only on q and d. Thus on the arc $|z| = R$ with (8.5), which we may call a major arc on $|z| = R$, we can take the complex square root of the formula for $F_{a,q}(z)$ which yields

$$F_{a,q}(z) = \pm\frac{1}{(1-z)\varphi(q)} + O_q(|1-z|^2 N^C) \tag{8.6}$$

as an asymptotic formula for all z on the major arc. Here, the same sign $\pm$ is kept on the whole major arc since $F_{a,q}(z)$ is continuous. Since $F_{a,q}(z)$ has only non-negative coefficients, the left-hand side in (8.6) is non-negative for the choice $z = e^{-1/N}$. With this choice, the main term in (8.6) is real and therefore must also be non-negative. Therefore, the sign $\pm$ on the right-hand side of (8.6) is $+$.

At this point we notice that we are unable to obtain a similar asymptotic formula when $a \neq b$. The square root step here shows that using this method we can prove the exclusion of $B_q = 1$ in Theorem 1 (2) only when $a = b$.

Step 2. Now we use the kernel

$$K(z) = z^{-N-1} + z^{-N} + \cdots + z^{-2} = z^{-N-1}\frac{1 - z^N}{1 - z}. \tag{8.7}$$

Then by using Cauchy's integral formula, we obtain

$$\psi(N; q, a) = \frac{1}{2\pi i}\int_{|z|=R} F_{a,q}(z)K(z)dz, \quad N = \int_{|z|=R} \frac{1}{1-z}K(z)dz.$$

Thus we deduce that

$$\psi(N; q, a) = \frac{N}{\varphi(q)} + \frac{1}{2\pi i}\int_{|z|=R}\left(F_{a,q}(z) - \frac{1}{(1-z)\varphi(q)}\right)K(z)dz. \tag{8.8}$$

By the second expression of $K(z)$ in (8.7), we see that $K(z) \ll |1-z|^{-1}$. Therefore, on the major arc (8.5) of length $O(N^{-C/3})$ we insert the asymptotic formula (8.6) and the contribution to this integral is $O(N^{C-2C/3}) = O(N^{C/3})$ with $C/3 < 1$.

Step 3. For the rest of the circle the minor arc where $|1 - z| > cN^{-C/3}$ we proceed with the Cauchy–Schwarz inequality and apply the Parseval identity.

By the Parseval identity, we obtain the estimate over the full arc

$$\int_{|z|=R}\left|F_{a,q}(z) - \frac{1}{(1-z)\varphi(q)}\right|^2 dz \ll \sum_{n\geq 0}(\Lambda(n) + 1)^2 e^{-2n/N} \ll N^{1+\varepsilon}, \tag{8.9}$$

where we used the estimate

$$\sum_{n\leq x}(\Lambda(n) + 1)^2 \ll x\log x \ll x^{1+\varepsilon}$$

and Lemma 14.

On the other hand, we use the decay of the kernel $K(z)$ on the minor arc. By using the estimate $K(z) \ll |1 - z|^{-1}$, we start with

$$I = \int_{\substack{|z|=R \\ |1-z|>cN^{-C/3}}} |K(z)|^2 dz \ll \int_{\substack{|z|=R \\ |1-z|>cN^{-C/3}}} \frac{dz}{|1-z|^2}.$$

Now we use the parametrization $z = Re^{i\alpha} = e^{-1/N+i\alpha}$ with $-\pi \leq \alpha \leq \pi$. On the minor arc, we have

$$N^{-C/3} \ll |1 - e^{-1/N+i\alpha}| \ll \left|-\frac{1}{N} + i\alpha\right| \ll \frac{1}{N} + |\alpha|$$

so, by recalling $C < 3$, we have $|\alpha| \geq c_1 N^{-C/3}$ with some small $c_1 > 0$ depending only on q and d. Also, note that

$$|1 - z|^2 = |1 - e^{-1/N}\cos\alpha + ie^{-1/N}\sin\alpha|^2 = 1 + e^{-2/N} - 2e^{-1/N}\cos\alpha. \tag{8.10}$$

By using the inequality of the arithmetic and geometric mean

$$2\left|e^{-1/N}\cos\alpha\right| \leq e^{-2/N} + (\cos\alpha)^2,$$

we find that

$$|1-z|^2 \geq 1-(\cos\alpha)^2 = (\sin\alpha)^2.$$

When $|\alpha| \geq \pi/2$, then $\cos\alpha < 0$ so (8.10) implies $|1-z|^2 \geq 1$. This yields

$$I \ll \int_{c_1 N^{-C/3}<|\alpha|\leq\pi} \frac{d\alpha}{|1-Re^{i\alpha}|^2} \ll \int_{c_1 N^{-C/3}<|\alpha|\leq\pi/2} \frac{d\alpha}{(\sin\alpha)^2} + 2\int_{\pi/2<|\alpha|\leq\pi} d\alpha$$
$$\ll \int_{c_1 N^{-C/3}<|\alpha|\leq\pi/2} \frac{d\alpha}{\alpha^2} + 1 \ll N^{C/3}.$$

Putting everything together the Cauchy–Schwarz inequality gives the minor arc estimate for the integral in (8.8) as

$$\ll \left(\int_{|z|=R} \left|F_{a,q}(z) - \frac{1}{(1-z)\varphi(q)}\right|^2 dz\right)^{1/2} I^{1/2} \ll N^{1/2+C/6+\varepsilon}$$

with $1/2+C/6 < 1$. This together with the major arc estimate allows us to conclude that

$$\psi(N;q,a) - \frac{N}{\varphi(q)} \ll N^{\varepsilon}(N^{C/3} + N^{1/2+C/6}).$$

The exponent of N is < 1 for small $\varepsilon > 0$. In the explicit formula for

$$\psi_{a,q}(N) = \frac{1}{\varphi(q)} \sum_{\chi(q)} \bar{\chi}(a)\psi(N,\chi)$$

that we obtain by inserting the explicit formula for $\psi(N,\chi)$ from Lemma 5 with $T = N$ and assuming DZC, no two terms N^{ρ_χ}/ρ_χ will cancel out for different characters. We conclude that $B_q < 1$. □

References

[1] C. Bauer, Goldbach's conjecture in arithmetic progressions: number and size of exceptional prime moduli, Arch. Math. **108** (2016), 159–172.

[2] G.Bhowmik and I.Z. Ruzsa, Average Goldbach and the Quasi-Riemann Hypothesis, Analysis Mathematica **44(1)** (2018), 51–56.

[3] G. Bhowmik and J.-C. Schlage-Puchta, Mean representation number of integers as the sum of primes, Nagoya Math. J. **200** (2010), 27–33.

[4] G. Bhowmik and J.-C. Schlage-Puchta, Meromorphic continuation of the Goldbach generating function, Funct. Approx. Comment. Math. **45** (2011), 43–53.

[5] J. B. Conrey, The Riemann hypothesis, Notices Amer. Math. Soc. **50** (2003), 341–353.

[6] S. Egami and K. Matsumoto, Convolutions of the von Mangoldt function and related Dirichlet series, in "Number Theory. Sailing on the Sea of Number Theory", S. Kanemitsu and J.-Y. Liu (eds.), Ser. Number Theory and Its Appl. **2**, World Scientific, 2007, pp.1–23.

[7] K. Ford, K. Soundararajan and A. Zaharescu, On the distribution of imaginary parts of zeros of the Riemann zeta function II, Math. Ann. **343** (2009), 487–505.

[8] A. Fujii, An additive problem of prime numbers, Acta Arith. **58** (1991), 173–179.

[9] P. X. Gallagher, A large sieve density estimate near $\sigma = 1$, Invent. Math. **11** (1970), 329–339.

[10] S. M. Gonek, An explicit formula of Landau and its applications to the theory of the zeta-function, Contemp. Math. **143** (1993), 395–413.

[11] A. Granville, Refinements of Goldbach's conjecture, and the generalized Riemann hypothesis, Funct. Approx. Comment. Math. **37** (2007), 159–173; Corrigendum, ibid. **38** (2008), 235–237.

[12] G. H. Hardy and J. E. Littlewood, Some problems of 'Partitio Numerorum'; III: On the expression of a number as a sum of primes, Acta Math. **44** (1923), 1–70.

[13] M.-C. Liu and T. Zhan, The Goldbach problem with primes in arithmetic progressions, in "Analytic Number Theory", Y. Motohashi (ed.), London Math. Soc. Lecture Note Ser. **247**, Cambridge Univ. Press, 1997, pp.227–251.

[14] H. L. Montgomery, *Ten lectures on the interface between analytic number theory and harmonic analysis*, Conference Board of the Mathematical Sciences **84**, Amer. Math. Soc., 1994.
[15] H. L. Montgomery and R. C. Vaughan, Multiplicative Number Theory I, Classical Theory, Cambridge, 2007.
[16] F. Rüppel, Convolutions of the von Mangoldt function over residue classes, Šiauliai Math. Semin. **7** (15) (2012), 135–156.
[17] Y. Suzuki, A mean value of the representation function for the sum of two primes in arithmetic progressions, Int. J. Number Theory **13** (4) (2017), 977–990.

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