Some Automorphism Invariance Properties for Multicontractions

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ABSTRACT. In the theory of row contractions on a Hilbert space, as initiated by Popescu, two important objects are the Poisson kernel and the characteristic function. We determine their behaviour with respect to the action of the group of unitarily implemented automorphisms of the algebra generated by creation operators on the Fock space. The case of noncommutative varieties, introduced recently by Popescu, is also discussed.

1. INTRODUCTION

Among the attempts to extend the dilation theory of contractions on a Hilbert space, as developed in [22], to multivariable operator theory, a most notable achievement is the theory of row contractions (which we will call below *multicontractions*), initiated by the work of Gelu Popescu [9–11]. It has been pursued in the last two decades by Popescu and others (see for instance, [2, 3, 7, 8, 13, 15]); a good presentation of the theory can be found in Popescu's memoir [19]. Popescu's theory is essentially noncommutative; later, starting with [4], interest has developed around the case of commuting multioperators. This presents some specific features; on the other hand, many properties of the commuting case can be obtained from the noncommuting situation. The two recent papers of Popescu [17, 18] have pursued systematically the development of the commutative situation from the noncommutative one, putting it into the more general framework of *constrained multioperators* (see Section 6).

Two objects related to a multicontraction play a significant role in Popescu's theory: the *Poisson kernel* and the *characteristic function*. The Poisson kernel is an important tool used by Popescu in order to prove the von Neumann inequality for row contractions; the characteristic function is essential in the model theory of completely noncoisometric multicontractions, while its commutative counterpart is related to Arveson's curvature (see, for instance, [1, 4, 6, 12, 16, 17]).

On the other hand, as the Sz.-Nagy–Foias theory of single contractions is related to classical function theory in the unit circle, an analoguous role in Popescu's theory is played by algebras generated by creation operators on the Fock space. There is a distinguished group of automorphisms of these algebras, that have been introduced by Voiculescu in [24] (and discussed recently in [7]); they are the non-commutative analogues of the analytic automorphisms of the unit ball. These automorphisms act on multicontractions, and it is interesting to see what is the effect of this transformations on the Poisson kernel and on the characteristic function. The purpose of this paper is to show, firstly, that these objects obey natural rules of transformation, and secondly, that the rules of transformation also extend to the case of constrained multicontractions. In particular, the relation between the transformation rules for commuting and for noncommuting multicontractions is clarified.

The first three sections following the introduction contain mostly preliminary material. The main results, Theorems 5.1 and 6.2, are proved in Sections 5 and 6. In connection to constrained objects, the last section discusses invariant ideals of the noncommutative Toeplitz algebra.

The method of proof uses, in order to avoid complicated computations, the machinery of Redheffer products. This machinery may seem unfamiliar, but it provides a simple and short way to reach the main results. Up to a certain point, using Redheffer products is equivalent to composing J-unitary operators, but there is a slightly larger level of generality that happens to be important in our context. For an illuminating discussion of these facts, see the first two sections of [25].

2. PRELIMINARIES

2.1. The Fock space A main object of study is formed by the Fock space and the non-commutative Toeplitz algebras that act on it. We will follow mainly the work of Popescu [9, 10, 12, 13], as well as [7].

In the whole paper we will fix a positive integer n. We denote by $\mathbb{F}_n^+$ the free semigroup with the n generators $1, \dots, n$ and unit $\varnothing$. An element $w = i_1 \cdots i_k \in \mathbb{F}_n^+$ is called a *word* in the letters $1, \dots, n$, and its *length* is $|w| = k$.

If $A = (A_1, \dots, A_n) \in \mathcal{L}(\mathcal{H})^n$ is a (not necessarily commuting) multioperator, we denote $A_\varnothing = I_{\mathcal{H}}$ and, if, $w = i_1 \cdots i_k \in \mathbb{F}_n^+$, then $A_w = A_{i_1} \cdots A_{i_k} \in \mathcal{L}(\mathcal{H})$.

Consider an n-dimensional complex Hilbert space $\mathbf{h}_n$, with basis vectors $e_1, \dots, e_n$. The *full Fock space* is then

$$\mathfrak{F}_n = \bigoplus_{k \geq 0} \mathbf{h}_n^{\otimes k}$$

where $\mathbf{h}_n^{\otimes 0} = \mathbb{C}\mathbf{1}$ and $\mathbf{h}_n^{\otimes k}$ is the tensor product of k copies of $\mathbf{h}_n$. An orthonormal basis of $\mathfrak{F}_n$ is given by $(e_w)_{w \in \mathbb{F}_n^+}$, where $e_\varnothing = \mathbf{1} \in \mathbf{h}_n^{\otimes 0}$, while, if $w = i_1 \cdots i_k \in \mathbb{F}_n^+$, then $e_w = e_{i_1} \otimes \cdots \otimes e_{i_k} \in \mathbf{h}_n^{\otimes k}$.

The left creation operators $L_i \in \mathcal{L}(\mathfrak{F}_n)$, $i = 1, \dots n$, are defined by

$$L_i \xi = e_i \otimes \xi, \quad \xi \in \mathfrak{F}_n.$$

The norm closed algebra generated by $L_1, \dots, L_n$ is denoted by $\mathfrak{l}_n$, and the weakly closed algebra by $\mathfrak{L}_n$; these algebras have been introduced (with a different notation) in [12].

Similarly, we have right creation operators R_i given by

$$R_i \xi = \xi \otimes e_i,$$

while the norm closed and weakly closed algebras they generate are denoted by $\mathfrak{r}_n$ and $\mathfrak{R}_n$ respectively. Each of the algebras $\mathfrak{L}_n$ and $\mathfrak{R}_n$ is the commutant of the other.

We can write any $f \in \mathfrak{L}_n$ as a formal series $f = \sum_w \hat{f}_w L_w$. For each $r < 1$ the series

$$f_r := \sum_w \hat{f}_w r^{|w|} L_w \tag{2.1}$$

converges uniformly, and thus $f_r \in \mathfrak{l}_n$; then $f = \text{SOT} - \lim_{r \to 1} f_r$. A similar statement is valid for $\mathfrak{R}_n$.

The *flip* operator is the involutive unitary $F \in \mathcal{L}(\mathfrak{F}_n)$ which acts on simple tensors by reversing the order of the components:

$$F(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}) = e_{i_n} \otimes \cdots \otimes e_{i_2} \otimes e_{i_1}.$$

We have then $R_i = F L_i F$.

If $\mathcal{E}, \mathcal{E}_*$ are Hilbert spaces, then a linear operator $M : \mathfrak{F}_n \otimes \mathcal{E} \to \mathfrak{F}_n \otimes \mathcal{E}_*$ is called *multianalytic* if

$$M(L_i \otimes I_{\mathcal{E}}) = (L_i \otimes I_{\mathcal{E}_*})M \quad \forall i = 1, \dots, n.$$

M is then uniquely determined by the "coefficients" $m_w \in \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$, defined by

$$\langle m_{\tilde{w}} k, k' \rangle = \langle M(\mathbf{1} \otimes k), e_w \otimes k_* \rangle, \quad k \in \mathcal{E}, k_* \in \mathcal{E}_*, w \in \mathbb{F}_n^+,$$

where $\tilde{w}$ is the reverse of w, i.e., $\tilde{w} = i_k \cdots i_1$ if $w = i_1 \cdots i_k$; we can then associate with M the formal Fourier expansion

$$\hat{M}(R_1, \dots, R_n) = \sum_{w \in \mathbb{F}_n^+} R_w \otimes m_w. \tag{2.2}$$

2.2. Redheffer products Several computations that appear in the sequel can be gathered in a simple uniform framework if we use the formalism of Redheffer products. The basic reference is [20]; we will follow the exposition in [23].

Suppose

$$\mathbf{L} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \mathbf{L}_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$$

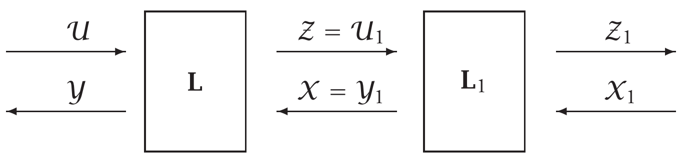
are bounded operators mapping $\mathcal{X} \oplus \mathcal{U}$ (respectively $\mathcal{X}_1 \oplus \mathcal{U}_1$) into $\mathcal{Y} \oplus \mathcal{Z}$ (respectively $\mathcal{Y}_1 \oplus \mathcal{Z}_1$); also $\mathcal{U}_1 = \mathcal{Z}$ and $\mathcal{X} = \mathcal{Y}_1$. Under the assumption

$$(*) \qquad I - B_1 C \text{ is invertible}$$

it follows that $I - CB_1$ is also invertible, and we define the *Redheffer product* by

$$\mathbf{M} = \mathbf{L} \circ \mathbf{L}_1 = \begin{pmatrix} A(I - B_1 C)^{-1} A_1 & B + A(I - B_1 C)^{-1} B_1 D \\ C_1 + D_1 C(I - B_1 C)^{-1} A_1 & D_1 (I - CB_1)^{-1} D \end{pmatrix}. \tag{2.3}$$

$\mathbf{M}$ is an operator from $\mathcal{X}_1 \oplus \mathcal{U}$ to $\mathcal{Y} \oplus \mathcal{Z}_1$. It is useful to visualize the interlacing of spaces by input-output boxes, in a manner suggested by system theory:



We will write also $\beta_{\mathbf{L}}(A_1, B_1)$ and $\alpha_{\mathbf{L}}(B_1)$ for the entries in the first row of $\mathbf{L} \circ \mathbf{L}_1$ (as given by (2.3)).

The basic properties of the Redheffer product are gathered in the following proposition. In its statement it is tacitly assumed that condition (*) is satisfied, when necessary.

Proposition 2.1. (i) *The identities matrices (on the corresponding spaces) act as unit elements also for the Redheffer products.*

(ii) *If* $\mathbf{L}$ *is invertible,* $\mathbf{L}_1 = \mathbf{L}^{-1}$, *and one can form* $\mathbf{L} \circ \mathbf{L}_1$, *then* $\mathbf{L}_1$ *is also the inverse of* $\mathbf{L}$ *with respect to the Redheffer product.*

(iii) *The Redheffer product is associative: if* $\mathbf{L}, \mathbf{L}_1, \mathbf{L}_2$ *are given, and all Redheffer products in* (2.4) *can be formed, then*

$$\mathbf{L} \circ (\mathbf{L}_1 \circ \mathbf{L}_2) = (\mathbf{L} \circ \mathbf{L}_1) \circ \mathbf{L}_2. \tag{2.4}$$

(iv) $\mathbf{L}, \mathbf{L}_1$ *contractions* (*isometries, coisometries, unitaries*) *imply* $\mathbf{L} \circ \mathbf{L}_1$ *contraction* (*isometry, coisometry, unitary respectively*). *In particular, if* $\mathbf{L}$ *and* B_1 *are contractions, then* $\alpha_{\mathbf{L}}(B_1)$ *is also a contraction.*

A particular case that will be useful is $\mathcal{Z}_1 = \{0\}$ (and thus $C_1 = D_1 = 0$).

In connection with Redheffer products, we need also a lemma concerning the structure of unitary 2×2 matrices. To state it, remember that if $\mathcal{E}_1, \mathcal{E}_2$ are two Hilbert spaces, and $C : \mathcal{E}_1 \to \mathcal{E}_2$ is a contraction, one defines the *defect operator* $D_C = (\mathbf{1}_{\mathcal{E}_1} - C^*C)^{1/2} \in \mathcal{L}(\mathcal{E}_1)$ and the *defect space* $\mathcal{D}_C = \overline{D_C\mathcal{E}_1} \subset \mathcal{E}_1$.

Lemma 2.2. *A 2×2 operator matrix from $\mathcal{E}_1 \oplus \mathcal{E}_2$ to $\mathcal{E}_1' \oplus \mathcal{E}_2'$ that has A^* as its $(2,1)$ entry, while the $(1,1)$ entry has dense range, has the form*

$$\mathbf{J} = \begin{pmatrix} Z_* D_{A^*} & -Z_* A Z^* \\ A^* & D_A Z^* \end{pmatrix}, \tag{2.5}$$

$Z_* : \mathcal{D}_{A^*} \to \mathcal{E}_1'$ and $Z : \mathcal{D}_A \to \mathcal{E}_2$ being unitary operators.

3. Automorphisms

The analytic automorphisms of the unit ball $\mathbb{B}^n$ act by composition on any Hilbert space of functions on $\mathbb{B}^n$. There exist corresponding unitarily implemented automorphisms on the non-commutative Toeplitz algebras on the Fock space.

3.1. The commutative case: automorphisms of the unit ball There are two different descriptions of the automorphisms of the unit ball $\mathbb{B}^n$. In view of further extensions, we identify elements in $\mathbb{B}^n$ with row $1 \times n$ contractive matrices. Naturally, the action of an $n \times n$ matrix on such an element will be done by multiplication on the right.

First form. Start with the group $U(1,n)$ of $(n+1) \times (n+1)$ matrices X that are J-unitary, where $J = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix}$; that is, $X^*JX = J$. According to the decomposition $\mathbb{C}^{n+1} = \mathbb{C} \oplus \mathbb{C}^n$, one writes $X = \begin{pmatrix} x & y \\ z^t & X' \end{pmatrix}$; note that with these conventions x is a scalar, while y and z are row matrices. Accordingly, there is a corresponding map $\phi_X : \mathbb{B}^n \to \mathbb{B}^n$, defined by

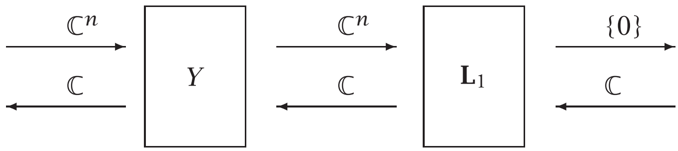
$$\phi_X(\lambda) = (x - \lambda z^t)^{-1}(\lambda X' - y). \tag{3.1}$$

Then the map $X \mapsto \phi_X$ is a group antihomomorphism from $U(1,n)$ to the group af automorphisms of $\mathbb{B}^n$ (the "anti" being due to our decision to see elements of $\mathbb{B}^n$ as row matrices and write the action of the group on the right); this antihomorphism is onto, and its kernel is formed by scalar unitaries.

Second form. A variant of (3.1) which uses a unitary instead of a J-unitary matrix is more natural in the context of Redheffer products. Namely, if $Y = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a unitary $(n+1) \times (n+1)$ matrix, then we can consider the map (see (2.3))

$$\alpha_Y(\lambda) = b + a\lambda(I - c\lambda)^{-1}d. \tag{3.2}$$

The corresponding diagram is



where $\mathbf{L}_1 = \begin{pmatrix} 1 & \lambda \\ 0 & 0 \end{pmatrix} : \mathbb{C} \oplus \mathbb{C}^n \to \mathbb{C} \oplus \{0\}$.

By Proposition 2.1 (iv), for λ contractive, $\alpha_Y(\lambda)$ is also contractive. Thus α_Y is an analytic map from $\mathbb{B}^n$ to $\mathbb{B}^n$; it is even an automorphism, since again Proposition 2.1 (ii) implies $\alpha_Y^{-1} = \alpha_{Y^*}$.

The passage from (3.1) to (3.2) is done by the formulas

$$a = x^{-1}, \qquad b = -x^{-1}y, \qquad c = x^{-1}z^t, \qquad d = X' - x^{-1}z^t y. \tag{3.3}$$

These formulas can be inverted, provided $a \neq 0$.

Working in the context of Redheffer products, (3.2) is more convenient; however, (3.1) is related to the automorphisms in Subsection ***3.2***. Also, while in (3.1) any J-unitary produces an automorphism, in (3.2) we must require $a \neq 0$.

3.2. The noncommutative case: the Fock space We shall introduce some facts and notations from [7]; in Section 4 therein the automorphisms of the algebra $\mathfrak{L}_n$ are investigated. It is shown that all contractive automorphisms of $\mathfrak{L}_n$ are actually unitarily implemented, and they are also automorphisms of the C^*-algebra $\mathfrak{l}_n$.

A detailed description of these automorphisms can be obtained following [24]. As in Subsection ***3.1***, take $X \in U(1,n)$, $X = \begin{pmatrix} x & y \\ z^t & X' \end{pmatrix}$. Write also $L[\zeta] = \sum_{i=1}^n \zeta_i L_i$ for $\zeta \in \mathbb{C}^n$. Then there is an automorphism Φ_X of $\mathfrak{L}_n$ such that the restriction to the generators is given by

$$\Phi_X(L[\zeta]) = (xI - L[z])^{-1}(L[X'\zeta] - (\zeta \cdot y^t)I). \tag{3.4}$$

This automorphism is implemented by a unitary $U_X \in \mathcal{L}(\mathfrak{F}_n)$, which satisfies

$$U_X(Ae_\varnothing) = \Phi_X(A)(xI - L[z])^{-1}e_\varnothing \tag{3.5}$$

for all $A \in \mathfrak{L}_n$; this means that $\Phi_X(A) = U_X A U_X^*$ for all $A \in \mathfrak{L}_n$. The map $X \mapsto \Phi_X$ from $U(1,n)$ to the automorphisms of $\mathfrak{L}_n$ has as image all unitarily implemented automorphisms (which actually coincide with all contractive automorphisms), and its kernel consists of the scalar matrices xI_{n+1}, with $x \in \mathbb{T}$.

To make the connection with ***3.1***, apply (3.4) for ζ a basis vector; one obtains

$$\Phi_X(L_i) = (xI - L[z])^{-1}\Big(\sum_{j=1}^n X'_{ji}L_j - y_i I\Big),$$

while writing (3.1) on coordinates yields

$$(\phi_X(\lambda))_i = \Big(x - \sum_{j=1}^{n} z_j\lambda_j\Big)^{-1} \Big(\sum_{j=1}^{n} \lambda_j X'_{ji} - y_i\Big).$$

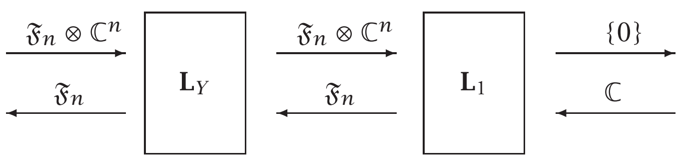
Consequently, (3.4) can be obtained by formally replacing λ_i in (3.1) with L_i.

One can interpret also these automorphisms in terms of Redheffer products. Suppose that, as in ***3.1***, one defines the unitary matrix $Y = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{L}(\mathbb{C} \oplus \mathbb{C}^n)$ by (3.3); denote by $\iota : \mathbb{C} \to \mathfrak{F}_n$ the inclusion map that sends 1 to $e_\varnothing$, and

$$\mathbf{L}_Y = \begin{pmatrix} aI_{\mathfrak{F}_n} & b \otimes I_{\mathfrak{F}_n} \\ c \otimes I_{\mathfrak{F}_n} & d \otimes I_{\mathfrak{F}_n} \end{pmatrix} : \mathfrak{F}_n \oplus (\mathfrak{F}_n \otimes \mathbb{C}^n) \to \mathfrak{F}_n \oplus (\mathfrak{F}_n \otimes \mathbb{C}^n)$$

$$\mathbf{L}_1 = \begin{pmatrix} \iota & L \\ 0 & 0 \end{pmatrix} : \mathbb{C} \oplus (\mathfrak{F}_n \otimes \mathbb{C}^n) \to (\mathfrak{F}_n) \oplus \{0\}$$

(we have implicitely used the fact that we can identify $\mathfrak{F}_n \otimes \mathbb{C}^n$ with $\mathfrak{F}_n^n$).



Then it follows immediately from the discussion above that:

$$\alpha_{\mathbf{L}_Y}(L) = \Phi_X(L) = U_X L U_X^*. \tag{3.6}$$

Moreover, from (3.5) and (3.3) we have

$$\begin{aligned} U_X(e_\varnothing) &= x^{-1}\Big(I - x^{-1}\sum_{j=1}^{n} z_j L_j\Big)^{-1} e_\varnothing \\ &= a\Big(I - \sum_{j=1}^{n} c_j L_j\Big)^{-1} e_\varnothing, \end{aligned}$$

whence

$$\beta_{\mathbf{L}_Y}(\iota, L) = U_X \iota.$$

Let us also note that, if X and Y are related by (3.3), then

$$\Phi_X^{-1}(L) = \Phi_{X^{-1}}(L) = \alpha_{\mathbf{L}_{Y^*}}(L). \tag{3.7}$$

Suppose now that we want to obtain similar relations with R instead of L. We may immediately note that $R_i = FL_iF$, which leads to

$$\alpha_{\mathbf{L}_Y}(R) = FU_X FRFU_X^* F.$$

But we can actually say more. Since $\mathfrak{R}_n$ is the commutant of $\mathfrak{L}_n$, it follows that $U_X B U_X^* \in \mathfrak{R}_n$ for all $B \in \mathfrak{R}_n$; this can be made precise using the following lemma.

Lemma 3.1. *For all $X \in U(1,n)$, we have $U_X F = FU_X$.*

Proof. We have to check the relation on simple tensors e_w, where $w = i_1 \cdots i_k$. Denote also $\tilde{w} = i_k \cdots i_1$. According to (3.5), we have

$$\begin{aligned} U_X F(e_w) &= U_X(e_{\tilde{w}}) = U_X(L_{i_k} \cdots L_{i_1} e_\varnothing) \\ &= \Phi_X(L_{i_k} \cdots L_{i_1})(xI - L[z])^{-1} e_\varnothing \\ &= \Phi_X(L_{i_k}) \cdots \Phi_X(L_{i_1})(xI - L[z])^{-1} e_\varnothing. \end{aligned}$$

If we write $\frac{1}{x} z = (a_1, \ldots, a_n)$, we have

$$(xI - L_z)^{-1} = x \sum_{\text{all words } j = j_1 \ldots j_s} a_{j_1} L_{j_1} \ldots a_{j_s} L_{j_s},$$

where the sum is norm convergent.

To simplify notations, define the map $\tilde{F} : \mathfrak{L}_n \to \mathfrak{L}_n$ by the formula $\tilde{F}(L_v) = L_{\tilde{v}}$. Then:

1. $\tilde{F}(AB) = \tilde{F}(B)\tilde{F}(A)$;
2. $\tilde{F}(L_v) = L_v$ if v has length 0 or 1;
3. $\tilde{F}((xI - L[z])^{-1}) = (xI - L[z])^{-1}$;
4. $FL_v e_\varnothing = \tilde{F}(L_v) e_\varnothing$.

Therefore, applying (3.4), we have

$$\begin{aligned} &\Phi_X(L_{i_k}) \cdots \Phi_X(L_{i_1})(xI - L[z])^{-1} e_\varnothing \\ &\quad = (xI - L[z])^{-1}(L[X'\zeta_{i_k}] - \langle \zeta_{i_k}, y\rangle I)(xI - L[z])^{-1} \ldots \\ &\qquad\quad (xI - L[z])^{-1}(L[X'\zeta_{i_1}] - \langle \zeta_{i_1}, y\rangle I)(xI - L[z])^{-1} e_\varnothing \\ &\quad = \tilde{F}[(xI - L[z])^{-1}(L[X'\zeta_{i_1}] - \langle \zeta_{i_1}, y\rangle I)(xI - L[z])^{-1} \ldots \\ &\qquad\quad (xI - L[z])^{-1}(L[X'\zeta_{i_k}] - \langle \zeta_{i_k}, y\rangle I)(xI - L[z])^{-1}] e_\varnothing \\ &\quad = F(xI - L[z])^{-1}(L[X'\zeta_{i_1}] - \langle \zeta_{i_1}, y\rangle I)(xI - L[z])^{-1} \ldots \\ &\qquad\quad (xI - L[z])^{-1}(L[X'\zeta_{i_k}] - \langle \zeta_{i_k}, y\rangle I)(xI - L[z])^{-1} e_\varnothing \\ &\quad = F\Phi_X(L_{i_1}) \cdots \Phi_X(L_{i_k})(xI - L[z])^{-1} e_\varnothing \\ &\quad = FU_X(e_w). \end{aligned}$$

The lemma is proved. ∎

As a consequence, $FU_XF = U_X$, and we have

$$\alpha_{\mathbf{L}_Y}(R) = U_X R U_X^*.$$

4. Multicontractions

Suppose $T = (T_1, \dots, T_n) \in \mathcal{L}(\mathcal{H})^n$ is a multicontraction; that is,

$$\sum_{i=1}^{n} T_i T_i^* \le 1_{\mathcal{H}}.$$

This is the same as requiring the row operator $T = (T_1 \ \cdots \ T_n) : \mathcal{H}^n \to \mathcal{H}$ to be a contraction. (We will denote by the same letter T the multioperator and the associated row contraction.) Accordingly, we have the operators $D_T = (\mathbf{1}_{\mathcal{H}^n} - T^*T)^{1/2}$ and $D_{T^*} = (\mathbf{1}_{\mathcal{H}} - TT^*)^{1/2}$, and the spaces $\mathcal{D}_T = \overline{D_T \mathcal{H}^n} \subset \mathcal{H}^n$, $\mathcal{D}_{T^*} = \overline{D_{T^*}\mathcal{H}} \subset \mathcal{H}$. If the row operator T is a strict contraction, we will say that T is a *strict multicontraction*.

A functional calculus for multicontractions has been developed by Popescu (see [12–14]). First, for a general multicontraction T we have an $\mathfrak{l}_n$-functional calculus; it is the unique completely contractive homomorphism $\rho : \mathfrak{l}_n \to \mathcal{L}(\mathcal{H})$ for which $\rho(L_i) = T_i$. This homomorphism can be extended to $\mathfrak{L}_n$ in an important particular case. Namely, T is called *completely noncoisometric (c.n.c.)* if there is no $h \in \mathcal{H}$, $h \neq 0$, such that

$$\sum_{|w|=k} \|T_w^* h\|^2 = \|h\|^2 \quad \text{for all } k \ge 0.$$

If T is c.n.c., then ρ can be extended to a completely contractive homomorphism defined on $\mathfrak{L}_n$, that we will denote with the same letter, $\rho : \mathfrak{L}_n \to \mathcal{L}(\mathcal{H})$. If $f \in \mathfrak{L}_n$, then $f_r \in \mathfrak{l}_n$, and we may apply ρ to obtain

$$\rho(f_r) = \sum_w \hat{f}_w r^{|w|} T_w$$

with the sum on the right converging absolutely. If T is c.n.c., then we have also

$$\rho(f) = \mathrm{SOT} - \lim_{r \to 1} \rho(f_r).$$

Similar results are valid for $\mathfrak{r}_n$ and $\mathfrak{R}_n$, the corresponding functional calculus being denoted by ρ'.

The next definition introduces two basic objects that appear in Popescu's theory of multicontractions (see [15, 17]).

Definition 4.1. Suppose T is a multicontraction. Then:

(a) The *Poisson kernel* K_T is the operator $K_T : \mathcal{H} \to \mathfrak{F}_n \otimes \mathcal{D}_{T^*}$ defined by

$$K_T h = \sum_w e_w \otimes D_{T^*} T_w^* h.$$

(b) The *characteristic function* Θ_T is the multianalytic operator

$$\Theta_T : \mathfrak{F}_n \otimes \mathcal{D}_T \to \mathfrak{F}_n \otimes \mathcal{D}_{T^*}$$

having the formal Fourier representation

$$\hat{\Theta}_T(R_1, \cdots, R_n) = -I_{\mathfrak{F}_n} \otimes T + (I_{\mathfrak{F}_n} \otimes D_{T^*})\Big(I_{\mathfrak{F}_n \otimes \mathcal{H}} - \sum_{i=1}^n R_i \otimes T_i^*\Big)^{-1} \times [R_1 \otimes I_{\mathcal{H}}, \dots, R_n \otimes I_{\mathcal{H}}](I_{\mathfrak{F}_n} \otimes D_T) \mid \mathfrak{F}_n \otimes \mathcal{D}_T. \tag{4.1}$$

The following proposition gathers several results from [17].

Proposition 4.2. (i) *The Poisson kernel and the characteristic function are contractions, and* $K_T K_T^* + \Theta_T \Theta_T^* = I_{\mathfrak{F}_n \otimes \mathcal{D}_{T^*}}$.

(ii) *If we define, for* $0 < r \le 1$,

$$\begin{aligned} K_{T,r} h &= \sum_w r^{|w|} e_w \otimes D_{T^*} T_w^* h \\ &= (I_{\mathfrak{F}_n} \otimes D_{T^*})\Big(I_{\mathfrak{F}_n \otimes \mathcal{H}} - \sum_{i=1}^n r R_i \otimes T_i^*\Big)^{-1} (e_\varnothing \otimes h), \end{aligned}$$

then

$$K_T = \mathrm{SOT} - \lim_{r \to 1} K_{T,r}. \tag{4.2}$$

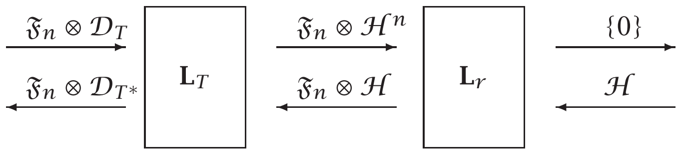
(iii) *If we replace* R_i *with* rR_i, $0 < r < 1$, *in* (4.1), *then the inverse in the right hand side exists, the equation can be used to define* $\Theta_T(rR)$, *and*

$$\Theta_T = \mathrm{SOT} - \lim_{r \to 1} \Theta_T(rR). \tag{4.3}$$

We can interpret the Poisson kernel and the characteristic function by means of the Redheffer product (see Section **2.2**). Remember that $\iota : \mathbb{C} \to \mathfrak{F}_n$ is the embedding $z \mapsto z e_\varnothing$. Take $r < 1$, and define then

$$\mathbf{L}_T = \begin{pmatrix} I_{\mathfrak{F}_n} \otimes D_{T^*} & -I_{\mathfrak{F}_n} \otimes T \\ I_{\mathfrak{F}_n} \otimes T^* & I_{\mathfrak{F}_n} \otimes D_T \end{pmatrix} : (\mathfrak{F}_n \otimes \mathcal{H}) \oplus (\mathfrak{F}_n \otimes \mathcal{D}_T) \to (\mathfrak{F}_n \otimes \mathcal{D}_{T^*}) \oplus (\mathfrak{F}_n \otimes \mathcal{H}^n), \tag{4.4}$$

$$\mathbf{L}_r = \begin{pmatrix} \iota \otimes I_{\mathcal{H}} & rR \otimes I_{\mathcal{H}} \\ 0 & 0 \end{pmatrix} : \tag{4.5}$$
$$\mathcal{H} \oplus (\mathfrak{F}_n \otimes \mathcal{H}^n) \to (\mathfrak{F}_n \otimes \mathcal{H}) \oplus \{0\}.$$



Then from (2.3) it follows that

$$\mathbf{L}_T \circ \mathbf{L}_r = \begin{pmatrix} K_{T,r} & \Theta_T(rR) \\ 0 & 0 \end{pmatrix} : \mathcal{H} \oplus (\mathfrak{F}_n \otimes \mathcal{D}_T) \to (\mathfrak{F}_n \otimes \mathcal{D}_{T^*}) \oplus \{0\}.$$

Otherwise stated,

$$\begin{aligned} \Theta_T(rR) &= \alpha_{\mathbf{L}_T}(rR \otimes I_{\mathcal{H}}), \\ K_{T,r} &= \beta_{\mathbf{L}_T}(\iota \otimes I_{\mathcal{H}}, rR \otimes I_{\mathcal{H}}). \end{aligned} \tag{4.6}$$

5. Multicontractions and automorphisms

Recall now (Subsection ***3.2***) that for any $X \in U(1,n)$ there exists a corresponding unitarily implemented automorphism Φ_X of the non-commutative Toeplitz algebras. By using the functional calculus ρ (see Section 4), we can extend the action of the automorphisms Φ_X to a multicontraction T. This is done by defining

$$\Phi_X(T) = \rho(\Phi_X(L)),$$

and it follows from (3.6) that we have then also

$$\Phi_X(T) = \alpha_{\mathbf{L}_Y}(T),$$

where, as usually, Y is connected to X by formulas (3.3). Since the functional calculus ρ is completely contractive, $\Phi_X(T)$ is also a multicontraction. According to (3.7), we have also

$$\Phi_X^{-1}(T) = \alpha_{\mathbf{L}_{Y^*}}(T). \tag{5.1}$$

The main result of this section is given by the next theorem. It shows the behaviour of the characteristic function and the Poisson kernel with respect to the unitarily implemented automorphisms Φ_X.

Theorem 5.1. *For each $X \in U(1,n)$ there exist unitary operators $Z : \mathcal{D}_{\Phi_X^{-1}(T)} \to \mathcal{D}_T$ and $Z_* : \mathcal{D}_{\Phi_X^{-1}(T)^*} \to \mathcal{D}_{T^*}$, such that:*

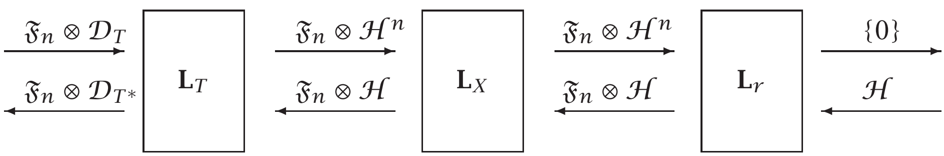
(i) $\Theta_{\Phi_X^{-1}(T)} = (U_X \otimes Z_*^*)\Theta_T(U_X^* \otimes Z)$.

(ii) $K_{\Phi_X^{-1}(T)} = (U_X \otimes Z_*^*)K_T$.

Proof. Let us define $\mathbf{L}_T$ and $\mathbf{L}_r$ by formulas (4.4) and (4.5) respectively, and $\mathbf{L}_X$ by

$$\mathbf{L}_X = \begin{pmatrix} a \otimes I_{\mathfrak{F}_n \otimes \mathcal{H}} & b \otimes I_{\mathfrak{F}_n \otimes \mathcal{H}} \\ c \otimes I_{\mathfrak{F}_n \otimes \mathcal{H}} & d \otimes I_{\mathfrak{F}_n \otimes \mathcal{H}} \end{pmatrix} : \\ (\mathfrak{F}_n \otimes \mathcal{H}) \oplus (\mathfrak{F}_n \otimes \mathcal{H}^n) \to (\mathfrak{F}_n \otimes \mathcal{H}) \oplus (\mathfrak{F}_n \otimes \mathcal{H}^n)$$

where a, b, c, d are related to X by formulas (3.3).



We want to apply the associativity of the Redheffer product, as stated in Proposition 2.1 (iii):

$$(\mathbf{L}_T \circ \mathbf{L}_X) \circ \mathbf{L}_r = \mathbf{L}_T \circ (\mathbf{L}_X \circ \mathbf{L}_r). \tag{5.2}$$

First, we have

$$\mathbf{L}_X \circ \mathbf{L}_r = \begin{pmatrix} (U_X \iota) \otimes I_{\mathcal{H}} & (U_X r R U_X^*) \otimes I_{\mathcal{H}} \\ 0 & 0 \end{pmatrix}.$$

It follows that

$$\begin{aligned} \alpha_{\mathbf{L}_T}((U_X R U_X^*) \otimes I_{\mathcal{H}}) &= -I_{\mathfrak{F}_n} \otimes T + (I_{\mathfrak{F}_n} \otimes D_{T^*})\Big(I_{\mathfrak{F}_n \otimes \mathcal{H}} - \sum_{i=1}^n (U_X r R_i U_X^*) \otimes T_i^*\Big)^{-1} \\ &\qquad [U_X r R_1 U_X^* \otimes I_{\mathcal{H}}, \dots, U_X r R_n U_X^* \otimes I_{\mathcal{H}}](I_{\mathfrak{F}_n} \otimes D_T) \big| \mathfrak{F}_n \otimes \mathcal{D}_T \\ &= (U_X \otimes I_{\mathcal{D}_{T^*}})\Theta_T(rR)(U_X^* \otimes I_{\mathcal{D}_T}) \end{aligned}$$

and

$$\begin{aligned}
&\beta_{\mathbf{L}_T}((U_X\iota)\otimes I_{\mathcal{H}},(U_X rRU_X^*)\otimes I_{\mathcal{H}})\\
&\quad=(I_{\mathfrak{F}_n}\otimes D_{T^*})\Big(I_{\mathfrak{F}_n\otimes\mathcal{H}}-\sum_{i=1}^n(U_X rR_iU_X^*)\otimes T_i^*\Big)^{-1}(U_X\iota)\otimes I_{\mathcal{H}}\\
&\quad=(U_X\otimes I_{\mathcal{D}_{T^*}})K_{T,r}.
\end{aligned}$$

Thus

$$\text{(5.3)}\quad \mathbf{L}_T\circ(\mathbf{L}_X\circ\mathbf{L}_r)=\begin{pmatrix}(U_X\otimes I_{\mathcal{D}_{T^*}})K_{T,r} & (U_X\otimes I_{\mathcal{D}_{T^*}})\Theta_T(rR)(U_X^*\otimes I_{\mathcal{D}_T})\\ 0 & 0\end{pmatrix},$$

and we have thus computed the right hand side of (5.2).

As for the left hand side, let us first remark that, computing $\mathbf{L}_T\circ\mathbf{L}_X$ according to (2.3), we obtain as $(2,1)$ entry $I_{\mathfrak{F}_n}\otimes(\alpha_{\mathbf{L}_{Y^*}}(T))^*$. To avoid messy computations, we will use Lemma 2.2 to obtain its other entries.

Noting that in $\mathbf{L}_T$ and $\mathbf{L}_X$ all spaces have $\mathfrak{F}_n$ as a tensor factor, and all operators have $I_{\mathfrak{F}_n}$ as a factor, we shall write (a slight abuse of notation) $\mathbf{L}_T=I_{\mathfrak{F}_n}\otimes\mathbf{L}'_T$, $\mathbf{L}_X=I_{\mathfrak{F}_n}\otimes\mathbf{L}'_X$. Since both $\mathbf{L}'_T$ and $\mathbf{L}'_X$ are unitary operators, the same is true of $\mathbf{L}'_T\circ\mathbf{L}'_X$. Its $(2,1)$ entry is

$$c+dT^*(I-bT^*)^{-1}a=\Big(c^*+a^*T(I-b^*T)^{-1}d^*\Big)^*=(\alpha_{\mathbf{L}_{Y^*}}(T))^*,$$

while its $(1,1)$ entry is $D_{T^*}(I-bT^*)^{-1}a$. This last operator has obviously dense range from $\mathcal{H}$ to $\mathcal{D}_{T^*}$ (remember that $a\neq 0$), and we may therefore apply Lemma 2.2. Consequently, the operators $Z_*:\mathcal{D}_{\alpha_{\mathbf{L}_{Y^*}}(T)^*}\to\mathcal{D}_{T^*}$ and $Z:\mathcal{D}_{\alpha_{\mathbf{L}_{Y^*}}(T)}\to\mathcal{D}_T$, defined by

$$\begin{aligned}
Z_*D_{\alpha_{\mathbf{L}_{Y^*}}(T)^*}&=D_{T^*}(I-bT^*)^{-1}a\\
ZD_{\alpha_{\mathbf{L}_{Y^*}}(T)}&=D_T(I-b^*T)d^*
\end{aligned}$$

are unitary, and

$$\mathbf{L}'_T\circ\mathbf{L}'_X=\begin{pmatrix}Z_*D_{\alpha_{\mathbf{L}_{Y^*}}(T)^*} & -Z_*\alpha_{\mathbf{L}_{Y^*}}(T)Z^*\\ \alpha_{\mathbf{L}_{Y^*}}(T)^* & D_{\alpha_{\mathbf{L}_{Y^*}}(T)}Z^*\end{pmatrix}.$$

Therefore

$$\mathbf{L}_T\circ\mathbf{L}_X=\begin{pmatrix}I_{\mathfrak{F}_n}\otimes(Z_*D_{\alpha_{\mathbf{L}_{Y^*}}(T)^*}) & -I_{\mathfrak{F}_n}\otimes(Z_*\alpha_{\mathbf{L}_{Y^*}}(T)Z^*)\\ I_{\mathfrak{F}_n}\otimes\alpha_{\mathbf{L}_{Y^*}}(T)^* & I_{\mathfrak{F}_n}\otimes(D_{\alpha_{\mathbf{L}_{Y^*}}(T)}Z^*)\end{pmatrix}=\mathbf{L}''\circ\mathbf{L}_{\alpha_{\mathbf{L}_{Y^*}}(T)},$$

where $\mathbf{L}'' = \begin{pmatrix} I_{\mathfrak{F}n}\otimes Z_* & 0 \\ 0 & I_{\mathfrak{F}n}\otimes Z^* \end{pmatrix}$. Thus

$$(5.4)\quad (\mathbf{L}_T \circ \mathbf{L}_X) \circ \mathbf{L}_r = (\mathbf{L}'' \circ \mathbf{L}_{\alpha_{\mathbf{L}_{Y*}}(T)}) \circ \mathbf{L}_r = \mathbf{L}'' \circ (\mathbf{L}_{\alpha_{\mathbf{L}_{Y*}}(T)} \circ \mathbf{L}_r)$$

$$= \mathbf{L}'' \circ \begin{pmatrix} K_{\alpha_{\mathbf{L}_{Y*}}(T),r} & \Theta_{\alpha_{\mathbf{L}_{Y*}}(T)}(rR) \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} (I_{\mathfrak{F}n} \otimes Z_*)K_{\alpha_{\mathbf{L}_{Y*}}(T),r} & (I_{\mathfrak{F}n} \otimes Z_*)\Theta_{\alpha_{\mathbf{L}_{Y*}}(T)}(rR)(I_{\mathfrak{F}n} \otimes Z^*) \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} (I_{\mathfrak{F}n} \otimes Z_*)K_{\Phi_X^{-1}(T),r} & (I_{\mathfrak{F}n} \otimes Z_*)\Theta_{\Phi_X^{-1}(T)}(rR)(I_{\mathfrak{F}n} \otimes Z^*) \\ 0 & 0 \end{pmatrix}$$

(we have used (5.1) for the last equality. We compare now (5.3) with (5.4), and make $r \to 1$. The resulting limits exist by (4.2) and (4.3), and we obtain the assertion of the theorem. □

6. CONSTRAINED ROW CONTRACTIONS

We introduce now some definitions from [17, 18], where the notion of *constrained* objects appears. Let $\mathfrak{J}$ be a WOT-closed two-sided ideal in $\mathfrak{L}_n$, $\mathfrak{J} \neq \mathfrak{L}_n$. We define two subspaces of $\mathfrak{F}_n$ by

$$\mathcal{M}_{\mathfrak{J}} = \overline{\mathfrak{J}\mathfrak{F}_n}, \quad \mathcal{N}_{\mathfrak{J}} = \mathfrak{F}_n \ominus \mathcal{M}_{\mathfrak{J}}.$$

Then $\mathcal{M}_{\mathfrak{J}}$ and $F\mathcal{M}_{\mathfrak{J}}$ are invariant to L and to R, while $\mathcal{N}_{\mathfrak{J}}$ and $F\mathcal{N}_{\mathfrak{J}}$ are invariant to L^* and R^* (remember F is the flip operator as defined by (***2.1***)).

The *constrained left and right creation operators* belong to $\mathcal{L}(\mathcal{N}_{\mathfrak{J}})$ and are given by

$$L_i^{\mathfrak{J}} = P_{\mathcal{N}_{\mathfrak{J}}} L_i \mid \mathcal{N}_{\mathfrak{J}},$$
$$R_i^{\mathfrak{J}} = P_{\mathcal{N}_{\mathfrak{J}}} R_i \mid \mathcal{N}_{\mathfrak{J}}.$$

An operator $M \in \mathcal{L}(\mathcal{N}_{\mathfrak{J}} \otimes \mathcal{E}, \mathcal{N}_{\mathfrak{J}} \otimes \mathcal{E}_*)$ is called *multianalytic* if

$$M(L_i^{\mathfrak{J}} \otimes I_{\mathcal{E}}) = (L_i^{\mathfrak{J}} \otimes I_{\mathcal{E}_*})M.$$

We want to define *constrained row contractions* by using the functional calculus with respect to elements of the ideal. A problem appears, since for a general multicontraction the functional calculus is only defined for elements in $\mathfrak{l}_n$; it can be extended to $\mathfrak{L}$ only for completely noncoisometric contractions. Thus, if T is a general multicontraction, and $\mathfrak{j} \subset \mathfrak{l}_n$ is a two-sided norm closed ideal, we say that T is $\mathfrak{j}$-constrained if $f(T) = 0$ for all $f \in \mathfrak{j}$. If T is c.n.c., and $\mathfrak{J} \subset \mathfrak{L}_n$, we say that

T is $\mathfrak{J}$-constrained if $f(T) = 0$ for all $f \in \mathfrak{J}$. If $\mathfrak{J}$ is the wot-closure of $\mathfrak{j}$, and T is c.n.c., then it is $\mathfrak{j}$-constrained iff it is $\mathfrak{J}$-constrained.

The next result connects the constraints with the automorphisms.

Proposition 6.1. *If T is $\mathfrak{j}$-constrained, then $\Phi_X^{-1}(T)$ is $\Phi_X(\mathfrak{j})$-constrained (and similarly for $\mathfrak{J}$-constraints, in case T is c.n.c.).*

Proof. If we denote $T' = \Phi_X^{-1}(T)$, and $\rho : \mathfrak{l}_n \to \mathcal{L}(\mathcal{H})$ is, as above, the functional calculus for T', then

$$T_i' = \rho((\Phi_X^{-1}(L))_i) = \rho(U_X^* L_i U_X) = \rho(\Phi_X^{-1}(L_i)).$$

Since the functional calculus for T' is the unique homomorphism algebra that maps L_i into T_i', it must be $\rho \circ \Phi_X^{-1}$, and therefore

$$f(T') = \rho(\Phi_X^{-1}(f)) = (\Phi_X^{-1}(f))(T).$$

Thus $f(T') = 0$ is equivalent to $(\Phi_X^{-1}(f))(T) = 0$, whence the statement of the proposition follows. $\square$

Now, if T is a $\mathfrak{j}$-constrained contraction, and $\mathfrak{J}$ is the wot-closure of $\mathfrak{j}$, we define as in [17]:

(a) the *constrained Poisson kernel* $K_{\mathfrak{J},T} : \mathcal{H} \to \mathcal{N}_{\mathfrak{J}} \otimes \mathcal{D}_{T^*}$ by

$$K_{\mathfrak{J},T} = P_{\mathcal{N}_{\mathfrak{J}} \otimes \mathcal{D}_{T^*}} K_T;$$

(b) the *constrained characteristic function* $\Theta_{\mathfrak{J},T} : \mathcal{N}_{\mathfrak{J}} \otimes \mathcal{D}_T \to \mathcal{N}_{\mathfrak{J}} \otimes \mathcal{D}_{T^*}$ by

$$\Theta_{\mathfrak{J},T} = P_{\mathcal{N}_{\mathfrak{J}} \otimes \mathcal{D}_{T^*}} \Theta_T | \mathcal{N}_{\mathfrak{J}} \otimes \mathcal{D}_T.$$

We can then give the following consequence of Theorem 5.1 for $\mathfrak{j}$-constrained multicontractions.

Theorem 6.2. *Suppose T is $\mathfrak{j}$-constrained, and denote $\mathfrak{j}' = \Phi_X(\mathfrak{j})$, $\mathfrak{J}' = \Phi_X(\mathfrak{J})$, $T' = \Phi_X^{-1}(T)$. Then $\Theta_{\mathfrak{J}',\mathfrak{T}'} = (U_X \otimes Z_*^*)\Theta_{\mathfrak{J},\mathfrak{T}}(U_X^* \otimes Z)$ and $K_{\mathfrak{J}',T'} = (U_X \otimes Z_*^*) K_{\mathfrak{J},T}$.*

Proof. By Theorem 5.1, we have

$$\Theta_{T'} = (U_X \otimes Z_*^*)\Theta_T(U_X^* \otimes Z),$$

where $Z : \mathcal{D}_{T'} \to \mathcal{D}_T$ and $Z_* : \mathcal{D}_{T'^*} \to \mathcal{D}_{T^*}$ are unitary operators.

We have

$$P_{\mathcal{N}_{\mathfrak{j}'}} = U_X P_{\mathcal{N}_{\mathfrak{j}}} U_X^*, \qquad P_{\mathcal{D}_{T'}} = Z^* P_{\mathcal{D}_T} Z, \qquad P_{\mathcal{D}_{T'^*}} = Z_*^* P_{\mathcal{D}_{T^*}} Z_*$$

whence

$$P_{\mathcal{N}_{\mathfrak{j}'}\otimes\mathcal{D}_{T'}} = (U_X \otimes Z^*)P_{\mathcal{N}_{\mathfrak{j}}\otimes\mathcal{D}_T}(U_X^*\lambda \otimes Z),$$
$$P_{\mathcal{N}_{\mathfrak{j}'}\otimes\mathcal{D}_{T'*}} = (U_X \otimes Z_*^*)P_{\mathcal{N}_{\mathfrak{j}}\otimes\mathcal{D}_{T*}}(U_X^* \otimes Z_*).$$

Therefore

$$\begin{aligned}\Theta_{\mathfrak{j}',\mathfrak{T}'} &= P_{\mathcal{N}_{\mathfrak{j}'}\otimes\mathcal{D}_{T'*}}\Theta_{T'}P_{\mathcal{N}_{\mathfrak{j}'}\otimes\mathcal{D}_{T'}}\\ &= (U_X\otimes Z_*^*)P_{\mathcal{N}_{\mathfrak{j}}\otimes\mathcal{D}_{T*}}(U_X^*\otimes Z_*)(U_X\otimes Z_*^*)\Theta_T(U_X^*\otimes Z)(U_X\otimes Z^*)P_{\mathcal{N}_{\mathfrak{j}}\otimes\mathcal{D}_T}(U_X^*\otimes Z)\\ &= (U_X \otimes Z_*^*)P_{\mathcal{N}_{\mathfrak{j}}\otimes\mathcal{D}_{T*}}\Theta_T P_{\mathcal{N}_{\mathfrak{j}}\otimes\mathcal{D}_T}(U_X^* \otimes Z)\\ &= (U_X \otimes Z_*^*)\Theta_{\mathfrak{j},\mathfrak{T}}(U_X^* \otimes Z).\end{aligned}$$

The computations for the Poisson kernel are similar. ◻

If $\mathfrak{j}$ is the commutator ideal $\mathfrak{c} = [\mathfrak{l}_n, \mathfrak{l}_n] \subset \mathfrak{l}_n$ (and correspondingly $\mathfrak{C} = [\mathfrak{L}_n, \mathfrak{L}_n] \subset \mathfrak{L}_n$), then the constraint becomes just commutativity. Then $\Phi_X(\mathfrak{c}) = \mathfrak{c}$, $\Phi_X(\mathfrak{C}) = \mathfrak{C}$ for all X, which translates in the fact that applying the automorphism Φ_X to a commuting multicontraction produces also a commuting multicontraction. Theorem 6.2 yields then the transformation rule of the commutative characteristic function with respect to automorphisms of the ball, as shown in [5] (see Theorem 6.3 therein).

7. INVARIANT IDEALS

In connection to Theorem 6.2, it is interesting to discuss bilateral ideals $\mathfrak{J}$ of $\mathfrak{L}_n$ which are invariant with respect to all automorphisms Φ_X. They have the property that, if T is a $\mathfrak{J}$-constrained multicontraction, $\alpha_X(T)$ is then also a $\mathfrak{J}$-constrained multicontraction.

We have already encountered the commutator ideal $\mathfrak{C}$. Other examples of invariant ideals are given by the iterated commutators $\mathfrak{C}^k$, defined by $\mathfrak{C}^{k+1} = [\mathfrak{L}_n, \mathfrak{C}^k]$. These form a decreasing sequence contained in $\mathfrak{C}$. We will prove below that there are no invariant ideals larger than $\mathfrak{C}$. But we need for this some more preparatory results.

First, it is shown in [10, 11] that any $\mathfrak{L}_n$-invariant subspace of $\mathfrak{F}_n$ is of the form $\Theta(\mathfrak{F}_n \otimes \mathcal{E})$, for $\mathcal{E}$ a Hilbert space and $\Theta : \mathfrak{F}_n \otimes \mathcal{E} \to \mathfrak{F}_n$ a multianalytic operator that is also an isometry (such a Θ is called *inner*). This multianalytic operator is essentially uniquely determined by the subspace: if $\Theta' : \mathfrak{F}_n \otimes \mathcal{E}' \to \mathfrak{F}_n$ satisfies $\Theta'(\mathfrak{F}_n \otimes \mathcal{E}') = \Theta(\mathfrak{F}_n \otimes \mathcal{E})$, then there exists a unitary $V : \mathcal{E}' \to \mathcal{E}$ such that $\Theta' = \Theta(I_{\mathfrak{F}_n} \otimes V)$. Based on these results, one proves in [7] that the map $\mathfrak{J} \mapsto \mathcal{M}_{\mathfrak{J}} = \overline{\mathfrak{J}e_\varnothing}(= \overline{\mathfrak{J}\mathfrak{F}_n})$ is a one to one map from the set of all bilateral ideals in $\mathfrak{L}_n$ onto the set of subspaces in $\mathfrak{F}_n$ invariant both to $\mathfrak{L}_n$ and to $\mathfrak{R}_n$.

Finally, in [2,8] the eigenvectors of $\mathfrak{L}_n^*$ are identified. Namely, for any $\lambda \in \mathbb{B}^n$, one defines

$$\nu_\lambda = (1 - \|\lambda\|^2)^{1/2}(I - L[\bar{\lambda}])^{-1} e_\varnothing. \tag{7.1}$$

Then $L_i^* \nu_\lambda = \bar{\lambda}_i \nu_\lambda$, whence $\langle L_w \nu_\lambda, \nu_\lambda \rangle = \lambda_w$ for any $w \in \mathbb{F}_n^+$. Note that ν_λ are also eigenvectors of $\mathfrak{R}_n^*$ (corresponding to the same eigenvalues). The space spanned by all ν_λ ($\lambda \in \mathbb{B}^n$) is $\mathcal{M}_{\mathfrak{C}}^\perp$. This last space is the symmetric Fock space, which we will denote by $\mathfrak{F}_n^s$, and the map $e_w \mapsto \lambda_w$ identifies it with a space of functions on $\mathbb{B}^n$. If $A \in \mathfrak{R}_n$, then the projection of $A\mathbf{1}$ onto $\mathfrak{F}_n^s$ is identified with the function $\langle A\nu_\lambda, \nu_\lambda \rangle$.

Theorem 7.1. *If $\mathfrak{J} \supset \mathfrak{C}$ is a bilateral ideal in $\mathfrak{L}_n$, and $\Phi_X(\mathfrak{J}) = \mathfrak{J}$ for all X, then either $\mathfrak{J} = \mathfrak{C}$ or $\mathfrak{J} = \mathfrak{L}_n$.*

Proof. Let $\mathcal{M} = \overline{\mathfrak{J}\mathfrak{F}_n}$ be the invariant subspace determined by $\mathfrak{J}$; then $\Phi_X(\mathfrak{J}) = \mathfrak{J}$ implies $U_X \mathcal{M} = \mathcal{M}$. Suppose $\mathcal{M} = \Theta(\mathfrak{F}_n \otimes \mathcal{G})$; define $\Gamma : \mathbb{B}^n \otimes \mathcal{E} \to \mathbb{C}$ by the formula

$$\Gamma(\lambda, h) = \langle \Theta(\nu_\lambda \otimes h), \nu_\lambda \rangle. \tag{7.2}$$

If $\Theta = \sum_w R_w \otimes m_w$, then

$$\begin{aligned}\Gamma(\lambda, h) &= \lim_{r \to 1} \sum_w \langle r^{|w|}(R_w \otimes m_w)(\nu_\lambda \otimes h), \nu_\lambda \rangle \\ &= \lim_{r \to 1} \sum_w \langle R_w \nu_\lambda, \nu_\lambda \rangle \langle m_w h, e_\varnothing \rangle = \lim_{r \to 1} \sum_w \lambda^w \langle m_w h, e_\varnothing \rangle.\end{aligned}$$

For $r < 1$ the series on the right is uniformly convergent and thus defines an analytic function $\lambda \in \mathbb{B}^n$. It follows then that $\Gamma(\lambda, h)$ is analytic in λ. It is obviously linear in h; so we may consider $\lambda \mapsto \Gamma(\lambda, \cdot)$ as an analytic map $\tilde{\Gamma}$ from $\mathbb{B}^n$ into $\mathcal{E}$ (actually, in the dual of $\mathcal{E}$, which can be identified with $\mathcal{E}$).

On the other hand, since U_X implements an automorphism of $\mathfrak{R}_n$, one checks easily that $\Theta_X = U_X \Theta (U_X^* \otimes I_{\mathcal{E}})$ is also a multianalytic inner operator. The invariance of $\mathcal{M}$ with respect to U_X implies that $\mathcal{M} = \Theta_X(\mathfrak{F}_n \otimes \mathcal{E})$. The essential uniqueness of this representation implies then that for any $X \in U(1,n)$ there exists $V_X \in \mathcal{L}(\mathcal{E})$ such that

$$U_X \Theta (U_X^* \otimes I_{\mathcal{E}}) = \Theta (I_{\mathfrak{F}_n} \otimes V_X).$$

Let us take now X such that $x > 0$ (remember that the mappings $X \mapsto \Phi_X$ and $X \mapsto \phi_X$ have as kernel the constant unitaries). From (3.5) and (7.1) it follows then that

$$U_X^* e_\varnothing = U_X^* \nu_0 = \nu_{\phi_X(0)}.$$

Therefore

$$\begin{aligned}\Gamma(\phi_X(0),h) &= \langle \Theta(\nu_{\phi_X(0)}\otimes h),\nu_{\phi_X(0)}\rangle\\ &= \langle U_X\Theta(U_X^*\otimes I_{\mathcal{E}})(e_\varnothing\otimes h),e_\varnothing\rangle\\ &= \langle \Theta(I_{\mathfrak{F}_n}\otimes V_X)(e_\varnothing\otimes h),e_\varnothing\rangle = \Gamma(0,V_X(h)).\end{aligned}$$

This last relation can be rewritten as $\tilde{\Gamma}\circ\phi_X(0) = V_X^*\tilde{\Gamma}(0)$. Since V_X is unitary, we obtain that $\|\tilde{\Gamma}(\phi_X(0))\| = \|\tilde{\Gamma}(0)\|$. The image of $\{X\in U(1,n) : x>0\}$ under the mapping $X\mapsto\phi_X(0)$ is the whole $\mathbb{B}^n$; therefore $\tilde{\Gamma}$ is an analytic function on $\mathbb{B}^n$ with values in the Hilbert space $\mathcal{E}$, of constant norm, which must be actually constant.

For any $h\in\mathcal{E}$ we can define an element $\Theta_h\in\mathfrak{R}$ by the formula $\Theta_h\xi = \Theta(\xi\otimes h)$. We have then, by (7.2) and the remarks before the statement of the theorem,

$$\Gamma(\lambda,h) = \langle\Theta_h\nu_\lambda,\nu_\lambda\rangle = (P_{\mathfrak{F}_n^s}(\Theta_h e_\varnothing))(\lambda).$$

Two cases present now. If $\tilde{\Gamma}$ is identically 0, then $(P_{\mathfrak{F}_n^s}(\Theta_h e_\varnothing)) = 0$ for all $h\in\mathcal{E}$, and thus the image of Θ is included in $\mathfrak{C}$. It follows that $\mathfrak{J}\subset\mathfrak{C}$; and then the assumption implies $\mathfrak{J}=\mathfrak{C}$.

In the opposite case, take $h\in\mathcal{E}$ such that $\Gamma(\lambda,h)$ is a nonnull constant. Then $(P_{\mathfrak{F}_n^s}(\Theta_h e_\varnothing))(\lambda)$ is a nonnull multiple of $e_\varnothing$. Thus $\mathcal{M}$ contains a vector of the form $ae_\varnothing+\xi_0$, with $a\neq 0$ and $\xi_0\in\mathcal{M}_{\mathfrak{C}}$. But the assumption $\mathfrak{C}\subset\mathfrak{J}$ implies $\mathcal{M}_{\mathfrak{C}}\subset\mathcal{M}$; therefore $e_\varnothing\in\mathcal{M}$. Since $\mathcal{M}$ is invariant, it follows that $\mathcal{M}=\mathfrak{F}_n$, whence $\mathfrak{J}=\mathfrak{L}$. □

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