On the determinacy of complex Jacobi matrices

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Abstract

The aim of this paper is to explore to which extend the theory of the Hamburger moment problem for real Jacobi matrices generalizes to the case of complex Jacobi matrices. In particular, we characterize the indeterminacy in terms of uniqueness of closed extensions of Jacobi matrices, and discuss the link to the growth of the smallest singular values of the underlying Hankel matrices. As a byproduct, we give a positive answer to the open question whether determinacy is preserved under bounded perturbations.

Keywords: Difference operator, Complex Jacobi matrix, Formal orthogonal polynomials, Indeterminacy, Hankel matrices.

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1 Introduction

Given $a_n \in \mathbb{C} \setminus \{0\}$ and $b_n \in \mathbb{C}$, we consider the following complex Jacobi matrix

$$\mathcal{A} = \begin{pmatrix} b_0 & a_0 & 0 & \dots \\ a_0 & b_1 & a_1 & \ddots \\ 0 & a_1 & b_2 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{1}$$

We denote by ℓ^2 the Hilbert space of complex square-summable sequences, with the usual scalar product $(u, v) = \sum \overline{u_j} v_j$. Furthermore, for a linear operator T in ℓ^2, we denote by $\mathcal{D}(T)$, $\mathcal{R}(T)$, and $\sigma(T)$ its domain of definition, its range and its spectrum, respectively. In what follows we identify $\mathcal{A}$ with the operator acting the set $\mathcal{C}_0$ of finite linear combinations of the canonical vectors $e_0, e_1, ... \in \ell^2$.

Complex Jacobi matrices occur naturally in the study of formal orthogonal polynomials and Jacobi continued fractions: define the sequences of polynomials $p(z) := (p_n(z))_{n\geq 0}$ and $q(z) := (q_n(z))_{n\geq 0}$ as solutions of the recurrence relation

$$a_n y_{n+1} + b_n y_n + a_{n-1} y_{n-1} = z y_n, \qquad n = 0, 1, 2, \dots, \tag{2}$$

with initial conditions

$$\begin{cases} q_{-1}(z) = 0, & q_0(z) = 1, \\ p_0(z) = 0, & p_1(z) = 1/a_0. \end{cases} \tag{3}$$

Then $(p_n(z)/q_n(z))_{n\neq 0}$ is a sequence of convergents of the Jacobi continued fraction associated to $\mathcal{A}$. Moreover, $(q_n(z))_{n\geq 0}$ is a sequence of formal orthogonal polynomials with respect to the linear functional c acting on the set of polynomials via $c_n := c(x^n) = (e_0, \mathcal{A}^n e_0)$, $n \geq 0$. It is known that the corresponding sequence of Hankel matrices

$$H_n = \begin{pmatrix} c_0 & c_1 & \dots & c_n \\ c_1 & c_2 & \dots & c_{n+1} \\ \vdots & \vdots & & \vdots \\ c_n & c_{n+1} & \dots & c_{2n} \end{pmatrix} \tag{4}$$

is strongly regular, that is, $\det(H_n) \neq 0$ for all $n \geq 0$. Conversely, the Shohat-Favard Theorem tells us that, for any linear functional c with strongly regular sequence of Hankel matrices and normalization $c(1) = c_0 = 1$, there exists a sequence of formal orthogonal polynomials $(q_n(z))_{n\geq 0}$ verifying the three term recurrence relation (2) and (3) for some complex Jacobi matrix.

Notice that a Jacobi matrix is real (i.e., $a_n, b_n \in \mathbb{R}$) if and only if all Hankel matrices are symmetric positive definite. Here many equivalent conditions are known to characterize the uniqueness of the corresponding moment problem. For convenience we state them in the following theorem, here $\mathbb{P}$ denotes the set of polynomials with complex coefficients.

Theorem 1.1 *For $a_n, b_n \in \mathbb{R}$, there is equivalent:*

(A) *There is a unique solution for the moment problem: find a positive probability measure μ supported on the real line such that $c_n = \int x^n \, d\mu(x)$ for all $n \geq 0$.*

(B) *The Jacobi matrix is determinate, that is, for any $z \in \mathbb{C}$, at least one of the sequences $p(z)$ and $q(z)$ is not in ℓ^2.*

(C) *There is a unique close extension in ℓ^2 of the Jacobi matrix $\mathcal{A}$ which is defined by matrix product.*

(D) *The linear operator $T : \mathbb{P} \mapsto \mathbb{P}$ defined by $T(x^n) = q_n$, $n \geq 0$, is an unbounded operator with respect to the norm $||p||^2 = \frac{1}{2\pi}\int_{|z|=1} |p(z)|^2 \, |dz|$ of the Hardy space H^2.*

(E) *The inverses of the Hankel matrices H_n, $n \geq 0$, are not uniformly bounded.*

(F) *The numerical range of the Hankel matrices is not bounded away from zero:* $\inf\{|(y, H_n y)/(y, y)| : y \in \mathbb{C}^n, n \geq 0\} = 0$.

The equivalence of the first three properties is very classical, see for instance [1, 12] or the more recent paper [11, Theorem 2 and Theorem 3]. The equivalence of condition **(E)** was the content of the recent Berg-Chen-Ismael Theorem [5]. Since H_n is symmetric positive definite in the real case, both properties **(E)** and **(F)** are equivalent to the fact that the smallest eigenvalue of H_n tends to zero as $n \to \infty$. For property **(D)** in more general settings we refer the reader to [6].

The aim of the present paper is to explore possible generalizations of Theorem 1.1 to the setting of complex Jacobi matrices. Condition **(A)** is out of the scope of the present paper. The notion of indeterminacy for complex Jacobi matrices (see property **(B)**) was introduced by Wall [13, Def. 22.1] who showed in the theorem of invariability [13, Theorem 22.1] that both $p(z)$ and $q(z)$ are elements of ℓ^2 for all $z \in \mathbb{C}$ if this condition holds for just one $z \in \mathbb{C}$. With respect to condition **(C)** we recall from [3, § 2.1] that there is a minimal and a maximal closed extension of $\mathcal{A}$ defined by matrix product, namely the operators $A = A_{\min}$ and $A_{\max}$, with $A_{\min}$ being the closure of $\mathcal{A}$, and thus

$$\mathcal{D}(A_{min}) = \{y \in \ell^2 : \exists \ (y^{(n)})_{n\geq 0} \subset \mathcal{C}_0 \ : \ y^{(n)} \underset{n}{\to} \ y, \ (\mathcal{A}y^{(n)})_{n\geq 0} \underset{n}{\to} \ \mathcal{A}y\},$$

whereas $D(A_{\max}) = \{y \in \ell^2 : \ \mathcal{A}y \in \ell^2\}$. A complex Jacobi matrix is called proper [3, Def. 2.2] if these operators coincide, or, in other words, if condition **(C)** of Theorem 1.1 holds. Concerning the equivalence of conditions **(B)** and **(C)** , it was shown in [3, Theorem 2.6(b)] that any proper complex Jacobi matrix is determinate. The converse was only shown to be true for Jacobi matrices with non-empty essential spectrum $\sigma_{ess}(A)$ [3, Theorem 2.6(b)], and remained an open question for general complex Jacobi matrices. In this context, recall from [3, Theorem 2.6(c)] that a Jacobi operator may only have a spectrum $\sigma(A) \neq \mathbb{C}$ if $\mathcal{A}$ is proper. Also, for both cases of proper and indeterminate complex Jacobi matrices, the problem of characterizing the resolvent [3, Theorems 2.10 and 2.11] and the problem of finite section resolvent convergence [3, Theorems 4.1 and 4.2] are well understood.

The paper is organized as follows: in §2 we establish the equivalence between proper and determinate Jacobi matrices, and draw some conclusions. In §3 we show that condition **(B)** implies property **(D)** , and give classes of complex Jacobi matrices where the reciprocal is true. The other implications are discussed in §4.

2 Proper and determinate Jacobi matrices

The aim of this section is to show the following statement.

Theorem 2.1 *A complex Jacobi matrix is proper iff it is determinate.*

Indeed, for an indeterminate Jacobi matrix $\mathcal{A}$ one may show that $\mathcal{N}(A_{\max}) \neq \{0\}$ and that $\mathcal{R}(A_{\max})$ is dense in ℓ^2, implying that $\mathcal{A}$ is not proper (compare with [3, Theorem 2.6(b)]). Hence, according to [13, Theorem 22.1], we only have to show that for complex a Jacobi matrix with $A_{\min} \neq A_{\max}$ there holds $p(0) \in \ell^2$ and $q(0) \in \ell^2$.

Before entering into details, let us mention the following consequence answering a question mentioned in [2, Remark 1].

Corollary 2.2 *Let $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ be two complex Jacobi matrices, with $\mathcal{A} - \widetilde{\mathcal{A}}$ being bounded (that is, $\sup_n |a_n - \widetilde{a}_n| + \sup_n |b_n - \widetilde{b}_n| < \infty$). Then $\mathcal{A}$ is indeterminate iff $\widetilde{\mathcal{A}}$ is indeterminate.*

The preservation of indeterminacy under diagonal bounded perturbations has been discussed in [13, Theorem 22.1]. For a real Jacobi matrix $\mathcal{A}$, the statement of Corollary 2.2 has been obtained already in [7] (though not stated there, the techniques presented in [7] do cover a more general frame). A proof of Corollary 2.2 is based on Theorem 2.1 and the observation that a proper Jacobi matrix remains proper after bounded perturbations (since the domains of definition of $A_{\max}$ and $A_{\min}$ remain invariant).

Example 2.3 *Let $(b_n)_{n\geq 0}$ be bounded. Then $\mathcal{A}$ is indeterminate iff*

$$\sum_{n=0}^{\infty} \Big| \prod_{j=0}^{n} \frac{a_{2j}}{a_{2j+1}} \Big|^2 + \sum_{n=0}^{\infty} \Big| \prod_{j=0}^{n} \frac{a_{2j+1}}{a_{2j+2}} \Big|^2 < \infty.$$

Moreover, the finiteness of the sums is invariant under bounded perturbation of the a_n. To see these two statements, it is sufficient to consider only the case $b_n = 0$ for all n by Corollary 2.2. In this case, we find that $p_{2n}(0) = q_{2n+1}(0) = 0$ for all $n \geq 0$, whereas for $q_{2n}(0)$ and $p_{2n+1}(0)$ we essentially find the products mentioned in the above formula.

Returning to the proof of Theorem 2.1, let $\mathcal{A}$ be a complex Jacobi matrix with $A_{\min} \neq A_{\max}$, and consider the block tridiagonal matrix

$$\mathcal{B} = \begin{pmatrix} B_0 & A_0 & 0 & \dots \\ A_0 & B_1 & A_1 & \ddots \\ 0 & A_1 & B_2 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

where

$$A_k = \begin{pmatrix} 0 & a_k \\ \overline{a_k} & 0 \end{pmatrix} = A_k^*, \qquad B_k = \begin{pmatrix} 0 & b_k \\ \overline{b_k} & 0 \end{pmatrix} = B_k^*.$$

Notice that $\mathcal{B}$ acting on $\mathcal{C}_0$ via matrix product is now a symmetric operator (since its matrix representation is hermitian). As before, we may consider $B := B_{\min}$, the closure of $\mathcal{B}$, as well as $B_{\max}$ with $\mathcal{D}(B_{\max}) = \{y \in \ell^2 : \mathcal{B} \cdot y \in \ell^2\}$, the maximal closed operator defined by matrix product, compare with [3, § 2.1]. It follows from [3, Lemma 2.1] that $B_{\max} = B^*$.

There is some generalization of Theorem 1.1 for hermitian block Jacobi matrices, for completeness we will show the implications required for the proof of Theorem 2.1. Let us first show the following result.

Lemma 2.4 *Let $\mathcal{A}$ be a complex Jacobi matrix with $A_{\min} \neq A_{\max}$, and consider the 2×2 matrix polynomials being defined by ($n = 0, 1, 2, \ldots$)*

$$\begin{aligned} &A_n Q_{n+1}(z) + B_n Q_n(z) + A_{n-1} Q_{n-1}(z) = z Q_n(z), \\ &A_n P_{n+1}(z) + B_n P_n(z) + A_{n-1} P_{n-1}(z) = z P_n(z), \\ &Q_{-1}(z) = 0, \quad Q_0(z) = I_2, \quad P_0(z) = 0, \quad P_1(z) = A_0^{-1}. \end{aligned}$$

Then, for each purely imaginary $z \in i \cdot \mathbb{R} \setminus \{0\}$,

$$\sum_{n=0}^{\infty} ||P_n(z)||^2 + \sum_{n=0}^{\infty} ||Q_n(z)||^2 < \infty. \tag{5}$$

Proof. First, after a suitable simultaneous permutation of rows and columns (we first take the even indices), the matrix $\mathcal{B}$ becomes the matrix

$$\begin{pmatrix} 0 & \mathcal{A} \\ \overline{\mathcal{A}} & 0 \end{pmatrix},$$

$\overline{\mathcal{A}}$ obtained from $\mathcal{A}$ by conjugating each entry, showing that

$$\begin{aligned} &\mathcal{D}(B) = \{(y_n)_{n\geq 0} \in \ell_2 : (y_{2k+1})_{k\geq 0}, (\overline{y_{2k}})_{k\geq 0} \in \mathcal{D}(A_{\min})\} \\ &\neq \mathcal{D}(B^*) = \{(y_n)_{n\geq 0} \in \ell_2 : (y_{2k+1})_{k\geq 0}, (\overline{y_{2k}})_{k\geq 0} \in \mathcal{D}(A_{\max})\}. \end{aligned}$$

Hence B is not self-adjoint. Fix $z \in i \cdot \mathbb{R} \setminus \{0\}$ as in the assertion of the

Lemma. We consider the reduced modulus [8, § IV.5.1]

$$\begin{aligned}\gamma(B) &= \min_{y\in\mathcal{D}(B)} \frac{||(zI-B)y||}{\operatorname{dist}(y,\mathcal{N}(zI-B))} \geq \min_{y\in\mathcal{D}(B)} \frac{||(zI-B)y||}{||y||} \\ &= \min_{y\in\mathcal{C}_0} \frac{||(zI-B)y||}{||y||} = \min_{y\in\mathcal{C}_0} \sqrt{|\mathrm{Im}(z)|^2 + \frac{(By,By)}{(y,y)}} \\ &\geq |z| > 0.\end{aligned}$$

As a consequence, $\mathcal{N}(zI-B) = \{0\}$, and both images $\mathcal{R}((zI-B)$ and $\mathcal{R}((zI-B)^*) = \mathcal{R}(\overline{z}I - B^*)$ are closed according to [8, Theorem IV.5.13]. In particular, by replacing z by $\overline{z} = -z$ we obtain

$$\mathcal{R}(zI-B^*) = \mathcal{N}(\overline{z}I-B)^\perp = \ell^2.$$

We claim that $\mathcal{N}(zI-B^*) \neq \{0\}$. Indeed, otherwise $\mathcal{R}(zI-B) = \mathcal{N}(\overline{z}I - B^*)^\perp = \ell^2$, and both $zI-B$ and $zI-B^*$ are injective, a contradiction to the fact that B is not self-adjoint, i.e., that B^* is a proper extension of B.

We may even determine the form of $\mathcal{N}(zI-B^*)$. By writing down explicitly the matrix product and comparing it with the above recurrence relations for $(Q_n(z))_{n\geq 0}$, we see that for any solution of $(zI-B^*)y = 0$ there necessarily exist a vector $d \in \mathbb{C}^2$ with $y = (Q_n(z)d)_{n\geq 0} \in \ell^2$. Hence $1 \leq \dim \mathcal{N}(zI-B^*) \leq 2$. However, $y \in \mathcal{N}(zI-B^*)$ if and only if

$$zy' = \mathcal{A}y'', \qquad zy'' = \overline{\mathcal{A}}y',$$

where $y' = (y_{2k})_{k\geq 0} \in \ell^2$ and $y'' = (y_{2k+1})_{k\geq 0} \in \ell^2$. Taking conjugates leaves to $-z\overline{y'} = \overline{z}\overline{y'} = \overline{\mathcal{A}}\overline{y''}$ and $-z\overline{y''} = \overline{z}\overline{y''} = \mathcal{A}\overline{y''}$. Consequently, with $0 \neq y \in \mathcal{N}(zI-B^*)$, also the orthogonal vector $\widetilde{y}$ defined by $\widetilde{y}_{2k+1} = -\overline{y_{2k}}$, $\widetilde{y}_{2k} = \overline{y_{2k+1}}$ is an element of $\mathcal{N}(zI-B^*)$, showing that $\mathcal{N}(zI-B^*) = 2$, i.e., the block Jacobi matrix $\mathcal{B}$ has the deficiency indices $(2,2)$. Thus both columns of $Q(z) := (Q_n(z))_{n\geq 0}$ are elements of ℓ^2. In addition, denoting by $||\cdot||_F$ the Froebenius norm, we have that

$$\begin{aligned}&\sum_{n=0}^{\infty} ||Q_n(z)||^2 \leq \sum_{n=0}^{\infty} ||Q_n(z)||_F^2 \\ &= \sum_{n=0}^{\infty} ||Q_n(z)\cdot\begin{pmatrix}1\\0\end{pmatrix}||^2 + ||Q_n(z)\cdot\begin{pmatrix}0\\1\end{pmatrix}||^2 \\ &= ||Q(z)\begin{pmatrix}1\\0\end{pmatrix}||^2 + ||Q(z)\begin{pmatrix}0\\1\end{pmatrix}||^2 < \infty,\end{aligned}$$

that is, the first part of the assertion of the Lemma. In order to show the second part, recall from above that $e_0, e_1 \in \mathcal{R}(zI - B^*)$. Again by writing down explicitly the matrix product for the pre-images of (e_0, e_1) and comparing it with the above recurrence relations for $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$, we see that there exists a $D \in \mathbb{C}^{2\times 2}$ with $(zI - B^*)Y = (e_0, e_1)$, $Y = (Q_n(z)D - P_n(z))_{n\geq 0} \in (\ell^2)^2$. Consequently, also both columns of $(P_n(z))_{n\geq 0}$ are elements of ℓ^2, and we obtain as above that $\sum_{n=0}^{\infty} ||P_n(z)||^2 < \infty$. $\square$

It is not difficult to see that (5) implies that

$$R(z) := \lim_{n\to\infty} (\sum_{n=0}^{\infty} Q_n(z)^* Q_n(z))^{-1}$$

exists and is invertible for any $z \in i \cdot \mathbb{R} \setminus \{0\}$. Therefore it follows from [9, Theorem 3.1] (see also [4, Theorem 2.6, p. 570]) that this limit exists and is invertible for any $z \in \mathbb{C} \setminus \mathbb{R}$. Such a case is usually referred to as the completely indeterminate case.

By comparing the two recurrence relations, one easily verifies that

$$Q_n(0) = \begin{pmatrix} \overline{q_n(0)} & 0 \\ 0 & q_n(0) \end{pmatrix}, \qquad P_n(0) = \begin{pmatrix} 0 & \overline{p_n(0)} \\ p_n(0) & 0 \end{pmatrix}, \qquad n \geq 0. \tag{6}$$

Hence if we are able to show that (5) also holds for $z = 0$, then the indeterminate case holds for $\mathcal{A}$, as claimed in Theorem 2.1.

Indeed, similar to the scalar case one may give a theorem of invariability in the matrix setting, compare for instance with the related result of [9, Theorem 3.2]. For our purpose it is sufficient to show the following result.

Lemma 2.5 *Let $\mathcal{A}$ be a complex Jacobi matrix with $A_{\min} \neq A_{\max}$. Then, for each $z \in \mathbb{C}$,*

$$\sum_{n=0}^{\infty} ||P_n(z)||^2 + \sum_{n=0}^{\infty} ||Q_n(z)||^2 < \infty. \tag{7}$$

Proof. Denote by $Q_n^*(z)$ (and by $P_n^*(z)$, respectively), the 2×2 matrix polynomial $Q_n(\overline{z})^*$ (and $P_n(\overline{z})^*$, respectively), and by B_{2n+2} the principal submatrix of order $2n+2$ of $\mathcal{B}$. One easily shows by recurence on n

$$\begin{pmatrix} P_n^*(z) & P_{n+1}^*(z) \\ Q_n^*(z) & Q_{n+1}^*(z) \end{pmatrix} \cdot \begin{pmatrix} 0 & -A_n \\ A_n & 0 \end{pmatrix} \cdot \begin{pmatrix} Q_n(z) & -P_n(z) \\ Q_{n+1}(z) & -P_{n+1}(z) \end{pmatrix} = I_4, \tag{8}$$

and the recurrence relations

$$(zI_{2n+2} - B_{2n+2}) \begin{pmatrix} Q_0(z) \\ \vdots \\ Q_n(z) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ A_n Q_{n+1}(z) \end{pmatrix},$$

$$(Q_0^*(x), ..., Q_n^*(x))(xI_{2n+2} - B_{2n+2}) = (0, ..., 0, Q_{n+1}^*(x)A_n),$$

$$(P_0^*(x), ..., P_n^*(x))(xI_{2n+2} - B_{2n+2}) = (-I_2, 0, ..., 0, P_{n+1}^*(x)A_n).$$

The last two relations together with (8) give

$$\Big(Q_n(x)(P_0^*(x), ..., P_n^*(x)) - P_n(x)(Q_0^*(x), ..., Q_n^*(x))\Big)(zI_{2n+2} - B_{2n+2})$$
$$= (-Q_n(x), 0, ..., 0, I_2),$$

where we notice that, by (8), the last block component of the vector on the left-hand side vanishes. Multiplying this relation on the right by the first components of $Q(z)$ gives for $x \neq z$

$$Q_n(z) - Q_n(x) = (z - x) \sum_{k=0}^{n-1} [Q_n(x)P_k^*(x)Q_k(x) - P_n(x)Q_k(x)Q_k(z)].$$

Choosing $x \in i \cdot \mathbb{R} \setminus \{0\}$ and $z \in \mathbb{C}$, we see that

$$Q(z) = Q(x) + (z - x)\mathcal{H}Q(z),$$

where, according to Lemma 2.4, $\mathcal{H}$ is a bounded operator of Schmidt class with a matrix description having only non-zero entries below the main diagonal. Since $Q(x)$ has columns in ℓ^2, we may conclude from [10, Lemma II.7.3] that that the columns of $Q(z)$ are in ℓ^2. Finally, in order to obtain the same result for $P(z)$, we consider the associated block Jacobi matrix obtained from $\mathcal{B}$ by dropping the first two rows and columns, and replace $Q_n(z), P_n(z)$ by $P_{n-1}(z) \cdot A_0$, and $P_{n-1}(z)Q_1^*(z) - Q_{n-1}(z)P_1^*(z)$, respectively. □

3 Indeterminacy and the map T

In this section we relate the determinacy of a complex Jacobi matrix to property **(D)** of Theorem 1.1. Write $q_n(z) = q_{0,n} + q_{1,n}z + \ldots + q_{n,n}z^n$, and consider the triangular matrix

$$T_n = \begin{pmatrix} q_{00} & q_{01} & q_{02} & \cdots & q_{0n} \\ 0 & q_{11} & q_{12} & & \vdots \\ 0 & 0 & q_{22} & & \\ \vdots & & \ddots & \ddots & \vdots \\ & & & & \\ 0 & \cdots & 0 & & q_{nn} \end{pmatrix}. \tag{9}$$

Notice that T_n is a matrix description of the restriction of the application T to the set $\mathbb{P}_n$ of polynomials of degree at most n: for $p(z) = a_0 + a_1 z + ... + a_n z^n$ we have $(Tp)(z) = a_0 q_0(z) + a_1 q_1(z) + ... + a_n q_n(z) = b_0 + b_1 z + ... + b_n z^n$, with

$$b = T_n a, \qquad a = (a_0, ..., a_n)^t, \qquad b = (b_0, ..., b_n)^t,$$

and moreover

$$||T|_{\mathbb{P}_n}||_{H^2 \mapsto H^2} = \sup_{p \in \mathbb{P}_n} \frac{||Tp||_{H^2}}{||p||_{H^2}} = \sup_{a \in \mathbb{C}^{n+1}} \frac{||T_n a||}{||a||} = ||T_n||.$$

Thus, we have the following equivalent formulation of condition **(D)**

$$||T||_{H^2 \mapsto H^2} < \infty \quad \text{if and only if} \quad \sup_{n \geq 0} ||T_n|| < \infty. \tag{10}$$

Denote by $\mathbb{D}$ the open unit disk. We have the following result which in Theorem 3.3 below will be further generalized.

Theorem 3.1 *For any indeterminate complex Jacobi matrix there holds* $||T||_{H^2 \mapsto H^2} < \infty$.

Conversely, a complex Jacobi matrix is indeterminate provided that $||T||_{H^2 \mapsto H^2} < \infty$, *and in addition*

$$\exists \zeta \in \mathbb{D} : \qquad \sup_{y \in \mathcal{C}_0} \frac{||y||}{||(\zeta I - A)y||} < \infty. \tag{11}$$

Proof. We start by establishing the following formulas (compare with [5, 6])

$$||T_n||^2 \leq ||T_n||_F^2 = \frac{1}{2\pi} \int_0^{2\pi} \sum_{j=0}^{n} |q_j(e^{it})|^2 \, dt < \infty, \tag{12}$$

$||\cdot||_F$ denoting the Froebenius (or Hilbert-Schmidt) norm, and for $z \in \mathbb{D}$

$$\sum_{j=0}^{n} |q_j(z)|^2 \leq \frac{1}{1-|z|^2} \cdot ||T_n||^2. \tag{13}$$

Formula (12) follows from the following simple observation

$$\begin{aligned} ||T_n||^2 \leq ||T_n||_F^2 &= \sum_{k=0}^{n} \sum_{j=0}^{n} |(T_n)_{j,k}|^2 \\ = \sum_{k=0}^{n} \sum_{j=0}^{k} |q_{j,k}|^2 &= \sum_{k=0}^{n} \frac{1}{2\pi} \int_0^{2\pi} |q_k(e^{it})|^2 \, dt. \end{aligned}$$

For a proof of (13), notice that for $z \in \mathbb{D}$

$$\begin{aligned} &\sum_{j=0}^{n} |q_j(z)|^2 = ||(q_0(z), ..., q_n(z))||^2 = ||(1, z, ..., z^n) T_n||^2 \\ &\leq ||T_n||^2 \cdot ||(1, z, ..., z^n)||^2 \leq \frac{1}{1-|z|^2} \cdot \sup_n ||T_n||^2. \end{aligned}$$

Suppose now that $\mathcal{A}$ is indeterminate. We remark that, according to the theorem of invariability [13, Theorem 22.1], the series $\sum_{j=0}^{\infty} |q_j(z)|^2$ converges uniformly on compact sets in the indeterminate case. As a consequence, here the function $z \mapsto ||q(z)||_{\ell^2}$ mapping $\mathbb{C}$ to $(0, +\infty)$ is continuous, and $||T||_{H^2 \mapsto H^2} < \infty$ is a consequence of (10) and (12).

Conversely, suppose that $||T||_{H^2 \mapsto H^2} < \infty$, and let $\zeta \in \mathbb{D}$ be as in the statement of the Theorem. Then from (10) and (13) we get that $q(\zeta) \in \ell^2$, and thus $q(\zeta) \in \mathcal{N}(zI - A_{\max})$. Thus, by Theorem 2.1, the indeterminacy will follow by showing by contradiction that $q(\zeta) \notin \mathcal{N}(zI - A_{\min})$. Indeed, $q(\zeta) \in \mathcal{N}(zI - A_{\min})$ implies that there is a sequence $(v_n) \subset \mathcal{C}_0$, with $v_n \to q(\zeta)$ and $(\zeta I - A) v_n \to 0$ for $n \to \infty$, in contradiction with the assumption on ζ. □

In the indeterminate case, all principal submatrices $(\zeta I - \mathcal{A})_{n+1,n}$ of $\zeta I - \mathcal{A}$ of size $(n+2) \times (n+1)$ can be shown to have a bounded left inverse, implying that condition (11) holds for all $\zeta \in \mathbb{C}$. A simple sufficient condition for (11) is that there exists an infinite set Λ such that the sequence $([(\zeta I - \mathcal{A})_{n,n}]^{-1})_{n \in \Lambda}$ of inverses of finite subsections of $\zeta I - \mathcal{A}$ is bounded. A class of complex

Jacobi matrices having property (11) was discussed by Wall: $\mathcal{A}$ is called positive definite if its numerical range satisfies $\Theta(A_{min}) \subset \{\mathrm{Im}(w) \leq 0\}$. We remind the reader that, for a linear operator S defined on some Hilbert space, the numerical range (or field of values) is defined by

$$\Theta(S) := \{(y, Sy) : \ y \in \mathcal{D}(S), \ ||y|| = 1\},$$

see, e.g., [8, Section V. 3.2].

Corollary 3.2 *A positive definite complex Jacobi matrix is indeterminate if and only if* $||T||_{H^2 \mapsto H^2} < \infty$.

Proof. We only need to show that (11) holds for all $\zeta \in \mathbb{C}$ with $\mathrm{Im}(\zeta) > 0$. Indeed, we have for all $y \in \mathcal{C}_0$

$$\frac{||y||}{||(\zeta I - A)y||} \leq \frac{||y||^2}{|(y, (\zeta I - A)y)|} \leq \frac{1}{\mathrm{dist}(\zeta, \Theta(A_{min}))} \leq \frac{1}{\mathrm{Im}(\zeta)} < \infty.$$

□

Before giving a generalization of Theorem 3.1, we mention some preliminary remarks. The orthogonality relation $c(q_j q_k) = \delta_{j,k}$ may be rewritten as $T_n^t H_n T_n = I_{n+1}$ (^{t} denoting the transposed without taking conjugates), or

$$H_n^{-1} = T_n T_n^t. \tag{14}$$

Notice also that $(1, x, ..., x^n)T_n = (q_0(x), ..., q_n(x))$, and thus

$$(1, x, ..., x^n) H_n^{-1} (1, y, ..., y^n)^t = \sum_{j=0}^{n} q_j(x) q_j(y) =: K_n(x, y).$$

If now $||T||_{H^2 \mapsto H^2} < \infty$, then $q(x), q(y) \in \ell^2$ by (13), and hence $K_n(x, y) \to K(x, y)$ uniformly in compact subsets of $\mathbb{D} \times \mathbb{D}$, with K being analytic in both arguments. As a consequence,

$$\lim_{n \to \infty} (e_j, H_n^{-1} e_k) = (e_j, TT^t e_k) = \frac{1}{j!} (\frac{d}{dx})^j \frac{1}{k!} (\frac{d}{dy})^k K(x, y)|_{(x,y)=(0,0)}.$$

Thus $K(x, y) = 0$ for all $x, y \in \mathbb{D}$ if and only if the (j, k) entry of H_n^{-1} tends to zero for all j, k if and only if the bounded operator TT^t is the zero operator. Clearly, this last property cannot be true for the case of real Jacobi matrices, since here $K(0, 0) \geq q_0(0)^2 = 1$. It remains the open question whether $TT^t = 0$ can be valid for complex Jacobi matrices.

Theorem 3.3 *Let $\mathcal{A}$ be a complex Jacobi matrix such that there exist $x, y \in \mathbb{D}$ with*

$$\limsup_{n\to\infty} |\sum_{j=0}^{n} q_j(x)q_j(y)| > 0. \tag{15}$$

Then $\mathcal{A}$ is indeterminate if and only if $||T||_{H^2\mapsto H^2} < \infty$.

Proof. We only need to consider the case $||T||_{H^2\mapsto H^2} < \infty$, and hence $K(x, y) \neq 0$. Notice that

$$\begin{aligned}(1, z, ..., z^{n+1})T_{n+1}(xI - \mathcal{A})_{n+1,n} &= (q_0(z), ..., q_{n+1}(z))(xI - \mathcal{A})_{n+1,n} \\ &= (x - z)(1, z, ..., z^n)T_n.\end{aligned}$$

Comparing powers of z we obtain

$$T_{n+1}(xI - \mathcal{A})_{n+1,n} = M_n(x)T_n,$$

where

$$M_n(x) := \begin{bmatrix} x & 0 & \cdots & 0 \\ -1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & x \\ 0 & \cdots & 0 & -1 \end{bmatrix},$$

with $||M_n(x)v|| \geq (1 - |x|)||v||$ for any vector v. Consequently,

$$\begin{aligned} 0 < \frac{1 - |x|}{||T||_{H^2\mapsto H^2}} &\leq (1 - |x|) \min_n \min_{v\in\mathbb{C}^{n+1}} \frac{||T_{n+1}^{-1}v||}{||v||} \\ &\leq (1 - |x|) \min_n \min_{v\in\mathbb{C}^n} \frac{||T_{n+1}^{-1}M_n(x)v||}{||M_n(x)v||} \\ &\leq \min_n \min_{v\in\mathbb{C}^n} \frac{||T_{n+1}^{-1}M_n(x)v||}{||v||} \\ &= \min_n \min_{v\in\mathbb{C}^n} \frac{||(xI - \mathcal{A})_{n+1,n}v||}{||T_n v||} \\ &= \min_{v\in\mathcal{C}_0} \frac{||(xI - A)v||}{||Tv||} \\ &\leq \frac{1}{\sqrt{1 - |y^2|}} \min_{v\in\mathcal{C}_0} \frac{||(xI - A)v||}{|(1, y, y^2, ...)Tv|}. \end{aligned}$$

As before we get $q(x) \in \ell^2$ from (13), and thus $q(x) \in \mathcal{N}(xI - A_{\max})$. If $\mathcal{A}$ is determinate, then also proper by Theorem 2.1, and hence $q(x) \in \mathcal{N}(xI - A_{\min})$. As a consequence, there is a sequence $(v_n) \subset \mathcal{C}_0$, with $||(xI - A)v_n|| \to 0$ and $v_n \to q(x)$ for $n \to \infty$. Consequently,

$$|(1, y, y^2, ...)Tv_n| = |(q_0(y), q_1(y), ...)v_n| \to |K(x,y)| \neq 0,$$

a contradiction to the above chain of inequalities. Hence $\mathcal{A}$ is indeterminate, as claimed in the statement of Theorem 3.3. □

4 Theorem 1.1 revisited for complex Jacobi matrices

In §2 we have seen that the properties **(B)** and **(C)** of Theorem 1.1 remain equivalent for complex Jacobi matrices. Also, property **(D)** implies property **(B)** by Theorem 3.1, and we have given large classes of complex Jacobi matrices in §3 where also the reciprocal is true. The following example shows that property **(F)** does no longer imply property **(B)** .

Example 4.1 *Let $\mathcal{A}$ be any indeterminate complex Jacobi matrix. From, e.g., Corollary 2.2 we see that $\mathcal{A}$ remains indeterminate after changing a_0 and b_0. For $a_0 = i$, $b_0 = -1$ we have*

$$c_0 = 1, \qquad c_1 = (e_0, \mathcal{A}e_0) = b_0 = -1, \qquad c_2 = (e_0, \mathcal{A}^2 e_0) = b_0^2 + a_0^2 = 0,$$

and

$$H_1 = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$$

has the eigenvalues $(1 \pm \sqrt{3})/2$, of different sign. Taking into account that H_1 is a principal minor, we get for all $n \geq 1$

$$0 \in conv(\sigma(H_1)) = \Theta(H_1) \subset \Theta(H_n),$$

i.e., property **(F)** *of Theorem 1.1 is true.*

From (14) we obtain that $||H_n^{-1}|| \leq ||T_n||^2$. As a consequence, property **(E)** implies property **(D)** , but we do not know whether the reciprocal is false. Finally, notice also that property **(E)** trivially implies property **(F)** (but not the reciprocal, see Example 4.1).

$$\begin{array}{ccccc} (\mathbf{B}) & \longleftrightarrow & (\mathbf{C}) & \nwarrow & \\ \not\uparrow & & & & (\mathbf{D}) \\ (\mathbf{F}) & \longleftarrow & (\mathbf{E}) & \nearrow & \end{array}$$

Figure 1: *The implications of Theorem 1.1 remaining valid for complex Jacobi matrices*

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