Some spectral properties of infinite band matrices.

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Dedicated to Claude Brezinski on the occasion of his 60th birthday.

Abstract

For operators generated by a certain class of infinite band matrices we establish a characterization of the resolvent set in terms of polynomial solutions of the underlying higher order recurrence relations. This enables us to describe some asymptotic behaviour of the corresponding systems of vector orthogonal polynomials. Finally we provide some new convergence results for Matrix–Padé approximants.

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1 Introduction

One of the main tools in the spectral analysis of nonsymmetric difference operators generated by infinite band matrices (band operators) is the study of the behaviour of polynomials defined by a systems of orthogonality relations and connected to *Hermite-Padé approximants* [9] or Matrix Padé approximants. In case of tridiagonal band matrices (so-called complex Jacobi matrices), this link has been successfully exploited in the last years not only to provide new characterizations of the resolvent set, but also to derive new convergence results for Padé approximants in terms of their J continued fraction coefficients, see for instance the survey paper [6] and the references therein. In this paper we generalize some results of [9, 2, 4, 5] to the case of a band operator of arbitrary order which is not assumed to be bounded. We mention here the importance of such studies to the spectral theory of differential operators. For example, various classes of such operators in $L^2(-\infty,\infty)$ with polynomial coefficients admit a band matrix representation in the basis of Hermite polynomials [11].

Consider some infinite nonsymmetric band matrix $A = (A_{k,l})_{k,l=0}^{\infty}$ with complex entries satisfying for all k and for all $\ell < k-s$ or $\ell > k+r$

$$A_{k,\ell} = 0, \quad A_{k,k+r} \neq 0, \quad A_{k+s,k} \neq 0. \tag{1}$$

That is, A takes the form

$$A = \begin{pmatrix} A_{0,0} & \dots & A_{0,r} & 0 & 0 & \dots & \dots & \dots \\ A_{1,0} & A_{1,1} & \dots & A_{1,r+1} & 0 & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & & \\ A_{s,0} & A_{s,1} & A_{s,2} & \dots & \dots & A_{s,s+r} & 0 & \dots \\ 0 & A_{s+1,1} & A_{s+1,2} & A_{s+1,3} & \dots & \dots & A_{s+1,s+r+1} & \dots \\ 0 & 0 & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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with $s+r+1$ nontrivial diagonals, where r and s are some fixed natural numbers. To the matrix A we assign finite-difference equations

$$A_{k,k-s}Y_{k-s} + A_{k,k-s+1}Y_{k-s+1} + \cdots + A_{k,k+r}Y_{k+r} = \lambda Y_k, \tag{2}$$

$$Y^+_{k-r}A_{k-r,k} + Y^+_{k-r+1}A_{k-r+1,k} + \cdots + Y^+_{k+s}A_{k+s,k} = \lambda Y^+_k, \tag{3}$$

where $k \geq 0$, $\lambda \in \mathbb{C}$ is some parameter, and the quantities $A_{k,\ell}$ with negative indices are arbitrary numbers satisfying assumption (1). We consider some particular fundamental systems of solutions $\{P^L(\lambda), Q^L(\lambda)\}$ of (2), and $\{P^R(\lambda), Q^R(\lambda)\}$ of (3), respectively, with elements being polynomials with respect to λ: Denote by

$$Q^L(\lambda) = (Q^L_k(\lambda))_{k=-s}^{\infty} = (Q^{1,L}_k(\lambda), \ldots, Q^{r,L}_k(\lambda))_{k=-s}^{\infty};$$
$$P^L(\lambda) = (P^L_k(\lambda))_{k=-s}^{\infty} = (P^{1,L}_k(\lambda), \ldots, P^{s,L}_k(\lambda))_{k=-s}^{\infty};$$

solutions of (2) satisfying the initial conditions

$$Q^L_{0:r-1} = I_r,\ Q^L_{-s:-1} = 0_{s\times r},\ P^L_{-s:-1} = (A_{0:s-1,-s:-1})^{-1},\ P^L_{0:r-1} = 0_{r\times s}.$$

Furthermore, denote by

$$Q^R(\lambda) = \begin{pmatrix} Q^{1,R}_k(\lambda) \\ \vdots \\ Q^{s,R}_k(\lambda) \end{pmatrix}_{k=-r}^{\infty}, \quad P^R(\lambda) = \begin{pmatrix} P^{1,R}_k(\lambda) \\ \vdots \\ P^{r,R}_k(\lambda) \end{pmatrix}_{k=-r}^{\infty}$$

solutions of the dual recurrence relation (3) satisfying the initial conditions

$$Q^R_{0:s-1} = I_s,\ Q^R_{-r:-1} = 0_{s\times r},\ P^R_{-r:-1} = (A_{-r:-1,0:r-1})^{-1},\ P^R_{0:r-1} = 0_{r\times s}.$$

Here and in what follows we use the following notations: $Q^L_{j:k}$ (and $Q^R_{j:k}$, respectively) for the stacked matrix with rows Q^L_ℓ, $\ell = j, j+1, ..., k$ (with columns Q^R_ℓ, $\ell = j, j+1, ..., k$), etc; $I_i, 0_{i\times j}$ for the identity and zero matrices of sizes i and $i\times j$. We shall also use $A_{j:k,m:n}$ for the submatrix of A composed of its rows labeled j to k, and its colums labeled m to n. Finally, we use block matrix notations like $[M\ N]$.

Algebraic aspects of these vector-valued polynomials have been discussed in detail by Sorokin and Van Iseghem [13, 14], see also the unpublished manuscript [3]. By the matrix version of the Shohat-Favart Theorem [13, Theorem 6], the systems $Q^L(\lambda)$ and $Q^R(\lambda)$ are satisfying the biorthogonality relation $< Q^L_n(\lambda), Q^R_k(\lambda) >= \delta_{n,k}$ with respect to some bilinear form $< \cdot, \cdot >$, see also Remark 1 below. In [14, §6, Eqn (11)] it is shown that the quantities

$$\pi_k(\lambda) = Q^L_{k:k+r-1}(\lambda)^{-1}P^L_{k:k+r-1}(\lambda) = P^R_{k:k+s-1}(\lambda)Q^R_{k:k+s-1}(\lambda)^{-1} \tag{4}$$

are convergents of a matrix continued fraction. These convergents are shown in [14, Theorem 1] to be Matrix Padé approximants of order k of some formal power series F at infinity with $r\times s$–valued coefficients (with respect to so-called regular multi-indices), the power series F being *weakly perfect* by [14, Theorem 6]. The above-mentioned authors finally show in [14, Theorem 5] that (P^R_n, Q^R_n) may be considered as vector Hermite-Padé approximants of F, including the classical Hermite-Padé approximants of type 1 for $r = 1$ and of type 2 for $s = 1$.

Duality relations like (4) or the Christoffel-Darboux formula of [13, Section 7] are closely connected to the fundamental work of Mahler [12] on duality between Hermite-Padé approximants of the types 1 and 2. In our setting the complete duality relations can be written as follows.

Lemma 1. *For all $k \geq 0$*

$$I_{r+s} = \begin{bmatrix} P^R_{k-r:k+s-1} \\ Q^R_{k-r:k+s-1} \end{bmatrix} \cdot \begin{bmatrix} 0_{r\times s} & -A_{k-r:k-1,k:k+r-1} \\ A_{k:k+s-1,k-s:k-1} & 0_{s\times r} \end{bmatrix} \cdot \begin{bmatrix} Q^L_{k-s:k+r-1}, -P^L_{k-s:k+r-1} \end{bmatrix}$$

This lemma is proved by induction with respect to k using (2) and (3).

In the present paper, we are less interested in algebraic aspects but more in analytic aspects of band operators and their associated polynomial solutions of the recurrence relations (2) and (3). We show in §2 by generalizing previous work [9, 5] for $r = 1$ that these vector polynomials enable us to derive a criterion for the resolvent set (see Theorem 1). As a consequence, we obtain in Theorem 2 some asymptotic behavior of vector orthogonal polynomials outside the spectrum of A. Section 3 is devoted to some convergence result for Matrix Padé approximants generalizing results for scalar Padé approximation from [2, 4].

2 Band operators and their resolvent

Denote by ℓ^2 the set of quadratic summable sequences, by $(e_n)_{n\geq 0}$ its usual canonical basis, and by $L(\ell^2)$ the set of bounded operators acting on ℓ^2. In what follows we will identify the (possibly unbounded) matrix A with the closure of the (densely defined) operator acting on finite linear combinations of the basis vectors e_k by the formula

$$Ae_k = \sum_{\max\{0,k-r\}}^{k+s} e_j A_{j,k}, \quad k \geq 0.$$

We notice that the action of this operator A is described via matrix calculus.

Recall that λ is an element of the resolvent set $\Omega(A)$ if there exists an operator $R(\lambda) = (\lambda I - A)^{-1} \in L(\ell^2)$ (I being the identity in $L(\ell^2)$) referred to as the resolvent of A such that $AR(\lambda)X = X$ and $R(\lambda)AY = Y$ for any $X \in \ell^2$ and $Y \in D(A)$, the domain of A. In [9], Kaliaguine considered the particular case $r = 1$ of bounded band operators. He showed that $\lambda \in \Omega(A)$ if and only if there is some exponential decay rate for particular fundamental systems of solutions of (2) and (3). In the present paper we consider the case of possibly unbounded general band operators A, with $(a_k)_{k\geq 0}$ defined by

$$a_k := \max\{||A_{k-r:k-1,k:k+r-1}||, ||A_{k:k+s-1,k-s:k-1}||\}, \quad k \geq 0,$$

containing a sufficiently dense bounded subsequence. Generalizing [5, Theorem 2.1] for the case $r = s = 1$ of a complex Jacobi matrix, we may establish the following criterion for $\Omega(A)$ together with an exponential decay rate.

Theorem 1. *Suppose that the band operator A with matrix representation (1) satisfies the following condition: there exists $\alpha > 0$, $p \in \mathbb{N}$ and a strictly increasing sequence of nonnegative integers $(k_n)_{n\geq 0}$ such that*

(5) $$k_{n+1} - k_n \leq p \quad \text{and} \quad a_{k_n} \leq \alpha, \quad n \geq 0.$$

Then $\lambda \in \mathbb{C}$ belongs to the resolvent set of A if and only if there exist positive constants C, $q < 1$ and a matrix $\mathfrak{M} = (\mathfrak{M}_{i,j})_{i=1,\ldots,r}^{j=1,\ldots,s} \in \mathbb{C}^{r\times s}$ such that

(6) $$\begin{aligned} |Q^L_k(\lambda)R^R_n(\lambda)| &\leq Cq^{n-k}, \quad 0 \leq k < n+r; \\ |R^L_k(\lambda)Q^R_n(\lambda)| &\leq Cq^{k-n}, \quad 0 \leq n < k+s, \quad k,n \in \mathbb{Z}_+, \end{aligned}$$

where $R_k^L(\lambda) = Q_k^L(\lambda)\mathfrak{M} - P_k^L(\lambda)$ *and* $R_n^R(\lambda) = \mathfrak{M}Q_n^R(\lambda) - P_n^R(\lambda)$. *In this case, the matrix* $\mathfrak{M} = \mathfrak{M}(\lambda)$ *is unique, and its elements are calculated by the formula*

$$\mathfrak{M}_{i,j}(\lambda) = (R(\lambda)e_{j-1}, e_{i-1}). \tag{7}$$

Proof. First of all, consider an arbitrary matrix $\mathfrak{M} \in \mathbb{C}^{r\times s}$. For $k, n \in \mathbb{Z}_+$ we define

$$R_{k,n} = \begin{cases} Q_k^L(\lambda)R_n^R(\lambda), & 0 \le k < n + r, \\ R_k^L(\lambda)Q_n^R(\lambda), & 0 \le n < k + s \end{cases} \tag{8}$$

(notice that the two different definitions for $n - s < k < n + r$ give the same value by Lemma 1, see also formula (15) below). We claim that for the infinite matrix $R = (R_{k,n})_{k,n=0}^{\infty}$ we have the matrix identities (formal products between infinite matrices)

$$(\lambda I - A)R = I, \quad R(\lambda I - A) = I. \tag{9}$$

By symmetry (replace A by its transposed), it is sufficient to show the first identity. Since $(Q_n^L(\lambda))_{n\ge 0}$ is a solution of (2), we have

$$(\lambda I - A)\begin{bmatrix} R_{0:k-1,n} \\ 0_{\infty,1} \end{bmatrix} = \begin{bmatrix} 0_{(k-s)\times 1} \\ A_{k-r:k-1,k:k+r-1}R_{k:k+r-1,n} \\ -A_{k:k+s-1,k-s:k-1}R_{k-s:k-1,n} \\ 0_{\infty\times 1} \end{bmatrix}, \quad s \le k \le n. \tag{10}$$

Similarly, $(R_n^L(\lambda))_{n\ge 0}$ is a solution of (2), and hence we have the (formal) identity

$$(\lambda I - A)\begin{bmatrix} 0_{n,1} \\ R_{n:\infty,n} \end{bmatrix} = \begin{bmatrix} 0_{(n-s)\times 1} \\ -A_{n-r:n-1,n:n+r-1}R_{n:n+r-1,n} \\ A_{n:n+s-1,n-s:n-1}R_{n-s:n-1}^L(\lambda)Q_n^R(\lambda) \\ 0_{\infty\times 1} \end{bmatrix}. \tag{11}$$

Combining identity (10) for $k = n$ with (11) leads to the (formal) identity

$$(\lambda I - A)R_{0:\infty,n} = \begin{bmatrix} 0_{n\times 1} \\ A_{n:n+s-1,n-s:n-1}[R_{n-s:n-1}^L(\lambda)Q_n^R(\lambda) - R_{n-s:n-1,n}] \\ 0_{\infty\times 1} \end{bmatrix} = e_n,$$

(here we used that $R_{n-s}^L(\lambda)Q_n^R(\lambda) - Q_{n-s}^L(\lambda)R_n^R(\lambda) = A_{n,n-s}^{-1}$, following again from Lemma 1). This shows claim (9). Notice that in the above reasoning we require that $n \ge s$. A proof for the case $0 \le n < s$ is similar, we omit the technical details.

Now let $\lambda \in \Omega(A)$ and $\mathfrak{M}(\lambda)$ as in (7). From the resolvent identity $(\lambda I - A)R(\lambda) = I$ together with (9) we see that the first s columns of the matrix representation of $R(\lambda)$ and of the matrix R satisfy the same recurrence relation and the same initializations (7); hence

$$R_{j,k} = (R(\lambda)e_k, e_j), \quad j \ge 0, \quad 0 \le k < s.$$

In other words, the first s terms in corresponding rows of both matrices are equal. Observing that the recurrence relations for the corresponding rows also coincide, we may conclude that R is the matrix representation of $R(\lambda)$. It remains to show the exponential decay rate (6) which for bounded operators has been established already in [8]. By symmetry it is sufficient to show again only the first part. Define

$$w_{k,n} := ||R_{k-s:k+r-1,n}||^2 \le ||R(\lambda)||^2, \quad 0 \le k \le n. \tag{12}$$

By noticing that the vector on the left-hand side of (10) is a finite combination of the basis elements $e_0, ..., e_{k-1}$ and hence an element of $D(A)$, we obtain from (10) the inequality

$$\sum_{j=s}^{k-1} w_{j,n} \leq (s+r) \cdot \Big(||R_{0:k-1,n}||^2 + w_{k,n} \Big) \leq a'_k \cdot w_{k,n}, \quad s \leq k \leq n,$$

where $a'_k := (s+r) \cdot (a_k^2 \cdot ||R(\lambda)||^2 + 1)$. Recall that, because of (5), the sequence $(k_{n+1} - k_n)$ is bounded by p and the sequence (a'_{k_n}) is bounded by some α'. Therefore, according to [5, Lemma 2] there exist $\tilde{C} > 0$ and $\tilde{q} \in (0,1)$ depending only on a'_k but not on the $w_{k,n}$ such that

$$s \leq j < k \leq n : \quad w_{j,n} \leq \tilde{C} \cdot \tilde{q}^{k-j} \cdot a'_k \cdot w_{k,n}.$$

Since for each n we find an $k \in \{n-p+1, ..., n\}$ being an element of (k_n), we may deduce that for $s \leq j \leq n-p$ there holds

$$|R_{j-s,n}|^2 \leq w_{j,n} \leq \tilde{C} \cdot \tilde{q}^{k-j} \cdot \alpha' \cdot w_{k,n} \leq (\tilde{C} \cdot \alpha' \cdot ||R(\lambda)||^2 \cdot \tilde{q}^{-p-s}) \cdot \tilde{q}^{n-(j-s)},$$

where for the last estimate we have used (12). On the other hand, for $j = n-s-p+1, ..., n+r-1$ we obtain directly from (12) that

$$|R_{j,n}|^2 \leq \max\{w_{n-p,n}, ..., w_{n,n}\} \leq (||R(\lambda)||^2 \cdot \tilde{q}^{-s-p}) \cdot \tilde{q}^{n-j}.$$

Therefore the claimed relation (6) is true with $q^2 = \tilde{q}$ and a suitable constant $C > 0$. Hence the necessity of Theorem 1 is proved.

Now let (6) be fulfilled. We build up the infinite matrix $R = (R_{k,n})_{k,n=0}^{\infty}$, where $R_{k,n}$ are defined by (8). Then as in Theorem 2 from [9] we obtain that such a matrix generates a (minimal) bounded operator (also denoted by R) with domain $D(R) = \ell^2$. From the second identity of (9) it follows that $R(\lambda I - A)Y = Y$ for any finite linear combination Y of the basis vectors, and hence also for any $Y \in D(A)$ by definition of the operator A. Therefore R is a bounded left inverse operator for $\lambda I - A$. We also have that $Im(\lambda I - A)$ is closed as the preimage of the closed set $D(A)$ under the continuous mapping. Hence $\lambda \in \Omega(A)$ together with the relation $R = R(\lambda)$ follow by showing that $Re_n \in D(A)$ and $(\lambda I - A)(Re_n) = e_n$ for all $n \geq 0$. In order to show this latter claim, we observe first that, similarly to the reasoning of (10) and (11), one finds that

$$(\lambda I - A)U^k = \begin{bmatrix} 0_{(k-s)\times 1} \\ A_{k-r:k-1,k:k+r-1} R_{k:k+r-1,n} \\ -A_{k:k+s-1,k-s:k-1} R_{k-s:k-1,n} \\ 0_{\infty \times 1} \end{bmatrix} + e_n, \quad U^k := \begin{bmatrix} R_{0:k-1,n} \\ 0_{\infty,1} \end{bmatrix}, \quad k > n.$$

Here $U^k \in D(A)$ as a finite combination of basis elements, with $U^k \to Re_n \in \ell^2$ for $k \to \infty$, and $||(\lambda I - A)U^k - e_n|| \leq a_k ||R_{k-s:k+r-1,n}||$ tending to zero for $k = k_j$, $j \to \infty$. Therefore, $Re_n \in D(A) = D(\lambda I - A)$, with $(\lambda I - A)Re_n = e_n$. Consequently, $\lambda \in \Omega(A)$, and $R = R(\lambda)$. In particular, the matrix $\mathfrak{M}$ being the $r \times s$ principal submatrix of the matrix R needs to be unique, and formula (7) follows. □

Corollary 1. *If $\lambda \in \Omega(A)$ and A satisfies the conditions of Theorem 1, then*

$$\limsup_{k\to\infty} ||R_k^L(\lambda)||^{\frac{1}{k}} < 1, \quad \limsup_{n\to\infty} ||R_n^R(\lambda)||^{\frac{1}{n}} < 1. \tag{13}$$

The matrix $\mathfrak{M} = \mathfrak{M}(\lambda)$ of Theorem 1 is called the *Weyl matrix* of the band operator A. For the particular case $r = s = 1$ of a Jacobi matrix A we recover the classical Weyl function as introduced by Berezanskii [7]. From (7) we see that $\mathfrak{M}$ is analytic in $\Omega(A)$.

Remark 1. It is not difficult to show that the power series F mentioned in the introduction in the context of formula (4) coincides with the formal asymptotic expansion at infinity

$$M_\infty(\lambda) = \sum_{k=0}^{\infty} z^{-1-k}\Big((A^k e_j, e_i)\Big)_{i=0,...,r-1;j=0,...,s-1}$$

coinciding with the Laurent expansion of $\mathfrak{M}$ at infinity in case of bounded A. As a consequence, the quantity π_n of (4) gives a Matrix Padé approximant of $\mathfrak{M}_\infty$ at infinity of order n. Moreover, the bilinear form

$$< P, Q >= \text{coeff}(\lambda^{-1}, P(\lambda)\mathfrak{M}_\infty(\lambda)Q(\lambda)), \quad P \in \mathbb{C}^{1\times r}[\lambda], \quad Q \in \mathbb{C}^{s\times 1}[\lambda]$$

is easily shown to verify the biorthogonality relations $< Q_n^L, Q_n^R >= \delta_{n,k}$ as required for the matrix version of the Shohat-Favard Theorem mentioned in the introduction. Following Kishakevich [10] who discussed the case of block tridiagonal matrices, we call such a bilinear form a *generalized spectral function* of A. Notice also that the knowledge of $\mathfrak{M}_\infty$ allows to reconstruct the operator A (up to normalization) either by using the orthogonalization procedure described in [16] or by expansion into a matrix continued fraction as proposed in [14].

Remark 2. Using the Cauchy formula, it is possible to give some analytic formulas, at least in the case of bounded operators A: for instance, the bilinear form is given by

$$< P, Q >= \frac{1}{2\pi i}\int_{\mathcal{C}} P(\lambda)\mathfrak{M}(\lambda)Q(\lambda)\, d\lambda,$$

where $\mathcal{C}$ be some contour in $\Omega(A)$ surrounding once ∞ in a clockwise direction. Furthermore, for the resolvent we have the following integral representation

$$(R(\lambda)e_k, e_n) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{Q_n^L(\lambda')\mathfrak{M}(\lambda')Q_k^R(\lambda')}{\lambda - \lambda'}\, d\lambda', \quad \lambda \in \Omega(A)$$

where the contour $\mathcal{C} \subset \Omega(A)$ surrounds once ∞ and λ. This latter formula includes for $k = 0, ..., s-1$ (and for $n = 0, ..., r-1$) the classical integral representation for functions of the second kind $R_n^L(\lambda)$ (and $R_k^R(\lambda)$, respectively). More generally, for any function ψ being analytic in some neighborhood of the spectrum of A we deduce the following matrix representation

$$\psi(A) = \Big(\frac{1}{2\pi i}\int_{\mathcal{C}} Q_n^L(\lambda)\mathfrak{M}(\lambda)Q_k^R(\lambda)\psi(\lambda)\, d\lambda\Big)_{n,k=0}^{\infty}.$$

Remark 3. As mentioned in [6, Section 2.1], we may identify the infinite matrix A via matrix calculus with possibly two closed and densely defined operators acting on ℓ^2. Let us write more explicitly $[A]_{\min}$ for the closure of the (densely defined) operator acting on finite linear combinations of the basis vectors e_k as considered before. The operator $[A]_{\max}$ with domain of definition $D([A]_{\max}) = \{y \in \ell^2,\ Ay \in \ell^2\}$ is easily seen to be the maximal closed operator acting via matrix calculus, and hence a closed extension of $[A]_{\min}$. Writing $A^H = (\bar{A}_{l,k})_{k,l=0}^{\infty}$ for the conjugate matrix, one may show [6, Lemma 2.1] that $([A]_{\min})^* = [A^H]_{\max}$, and $([A^H]_{\max})^* = [A]_{\min}$. The matrix A is called proper if the operators $[A]_{\min}$ and $[A]_{\max}$ coincide. We refer the reader to [6, Section 2.2] for further details and especially for the link between proper and determinated Jacobi matrices.

It is possible to show following the reasoning of [6, Example 2.7] that the divergence of $\sum_k 1/a_k$ implies that A is proper. Thus for instance bounded A or more generally the matrices considered in Theorem 1 are proper. Indeed, it is possible to show by generalizing the reasoning of [6, Theorem 2.10] that $\lambda \in \Omega(A)$ if and only if A is proper, and there exists a bounded operator within the set of some formal left inverses built with help of the fundamental solutions (which are given in (8) and (9)). In this case, the corresponding formal left inverse is the matrix representation of the resolvent of A.

Theorem 2. *If $\lambda \in \Omega(A)$ and A satisfies the conditions of Theorem 1, then for $i = 1, \dots, r$, and $j = 1, \dots, s$ there holds*

$$\limsup_{k\to\infty} |Q_k^{i,L}(\lambda)|^{\frac{1}{k}} > 1, \quad \limsup_{k\to\infty} |Q_k^{j,R}(\lambda)|^{\frac{1}{k}} > 1. \tag{14}$$

Proof. Multiplying the equation of Lemma 1 on the left and on the right with

$$\begin{bmatrix} I_r & -\mathfrak{M} \\ 0 & I_s \end{bmatrix}, \quad \begin{bmatrix} I_r & \mathfrak{M} \\ 0 & I_s \end{bmatrix}$$

gives

$$\begin{aligned} I_{r+s} = & \begin{bmatrix} -R^R_{k-r:k+s-1} \\ Q^R_{k-r:k+s-1} \end{bmatrix} \\ & \cdot \begin{bmatrix} 0_{r\times s} & -A_{k-r:k-1,k:k+r-1} \\ A_{k:k+s-1,k-s:k-1} & 0_{s\times r} \end{bmatrix} \cdot \begin{bmatrix} Q^L_{k-s:k+r-1} R^L_{k-s:k+r-1} \end{bmatrix} \end{aligned} \tag{15}$$

and hence in particular

$$Q^R_{k-r:k+s-1} \cdot \begin{bmatrix} 0_{r\times s} & -A_{k-r:k-1,k:k+r-1} \\ A_{k:k+s-1,k-s:k-1} & 0_{s\times r} \end{bmatrix} \cdot R^L_{k-s:k+r-1} = I_s.$$

Hence for any vector $a, ||a|| \geq 1$ of size s and for indices $k = k_n$ where the above coefficients are bounded in norm

$$||a||^2 \leq ||a^T Q^R_{k-r:k+s-1}|| \cdot a_k \cdot ||R^L_{k-s:k+r-1} a||.$$

Since $||R^L_{k-s:k+r-1} a|| \leq ||R^L_{k-s:k+r-1}|| \cdot ||a||$, we deduce that

$$\begin{aligned} 1 & \leq \liminf_{n\to\infty} ||a^T \cdot Q^R_{k_n-r:k_n+s-1}(\lambda)||^{1/k_n} \cdot ||R^L_{k_n-s:k_n+r-1}||^{1/k_n} \\ & \leq \liminf_{n\to\infty} ||a^T \cdot Q^R_{k_n-r:k_n+s-1}(\lambda)||^{1/k_n} \cdot \limsup_{n\to\infty} ||R^L_{k_n-s:k_n+r-1}||^{1/k_n} \\ & < \liminf_{n\to\infty} ||a^T \cdot Q^R_{k_n-r:k_n+s-1}(\lambda)||^{1/k_n} \leq \limsup_{k\to\infty} ||a^T \cdot Q^R_k(\lambda)||^{1/k}. \end{aligned}$$

Taking as a a unit vector, we get asymptotics for each component. A similar result is obtained for the Q^L_k. □

Consider as an example for some $\beta > 0$ the operator A^β generated by the matrix A^β with $s = 2, r = 1$, and two nonzero diagonals:

$$\begin{aligned} (A^\beta_{n+2,n})_{n\geq 0} &= (1, 1, 2^\beta, 2^\beta, 1, 1, 3^\beta, 3^\beta, 1, 1, 4^\beta, 4^\beta, \dots), \\ (A^\beta_{n,n+1})_{n\geq 0} &= (1, 1, 1, (1/2)^\beta, 1, 1, 1, (1/3)^\beta, 1, 1, 1, (1/4)^\beta, \dots). \end{aligned}$$

Notice that $A^\beta = DA^0D^{-1}$, with the diagonal matrix $D = \text{diag}(d_0, d_1, ...)$, and $d_{4k} = d_{4k+1} = d_{4k+2} = d_{4k+3} = ((k+1)!)^\beta$ for $k \geq 0$, such that

$$a_{4k+1} = \max\{||A^\beta_{4k,4k+1}||, ||A^\beta_{4k+1:4k+2,4k-1:4k}||\} = 1, \quad k \geq 1.$$

At first, let $\beta = 0$. The operator A^0 is bounded, with two nonzero diagonals consisting of ones. It is known for instance from the theory of Toeplitz operators that the resolvent set of A^0 is the domain

$$\Omega(A^0) = \{\lambda : |w_1(\lambda)| > 1, |w_2(\lambda)| < 1, |w_3(\lambda)| < 1\}$$

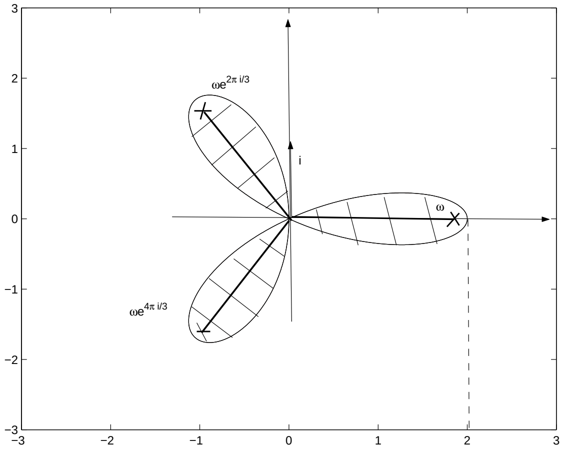

Figure 1: Spectrum of A^0

where $w_1(\lambda), w_2(\lambda), w_3(\lambda)$ are the branches of the algebraic function $w^3 - \lambda w^2 + 1 = 0$ in the complex plane λ with cuts along the segments $[0, \omega]$, $[0, \omega \exp(2\pi i/3)]$, and $[0, \omega \exp(4\pi i/3)]$, $\omega = \frac{3}{4^{1/3}}$, such that $|w_1(\lambda)| > |w_2(\lambda)|$, $|w_1(\lambda)| > |w_3(\lambda)|$. Thus the spectrum of A^0 is a three-petal rose, see Figure 1. Now consider the case $\beta > 0$. In order to make the dependence on β more explicitly, we write $q_{k,\beta}^{1,R}(\lambda) = Q_k^{1,R}(\lambda)$. The reader can easily check by recurrence on k that $q_{k,\beta}^{1,R}(\lambda) = q_{k,0}^{1,R}(\lambda)/d_k$. From (3) one easily sees that $q_{k,0}^{1,R}(\lambda)$ grows at most exponentially in k for any $\lambda \in \mathbb{C}$, whereas d_k grows faster than exponentially. Hence the second condition of (14) is not fulfilled for $q_{k,\beta}^{1,R}(\lambda)$ for any $\lambda \in \mathbb{C}$, implying that the spectrum of A^β is the whole plane $\mathbb{C}$ for all $\beta > 0$. We have therefore found a proper matrix A^β with empty resolvent set.

3 Convergence of Matrix Padé approximants

Once having established a criterion of the resolvent set similar to the case of complex Jacobi matrices, we now show that known convergence results of scalar Padé approximants in terms of the corresponding difference operator like [2, Theorem 3.11] or [4, Theorem 3.1 and Theorem 4.1] can be generalized to the matrix setting.

Theorem 3. *Let A be bounded. Then the sequence of Padé approximants $(\pi_k)_{k\geq 0}$ from (4) converges in capacity to the Weyl function in compact subsets of the unbounded component of the resolvent set of A. Moreover, the convergence is uniform in compact subsets of the complement of the numerical range of A.*

Proof. Suppose that $s \leq r$ (the case $r < s$ is similar). We first notice that the sequence of matrices

$$R^L_{n:n+s-1}(\lambda) Q^R_{n:n+s-1}(\lambda) = \Big(((\lambda I - A)^{-1} e_{n+j}, e_{n+k}) \Big)_{j,k=0,\ldots,s-1}$$

is uniformly bounded on compact subsets of $\Omega(A)$. According to the expansion of the resolvent at infinity, we obtain for such matrices the following expansion at infinity

$$R^L_{n:n+s-1}(\lambda) Q^R_{n:n+s-1}(\lambda) = \lambda^{-1} I_s + \mathcal{O}(\lambda^{-2}).$$

Consequently, the sequence of functions

$$h_n(\lambda) := \det(R^L_{n:n+s-1}(\lambda) Q^R_{n:n+s-1}(\lambda))$$

forms a normal family in $\Omega(A)$, with $h_n(\lambda) = \lambda^{-s} + \mathcal{O}(\lambda^{-s-1})$. This shows that any accumulation point of h_n in the unbounded component of the resolvent set is different from the function 0. Following [4, Lemma 2.4(c)], we may deduce that for any compact F a subset of the unbounded connected component of the resolvent set there exist constants $C_1, C_2 > 0$ and monic polynomials p_j of degree bounded by C_2 such that $|h_n(\lambda)| \geq C_1 \cdot |p_n(\lambda)|$ for all $n \geq 0$ and for all $\lambda \in F$. By noticing that the matrix of cofactors of $R^L_{n:n+s-1}(\lambda) Q^R_{n:n+s-1}(\lambda)$ is also bounded uniformly on F, we find a constant C_3 such that

$$\sup_{n\geq 0} \sup_{\lambda\in F} |p_n(\lambda)| \cdot ||[R^L_{n:n+s-1}(\lambda) Q^R_{n:n+s-1}(\lambda)]^{-1}|| \leq C_3.$$

On the other hand, it is easy to show that the constants $C = C(\lambda)$ and $q = q(\lambda)$ of Theorem 1 may be chosen to be continuous as a function of λ. Hence the limit relations of Corollary 1 are true uniformly in F, that is,

$$\limsup_{n\to\infty} \sup_{\lambda\in F} ||R^L_{n:n+s-1}(\lambda)||^{1/n} < 1, \quad \limsup_{n\to\infty} \sup_{\lambda\in F} ||R^R_{n:n+s-1}(\lambda)||^{1/n} < 1.$$

Consequently,

$$\begin{aligned} &\limsup_{n\to\infty} \sup_{\lambda\in F} [|p_n(\lambda)| \cdot ||\mathfrak{M}(\lambda) - Q^L_{k:k+r-1}(\lambda)^{-1} P^L_{k:k+r-1}(\lambda)||]^{1/n} \\ &\leq \limsup_{n\to\infty} \sup_{\lambda\in F} ||R^L_{n:n+s-1}(\lambda)||^{1/n} \cdot ||R^R_{n:n+s-1}(\lambda)||^{1/n} < 1, \end{aligned}$$

implying convergence in capacity in F (compare with the proof of [4, Theorem 3.1]). In order to establish the second part of the assertion, it is possible to show following the lines of the proof of Theorem 1 that $\pi_n(\lambda)$ is the Weyl function of the (finite dimensional) operator $A_{0:n-1,0:n-1}$. Since the numerical range of this operator is included in the numerical range of A, we may conclude that π_n is analytic outside the numerical range of A. Thus, the uniform convergence follows from the convergence of capacity and the Lemma of Gonchar (see, e.g., [4, Section 4]). □

Remark 4. It seems that only few results on the convergence of Matrix-Padé approximants have been established so far. In case of a symmetric operator with $r = s > 1$, one may easily show that our Theorem 3 contains a matrix version of the Markov convergence theorem since here the numerical range of A coincides with the convex hull of the union of the supports of the underlying measures of the (matrix–valued) Markov function.

Another class of bounded operators leading to so-called Stieltjes systems has been discussed in [1] for $r = 1$ and in [15] for general r, s (the proof in the latter paper is only given for $r = 2$ and $s = 3$). This subclass of operators (1) is described by the requirement that $A_{k,k+r} > 0$, $A_{k+s,k} > 0$ for all $k \geq 0$ and $A_{k,\ell} = 0$ else. Here one may establish uniform convergence in some set which in general is shown to be larger than the complement of the convex hull of the spectrum.

We finally mention that the results given in this paper can be generalized to study spectra of operators generated by band matrices with operator elements.

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