A RECURRENCE RELATION CONNECTED TO THE CONVERGENCE OF VECTOR S-FRACTIONS

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Abstract. Recently, some sufficient and necessary conditions have been given on the convergence of the so-called vector Stieltjes continued fraction of dimension p in terms of the coefficients. In the present paper we aim to continue this study for the case of dimension 2. In particular, we show that here the convergence is determined by the asymptotics of solutions of a particular three-term recurrence relation, which is closely analyzed.

As a consequence, several new results on the convergence problem for two-dimensional Stieltjes continued fractions are obtained. We finally describe the link to a vector moment problem.

Key words: Convergence of S-fractions, Stieltjes moment problem. Simultaneous approximation.
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1. Introduction

The results of this paper are essentially motivated by the convergence problem for the vector Stieltjes continued fraction (VSCF). In the simplest vector case, a VSCF is a vector continued fraction of the form

$$\frac{(1,1)|}{|(0,z)} + \frac{(1,-a_1)|}{|(0,1)} + \frac{(1,-a_2)|}{|(0,1)} + \frac{(1,-a_3)|}{|(0,z)} + \frac{(1,-a_4)|}{|(0,1)} + \frac{(1,-a_5)|}{|(0,1)} + \cdots \tag{1}$$

where the division of vectors is defined according to the Jacobi-Perron rule. Vector Stieltjes continued fractions were introduced in [2] in connection with the study of non symmetric difference operators (see also [11] for general matrix continued fractions). If all $a_n > 0$ then the convergents of the VSCF (1) give a local approximation (at infinity) of two Stieltjes

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type formal power series

$$S_j(z) \sim \sum_{n=0}^{\infty} \frac{S_{n,j}}{z^{n+1}}, \quad \text{where} \quad S_{n,j} = \int_0^{\infty} x^n d\mu_j(x), \quad j = 1, 2$$

and μ_j are positive measures with a common support on $[0, \infty)$. The *convergence problem for vector Stieltjes continued fractions* is whether the VSCF converges everywhere in the complex plane outside the common support of the measures μ_j provided that all a_n are positive. Detailed analysis of the convergence problem (see [5] and Theorem 3.1 below) shows that the convergence of a VSCF at one point $z \in (-\infty, 0)$ is equivalent to a particular asymptotic behavior of the solutions of the following recurrence equation

$$y_n + c_n y_{n-1} + c_n y_{n-2} = 0, \quad n \geq 1, \tag{2}$$

where $c_n = c_n(z) \in (0, 1)$ are given real numbers. More precisely, the convergence of a VSCF is equivalent to the following property of the solutions of the recurrence equation (2)

Property (P): for any initial values y_0, y_{-1} one has $\lim_{n\to\infty} y_n = 0$.

We should mention in this context that, by Lemma 2.6, any solution of (2) is bounded.

The recurrence equation (2) can be written in the following equivalent form

$$\begin{bmatrix} y_n \\ y_{n-1} \end{bmatrix} = T_n \cdot \begin{bmatrix} y_{n-1} \\ y_{n-2} \end{bmatrix}, \quad T_n := \begin{bmatrix} -c_n & -c_n \\ 1 & 0 \end{bmatrix}, \tag{3}$$

We notice that a fundamental solution of (3) is given by the columns of the matrix $\tilde{T_n} := T_n \cdot T_{n-1} \cdot \ldots \cdot T_1$. Hence property (P) is equivalent to the question whether the unique fixed point $(0,0)^T$ of the dynamical system (3) is an attractive point, i.e., $||\tilde{T_n}|| \to 0$ for $n \to \infty$. In the latter case, the determinant of $\tilde{T_n}$ (which is easily calculated) needs to tend to zero. This gives the necessary condition

$$(P) \quad \Longrightarrow \quad \prod_{n=1}^{\infty} c_n = 0 \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} (1 - c_n) = +\infty. \tag{4}$$

As we will see in Corollary 2.3, this necessary condition (which using different techniques has been discovered earlier in [5]) turns out to be not sufficient. In the same paper [5], one of the authors has established the sufficient condition

$$(P) \quad \Longleftarrow \quad \sum_{n=1}^{\infty} (1 - c_n)(1 - c_{n+1}) = +\infty \tag{5}$$

(see also Subsection 3.2 below). In particular,

$$(P) \quad \Longleftarrow \quad \sup c_n < 1. \tag{6}$$

Here we will establish another sufficient condition, namely $\sum (1 - \max\{c_n, c_{n+1}, c_{n+2}\}) = +\infty$, see Theorem 2.1 below. The analysis of property (P) seems to be quite difficult if the sequence (c_n) tends to 1. However, under the additional condition $\sup(c_n + c_{n+1}) < 2$, we will show in Theorem 2.2 that (P) holds if and only if $\sum (1 - |c_n - c_{n+1}|) = \infty$.

The paper is organized as follows. Section 2 is devoted to the analysis of the recurrence relation (2). We first present in two Theorems some necessary and/or sufficient conditions on (c_n) in order to insure that property (P) holds. Subsequently, some illustrating examples as well as the proofs for these statements are given. One ingredient in our proofs are techniques known from the study of trace class perturbations of three term recurrence relations [4, 8].

Finally, in Section 3 we discuss the application of our results to the convergence of a VSCF. We show in detail how the convergence of such vector continued fractions is connected to the property (P) for recurrence equation (2). Then we use our findings of Section 2 in order to give answers to some open problems related with [3, 5], and to present some new sufficient conditions for the convergence of a VSCF. Finally we discuss the connection between the convergence of the VSCF and the determinacy of the vector Stieltjes moment problem.

2. The property (P)

2.1. **The statements.** As a complement of (5), we have the following sufficient condition for property (P).

Theorem 2.1. *If*

$$\sum_{n=1}^{\infty}(1-\max\{c_n,c_{n+1},c_{n+2}\})=+\infty \tag{7}$$

then (P) *holds true. If in addition*

$$\inf\frac{1-c_n}{1-c_{n+1}}>0 \quad \text{or} \quad \sup\frac{1-c_n}{1-c_{n+1}}<\infty \tag{8}$$

then condition (7) is also necessary for property (P).

In the next theorem we give a necessary condition for property (P) complementing (4). This condition turns out to be also sufficient under some additional assumption.

Theorem 2.2. *If property* (P) *holds then*

$$\sum_{n=1}^{\infty}(1-|c_n-c_{n+1}|)=\infty \tag{9}$$

If in addition

$$\sup_{n\in\mathbb{N}}\{c_n+c_{n+1}\}<2 \tag{10}$$

then (9) implies (P).

Before turning to the proof of these statements, let us shortly discuss some examples. For the sequence $c_n=1-1/n$, we obtain property (P) from Theorem 2.1 (but by none of the other necessary or sufficient criteria mentioned in this paper). Considering the two sequences $c_n=1-1/n^2$, and $c_{2k}=1-c_{2k+1}=1-1/(k+1)^2$, we see that the two necessary conditions of (4) and of Theorem 2.2 do not imply each other.

Theorem 2.2 is best illustrated by considering asymptotically periodic sequences: suppose that

$$\lim_{k\to\infty}c_{2k}=1, \quad \lim_{k\to\infty}c_{2k+1}=\gamma<1,$$

i.e., assumption (10) holds. Again, none of the previously found criteria (4), (5), (6), or (7) enable us to decide whether property (P) holds. If $\gamma \in (0,1)$ then the sum in (9) diverges, showing that (P) holds (indeed, we could also give a direct proof based on Perron's Theorem). If $\gamma = 0$, we may simplify the sum occurring in (9), leading to the following statement

Corollary 2.3. *Suppose that* (c_{2k}) *tends to* 1, *and* (c_{2k+1}) *tends to* 0. *Then*

$$(P) \quad \Longleftrightarrow \quad \sum_{k=1}^{\infty}(1-c_{2k}) = \infty \text{ or } \sum_{k=1}^{\infty} c_{2k-1} = \infty.$$

Taking $c_{2k} = 1 - 1/(k+1)^{3/2}$ and $c_{2k+1} = 1/(k+1)$, we know from Corollary 2.3 that (P) holds. Consequently, none of the sufficient conditions (5), (6), or (7) are necessary. Finally, the example $c_{2k} = 1 - c_{2k+1} = 1 - 1/(k+1)^2$ without property (P) shows that the converse of (4) is not true. Moreover, we also see that a Stieltjes–type condition of the form

$$\sum_{n=1}^{\infty} \sqrt{(1-c_n)(1-c_{n+1})} = +\infty \tag{11}$$

weakening (5) does no longer imply property (P).

2.2. **Some preliminary results.** Beside the application for vector continued fractions, the study of the recurrence relation (2) also attracted our attention since any solution (y_n) of (2) has the remarkable property that the sequence (δ_n), defined by

$$\delta_n = \delta_n(y_0, y_1) := \begin{cases} |y_{n-1}| + |y_n| & \text{if } y_n \cdot y_{n-1} \geq 0, \\ \max\{|y_{n-1}|, |y_n|\} & \text{if } y_n \cdot y_{n-1} \leq 0, \end{cases} \tag{12}$$

is decreasing for all values of y_0, y_{-1}, and thus has a limit $\delta(y_0, y_{-1}) \geq 0$, see Lemma 2.6 below. Hence (P) is equivalent to the fact that $\delta(y_0, y_{-1}) = 0$ for all $y_0, y_{-1} \in \mathbb{C}$. Notice that $y_{-1} = y_0 = 0$ gives the trivial solution $y_n \equiv 0$. Hence, in our further analysis this case will be excluded.

By definition (12), the quantity δ_n equals one of the three quantities $|y_{n-1}|$, $|y_n|$ or $|y_n| + |y_{n+1}|$. A key observation in our proofs of Theorem 2.1 and Theorem 2.2 is that any of these three cases occur in a quite regular manner while varying n; indeed, there are cycles of length 2 and 3, see Lemma 2.5. This will enable us to estimate in Lemma 2.6 and Lemma 2.7 the quantity δ_n in terms of the coefficients c_k and the previous quantities δ_k for $k < n$. However, since the above three cases are not mutually exclusive, we need to consider separately the cases where $y_n = 0$ or $y_{n-1} = 0$.

Definition 2.4. *We define the three sets* Λ_1, Λ_2, *and* Λ_3 *as follows*

$$\begin{aligned} \Lambda_1 &= \{n \in \mathbb{N} : y_n y_{n-1} < 0, \ |y_n| \geq |y_{n-1}|\}, \\ \Lambda_2 &= \{n \in \mathbb{N} : y_n y_{n-1} < 0, \ |y_n| < |y_{n-1}|\} \cup \{n : \ y_n = 0\}, \\ \Lambda_3 &= \{n \in \mathbb{N} : y_n y_{n-1} > 0\} \cup \{n : \ y_{n-1} = 0\}. \end{aligned}$$

Clearly, these three sets form a partition of $\mathbb{N}$, and $\delta_n = |y_n|$ if $n \in \Lambda_1$, $\delta_n = |y_{n-1}|$ if $n \in \Lambda_2$, and $\delta_n = |y_{n-1}| + |y_n|$ if $n \in \Lambda_3$. The following result shows that there are indeed cycles of length 2 or 3.

Lemma 2.5. *The following implications hold*

$$\begin{array}{lll} n \in \Lambda_1 & \Longrightarrow & n+1 \in \Lambda_2, \quad \textit{and} \quad \delta_{n+1} = \delta_n, \\ n \in \Lambda_2 & \Longrightarrow & n+1 \in \Lambda_3, \\ n \in \Lambda_3 & \Longrightarrow & n+1 \in \Lambda_1 \cup \Lambda_2. \end{array}$$

Proof: Let $n \in \Lambda_1$. Then $y_n y_{n-1} < 0$, $|y_n| \geq |y_{n-1}|$, and from (2) we get

$$y_{n+1} y_n = -c_{n+1}(|y_n|^2 + y_{n-1} y_n) \leq 0$$

If $y_{n+1} y_n = 0$ then $y_{n+1} = 0$ implying $n+1 \in \Lambda_2$ and $\delta_{n+1} = \delta_n = |y_n|$ as claimed above. If $y_{n+1} y_n < 0$ then writing

$$|y_{n+1}| = |c_{n+1}(y_n + y_{n-1})| = c_{n+1}(|y_n| - |y_{n-1}|) \leq c_{n+1}|y_n|$$

we obtain $\delta_{n+1} = \delta_n = |y_n|$, hence $n+1 \in \Lambda_2$ and the first implication is proved.

Now suppose that $n \in \Lambda_2$. If $y_n = 0$ then clearly $n+1 \in \Lambda_3$. Otherwise, we have $y_n y_{n-1} < 0$ and $|y_{n-1}| > |y_n|$, implying that

$$y_{n+1} y_n = -c_{n+1}(|y_n|^2 + y_{n-1} y_n) > 0,$$

and thus $n+1 \in \Lambda_3$.

In order to prove the last implication, we take $n \in \Lambda_3$ for which $y_n y_{n-1} \geq 0$ and $y_n > 0$. Hence

$$y_n y_{n+1} = -c_{n+1}(y_n^2 + y_{n-1} y_n) < 0,$$

showing that $n+1 \in \Lambda_1 \cup \Lambda_2$. □

We are now prepared to show that the sequence $(\delta_n)_{n \geq 0}$ is decreasing, and hence has a limit $\delta = \delta(y_0, y_{-1}) \geq 0$.

Lemma 2.6. *If $(y_n)_{n \geq -1}$ is a solution of (2) then the sequence $(\delta_n)_{n \geq 0}$ defined by (12) is decreasing for any value of the initial conditions y_0, y_{-1}.*

Proof: Let $n \in \mathbb{N}$. If $n \in \Lambda_1$ then $\delta_{n+1} = \delta_n$ by Lemma 2.5. If $n \in \Lambda_2$ then $y_n y_{n+1} \geq 0$. Without loss of generality, we may suppose that $y_n \geq 0$. Hence, in this case $y_{n-1} \leq 0$, $y_{n+1} \geq 0$ and we obtain

$$\delta_{n+1} = y_n + y_{n+1} = |y_n| - c_{n+1}(y_{n-1} + y_n) = c_{n+1}|y_{n-1}| + (1 - c_{n+1})|y_n| < |y_{n-1}| = \delta_n$$

Finally, if $n \in \Lambda_3$ then $\delta_{n+1} = \max\{|y_n|, |y_{n+1}|\}$ according to Lemma 2.5. Since

$$|y_{n+1}| = c_{n+1}|y_n + y_{n-1}| \leq \delta_n$$

and $y_n \leq \delta_n$, we obtain $\delta_{n+1} \leq \delta_n$. □

We terminate this subsection by establishing the following inequalities.

Lemma 2.7. *If $n \in \Lambda_2$ then*

$$\delta_n \leq c_{n-1} \delta_{n-2}, \tag{13}$$

with equality if in addition $n-1 \in \Lambda_1$. If $n \in \Lambda_3$ then we have the estimation

$$\delta_n \leq c_n \delta_{n-1} + (1 - c_n) c_{n-1} \delta_{n-2}, \tag{14}$$

with equality if $n-2 \in \Lambda_3$.

Proof: Let $n \in \Lambda_2$. Then $\delta_n = |y_{n-1}| = c_{n-1}|y_{n-2} + y_{n-3}| \leq c_{n-1}\delta_{n-2}$. If $n-1 \in \Lambda_1$, we get $n-2 \in \Lambda_3$ by Lemma 2.5, and hence $\delta_{n-2} = |y_{n-2} + y_{n-3}|$.

Now let $n \in \Lambda_3$, and thus $n-1 \in \Lambda_2$ by Lemma 2.5. We suppose without loss of generality that $y_n \geq 0$. Then $y_{n-1} \geq 0$ and $y_{n-2} \leq 0$. Furthermore,

$$\delta_n = y_n + y_{n-1} = (1-c_n)y_{n-1} - c_n y_{n-2} = -(1-c_n)c_{n-1}(y_{n-2} + y_{n-3}) + c_n|y_{n-2}|,$$

implying that

$$\delta_n \leq c_{n-1}(1-c_n)\delta_{n-2} + c_n\delta_{n-1}$$

If $n-2 \in \Lambda_3$ then $\delta_{n-2} = -(y_{n-2} + y_{n-3})$ and thus equality holds. □

2.3. **Proof of Theorem 2.1.** We first show that (7) implies (P), that is, given some complex numbers y_0, y_{-1}, we need to show that the corresponding quantity $\delta = \delta(y_0, y_{-1})$ equals zero. The case $y_0 = y_{-1} = 0$ is trivial. Otherwise, define Λ_1, Λ_2 and Λ_3 as in Definition 2.4, and write more explicitly $\Lambda_2 := \{n_0, n_1, n_2, ...\}$ with increasing n_j. From Lemma 2.5 we know that $n_{j-1} \in \{n_j - 2, n_j - 3\}$ for all $j \geq 1$; in particular, Λ_2 contains an infinite number of elements. Inequality (13) tells us that

$$\delta_{n_j} \leq c_{n_j-1}\delta_{n_j-2} \leq c_{n_j-1}\delta_{n_{j-1}},$$

the latter inequality being trivial if $n_{j-1} = n_j - 2$, and else following from Lemma 2.6. Hence,

$$\delta_{n_k} \leq \delta_{n_0} \prod_{j=1}^{k} c_{n_j-1}, \tag{15}$$

and $\delta(y_0, y_{-1}) = 0$ provided that the product on the right-hand side tends to 0 for $k \to \infty$. On the other hand,

$$\begin{aligned} [\prod_{j=1}^{k} c_{n_j-1}]^3 &\leq \prod_{j=1}^{k} [c_{n_j-1}]^{n_{j+1}-n_j} \\ &\leq \prod_{j=1}^{k} \prod_{n=n_{j-1}}^{n_j-1} \max\{c_n, c_{n+1}, c_{n+2}\} = \prod_{n=n_0}^{n_k-1} \max\{c_n, c_{n+1}, c_{n+2}\}, \end{aligned}$$

where the right-hand side tends to zero for $k \to \infty$ by (7). Hence $\delta(y_0, y_{-1}) = 0$.

In order to show that, under under assumption (8), condition (7) becomes also necessary, notice first that, for all n,

$$\begin{aligned} m := \inf_k \frac{1-c_k}{1-c_{k+1}} > 0 \quad &\Longrightarrow \quad 1 - \max\{c_n, c_{n+1}, c_{n+2}\} \geq \min\{1, m^2\}(1 - c_{n+2}), \\ M := \sup_k \frac{1-c_k}{1-c_{k+1}} < \infty \quad &\Longrightarrow \quad 1 - \max\{c_n, c_{n+1}, c_{n+2}\} \geq \min\{1, \frac{1}{M^2}\}(1 - c_n). \end{aligned}$$

Hence, in both cases described in (8), relation (7) holds if and only if $\sum(1 - c_n) = \infty$. However, according to (4), the latter condition is necessary for (P). □

2.4. **Proof of Theorem 2.2.** Our proof for Theorem 2.2 is divided into several parts. One implication is shown in the following Lemma. Here we apply classical techniques from operator theory where we adapt arguments used in the study of trace class perturbations of three term recurrence relations (see for instance [8] or [4, Section 3.5]).

Lemma 2.8. *Let the sequence $(c_n)_{n\geq 1}$ be such that*

$$\sum_{n=1}^{\infty}(1-|c_n-c_{n+1}|) \;<\; +\infty$$

Then property (P) does not hold.

Proof: The aim of the following considerations is to construct explicitly a solution of (2) with $((-1)^k y_{2k+\alpha})$ tending to some constant $\delta > 0$ different from zero for a fixed $\alpha \in \{0,1\}$.

In a first step we will show that our convergence assumption implies that

$$\lim_{k\to\infty} c_{2k+\alpha} = 1 \quad \text{and} \quad \lim_{k\to\infty} c_{2k+\alpha+1} = 0, \qquad \text{for a fixed } \alpha \in \{0,1\}. \tag{16}$$

Indeed, we have $\lim_{n\to\infty}|c_n - c_{n+1}| = 1$. Fixing an arbitrary $\epsilon > 0$, there exist an $N(\epsilon) \in \mathbb{N}$ such that $|c_n - c_{n+1}| > 1-\epsilon$ for $n \geq N$. Since in addition $c_n \in (0,1)$, we get $c_n \in (0,\epsilon)\cup(1-\epsilon,1)$ for $n \geq N$, and more precisely, $c_N \in (0,\epsilon)$ implies that $c_{N+2k} \in (0,\epsilon)$ and $c_{N+2k-1} \in (1-\epsilon,1)$ for all $k \geq 1$. Since ϵ is arbitrary, we obtain (16).

As a consequence, we may rewrite the assumption of the Lemma as

$$\sum_{k=1}^{\infty} c_{2k+\alpha+1} < \infty, \quad \text{and} \quad \sum_{k=1}^{\infty} 1 - c_{2k+\alpha} < \infty. \tag{17}$$

Since we are only interested in the asymptotic behavior of a solution of (2), we may drop without loss of generality the first terms in the recurrence relation (2), in particular we may assume without loss of generality that $\alpha = 0$, and that

$$\max\{sup_k|c_{2k-1}|, sup_k|c_{2k}-1|\} < 1/12. \tag{18}$$

Consider the infinite tridiagonal matrix

$$\mathcal{C} = \left(\begin{array}{cc|cc|cc|cc|c}
c_1 & 1 & 0 & \cdots & & & & & \\
c_2 & c_2 & 1/2 & 0 & \cdots & & & & \\
\hline
0 & 2c_3 & c_3 & 1 & 0 & \cdots & & & \\
\vdots & 0 & c_4 & c_4 & 2 & 0 & \cdots & & \\
\hline
 & & 0 & 2c_5 & c_5 & 1 & 0 & \cdots & \\
 & & & 0 & c_6 & c_6 & 1/2 & 0 & \cdots \\
\hline
 & & & & \ddots & \ddots & \ddots & \ddots & \ddots
\end{array}\right),$$

then a solution of (2) is given by

$$y_n = (-1)^n \det(\mathcal{C}_n),$$

where we adopt the convention that $\mathcal{A}_n$ denotes the principal submatrix of order n of some infinite matrix $\mathcal{A}$ (the scaling factors 2 and 1/2 are introduced in order to obtain later bounded operators).

Consider a second sequence of recurrence coefficients, namely $\widetilde{c}_{2k} = 1$, $\widetilde{c}_{2k-1} = 0$ for $k \geq 0$. The corresponding (block upper triangular) infinite matrix is given by

$$\widetilde{\mathcal{C}} = \begin{pmatrix} D & H & 0 & \cdots & & \\ 0 & D & H & 0 & \cdots & \\ 0 & 0 & D & H & 0 & \cdots \\ \vdots & & \ddots & \ddots & \ddots & \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 \\ 1/2 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Notice that $\widetilde{\mathcal{C}}_{2k}$ is invertible. More precisely, its inverse is given by $\mathcal{B}_{2k}$, with the infinite matrix

$$\mathcal{B} = \begin{pmatrix} D^{-1} & -D^{-1}HD^{-1} & D^{-1}(HD^{-1})^2 & -D^{-1}(HD^{-1})^3 & \cdots \\ 0 & D^{-1} & -D^{-1}HD^{-1} & D^{-1}(HD^{-1})^2 & \cdots \\ \vdots & 0 & D^{-1} & -D^{-1}HD^{-1} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

where

$$D^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } D^{-1}(HD^{-1})^n = \begin{pmatrix} -2^{-n} & 2^{-n} \\ 0 & 0 \end{pmatrix} \text{ for } \quad n \geq 1.$$

Moreover, we have

$$\begin{aligned} y_{2k} = \det(\mathcal{C}_{2k}) &= \det(\widetilde{\mathcal{C}}_{2k}) \cdot \det(\mathcal{I}_{2k} + (\mathcal{C}_{2k} - \widetilde{\mathcal{C}}_{2k})\widetilde{\mathcal{C}}_{2k}^{-1}) \\ &= (-1)^k \cdot \det(\mathcal{I}_{2k} + (\mathcal{C}_{2k} - \widetilde{\mathcal{C}}_{2k})\mathcal{B}_{2k}), \end{aligned}$$

and it remains to discuss the asymptotic of the determinant.

Let us shortly recall some notation and results concerning infinite matrices and operators. Given an infinite matrix $\mathcal{A}$ define the quantity

$$||\mathcal{A}|| := \sup_{n \geq 0} ||\mathcal{A}_n||.$$

It is well known that a infinite matrix $\mathcal{A}$ with $||\mathcal{A}|| < \infty$ can be identified with a bounded operator A acting on ℓ^2 via matrix calculus, and $||A|| = ||\mathcal{A}||$. We recall the known estimate (see [7, Example III.2.3])

$$||\mathcal{A}||^2 \leq \left[sup_j \sum_{k=0}^{\infty} |a_{j,k}| \right] \left[sup_k \sum_{j=0}^{\infty} |a_{j,k}| \right]. \tag{19}$$

Write $\mathcal{I}$ for the infinite identity matrix. We will also use the fact that, provided that the operator A associated to $\mathcal{A}$ is of trace class (see [7, Section X.1.3]) and of norm less than 1, then $\det(I + A) \neq 0$ is defined, and (see [6, Lemma XI 9.16])

$$\det(I + A) = \lim_{n \to \infty} \det(\mathcal{I}_n + \mathcal{A}_n).$$

Finally, we recall from [7, Section X.1.3] that a composition of a bounded operator and a trace class operator is of trace class.

Notice that the bidiagonal infinite matrix

$$\mathcal{C}-\widetilde{\mathcal{C}}=\left(\begin{array}{cc|cc|cc|cc|c}
c_1-1 & 0 & 0 & \cdots & & & & & \\
c_2 & c_2 & 0 & 0 & \cdots & & & & \\
\hline
0 & 2c_3-2 & c_3-1 & 0 & 0 & \cdots & & & \\
\vdots & 0 & c_4 & c_4 & 0 & 0 & \cdots & & \\
\hline
 & & 0 & 2c_5-2 & c_5-1 & 0 & 0 & \cdots & \\
 & & & 0 & c_6 & c_6 & 0 & 0 & \cdots \\
\hline
 & & & & \ddots & \ddots & \ddots & \ddots & \ddots
\end{array}\right)$$

represents a trace class operator $C-\widetilde{C}$ since, by (17), the sum of the absolute value of the entries is finite. According to (18) and (19), we know that

$$||C-\widetilde{C}||=||\mathcal{C}-\widetilde{\mathcal{C}}||\leq 3\cdot\max\{sup_k|c_{2k-1}|,sup_k|c_{2k}-1|\}<1/4,$$

and one easily verifies that $||B||=||\mathcal{B}||\leq 4$. Hence $(C-\widetilde{C})B$ is in fact a trace class operator of norm smaller than 1. Noticing that, according to its block triangular structure,

$$(\mathcal{C}_{2k}-\widetilde{\mathcal{C}}_{2k})\mathcal{B}_{2k}=((\mathcal{C}-\widetilde{\mathcal{C}})\mathcal{B})_{2k},$$

we may conclude that

$$\lim_{k\to\infty}(-1)^k y_{2k}=\det(I-(C-\widetilde{C})B)\neq 0,$$

and hence property (P) does not hold. □

In the proof of the second part of Theorem 2.2, we will make use of some further observations. Denote by $card(\Lambda)$ the cardinal of some set Λ. From Lemma 2.5 one can conclude that $card(\Lambda_2)=card(\Lambda_3)=\infty$. However, $card(\Lambda_1)=\infty$ (or, in other words, the existence of an infinite number of cycles of length 3) may already imply that (P) holds.

Lemma 2.9. *Under the assumption (10), if $card(\Lambda_1)=\infty$ then property (P) holds true.*

Proof: We suppose that (P) does not hold, and that $card(\Lambda_1)=\infty$. Then

$$\prod_{n\in\Lambda_2}c_{n-1}>0$$

by (15). We conclude using condition (10) that

$$\lim_{n\in\Lambda_2}c_{n-1}=1,\quad \sup_{n\in\Lambda_2}c_n<1,\quad \sup_{n\in\Lambda_2}c_{n-2}<1.$$

Also, from (5) we know that

$$\sum_{n\in\Lambda_2}(1-c_{n-2})(1-c_{n-3})<\infty,\ \text{and hence}\ \lim_{n\in\Lambda_2}c_{n-3}=1.$$

Recall from Lemma 2.5 that $n-1\in\Lambda_1$ implies that $n-2\in\Lambda_3$, and $n,n-3\in\Lambda_2$. Thus

$$\lim_{n-1\in\Lambda_1}c_{n-3}=\lim_{n\in\Lambda_2}c_{n-3}=1.$$

On the other hand, $(c_{n-3})_{n-1\in\Lambda_1}$ is a subsequence of $(c_n)_{n\in\Lambda_2}$, the latter having a supremum strictly smaller than 1, a contradiction. □

We are now prepared to show the second part of Theorem 2.2.

Proof of Theorem 2.2: The fact that (P) implies (9) follows directly from Lemma 2.8 proved above. It remains to show that (10) and (9) imply property (P). Let y_{-1}, y_0 be fixed. For the corresponding solution $(y_n)_{n\geq -1}$ of the difference equation (2) we may suppose that $card(\Lambda_1) < \infty$, otherwise property (P) follows directly from Lemma 2.9.

Thus there exists $k_0 \in \mathbb{N}$ and $\alpha \in \{0,1\}$ such that $\forall k \geq k_0$ there holds $2k+\alpha \in \Lambda_3$ and $2k+1+\alpha \in \Lambda_2$. Without loss of generality we may suppose that $\alpha = 0$. Then for all $k \geq k_0$

$$\delta_{2k+1} = |y_{2k}| \quad \text{and} \quad \delta_{2k} = |y_{2k}| + |y_{2k-1}|$$

with

$$y_{2k+1}y_{2k} \leq 0 \quad \text{and} \quad y_{2k}y_{2k-1} \geq 0.$$

Furthermore, from assuption (9) we know that

$$\begin{aligned} &2\sum_{k=1}^{\infty} c_{2k+1} + 2\sum_{k=1}^{\infty}(1-c_{2k}) \geq \sum_{k=1}^{\infty}(1-c_{2k}+c_{2k+1}) + \sum_{k=1}^{\infty}(1-c_{2k+2}+c_{2k+1}) \\ &\geq \sum_{n=2}^{\infty}(1-|c_n - c_{n+1}|) = \infty, \end{aligned}$$

and thus

$$\prod_{k=k_0}^{\infty} c_{2k} = 0 \text{ or } \prod_{k=k_0}^{\infty}(1-c_{2k+1}) = 0. \tag{20}$$

We write

$$|y_{2k+1}| = c_{2k+1}|y_{2k} + y_{2k-1}| = c_{2k+1}\delta_{2k}.$$

Consequently,

$$\delta_{2k+2} = |y_{2k+2}| + |y_{2k+1}| = \delta_{2k+3} + c_{2k+1}\delta_{2k} \tag{21}$$

Taking into account (14) (recall that $2k, 2k+2 \in \Lambda_3$), we obtain for δ_{2k+2} the following identity

$$\delta_{2k+2} = c_{2k+2}\delta_{2k+1} + (1-c_{2k+2})c_{2k+1}\delta_{2k}.$$

Hence, substituting the previous equality in (21) we obtain

$$\delta_{2k+3} = c_{2k+2}\delta_{2k+1} - c_{2k+1}c_{2k+2}\delta_{2k}$$

Since $\delta_{2k+1} \leq \delta_{2k}$, we get for all $k \geq k_0$ the following estimate

$$\delta_{2k+3} \leq c_{2k+2}(1-c_{2k+1})\delta_{2k+1}.$$

Using (20), we may conclude that $\delta_{2k+1} \to 0$ for $k \to \infty$, as required for the assertion of Theorem 2.2. $\square$

3. Vector Stieltjes continued fractions.

A Vector Stieltjes continued fraction (VSCF) of dimension p is a vector continued fraction of the form

$$\frac{(1,...,1)|}{|(0,...,0,z)} + \frac{(1,...,1,-a_1)|}{|(0,...,0,1)} + \cdots + \frac{(1,...,1,-a_p)|}{|(0,...,0,1)} + \frac{(1,...,1,-a_{p+1})|}{|(0,...,0,z)} + \cdots, \tag{22}$$

where $a_1, a_2, ...$ are complex numbers different from zero, and the product and the quotient of two vectors $c, b \in \mathbb{C}^p$ are defined according to the Jacobi-Perron rule by the formulas

$$c \cdot b = (c_1 b_1, ..., c_p b_p), \quad \frac{1}{c} = \left(\frac{1}{c_p}, \frac{c_1}{c_p}, ..., \frac{c_{p-1}}{c_p}\right)$$

The finite sections of the fraction (22) are usually called convergents; they give a local approximation at infinity of the formal power series

$$S_j(z) \sim \sum_{n=0}^{\infty} \frac{S_{n,j}}{z^{n+1}}, \qquad j = 1, 2, ..., p. \tag{23}$$

We recall from [2] that the coefficients $S_{n,j}$ of these formal power series have the following genetic sum's representation

$$S_{n,j} = \sum_{i_1=1}^{j} a_{i_1} \sum_{i_2=1}^{i_1+p} a_{i_2} \sum_{i_3=1}^{i_2+p} a_{i_3} \cdots \sum_{i_n=1}^{i_{n-1}+p} a_{i_n}, \qquad S_{0,j} = 1. \tag{24}$$

In this paper we will always suppose that $a_n > 0$ for all n. Under this condition, it is shown in [2] that there exists a vector of positive measures $(\mu_1, \mu_2, ..., \mu_p)$ with a common support on $[0, \infty)$ such that

$$S_{n,j} = \int_0^\infty x^n d\mu_j(x), \qquad j = 1, 2, ..., p. \tag{25}$$

This allows to associate with a vector continued fraction (22) and formal power series (23) a Stieltjes system of functions

$$S_j(z) = \int_0^\infty \frac{d\mu_j(x)}{z - x} \qquad j = 1, 2, ..., p \tag{26}$$

The convergence problem for a VSCF is whether the fraction converge everywhere in the complex plane outside the interval $[0, \infty)$, and in this case to characterize its limit.

In this section we will first explain how the convergence problem is linked to the recurrence equation (2). Then we give some applications of the results of Section 2 for the case $p = 2$. At the end we present some general results on the convergence of VSCF in connection with the determinate vector Stieltjes moment problem.

We note that, after some equivalence transformation, the fraction (22) takes a following form

$$\frac{(1,...,1)|}{|(0,...,0,-b_1 z)} + \frac{(1,...,1,1)|}{|(0,...,0,b_2)} + \cdots + \frac{(1,...,1,1)|}{|(0,...,0,b_{p+1})} + \frac{(1,...,1,1)|}{|(0,...,0,-b_{p+2} z)} + \cdots \tag{27}$$

where $b_n > 0$ are related with a_n by the equations

$$a_n(b_n b_{n+1} \cdots b_{n+p}) = 1, \quad n \geq 1 \quad \text{and} \quad b_1 = b_2 = \cdots = b_p = 1. \tag{28}$$

The vector continued fractions (22) and (27) are equivalent in the sense that their nth convergents coincide for all $n \geq 1$ (see the proof of Theorem 3.1 below). For the case $p = 1$ we obtain from (27) the classical Stieltjes continued fraction in the form preferred by T. Stieltjes [12].

3.1. **Recurrence relations associated to a VSCF.** We establish in this subsection the connection between our convergence problem and the recurrence equation (2). For fixed $z \in (-\infty, 0)$, we define the sequence (c_n), $c_n = c_n(z)$, by

$$a_{n-p}(1 - c_{n-p})(1 - c_{n-p+1}) \cdots (1 - c_n) = (-z)c_n, \quad n > p \tag{29}$$

with $c_1 = c_2 = \cdots = c_p = 0$. Since $a_n > 0$, one can verify by induction that $0 < c_n < 1$ for all $n > p$.

Theorem 3.1. *For a given sequence (a_n) and $z \in (-\infty, 0)$, let the sequence (c_n) be defined by (29). Then the vector continued fraction (22) converges at the point z if and only if the property (P) holds, this is, any solution of the recurrence equation*

$$y_n = -c_n(y_{n-1} + y_{n-2} + \ldots + y_{n-p}), \quad n > p \tag{30}$$

tends to zero.

Proof: We proceed in two steps: first we recall the $(p + 2)$ term recurrence relations connected to a vector continued fraction (see [9, Section IV.5] and [2, Section 4]), and show that the fractions (22) and (27) are equivalent. Subsequently, we provide a proof for the assertion of Theorem 3.1.

By definition of a vector continued fraction, the numerators $A_{n,j}$ $(j = 1, 2, ..., p)$ and denominator $A_{n,0}$ of the nth convergent of the fraction (22) satisfy the recurrence equation

$$A_{n+1,j} = \epsilon_n A_{n,j} - a_{n-p+1} A_{n-p,j}, \quad j = 0, 1, 2, ..., p, \quad n \geq p, \tag{31}$$

where

$$\epsilon_n = \begin{cases} z & if\ n = k(p+1) \\ 1 & \text{otherwise}, \end{cases} \tag{32}$$

with the following matrix of initial conditions

$$\begin{array}{cccccc} & n=0 & n=1 & n=2 & \ldots & n=p \\ A_{n,0} & 1 & z & z & \ldots & z \\ A_{n,1} & 0 & 1 & 1 & \ldots & 1 \\ A_{n,2} & 0 & 0 & 1 & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ A_{n,p} & 0 & 0 & 0 & \ldots & 1 \end{array} \tag{33}$$

For what follows it is suitable to consider a different normalization: we define sequences $(\Delta_{n,j})$ by

$$\Delta_{n,j} = (d_1 d_2 \cdots d_n) A_{n,j}, \quad n \geq 1, \quad \Delta_{0,j} = A_{0,j}.$$

For $\Delta_{n,j}$ we have the following recurrence relations

$$\Delta_{n+1,j} = d_{n+1}\epsilon_n\Delta_{n,j} - a_{n-p+1}(d_{n-p+1}d_{n-p+2}\cdots d_{n+1})\Delta_{n-p}, \quad n \geq p. \tag{34}$$

Now we choose d_n so that

$$a_{n-p+1}(d_{n-p+1}d_{n-p+2}\cdots d_{n+1}) = -1, \quad n \geq p.$$

It follows that the recurrences for $\Delta_{n,j}$ take the form

$$\Delta_{n+1,j} = d_{n+1}\epsilon_n\Delta_{n,j} + \Delta_{n-p,j}, \quad n \geq p.$$

Choosing $d_1 = -1$, $d_2 = 1$, ..., $d_p = 1$, we notice that $d_n < 0$ for $n = 1 + (p+1)k$, and $d_n > 0$ otherwise. Hence the quantities b_n defined by (28) satisfy

$$b_n = \begin{cases} -d_n, & n = 1 + k(p+1) \\ d_n, & \text{otherwise,} \end{cases}$$

and in particular they are strictly positive. Introducing $\tilde{\epsilon}_n = -z$ for $n = k(p+1)$, and $\tilde{\epsilon}_n = 1$ otherwise, we may rewrite (34) as follows

$$\Delta_{n+1,j} = b_{n+1}\tilde{\epsilon}_n\Delta_{n,j} + \Delta_{n-p,j}, \quad n \geq p. \tag{35}$$

By construction, the quantities $\Delta_{n,j}$, $j = 1, 2, ..., p$, and $\Delta_{n,0}$, respectively, are the numerators and the denominator of the nth convergent of the continued fraction (22). According to (35), they are also the numerators and denominator of the n-th convergent of the continued fraction (27), and hence these two vector continued fractions are equivalent.

Using (28) and (35) one may easily check that the quantities c_n defined in (29) are equal to

$$c_n = \frac{\Delta_{n-p-1,0}(z)}{\Delta_{n,0}(z)}, \quad n > p, \quad \text{and} \quad c_1 = c_2 = \cdots = c_p = 0. \tag{36}$$

Using again (35) we obtain the following recurrence relation for the convergents of the VSCF

$$\Pi_{n,j} = (1 - c_n)\Pi_{n-1,j} + c_n\Pi_{n-p-1,j}, \qquad n > p, j = 1, ..., p. \tag{37}$$

This convexity relation discovered first in [5] has been the starting point for several results on the convergence of the VSCF [3, 5]. Notice that the matrix of initial conditions for $\Pi_{n,j}$, $j = 1, 2, ..., p$, $n = 1, 2, ..., p$ is nonsingular. This implies that the convergence of a VSCF at the point z is equivalent to the existence of the limit for any solution of the recurrence equation (37).

Now we are prepared to finish the proof of Theorem 3.1. Suppose that the VSCF converges at the point z, and let (y_n) be a solution of the recurrence relation (30). Then the sequence (Y_n) defined by $Y_n = y_0 + ... + y_n$, $n \geq 0$, is a solution of (37), and hence has a limit. Consequently, $y_n = Y_n - Y_{n-1} \to 0$ for $n \to \infty$, as required for property (P).

Suppose now that property (P) holds, and let (Y_n) be a solution of the equation (37). Then the sequence (y_n) defined by $y_n = Y_n - Y_{n-1}$ is a solution of (30), and hence tends to zero by assumption. The convexity relations (37) imply that for the sequences

$$M_n = \max\{Y_n, Y_{n-1}, ..., Y_{n-p}\}, \quad m_n = \min\{Y_n, Y_{n-1}, ..., Y_{n-p}\}$$

one has

$$m_n \leq m_{n+1} \leq M_{n+1} \leq M_n.$$

From $\lim y_n = 0$ we conclude that $\lim M_n = \lim m_n$, and therefore the limit of (Y_n) exists. As mentioned above, this implies in particular that any sequence $(\Pi_{n,j}(z))$ admits a limit for $n \to \infty$, in other words, we obtain convergence pointwise of the VSCF. □

Remark 3.2. *Instead of fixing first the sequence (a_n), we could also define first a sequence (c_n) with $c_1 = c_2 = \cdots = c_p = 0$, and $c_n \in (0,1)$ for all $n > p$, fix a $z \in (-\infty, 0)$, and define the positive quantities $a_1, a_2, ...$ by (29). Hence, property (P) for a given sequence (c_n) is indeed equivalent to the convergence of a VSCF at a fixed $z \in (-\infty, 0)$.*

As a byproduct of the proof of Theorem 3.1, we should mention that any solution of the recurrence (30) is bounded provided that $c_n \in (0,1)$ for $n > p$. Notice that, for $p = 2$, equation (30) coincides with the basic recurrence relation (2) of Section 2.

Remark 3.3. *For positive $a_1, a_2, ...$ it is known (see* [2, Section 6.1] *and* [5, Proof of Theorem 1]*) that, for all j, the sequences $(\Pi_{n,j})_{n \geq 1}$ of a VSCF form a normal family in $\mathbb{C} \setminus [0, +\infty)$. Taking into account the Theorem of Vitali we see that for establishing local uniform convergence of the VSCF in $\mathbb{C} \setminus [0, +\infty)$ it is sufficient to show that the VSCF converges pointwise for all $z \in (-\infty, 0)$ (or some smaller interval).*

According to (29), the sequence (c_n) and hence the property (P) depends on the point $z \in (-\infty, 0)$ under consideration. In the scalar case $p = 1$ it is known that the Stieltjes CF converges at one point $z < 0$ if and only if it converges for all $z < 0$. It would be quite interesting to know whether a similar result remains valid in the vector case $p > 1$: is it true that property (P) holds at $z = -1$ if and only if (P) holds for all $z \in (-\infty, 0)$? In this context it could be helpful to use the fact that, for fixed n, the function $c_n(z)$ is increasing in $z \in (-\infty, 0)$ (see the Proof of Theorem 3.5).

3.2. **Convergence results for a VSCF for $p = 2$.** In order to relate more explicitly our findings of Section 2 to the problem of convergence of a VSCF, we will study in this subsection the case $p = 2$. From [5, Theorem 1] it is known that the VSCF of dimension $p = 2$ converges provided that

$$\sum_{k=1}^{\infty} \frac{1}{a_k + a_{k+1}} = +\infty. \tag{38}$$

In particular, the latter relation is true if (a_n) is bounded (this case was already covered by [2, Section 6.2]).

For our case $p = 2$, the relations (29) take the form

$$a_{n-2}(1 - c_{n-2})(1 - c_{n-1})(1 - c_n) = (-z)c_n. \tag{39}$$

It is easy to see from (39) that (a_n) is bounded if and only if $\sup_n c_n < 1$, which in the sequel will be excluded. Let us recall the following two results from [5]: it is shown in [5, Proof of Theorem 1] that condition (5) is sufficient for the convergence of a VSCF at z. Moreover, from [5, Lemma 3] we know that

$$\sum_n b_n = \infty \quad \Leftrightarrow \quad \sum_n (1 - c_n) = \infty \tag{40}$$

In the classical case $p = 1$, the Stieltjes Theorem tells us that the continued fraction (27) converges if and only if $\sum b_n = \infty$. For $p > 1$, this condition remains necessary but is no longer sufficient, as conjectured already in [3, Remark 2]. Indeed, take for instance $c_{2k} = 1/(k+1)^{1+\eta}$, $c_{2k+1} = 1 - 1/(k+1)^{1+\eta}$ for some $\eta > 0$, and $z = -1$, then clearly $\sum_n (1 - c_n) = \infty$. On the other hand, from Theorem 2.2 together with Theorem 3.1 we know that the corresponding VSCF does not converge at $z = -1$. Notice that, by (39), the recurrence coefficients for this example behave like $a_{2k} \to 1$ and $a_{2k-1}/k^{2+2\eta} \to 1$ for $k \to \infty$. Also, the same example with $\eta = 0$ allows us also to conclude that condition (38) is not necessary for the convergence of the VSCF at $z = -1$.

As a further application of Section 2 we have the following theorem.

Theorem 3.4. *Suppose that* $\sup\limits_{k\in\mathbb{N}} a_{2k+1+\alpha} < \infty$ *for some fixed* $\alpha \in \{0, 1\}$. *If*

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{a_{2k+\alpha}}} = +\infty \tag{41}$$

then the VSCF converges uniformly on every compact subset of $\mathbb{C} \setminus [0, \infty)$.

Proof: As mentioned in Remark 3.3, it is sufficient to prove pointwise convergence for any $z \in (-\infty, 0)$. Moreover, since the assumptions on (a_n) remain valid after replacing a_n by $a_n/(-z)$, we see from (39) and Theorem 3.1 that it is sufficient to show that property (P) holds for $z = -1$.

Without loss of generality we take $\alpha = 0$. From (39) we see that $\sup_{k\in\mathbb{N}} c_{2k+1} < 1$. If there exists a subsequence (k_j) such that $\sup_{j\in\mathbb{N}} c_{2k_j} < 1$ then from (5) we know that the VSCF converges. Hence we may suppose that

$$\lim_{k\to\infty} c_{2k} = 1$$

Thus, the sequence $(c_n)_{n\geq 1}$ satisfies the hypothesis of Theorem 2.2, and one concludes that the continued fraction converges if and only if

$$\sum_{k=1}^{\infty} c_{2k+1} = +\infty \quad \text{or} \quad \sum_{k=1}^{\infty} (1 - c_{2k}) = +\infty.$$

Rewriting (39) for $n = 2k + 1$ we obtain

$$\frac{1 - c_{2k}}{c_{2k+1}} = \frac{1}{a_{2k-1}} \frac{1}{(1 - c_{2k-1})(1 - c_{2k+1})}$$

which implies

$$\inf_{k\in\mathbb{N}} \frac{1 - c_{2k}}{c_{2k+1}} > 0.$$

Thus we have $\lim_{k\to\infty} c_{2k+1} = 0$, and the equivalence

$$(P) \quad \Leftrightarrow \quad \sum_{k=1}^{\infty} (1 - c_{2k}) = +\infty \tag{42}$$

Substituting $n = 2k$ in (39) we get

$$(1 - c_{2k})(1 - c_{2k-2}) = \frac{1}{a_{2k-2}} \frac{c_{2k}}{(1 - c_{2k-1})}$$

Since $\lim_k \frac{c_{2k}}{1-c_{2k-1}} = 1$, we obtain that (41) implies $\sum_{k=1}^{\infty} \sqrt{(1-c_{2k})(1-c_{2k-2})} = +\infty$, which at the same time implies the divergence of the serie on the right hand side of (42). Thus we obtain (P) and the theorem is proved. □

Comparing the assertion of Theorem 3.4 with the example mentioned just before, it seems that condition (41) cannot be improved.

For sequences of coefficients (a_n) having some regular behavior we may even be much more precise.

Theorem 3.5. *Suppose that the sequence* (a_n) *satisfies the following convexity condition*

$$\sum_{n=2}^{\infty} |1 - \frac{a_{n+1}a_{n-1}}{a_n^2}| < \infty,$$

and suppose that $\inf a_n > 0$. *Then the VSCF converges uniformly on every compact subset of* $\mathbb{C} \setminus [0, \infty)$ *if and only if*

$$\sum_{n=1}^{\infty} b_n = \infty \qquad \Longleftrightarrow \qquad \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{a_n}} = \infty. \tag{43}$$

Proof: We start by establishing the equivalence (43). By assumption, the expression

$$\prod_{k=2}^{n} \frac{a_{k+1}a_{k-1}}{a_k^2} = \frac{a_{n+1}}{a_n}\frac{a_1}{a_2}$$

converges to some constant different from zero. Similarly, using (28) we know that, for $\alpha \in \{0, \pm 1\}$, the expression

$$\prod_{k=1}^{n} \frac{a_{3k+1-\alpha}a_{3k-1-\alpha}}{a_{3k-\alpha}^2} = \prod_{k=1}^{n} \frac{b_{3k-\alpha}b_{3k+2-\alpha}}{b_{3k-1-\alpha}b_{3k+3-\alpha}} = \frac{b_{3n+2-\alpha}}{b_{3n+1-\alpha}}\frac{b_{3-\alpha}}{b_{2-\alpha}}$$

converges to some constant different from zero. In particular we may conclude that there exists a finite positive constant γ with

$$\frac{1}{\gamma} \leq \inf_n \frac{b_{n+1}}{b_n} \leq \sup_n \frac{b_{n+1}}{b_n} \leq \gamma. \tag{44}$$

Injecting this information in (28) we conclude that

$$a_n \cdot b_n^3 = \frac{b_n}{b_{n+1}}\frac{b_n}{b_{n+2}} \in [\frac{1}{\gamma^3}, \gamma^3],$$

showing the equivalence (43). Since in addition $\inf a_n > 0$ by assumption, we may also conclude that the sequence (b_n) is bounded.

We now turn to the proof of convergence. Notice first that, as mentioned in (40), the divergence property (43) is necessary for the convergence. In order to prove the converse, suppose for the sequel of the proof that (43) holds. The aim of the following considerations is to show pointwise convergence of the VSCF at $z = -1$. We start by establishing the property

$$\sup_n \frac{1 - c_{n+1}(-1)}{1 - c_n(-1)} < \infty. \tag{45}$$

Writing shorter $F_n := \Delta_{n,0}(-1)/\Delta_{n+1,0}(-1)$ and recalling from (36) that $1 - c_{n+1}(-1) = b_{n+1}F_n$, we see from (44) that it is sufficient to show the existence of some $\epsilon > 0$ such that $F_n \in [\epsilon, 1/\epsilon]$ for sufficiently large n. Using (35) one easily verifies the following development into a VSCF

$$\vec{u}_n = \frac{(1,1)}{(0, b_{n+1}) + \vec{u}_{n-1}}, \quad n \geq 2, \quad \vec{u}_n := (F_n, F_n F_{n-1}),$$

implying that

$$\vec{u}_n = \frac{(1,1)}{(0, b_{n+1})} + \frac{(1,1)}{(0, b_n)} + \frac{(1,1)}{(0, b_{n-1})} + \vec{u}_{n-3} = (T_{1,n}(\vec{u}_{n-3}), T_{2,n}(\vec{u}_{n-3})),$$

where

$$T_{1,n}(x,y) = \frac{x + yb_n + b_n b_{n-1}}{1 + b_{n+1}x + b_{n+1}b_n y + b_{n+1}b_n b_{n-1}},$$

$$T_{2,n}(x,y) = \frac{y + b_{n-1}}{1 + b_{n+1}x + b_{n+1}b_n y + b_{n+1}b_n b_{n-1}}.$$

Notice that $T_{1,n}(x,y)$ is increasing in x and in y, whereas $T_{2,n}(x,y)$ is decreasing in x and increasing in y. We now heavily use (44) for showing that

$$T_{1,n}([\underline{x}, \overline{x}], [\underline{y}, \overline{y}]) \subset [\underline{x}, \overline{x}], \quad \underline{x} = \epsilon, \quad \overline{x} = \frac{1}{\epsilon},$$

$$T_{2,n}([\underline{x}, \overline{x}], [\underline{y}, \overline{y}]) \subset [\underline{y}, \overline{y}], \quad \underline{y} = \gamma\epsilon^2, \quad \overline{y} = \frac{\gamma^2}{\epsilon},$$

with a suitable $\epsilon > 0$. Indeed the inclusions are true provided that (B denoting the upper bound for (b_n))

$$\begin{aligned} T_{1,n}(\underline{x}, \underline{y}) \geq \underline{x} &\quad \Longleftarrow \quad \gamma\underline{x}^2 = \underline{y} \text{ and } (\underline{y} + B)\underline{x}\gamma^2 \leq 1, \\ T_{1,n}(\overline{x}, \overline{y}) \leq \overline{x} &\quad \Longleftarrow \quad \gamma\overline{y} \leq \overline{x}^2 \text{ and } \overline{y}\overline{x} \geq \gamma^2, \\ T_{2,n}(\overline{x}, \underline{y}) \geq \underline{y} &\quad \Longleftarrow \quad \gamma^2\underline{y}[\overline{x} + B\underline{y} + B^2] \leq 1, \\ T_{2,n}(\underline{x}, \overline{y}) \leq \overline{y} &\quad \Longleftarrow \quad \underline{x}\overline{y} = \gamma^2. \end{aligned}$$

The reader may check that indeed all four sufficient conditions are true for sufficiently small ϵ. Since in addition $\vec{u}_1, \vec{u}_2, \vec{u}_3 \in [\underline{x}, \overline{x}] \times [\underline{y}, \overline{y}]$ for sufficiently small $\epsilon > 0$, we have shown our claim (45).

We recall from (40) that our assumption (43) together with (45) implies that

$$\sum_{n=1}^{\infty} [1 - \max\{c_n(-1), c_{n+1}(-1), c_{n+2}(-1)\}] = +\infty.$$

Hence Theorem 2.1 together with Theorem 3.1 gives convergence of the VSCF at $z = -1$.

It is now possible to deduce immediately pointwise convergence for all $z < -1$ (and hence locally uniform convergence) of the VSCF. Indeed, we show below that the function $c_n(z)$ increases in z for any fixed n. Therefore, for any $z < -1$,

$$\sum_{n=1}^{\infty} [1 - \max\{c_n(z), c_{n+1}(z), c_{n+2}(z)\}] \geq \sum_{n=1}^{\infty} [1 - \max\{c_n(-1), c_{n+1}(-1), c_{n+2}(-1)\}] = +\infty$$

and again Theorem 2.1 together with Theorem 3.1 gives convergence of the VSCF at z. In order to show our final claim, recall from [2, Lemma 7] that the zeros of $\Delta_{n,0}$ and $\Delta_{n+p+1,0} = \Delta_{n+3,0}$ are simple, positive, and interlace. Hence we have the partial fraction decomposition

$$c_n(z) = \sum_{j=1}^{m} \frac{\gamma_{n,j}}{x_{n,j} - z},$$

where the residuals $\gamma_{n,1}, ..., \gamma_{n,m}$ have all the same sign. Since $x_{n,j} \geq 0$ and $c_n(-1) > 0$, we may conclude that all $\gamma_{n,1}, ..., \gamma_{n,m}$ are positive, and hence $c_n(z)$ is increasing in z for fixed n. This shows our theorem. □

Remark 3.6. *We have shown implicitly in the proof of Theorem 3.5 that, provided that* (b_n) *is a bounded sequence satisfying (44), the VSCF (27) for* $p = 2$ *converges uniformly on every compact subset of* $\mathbb{C} \setminus [0, \infty)$ *if and only if* $\sum b_n = +\infty$.

The reader may easily check that the assumptions of Theorem 3.5 are verified for the following coefficients.

Corollary 3.7. *For some* $\lambda > 0$ *and some polynomial* P *with* $P([0,1]) \subset (0, +\infty)$, *let* $a_n = n^\lambda \cdot P(1/n)$, $n \geq 1$. *Then we have convergence of the VSCF if and only if* $\sum_n 1/\sqrt[3]{a_n} = \infty$.

Consequently, convergence takes place if $a_n = n^\lambda$ with $0 < \lambda \leq 3$ (but not for $\lambda > 3$).

3.3. **The vector moment problem and the convergence of the VSCF.** The classical Stieltjes theorem for the case $p = 1$ states that a scalar Stieltjes continued fraction converges if and only if the associated moment problem (25) is determinate, i.e., there exists one and only one measure μ with support on $[0, \infty)$ such that $S_n = \int_0^\infty x^n d\mu(x)$. The limit function in this case is the Stieltjes type function $S(z)$ defined by (26). The situation in the vector case seems to be much more involved.

Definition 3.8. *We call vector Stieltjes moment problem the task of finding a vector of measures* $(\mu_1, ..., \mu_p)$ *from a given sequence of moment vectors* $((S_{n,1}, ..., S_{n,p}))_{n\geq 0}$ *such that the integral representation (25) holds. The vector Stieltjes moment problem is said to be determinate if it admits only one solution, and otherwise non determinate.*

It is clear that the vector Stieltjes moment problem is determinate if and only if, for any $j = 1, 2, ..., p$, the scalar moment problem for $(S_{n,j})_{n\geq 0}$ is determinate. An interesting open question in this context is whether the determinacy of the vector Stieltjes moment problem is equivalent to the convergence of the corresponding VSCF as in the classical case. We are able to show the following partial result.

Theorem 3.9. *If the scalar Stieltjes moment problem (25) is determinate for some* j *for the moments defined by (24) then the* j*th component of VSCF (22) converges uniformly on compact subsets of the* $\mathbb{C} \setminus [0, \infty)$ *to the function* $S_j(z)$ *of (26).*

As a corollary of Theorem 3.9 we get the following sufficient condition of Carleman type for the convergence of a VSCF.

Corollary 3.10. *If*

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{\max\limits_{1\le i\le pn} a_i}} = \infty \tag{46}$$

then the VSCF (22) converges uniformly on every compact subset of $\mathbb{C}\setminus[0,\infty)$ *to the vector of functions (26).*

In particular, if (a_n) is an increasing sequence then the condition

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{a_{np}}} = \infty \tag{47}$$

is sufficient for the convergence of the VSCF. Note that, for $p = 1$, equation (47) becomes the well-known Carleman condition for the convergence of the scalar Stieltjes continued fraction (see for instance [1] or [9, Theorem II.7.3]). The question whether the convergence of a VSCF implies a determinate vector Stieltjes moment problem remains still open.

The rest of this subsection is devoted to the proof of the results.

Proof of Theorem 3.9: We are using some basic facts from [2] and some weak convergence arguments. Since $a_1, a_2, ... > 0$, we have for the jth component of the nth convergent $\Pi_n = (A_{n,1}/A_{n,0}, ..., A_{n,p}/A_{n,0})$ of (22) the following representation (see [2, Theorem 7])

$$\Pi_{n,j}(z) = \frac{A_{n,j}(z)}{A_{n,0}(z)} = \sum_{i=1}^{m} \frac{\mu_{n,i}^{j}}{(z - x_{n,i})} \tag{48}$$

where $m = m(n)$ is the degree of the denominator $A_{n,0}$, $x_{n,1}, ..., x_{n,m} \in [0,\infty)$ are the zeros of $A_{n,0}$, and (what is important) $\mu_{n,i}^{j} \geq 0$. At the same time,

$$S_j(z) - \Pi_{n,j}(z) = O(1/z^{m(n)+h_j(n)+1}), \tag{49}$$

where $h_j(n) \geq 0$. We construct a sequence of discrete measures $\nu_{n,j}$ with masses $\mu_{n,i}^{j}$ at the points $x_{n,i}$, $i = 1, ..., m$. Then (48) can be then written as

$$\Pi_{n,j}(z) = \int_0^\infty \frac{d\nu_{n,j}(x)}{z - x}.$$

The approximation property (49) gives

$$S_{k,j} = \int_0^\infty x^k d\nu_{n,j}(x), \quad k = 0, 1, ..., m(n) \tag{50}$$

Now suppose that that some subsequence of $\Pi_{n,j}(z)$ is convergent at some fixed point $z \in \mathbb{C} \setminus [0, \infty)$. By Helly's theorem (see [10, p. xii]) we can extract from the sequence of measures $\nu_{n,j}$ a weakly convergent subsequence, with limit measure ν. From (50) it follows that this limit measure is a solution of the same moment problem as the measure μ_j, and hence $\nu = \mu_j$ by assumption. Consequently, the sequence $(\mu_{j,n})_n$ converges weakly to μ_j. In particular, for any $z < 0$ we get

$$\lim_{n\to\infty} \Pi_{n,j}(z) = \lim_{n\to\infty} \int_0^\infty \frac{d\nu_{n,j}(x)}{z - x} = \int_0^\infty \frac{d\nu_j(x)}{z - x} = S_j(z).$$

Finally, the claimed uniform convergence follows from the pointwise convergence on $(-\infty, 0)$ together with the normality of $(\Pi_{n,j}(z))$. □

Proof of Corollary 3.10: From the genetic sum's representation (24) of moments it follows that

$$0 < S_{n,1} < S_{n,2} < \cdots < S_{n,p}.$$

On the other hand, from the same representation we get the estimates

$$S_{n,p} \leq M_n [\max\{a_1, a_2, ..., a_{pn}\}]^n$$

where M_n is the number of monomials in the sum (24). From [2, Lemma 8] it is known that

$$M_n \leq M\alpha^n, \quad \alpha = (p+1)(1+\frac{1}{p})^p.$$

This implies the relation

$$\sqrt[2n]{S_{n,p}} \leq M^{1/2n} \alpha^{1/2} [\max\{a_1, a_2, ..., a_{np}\}]^{1/2}.$$

Consequently, we obtain for $j = 1, ..., p$ the implications

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{\max\limits_{1 \leq i \leq pn} a_i}} = \infty \quad \Longrightarrow \quad \sum_{n=0}^{\infty} \frac{1}{\sqrt[2n]{S_{n,p}}} = \infty \quad \Longrightarrow \quad \sum_{n=0}^{\infty} \frac{1}{\sqrt[2n]{S_{n,j}}} = \infty.$$

By the classical Carleman condition [13, Theorem 88.1(b)], we may conclude that the scalar Stieltjes moment problem for each sequence of moments $(S_{n,j})_n$, $j = 1, 2, ..., p$, is determinate. Hence Theorem 3.9 gives the claimed convergence. □

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