On explicit error bounds for best rational and meromorphic approximation of Markov functions on the unit circle

by

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Abstract

We establish an explicit inequality for best uniform approximation on the unit circle of Markov functions both by meromorphic and by rational functions. Both bounds are sharp up to a constant. In the proof we make use of AAK theory, an essential ingredient are bounds for ratios of singular values of related Hankel operators. Finally we apply our results to the problem of optimal stable model reduction of scalar transfer functions in linear system theory, and to the problem of estimating the spectral condition number of scaled Cauchy matrices.

Key words: Best rational approximation, best meromorphic approximation, eigenvalues of Hankel operators, model reduction, Cauchy matrices.
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1 Introduction

Denote by H^p the Hardy space of analytic functions on the open unit disk $\mathbb{D}$ with circumference $\mathbb{T}$ (and especially by H^∞ the set of bounded analytic functions on $\mathbb{D}$), by $\mathcal{R}_k$ the set of rational functions p/q with $\deg p \leq \deg q \leq k$, and consider the set $H_k^\infty = H^\infty + \mathcal{R}_k$ of functions meromorphic in $\mathbb{D}$, with at most k poles. We will be interested to give explicit error bounds in best uniform approximation on $\mathbb{T}$ of Markov functions

$$f(z) = f_0 + \int_b^c \frac{d\mu(x)}{z-x}, \quad -1 < b < c < 1, \quad f_0 \in \mathbb{C}, \tag{1}$$

μ being some positive Borel measure with support included in $[b,c]$, by functions in H_k^∞, and in $\mathcal{R}_k$, respectively. This is, we want to estimate the distances

$$dist(f, H_k^\infty) = \inf_{r \in H_k^\infty} ||f-r||_{L^\infty(\mathbb{T})}, \quad dist(f, \mathcal{R}_k) = \inf_{r \in \mathcal{R}_k} ||f-r||_{L^\infty(\mathbb{T})}$$

where clearly $dist(f, H_k^\infty) \leq dist(f, \mathcal{R}_k)$. Starting with Walsh [Wal60], a number of authors contributed to this questions, recall for instance a conjecture of Gonchar [Gon84] solved by Parfenov [Par88]. For instance, from, e.g., [Par88] or [StTo92, Theorem 6.2.2]) we know that

$$\limsup_{k\to\infty} dist(f, \mathcal{R}_k)^{1/k} \leq \gamma := \exp\left(-\frac{2}{cap\,([b,c],\mathbb{T})}\right),$$

where $cap\,(\cdot,\cdot)$ denotes the logarithmic capacity of the condensor formed by two closed disjoint subsets of the complex plane. Recall that $cap\,([b,c],\mathbb{T})$ may be explicitly given in terms of elliptic

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integrals of the first kind, namely (see, e.g., [Akh90, SaTo97])

$$\frac{1}{cap\,([b,c],\mathbb{T})} = \frac{\pi}{2}\frac{K'(a)}{K(a)} =: \mu(a), \quad a = \frac{c-b}{1-bc}, \tag{2}$$

where for $\lambda \in [0,1]$

$$K(\lambda) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\lambda^2 t^2)}}, \quad K'(\lambda) = K(\sqrt{1-\lambda^2}).$$

The aim of this paper is to show the following statement.

Theorem 1.1. *Let f be as in* (1), *and* $\gamma = \exp(-\frac{2}{\text{cap}\,([b,c],\mathbb{T})})$. *Then for all $k \geq 0$ we have*

$$\text{dist}(f, H_k^\infty) \leq 4 \cdot \gamma^k \cdot \text{dist}(f, H^\infty), \tag{3}$$

$$\text{dist}(f, \mathcal{R}_k) \leq 4 \cdot \gamma^k \cdot \text{dist}(f, \mathcal{R}_0). \tag{4}$$

Moreover, for any for fixed b, c, the constant 4 *in* (3) *may not be replaced by any smaller constant.*

The optimality of the constant 4 is discussed in Remark 3.5 below, indeed we will determine explicitly the best constant in (3) for fixed b, c and k, and discuss possible improvements of (4). It is quite instructive to compare Theorem 1.1 with a recent result of Baratchart, Prokhorov and Saff [BPS01, Theorem 4] on strong asymptotics of $dist(f, H_k^\infty)$ for fixed f as in (1) with a measure satisfying a Szegő condition on $[b, c]$. From their result we deduce in Remark 3.6 below that, for such Markov functions f,

$$\lim_{k\to\infty} \frac{1}{\gamma^k} \frac{dist(f, H_k^\infty)}{dist(f, H^\infty)} \leq 2, \tag{5}$$

where again 2 is an optimal constant.

Our method of proof relies on the classical AAK theory [AAK71, AAK78] which states that $dist(f, H_k^\infty)$, $k = 0, 1, ...$, coincides with the $(k+1)$th singular value $\sigma_k(A_f)$ of the Hankel operator $A_f : H^2 \mapsto \overline{H^2}$, $A_f g = \Pi(f \cdot g)$, where $\overline{H^2}$ denotes the orthogonal complement of H^2 in $L^2(\mathbb{T})$, and $\Pi : L^2(\mathbb{T}) \mapsto \overline{H^2}$ the corresponding projection operator. Here we will estimate (ratios of) these singular values for Markov functions f in terms of the third Zolotarev problem, leading to our explicit bounds. Estimates for ratios of the extremal eigenvalues of positive definite Hankel *matrices* have been the subject of [Bec00].

Theorem 1.1 has some interesting implications for the the computation of certain n-widths, see [BPS01, Chapter 1.5] and the references therein. Also, we obtain related results in Corollary 2.4 below for the euclidean condition number of some classes of scaled Cauchy matrices. Let us here mention another application of Theorem 1.1 for stable model reduction of SISO transfer functions in linear system theory as occurring for instance in the context of RC circuits [Ob96, SrMy97] :

Corollary 1.2. *Given a stable transfer function of the form*

$$h(w) = \delta + e^T (wI - D)^{-1} d, \quad \delta \in \mathbb{C}, \quad d, e \in \mathbb{R}^n,$$

with a real symmetric matrix D of order n having all its eigenvalues in the left half plane, there exists a rational *approximant h_k of McMillan degree $k < n$ having all its poles in the left half plane, and*

$$\max_{w \in i\cdot\mathbb{R}} |h(w) - h_k(w)| \leq 2 \cdot \sqrt{|e^T D^{-1} e| \cdot |d^T D^{-1} d|} \cdot \gamma^{k'}, \quad \gamma := \exp(-\frac{2\pi K(1/\kappa(D))}{K(\sqrt{1 - 1/\kappa(D)^2})}), \tag{6}$$

with $\kappa(D) \geq 1$ denoting the euclidean condition number of D. Here $k' = k$ in the symmetric case $d = e$, and else k' is the largest integer $\leq k/2$.

The rest of the paper is organized as follows : In Lemma 2.1 we show that ratios of singular values of our Hankel operators connected to (1) may be bounded in terms of the third Zolotarev problem, leading to explicit bounds in terms of elliptic functions. The sharpness of these bounds is discussed in Lemma 2.3. In Section 3 we study AAK approximants of rational functions. Subsequently, we provide the proofs of our main assertions given above.

2 Estimates for ratios of singular values

Let us recall that, for Hankel operators of Markov functions (1), there is a simple integral representation. Indeed, for $g \in H^2$ and $|z| > 1$ we have

$$\begin{aligned}(A_f g)(z) &= \Pi(f \cdot g)(z) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(\zeta)g(\zeta)}{z-\zeta}\,d\zeta = \frac{1}{2\pi i}\int_{\mathbb{T}}\int \frac{d\mu(x)}{\zeta - x}\frac{g(\zeta)}{z-\zeta}\,d\zeta \\ &= \int d\mu(x)\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{g(\zeta)}{(\zeta-x)(z-\zeta)}\,d\zeta = \int \frac{g(x)}{z-x}\,d\mu(x),\end{aligned}$$

where for the last equality we used the Cauchy integral formula. Moreover, the operator $\widetilde{A}_f : H^2 \mapsto H^2$ defined by

$$(\widetilde{A}_f g)(z) = \frac{1}{z}(A_f g)(\frac{1}{z}) = \int \frac{g(x)}{1-zx}\,d\mu(x), \quad |z| < 1,$$

clearly is compact, self-adjoint and semi positive definite, with its eigenvalues coinciding with the singular values $\sigma_0(A_f) \geq \sigma_1(A_f) \geq ... \geq 0$ of A_f. Indeed, for $g, h \in H^2$ we obtain using again the Cauchy integral formula

$$(h, \widetilde{A}_f g) = \frac{1}{2\pi}\int_{\mathbb{T}} \overline{h(z)}(\widetilde{A}_f g)(z)\,|dz| = \int \overline{h(x)} g(x)\,d\mu(x), \tag{7}$$

showing in particular that

$$\mathrm{trace}(\widetilde{A}_f) = \sum_{j=0}^{\infty}(z^j, \widetilde{A}_f z^j) = \sum_{j=0}^{\infty}\int x^{2j}\,d\mu(x) = \int \frac{d\mu(x)}{1-x^2} = \frac{f(1)-f(-1)}{2} < \infty, \tag{8}$$

i.e., $\widetilde{A}_f$ is of trace class.

It is well known that singular values of a Hankel operator corresponding to a Markov function can be bounded by means of minimal Blaschke products of the form

$$B_k(z) = \prod_{j=1}^{k} \frac{z - \zeta_j}{1 - \overline{\zeta_j} z},$$

see, e.g., [BPS01, Par88]. Notice that $B_k \in H^2$ if $\zeta_1, ..\zeta_k \in \mathbb{D}$, and then $||B_k||_{L^2(\mathbb{T})} = 1$. In Lemma 2.1 below we show that also ratios of singular values may be bounded in a similar manner. Moreover, since the problem (10) of minimal Blaschke products for an interval is equivalent to the third Zolotarev problem, and since for the latter problem a solution is explictly known in terms of elliptic functions (see for instance [Akh90, §50 and §51]), we obtain some explicit upper bounds for ratios of singular values. Notice that, according to Lemma 2.3 below, the bound (9) may not be improved at least for $j = 0$.

Lemma 2.1. *Let f be as in (1), and $j,k \geq 0$. Then*

$$\sigma_{j+k}(A_f) \leq Z_k \cdot \sigma_j(A_f), \tag{9}$$

where

$$Z_k := \min\Big\{ \max_{z\in[b,c]} |\prod_{j=1}^{k} \frac{z-\xi_j}{1-\overline{\xi}_j z}| : \xi_1, ..., \xi_k \in \mathbb{C} \Big\}^2. \tag{10}$$

Moreover, with $\gamma = \exp(-\frac{2}{\mathrm{cap}\,([b,c],\mathbb{T})})$ there holds

$$\frac{4\gamma^k}{1+16\gamma^{2k}} \leq Z_k \leq 4\gamma^k. \tag{11}$$

Proof. Let us start by showing that the infimum in (10) is attained for some Blaschke product B_k with zeros $z_1, ..., z_k \in [b,c] \subset \mathbb{D}$. Indeed, writing $T_\zeta(z) = (z-\zeta)/(1-\overline{\zeta}z)$, one verifies by means of elementary computations that, for $z \in [b,c]$,

$$|T_\zeta(z)| \geq \begin{cases} |T_{1/\overline{\zeta}}(z)| & \text{if } |\zeta| > 1, \\ |T_{\mathrm{Re}(\zeta)}(z)| & \text{if } |\zeta| \leq 1, \\ |T_b(z)| & \text{if } \zeta \in [-1,b), \\ |T_c(z)| & \text{if } \zeta \in (c,1]. \end{cases}$$

Hence it is sufficient to consider in (10) only $\zeta_1, ..., \zeta_k$ out of the compact $[b,c]$. Since the maximum on $[b,c]$ of a Blaschke product is a continuous function of $\zeta_1, ..., \zeta_k$, our claim follows.

We are now prepared to show (9). Let B_k as above, and G an arbitrary subspace of H^2 of codimension j. Since B_k has k zeros in $\mathbb{D}$, the space $B_k G = \{B_k g : g \in G\}$ has codimension $k+j$, and hence

$$\begin{aligned} \sigma_{j+k}(A_f) &= \sigma_{j+k}(\widetilde{A}_f) = \inf_{\substack{G\subset H^2 \\ \mathrm{codim}(G)=k+j}} \sup_{g\in G} \frac{(g,\widetilde{A}_f g)}{(g,g)} \leq \inf_{\substack{G\subset H^2 \\ \mathrm{codim}(G)=j}} \sup_{g\in G} \frac{(B_k g,\widetilde{A}_f(B_k g))}{(B_k g, B_k g)} \\ &= \inf_{\substack{G\subset H^2 \\ \mathrm{codim}(G)=j}} \sup_{g\in G} \frac{\int_b^c |B_k(x)|^2 |g(x)|^2 \, d\mu(x)}{(B_k g, B_k g)} \leq Z_k \cdot \sigma_j(A_f). \end{aligned}$$

In order to establish the inequalities of (11), let us first show the following link with the third Zolotarev problem

$$Z_k = Z_k' := \min\Big\{ \frac{\max_{z\in E} |r(z)|}{\min_{z\in E^{-1}} |r(z)|} : r \in \mathcal{R}_k \Big\},$$

where $E = [b,c]$, and $E^{-1} := \{z : 1/z \in E\}$. Indeed, the inequality $Z_k' \leq Z_k$ is trivial, since the Blaschke product

$$r(z) := \prod_{j=1}^{k} \frac{z-\xi_j}{1-\overline{\xi}_j z}$$

is an element of $\mathcal{R}_k$, with $|r(1/z)| = 1/|r(z)|$, and thus $\max_{z\in E}|r(z)| = 1/\min_{z\in E^{-1}}|r(z)|$. Also, following for instance Akhieser [Akh90, §50] one shows without difficulties that there exists an extremal function r of the third Zolotarev problem which has the symmetry property $r(1/z) = 1/r(z)$ and which has all its zeros in E. Hence $Z_k = Z_k'$.

Following [Akh90, §50 and §51], let us write explicitly the value of Z_k in terms of elliptic functions. Define the quantities $\kappa, \widetilde{\kappa}$ by

$$\frac{\kappa+1}{2\sqrt{\kappa}} = \frac{1-bc}{c-b} = \frac{1}{a} > 1, \quad \kappa \in (0,1), \quad \widetilde{\kappa} = (\frac{1-\sqrt{\kappa}}{1+\sqrt{\kappa}})^2 = \frac{1-a}{1+a}. \tag{12}$$

We apply the variable transformation

$$y = T(z) = \frac{1}{\sqrt{\kappa}} \frac{z - \frac{\sqrt{\kappa}+b}{1+\sqrt{\kappa}b}}{1 - z\frac{\sqrt{\kappa}+b}{1+\sqrt{\kappa}b}}$$

which bijectively maps E on $[-1,1]$ (with $T(b) = -1$, $T(c) = +1$), and E^{-1} on $\{y \in \mathbb{R} : |y| \geq 1/\kappa\}$. Hence

$$Z_k = \min\Big\{ \frac{\max_{|y|\leq 1} |r(y)|}{\min_{|y|\geq 1/\kappa} |r(y)|} : r \in \mathcal{R}_k \Big\}.$$

Akhieser [Akh90, §50, p.144] shows the following link with Zolotarev's forth problem (the rational approximation of the sign function)

$$\frac{2 \cdot \sqrt{Z_k}}{1+Z_k} = \min\Big\{ \max\{ \max_{z\in[-1/\widetilde{\kappa},-1]} |r(z)+1|, \max_{z\in[1,1/\widetilde{\kappa}]} |r(z)-1| \} : r \in \mathcal{R}_k \Big\}.$$

Since there exists an extremal function ρ of the latter problem which is odd and has no poles in $[-1,1]$, we may write $\rho(z) = z \cdot p(z^2)/q(z^2)$ with p, q polynomials of degree bounded by $(k-1)/2$, and $k/2$, respectively, showing that

$$\frac{2 \cdot \sqrt{Z_k}}{1+Z_k} = \min\Big\{ \max_{z\in[1,1/\widetilde{\kappa}^2]} |1 - \sqrt{z} \cdot \frac{p(z)}{q(z)}| : \deg p \leq \frac{k-1}{2}, \deg q \leq \frac{k}{2} \Big\},$$

compare with [Akh90, §50, p.147]. The function $\mu : (0,1) \mapsto (0,+\infty)$ defined in (2) is bijective and strictly decreasing, one usually refers to it as the modulus of the *Grötzsch* ring. Obviously, $Z_0 = 1$. In the case $k > 0$ (here $Z_k < 1$), we know from [Akh90, p.150, Tables 1 and 2] that

$$\frac{2 \cdot \sqrt{Z_k}}{1+Z_k} = \frac{1-\kappa_k}{1+\kappa_k}, \quad \text{where} \quad \mu(\kappa_k) = \frac{\mu(\widetilde{\kappa})}{k}.$$

We obtain using Landen and Gauss transformations [Akh90, p.212, Tables XX and XXI]

$$\mu(Z_k) \quad = \quad 2\mu(\frac{2 \cdot \sqrt{Z_k}}{1+Z_k}) = 2\mu(\frac{1-\kappa_k}{1+\kappa_k}) = 4\mu(\sqrt{1-\kappa_k^2}) = \frac{\pi^2}{\mu(\kappa_k)} = \frac{\pi^2 \cdot k}{\mu(\widetilde{\kappa})}.$$

Using (2) and (12), we may conclude that

$$\mu(Z_k) \quad = \quad \frac{\pi^2 \cdot k}{\mu(\frac{1-a}{1+a})} = \frac{\pi^2 k}{2\mu(\sqrt{1-a^2})} = 2k\mu(a) = \frac{2k}{cap\,([b,c],\mathbb{T})}.$$

Lower and upper bounds of $\mu(r)$ in terms of r have been subject of a number of publications, see for instance [QVV98, Theorem 1.7]. We here use the rough estimates

$$\log(\frac{(1+\sqrt{1-r^2})^2}{r}) \leq \mu(r) \leq \log(4/r), \quad 0 < r < 1.$$

From the upper bound for $\mu(r)$ we may conclude that

$$Z_k \leq 4 \cdot \gamma^k, \quad \gamma = \exp(-\frac{2}{cap\,([b,c],\mathbb{T})}),$$

as claimed in (11). Moreover,

$$Z_k \cdot \gamma^{-k} \geq (1+\sqrt{1-Z_k^2})^2 \geq 4(1-Z_k^2),$$

leading to the lower bound for Z_k as claimed in (11). □

Remark 2.2. *After some elementary computations using the explicit formulas given in [Akh90, §50 and §51] one shows that the extremal Blaschke product in (10) may be expressed in terms of Jacobi elliptic functions as follows*

$$B(z) = \frac{1-\sqrt{\kappa_k}\ sn(\frac{K(\kappa_k)}{K(\widetilde{\kappa})}u(z);\kappa_k)}{1+\sqrt{\kappa_k}\ sn(\frac{K(\kappa_k)}{K(\widetilde{\kappa})}u(z);\kappa_k)}, \quad \text{where} \quad \frac{1-\sqrt{\widetilde{\kappa}}\ sn(u(z);\widetilde{\kappa})}{1+\sqrt{\widetilde{\kappa}}\ sn(u(z);\widetilde{\kappa})} = \frac{z-\frac{\sqrt{\kappa}+b}{1+\sqrt{\kappa}b}}{1-z\frac{\sqrt{\kappa}+b}{1+\sqrt{\kappa}b}},$$

κ and $\widetilde{\kappa}$ are defined in (12), and $\mu(\kappa_k)=\mu(\widetilde{\kappa})/k$. In particular, for

$$u_{j,2k} := u(z_{j,2k}) = K(\widetilde{\kappa}) + iK'(\widetilde{\kappa})\frac{2k-j}{2k}, \quad j=0,...,2k,$$

one shows that $z_{0,2k} = b < z_{1,2k} < ... < z_{2k,2k} = c$, that the zeros of B are given by $z_{1,2k}, z_{3,2k}, ..., z_{2k-1,2k}$, and the alternant by $z_{0,2k}, z_{2,2k}, ..., z_{2k,2k}$, with

$$(-1)^j \cdot B(z_{2k-2j,2k}) = \max_{z\in[b,c]} |B(z)| = \sqrt{Z_k} = \frac{1-\sqrt{\kappa_k}}{1+\sqrt{\kappa_k}}.$$

□

Following the reasoning of the above proof one easily shows that

$$Z_1 = \frac{(1-\sqrt{1-a^2})^2}{a^2} = \Big(\frac{1-bc-\sqrt{(1-b^2)(1-c^2)}}{c-b}\Big)^2 < 1.$$

In particular we deduce from Lemma 2.1 the well-known fact that the singular values of A_f with f as in (1) are (either zero or) simple.

Lemma 2.3. *For any $k \geq 0$ there exists a function $f \in \mathcal{R}_{k+1}$ of class (1) with*

$$\sigma_k(A_f) = Z_k \cdot \sigma_0(A_f).$$

Proof. Let us start by recalling the link to eigenvalues of Cauchy matrices: by identifying H^2 with the set ℓ^2 of quadratic summable sequences, we may identify $\widetilde{A}_f$ with a semi-infinite symmetric matrix. In particular, for rational f of type (1)

$$f(z) = \sum_{j=0}^{k} \frac{d_j^2}{z-x_j}, \quad b \leq x_0 < x_1 < ... < x_k \leq c, \quad d_j \in \mathbb{R}\setminus\{0\},$$

we obtain the matrix representation $\mathcal{A} = V^T \cdot D^2 \cdot V$, $V = (x_j^\ell)_{\ell=0,1,2,...;j=0,1,...,k}$, $D = diag\,(d_0,...,d_k)$. Notice that both A_f and $\mathcal{A}$ are of rank $k+1$. Hence the non-trivial distinct singular values of A_f are given by the eigenvalues of the positive definite matrix DCD, with the positive definite Cauchy matrix

$$C = VV^T = (\frac{1}{1-x_j x_\ell})_{j,\ell=0,...,k}.$$

Notice also that the inverse of C is explicitly known

$$C^{-1} = \Delta \cdot C \cdot \Delta, \quad \Delta = diag\,(\frac{1}{1-x_j^2} \cdot \prod_{\ell \neq j} \frac{x_j - x_\ell}{1 - x_j x_\ell})_{j=0,1,...,k}, \tag{13}$$

where the diagonal elements of Δ have alternating signs.

We now turn to the proof of Lemma 2.3. Let $y_1 < y_2 < ... < y_k$ be the (simple) zeros of the extremal Blaschke product in (10), and $x_0 < x_1 < ... < x_k$ its alternant (compare with Remark 2.2). Furthermore, define the functions

$$h_1(z) = \prod_{j=0}^{k} \frac{z - x_j}{1 - x_j z}, \quad h_2(z) = \prod_{j=1}^{k} \frac{z - y_j}{1 - y_j z}, \quad h(z) := h_1(z) \cdot h_2(z),$$

and $d_j = 1/\sqrt{|h_1'(x_j)|}$, that is,

$$f(z) = \sum_{j=0}^{k} \frac{1}{|h_1'(x_j)|} \frac{1}{z - x_j}.$$

From the above considerations we see easily that $M = DCD$ has the inverse

$$M^{-1} = D^{-1} \Delta C \Delta D^{-1} = EME, \quad E = diag\,(1, -1, ..., (-1)^k).$$

We may conclude that

$$\sigma_j(A_f) = \sigma_j(M) = \sigma_j(M^{-1}) = \frac{1}{\sigma_{k-j}(M)}, \quad j = 0, 1, ..., k.$$

Hence for the claim of Lemma 2.3 it is sufficient to show that $\sigma_k(A_f) = \sqrt{Z_k}$.

In order to show this last statement we will make use of AAK theory, more precisely, of the characterization of the kth AAK approximant of $f \in \mathcal{R}_{k+1}$ to be shown below in Lemma 3.2(b). Notice first that h_2 is just the extremal Blaschke product in (10), and thus $h_2(x_j) = (-1)^{k-j} \cdot \sqrt{Z_k}$ (compare with Remark 2.2). Furthermore, h has the simple zeros $x_{j-1} < y_j < x_j$ for $j = 1, ..., k$, with

$$h'(x_j) = h_1'(x_j) \cdot h_2(x_j) = |h_1'(x_j)| \cdot \sqrt{Z_k} > 0, \quad h'(y_j) = h_1(y_j) \cdot h_2'(y_j) < 0.$$

Define

$$g(z) = \sum_{j=1}^{k} \frac{\sqrt{Z_k}}{|h'(y_j)|} \frac{1}{z - y_j},$$

then clearly $g \in \mathcal{R}_k$, with all poles in $\mathbb{D}$ and distinct from those of f. Moreover,

$$f(z) - g(z) = \sqrt{Z_k} \cdot \Big(\sum_{j=0}^{k} \frac{1}{|h'(x_j)|(x - x_j)} - \sum_{j=1}^{k} \frac{1}{|h'(y_j)|(x - y_j)} \Big) = \frac{\sqrt{Z_k}}{h(z)}.$$

Since $1/h$ is a Blaschke product, it follows from Lemma 3.2(b) that g is the kth AAK approximant of f, and the required property $\sqrt{Z_k} = \sigma_k(A_f)$ follows from Lemma 3.2(a). □

Combining the proof of Lemma 2.3 with the estimate of Lemma 2.1 we have shown the following result on the euclidean condition number $||M|| \cdot ||M^{-1}|| = \sigma_0(M)/\sigma_k(M)$ of diagonally scaled Cauchy matrices. Here the second statement follows from the first by a simple linear transformation technique.

Corollary 2.4. **(a)** *Let $x_0, ..., x_k \in [b, c] \subset (-1, 1)$, D diagonal, and $M = D \cdot (\frac{1}{1-x_\ell x_j})_{\ell,j=0,...,k} \cdot D$, then*

$$||M|| \cdot ||M^{-1}|| \geq \frac{1}{Z_k},$$

and the lower bound may be attained.

(b) *Let $y_0, ..., y_k > 0$ such that $\frac{y_j - 1}{y_j + 1} \in [b, c] \subset (-1, 1)$ for all j, D diagonal, and $M = D \cdot (\frac{1}{y_\ell + y_j})_{\ell,j=0,...,k} \cdot D$, then*

$$||M|| \cdot ||M^{-1}|| \geq \frac{1}{Z_k},$$

and the lower bound may be attained.

We refer the reader to [FaOl01] for a different lower bound (in terms of the y_j) of the spectral condition number of Cauchy matrices $(\frac{1}{y_\ell + y_j})_{\ell,j=0,...,k}$ with positive y_j.

3 Proof of the main assertions

The assertion (3) of Theorem 1.1 for meromorphic approximation of Markov functions follows from the results of the preceding section; however, for proving assertion (4) it will be useful to know how well rational functions may be approximated on the unit circle by rational functions of smaller complexity.

Lemma 3.1. *Let $f_K \in \mathcal{R}_K$, with K poles in $\mathbb{D}$. Then, for all $0 \leq j \leq K$, there exists a $f_j \in \mathcal{R}_j$ with*

$$\mathrm{dist}(f_K, \mathcal{R}_j) \leq ||f_K - f_j||_{L^\infty(\mathbb{T})} \leq \sigma_j(A_{f_K}) + \sigma_{j+1}(A_{f_K}) + ... + \sigma_{K-1}(A_{f_K}).$$

Proof. First we recall from the Kronecker Theorem that A_{f_K} has rank K, and hence $\sigma_\ell(A_{f_K}) = 0$ for $\ell \geq K$. Let $k_0 = 0 < k_1 < k_2 < ... < k_{r+1} = K$ with

$$\sigma_{k_\ell}(A_{f_K}) = \sigma_{k_\ell+1}(A_{f_K}) = ... = \sigma_{k_{\ell+1}-1}(A_{f_K}) \neq \sigma_{k_{\ell+1}}(A_{f_K})$$

for $\ell = 0, ..., r$. For $k = K-1, K-2, ..., j$, denote by f_k the kth AAK approximant of f_{k+1}. We show by recurrence on k that, provided that $k_\ell \leq k < k_{\ell+1}$ ($k_{r+2} = \infty$), the function f_k is an element of $\mathcal{R}_{k_\ell}$, with k_ℓ poles lying all in $\mathbb{D}$, and $\sigma_m(A_{f_k}) = \sigma_m(A_{f_K})$ for $0 \leq m < k_\ell$.

This property is true for $k = K$ by assumption. If now $k+1 < k_{\ell+1}$, then $f_{k+1} \in H_k^\infty$, and hence $f_{k+1} = f_k$. It remains to discuss the case $k+1 = k_{\ell+1}$ where the first part of the claim follows from Lemma 3.2(a) (with $\kappa = k_\ell$), and the last part from Lemma 3.2(c). Consequently, provided that $k_{\ell^*} \leq j < k_{\ell^*+1}$, we get

$$\begin{aligned} ||f_K - f_j||_{L^\infty(\mathbb{T})} &= ||\sum_{\ell=\ell^*+1}^{r} (f_{k_\ell} - f_{k_\ell - 1})||_{L^\infty(\mathbb{T})} \leq \sum_{\ell=\ell^*+1}^{r} ||f_{k_\ell} - f_{k_\ell-1}||_{L^\infty(\mathbb{T})} \\ &= \sum_{\ell=\ell^*+1}^{r} \sigma_{k_\ell - 1}(A_{f_{k_\ell}}) = \sum_{\ell=\ell^*+1}^{r} \sigma_{k_\ell-1}(A_{f_K}) \leq \sum_{j=j}^{K-1} \sigma_j(A_{f_K}), \end{aligned}$$

as claimed in the assertion of Lemma 3.1. □

Forming AAK approximants of rational functions is a well-understood process in linear system theory. Here the kth AAK approximant is rational but may have undesirable ("unstable") poles outside the unit disk. The bound of Lemma 3.1 has been given before by Glover (see [Glo84] and [Ant98, Theorem 3.2]), at least up to a factor 2 and for real rational functions. The proof given in [Ant98] is also based on one-step AAK reductions as discussed in the proof of Lemma 3.1. Here one requires some deep properties of this reduction process, which are summarized in Lemma 3.2 below. For instance, f_k is a "stable all-pass dilatation" of f_{k+1}. This and other properties mentioned in Lemma 3.2 seem to be well-established in the linear system community (they are only partly shown in [Ant98]). Both for the sake of completeness and for trying to make these statements more transparent to people from other communities, we provide a proof of Lemma 3.2.

Lemma 3.2. *Let $f_{k+1} \in \mathcal{R}_{k+1}$, with $k+1$ poles (counting multiplicities) lying all in $\mathbb{D}$, and let $\kappa \in \{0, 1, ..., k\}$ with $\sigma_k(A_{f_{k+1}}) = ... = \sigma_\kappa(A_{f_{k+1}}) \neq \sigma_{\kappa-1}(A_{f_{k+1}})$ or else $\kappa = 0$. Then the following holds*

(a) *The kth AAK approximant $f_k \in H_k^\infty$ of f_{k+1} is a rational function, $f_k \in \mathcal{R}_\kappa$ with κ poles lying all in $\mathbb{D}$ and being distinct from those of f_{k+1}, and*

$$f_{k+1}(z) - f_k(z) = s \cdot B(z), \quad B \text{ a Blaschke product,} \tag{14}$$

with $s = \pm\sigma_k(A_{f_{k+1}})$.

(b) *Any rational function $f_k \in \mathcal{R}_k$ with poles which lie all in $\mathbb{D}$ and are distinct from those of f_{k+1}, and satisfying (14) for some $s \in \mathbb{C}$ equals the kth AAK approximant of f_{k+1}.*

(c) *There holds $\sigma_j(A_{f_{k+1}}) = \sigma_j(A_{f_k})$ for $j = 0, 1, ..., \kappa - 1$.*

Proof. From AAK theory it is known that $dist(f_{k+1}, H_k^\infty) = \sigma_k(A_{f_{k+1}})$, with the infimum being attained for a unique $f_k \in H_k^\infty$. This f_k has κ poles in $\mathbb{D}$, and satisfies

$$||f_{k+1} - f_k||_{L^\infty(\mathbb{T})} = \sigma_k(A_{f_{k+1}}), \quad f_{k+1}(z) - f_k(z) = \frac{u(z)}{v(z)}, \quad u(z) := (A_{f_{k+1}}v)(z),$$

where $v \in H^2$ is any singular vector corresponding to $\sigma_k(A_{f_{k+1}})$. In addition it is known that we may find a particular such singular vector v_0 having the additional symmetry property

$$u_0(z) := (A_{f_{k+1}}v_0)(z) = s \cdot \frac{1}{z}\overline{v}_0(\frac{1}{z}), \quad \text{where } \overline{v}(z) := \overline{v(\overline{z})}, \quad s = \pm\sigma_k(A_{f_{k+1}}). \tag{15}$$

These ingredients will enable us to show part (a). By assumption, there exist $b_{j,\ell} \in \mathbb{C}$, $b_{j,\ell_j-1} \neq 0$, and $z_1, ..., z_s \in \mathbb{D}$ with

$$f_{k+1}(z) = b_{0,0} + \sum_{j=1}^{s}\sum_{\ell=0}^{\ell_j-1} \frac{b_{j,\ell}}{(z - z_j)^{\ell+1}}, \quad \sum_{j=1}^{s} \ell_j = k + 1.$$

Then $u_0 = A_{f_{k+1}}v_0$ is rational, more precisely, for $|z| > 1$ we have

$$u_0(z) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_{k+1}(\zeta)v_0(\zeta)}{z - \zeta}\,d\zeta = \sum_{j=1}^{s}\sum_{\ell=0}^{\ell_j-1} b_{j,\ell} \sum_{m=0}^{\ell} \frac{v_0^{(m)}(z_j)}{m! \cdot (z - z_j)^{\ell+1-m}}.$$

Notice that u_0 is a rational function, with degree of the denominator equal to $k'+1$, and degree of the numerator $\leq k'$, where $k' \leq k$, and $k' < k$ if $v(z_j) = 0$ for at least one $j \in \{1, ..., s\}$. It follows from (15) that also v_0 is rational, and u_0/v_0 equals s times a Blaschke product, showing (14). Furthermore, since

$$f_{k+1}v_0 - A_{f_{k+1}}v_0 = f_{k+1}v_0 - u_0 = f_k v_0$$

is analytic in $\mathbb{D}$ and v_0 has at most k' zeros by (15), we may conclude that f_k is rational, with at most $k' \in \{\kappa, .., k\}$ poles, and these are within the zeros of v_0.

For a proof of (a) it remains to show that f_k has no zeros outside of $\mathbb{D}$ and that its zeros are distinct from those of f_{k+1}. We proceed by recurrence on $k \geq 0$. In case $k = 0$ we have $\kappa = k = 0$, and there is nothing to show since f_k is a constant. Suppose now that $k \geq 1$, and consider the case $\kappa = k$ (and thus all poles of f_k are in $\mathbb{D}$). Then $k' = k$, and hence $v_0(z_j) \neq 0$ for $j = 1, ..., s$. It follows that the poles of f_k are distinct from those of f_{k+1}, as claimed above. As second case we suppose that $\kappa < k$, and that f_{k+1} has a pole, say, z_1, being not a pole of f_k. Then $f_{k+1}(1/\overline{z}_1)$ is finite, and we may write

$$f_{k+1}(z) - f_k(z) = \frac{1 - \overline{z}_1 z}{z - z_1} \cdot (g_k(z) - g_{k-1}(z)), \quad \text{where} \tag{16}$$

$$f_{k+1}(z) - f_{k+1}(1/\overline{z}_1) = \frac{1 - \overline{z}_1 z}{z - z_1} \cdot g_k(z), \quad f_k(z) - f_{k+1}(1/\overline{z}_1) = \frac{1 - \overline{z}_1 z}{z - z_1} \cdot g_{k-1}(z).$$

Notice that $g_k \in \mathcal{R}_k$ has k poles in $\mathbb{D}$, and $g_{k-1} \in H^\infty_\kappa \subset H^\infty_{k-1}$. Since the Blaschke factor is of modulus 1 on the unit circle, we may conclude that g_{k-1} is the $(k-1)$th AAK approximant of g_k. Indeed, if $g \in H^\infty_{k-1}$ would be closer to g_k, then $f(z) = f_{k+1}(1/\overline{z}_1) + \frac{1-\overline{z}_1 z}{z - z_1} \cdot g(z)$ in H^∞_k would be closer to f_{k+1} than f_k, a contradiction. Our claim for this case follows now from the recurrence hypothesis.

As third and final case suppose that $\kappa < k$, and that the poles of f_{k+1} are also poles of f_k (with possibly smaller multiplicities). We want to show that this case is impossible. Since f_k has at least $k + 1 - \kappa \geq 2$ poles less than f_{k+1} in $\mathbb{D}$, we find a $z_1 \in \mathbb{D}$ being a pole both of f_{k+1} and of f_k, and having a strictly higher multiplicity for f_{k+1}. Then for g_k, g_{k-1} in (16) we obtain that $g_k \in \mathcal{R}_k$ has k poles in $\mathbb{D}$, and $g_{k-1} \in H^\infty_{\kappa-1} \subset H^\infty_{k-1}$. As before one shows that g_{k-1} is the $(k-1)$th AAK approximant of g_k. From the recurrence hypothesis we get that g_{k-1} has no poles in common with g_k. Since any of the poles of f_{k+1} are either z_1 or joint poles of g_{k-1} and g_k, we may conclude that f_{k+1} has only one pole at z_1 of multiplicity $k + 1 \geq 2$, and $f_k \in \mathcal{R}_\kappa$ has a simple pole at z_1 (plus possibly other poles). Furthermore, since there is invariance with respect to Moebius transformation of the argument, we may assume that $z_1 = 0$. Consider the set of right-hand singular vectors corresponding to $\sigma_{k+1}(A_{f_{k+1}})$

$$\mathcal{V} = \{v \in H^2 : ||A_{f_{k+1}}v||_{L^2(\mathbb{T})} = \sigma_{k+1}(A_{f_{k+1}}) \cdot ||v||_{L^2(\mathbb{T})}\}.$$

Notice that $A_{f_{k+1}}$ has a matrix representation containing as only nonzero entry a principal submatrix H of order $k+1$ being of Hankel structure and invertible, in particular $H_{k+1,1} = H_{k+1-j,j} \neq 0$, and $H_{j,\ell} = 0$ for $j + \ell > k+1$. Hence any element of $\mathcal{V}$ is a polynomial of degree at most k. Since $\mathcal{V}$ forms a vector space of dimension $k+1-\kappa$, we may find in this set a polynomial v_1 of degree $\leq \kappa$. Recall that, for any $v \in \mathcal{V}$, the function $f_{k+1}v - u = f_k v$, $u = A_{f_{k+1}}v$, is analytic in $\mathbb{D}$. Since f_k has κ poles in $\mathbb{D}$, we may conclude that v_1 has degree κ, having as zeros the poles of f_k (counting multiplicities), and that

$$\mathcal{V} = \{\alpha \cdot v_1 : \ \alpha \text{ is a polynomial of degree at most } k - \kappa\}.$$

Identify elements of $\mathcal{V}$ with vectors in $\mathbb{C}^{k+1}$, then, for any $v \in \mathcal{V}$, we have for $w = Hv$ the relation

$$||H^{-1}w|| = \frac{1}{\sigma_k(H)}||w|| = ||H^{-1}|| \cdot ||w||.$$

Conversely, any vector w satisfying this relation leads to some $v = H^{-1}w \in \mathcal{V}$. Since v_1 has a simple zero at 0, the first component of any vector in $\mathcal{V}$ and in particular of v_1 vanishes. Hence the vector corresponding to $v(z) = z^{k-\kappa}v_1(z)$ has the first $k - \kappa + 1$ components equal to zero, and the $(k - \kappa + 2)$th component different from zero. Hence the last $k - \kappa + 1$ components of $w = Hv$ are equal to zero, and the κth component different from 0. Denote by $w^* \in \mathbb{C}^{k+1}$ the vector obtained by shifting the first κ components of w downwards by $k - \kappa + 1$ positions, the other components being equal to zero. According to the (lower) Hankel structure of H^{-1} one easily verifies that the first κ components of $H^{-1}w^*$ coincide with the last κ entries of $v = H^{-1}w$ (which are the only nontrivial ones). Hence

$$||H^{-1}|| \cdot ||w|| = ||H^{-1}w|| \leq ||H^{-1}w^*|| \leq ||H^{-1}|| \cdot ||w^*|| = ||H^{-1}|| \cdot ||w||,$$

showing that $H^{-1}w^* \in \mathcal{V}$. However, by construction, the first component of $H^{-1}w^*$ is different from zero, a contradiction. Thus the third case may not occur, and we have shown part (a).

For a proof of part (b), suppose that f_k has κ' poles counting multiplicities. By assumption (b), the Blaschke product in (14) takes the form

$$B(z) = \prod_{j=1}^{k+\kappa'+1} \frac{1 - \overline{\zeta_j} z}{z - \zeta_j}, \quad \zeta_j \in \mathbb{D}.$$

For any $\ell \in \{1, ..., k + \kappa' + 1\}$, define $B_\ell \in H^2$ by

$$B_\ell(z) = \prod_{j=1}^{\ell} \frac{1}{1 - \overline{\zeta_j} z},$$

then, for $|z| > 1$,

$$(A_B B_\ell)(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{B(\zeta) B_\ell(\zeta)}{z - \zeta} \, d\zeta = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{B(1/t)}{t - 1/z} \frac{t^{\ell-1}/z}{\prod_{j=1}^{\ell}(t - \overline{\zeta_j})} \, dt = B(z) \cdot B_\ell(z)$$

by the Cauchy residual theorem. Notice also that, by the Kronecker Theorem, A_B has rank $k + \kappa' + 1$. Writing

$$\mathcal{C}_1 := \operatorname{span}\{\prod_{j=1}^{\ell} \frac{1}{1 - \overline{\zeta_j} z} : \ell = 1, ..., k + \kappa' + 1\} \subset H^2,$$

$$\mathcal{C}_2 := \operatorname{span}\{\prod_{j=1}^{\ell} \frac{1}{z - \zeta_j} : \ell = 1, ..., k + \kappa' + 1\} \subset \overline{H^2},$$

we may conclude that $||A_B g||_{L^2(\mathbb{T})} = ||g||_{L^2(\mathbb{T})}$ for each $g \in \mathcal{C}_1$, with the range of A_B given by $\mathcal{C}_2$, and its kernel by the orthogonal complement of $\mathcal{C}_1$ in H^2. Hence

$$\sigma_0(A_B) = \sigma_1(A_B) = ... = \sigma_{k+\kappa}(A_B) = 1, \quad \sigma_{k+\kappa+1}(A_B) = \sigma_{k+\kappa+2}(A_B) = ... = 0.$$

Consequently, for any $f \in H_k^\infty$ with $g = f - f_k \in H_{k+\kappa'}^\infty$ we obtain using (14)

$$||f_{k+1} - f||_{L^\infty(\mathbb{T})} = ||sB - g||_{L^\infty(\mathbb{T})} \geq \sigma_{k+\kappa'}(A_{sB}) = |s| \cdot \sigma_{k+\kappa'}(A_B) = |s|.$$

Moreover, according to the uniqueness of the $(k + \kappa')$th AAK approximant of sB, we have equality iff $g = 0$. Consequently, f_k is the kth AAK approximant of f_{k+1}, as claimed in part (b).

We finally turn to the proof of part (c). With the notations of the preceding paragraph (and in particular $\kappa' = \kappa \geq 1$ and $|s| = \sigma_k(A_{f_{k+1}})$), we first notice that $A_{f_{k+1}}$, and A_{f_k}, respectively, are of rank $k+1$, and κ, respectively, with their ranges being a subset of $\mathcal{C}_2$, and their kernels being subsets of the orthogonal complement of $\mathcal{C}_1$ in H^2. Dividing the operator identity $A_{f_{k+1}} - A_{f_k} = sA_B$ by $s = \pm\sigma_k(A_{f_{k+1}})$ and applying orthogonal projection operators on $\mathcal{C}_1$, and on $\mathcal{C}_2$, respectively, we may apply Lemma 3.3 below, telling us that

$$\frac{\sigma_j(A_{f_{k+1}})}{\sigma_k(A_{f_{k+1}})} = \frac{\sigma_j(A_{f_k})}{\sigma_k(A_{f_{k+1}})}, \quad j = 0, 1, ..., \kappa - 1,$$

as claimed in the assertion of Lemma 3.2. □

Lemma 3.3. *Let $C_1, C_2 \in \mathbb{C}^{(k+\kappa+1)\times(k+\kappa+1)}$, with C_1 of rank $k+1$, C_2 of rank $\kappa \leq k$, and $C_1 - C_2$ being unitary. Then $\sigma_k(C_1) = ... = \sigma_\kappa(C_1) = 1$, and $\sigma_j(C_1) = \sigma_j(C_2)$ for $j = 0, 1, ..., k-1$.*

Proof. Write $\Sigma_1 = diag\,(\sigma_j(C_1))_{j=0,1,...,k}$, $\Sigma_2 = diag\,(\sigma_j(C_2))_{j=0,1,...,\kappa-1}$, then we have the (incomplete) SVD decompositions

$$C_1 = U_1\Sigma_1V_1^*, \quad C_2 = U_2\Sigma_2V_2^*,, \quad U_1^*U_1 = V_1^*V_1 = I_{k+1}, \quad U_2^*U_2 = V_2^*V_2 = I_\kappa.$$

Let U_4 be such that the columns of (U_1, U_4) form an orthogonal basis of $\mathbb{C}^{k+\kappa+1}$. By assumption,

$$C_1 - C_2 = \left[\begin{array}{cc} U_1 & U_2 \end{array}\right] \left[\begin{array}{cc} \Sigma_1 & 0 \\ 0 & -\Sigma_2 \end{array}\right] \left[\begin{array}{cc} V_1 & V_2 \end{array}\right]^* =: \left[\begin{array}{cc} U_1 & U_4 \end{array}\right] \cdot V^* \tag{17}$$

is unitary, and hence V is unitary. Let us write $V = (V_3, V_4)$, with $V_3 \in \mathbb{C}^{(k+1)\times(k+\kappa+1)}$. Multiplying relation (17) by U_4^* gives

$$-U_4^*U_2\Sigma_2V_2^* = V_4^*,$$

showing in particular that $V_2^*V_3 = 0$, and that $W := -U_4^*U_2\Sigma_2 \in \mathbb{C}^{\kappa\times\kappa}$ is unitary. Hence, multiplying (17) on the left by $(U_1, U_4)^*$ and on the right by V leads to

$$\begin{aligned} I_{k+\kappa+1} &= \left[\begin{array}{cc} I_{k+1} & U_1^*U_2 \\ 0 & -W\Sigma_2^{-1} \end{array}\right] \left[\begin{array}{cc} \Sigma_1 & 0 \\ 0 & -\Sigma_2 \end{array}\right] \left[\begin{array}{cc} V_1^*V_3 & V_1^*V_4 \\ 0 & V_2^*V_4 \end{array}\right] \\ &= \left[\begin{array}{cc} I_{k+1} & -U_1^*U_2\Sigma_2 \\ 0 & W \end{array}\right] \left[\begin{array}{cc} \Sigma_1V_1^*V_3 & \Sigma_1V_1^*V_4 \\ 0 & V_2^*V_4 \end{array}\right], \end{aligned}$$

implying that

$$\left[\begin{array}{cc} \Sigma_1V_1^*V_3 & \Sigma_1V_1^*V_4 \\ 0 & V_2^*V_4 \end{array}\right] = \left[\begin{array}{cc} I_{k+1} & -U_1^*U_2\Sigma_2 \\ 0 & W \end{array}\right]^{-1} = \left[\begin{array}{cc} I_{k+1} & U_1^*U_2\Sigma_2W^* \\ 0 & W^* \end{array}\right].$$

In particular,

$$\Sigma_1V_1^* = (\Sigma_1V_1^*V_3)V_3^* + (\Sigma_1V_1^*V_4)V_4^* = V_3^* + U_1^*U_2\Sigma_2W^*V_4^*,$$

and

$$\Sigma_1^2 = \Sigma_1V_1^*V_1\Sigma_1 = I_{k+1} + U_1^*U_2\Sigma_2[U_1^*U_2\Sigma_2]^*.$$

Hence $\Sigma_1^2 - I_{k+1}$ is a semi positive definite matrix of rank $\leq \kappa$, showing that $\sigma_k(C_1) = ... = \sigma_\kappa(C_1) = 1$. Furthermore, the singular values of $U_1^*U_2\Sigma_2$ are given by $\sqrt{\sigma_j(C_1)^2 - 1}$, $j = 0, 1, ..., k-1$. On the other hand,

$$U_2\Sigma_2 = U_1U_1^*U_2\Sigma_2 + U_4U_4^*U_2\Sigma_2 = U_1U_1^*U_2\Sigma_2 - U_4W,$$

and

$$[U_2\Sigma_2]^*U_2\Sigma_2 = \Sigma_2^2 = [U_1^*U_2\Sigma_2]^*[U_1^*U_2\Sigma_2] + I_\kappa,$$

showing that the singular values of $U_1^*U_2\Sigma_2$ are given by $\sqrt{\sigma_j(C_2)^2 - 1}$, $j = 0, 1, ..., \kappa - 1$. Thus $\sigma_j(C_1) = \sigma_j(C_2)$ for $j = 0, 1, ..., \kappa - 1$, as claimed in Lemma 3.3. □

Before establishing Theorem 1.1, let us show the following result for Markov functions.

Lemma 3.4. *For any f as in (1) there holds*

$$\text{dist}(f, \mathcal{R}_0) = \frac{f(1) - f(-1)}{2} = \int_b^c \frac{d\mu(x)}{1 - x^2} \leq ||f||_{L^\infty(\mathbb{T})}. \tag{18}$$

Proof. We have for any $c \in \mathcal{R}_0 = \mathbb{C}$ and for any function f as in (1)

$$\begin{aligned} ||f - c||_{L^\infty(\mathbb{T})} &\geq \frac{|f(1) - c| + |f(-1) - c|}{2} \geq \frac{f(1) - f(-1)}{2} \\ &= \int_b^c (\frac{1/2}{1 - x} - \frac{1/2}{-1 - x})\, d\mu(x) = \int_b^c \frac{d\mu(x)}{1 - x^2}. \end{aligned}$$

On the other hand, for $c = (f(1) + f(-1))/2$ we get

$$f(z) - \frac{f(1) + f(-1)}{2} = \int_b^c (\frac{1}{z - x} - \frac{1/2}{1 - x} - \frac{1/2}{-1 - x})\, d\mu(x) = \int_b^c \frac{1 - zx}{z - x} \frac{d\mu(x)}{1 - x^2}$$

with

$$\max_{|z|=1} |f(z) - \frac{f(1) + f(-1)}{2}| \leq \int_b^c \max_{|z|=1} |\frac{1 - zx}{z - x}| \frac{d\mu(x)}{1 - x^2} = \int_b^c \frac{d\mu(x)}{1 - x^2},$$

as claimed in (18). □

Proof of Theorem 1.1. As mentioned before, a proof for assertion (3) is an immediate consequence of our preceding considerations: from AAK theory and Lemma 2.1 (for $j = 0$) we know that

$$dist(f, H_k^\infty) = \sigma_k(A_f) \leq Z_k \sigma_0(A_f) \leq 4 \cdot \gamma^k \cdot \sigma_0(A_f) = 4 \cdot \gamma^k \cdot dist(f, H^\infty),$$

implying (3). Moreover, from Lemma 2.3 we know that Z_k may not be replaced by any smaller constant for fixed k, b, and c. In addition, from (11) we know that, for any fixed b, c,

$$\lim_{k \to \infty} \frac{Z_k}{\gamma^k} = 4,$$

and hence the constant 4 in (3) is optimal.

In order to show (4), notice that the error of Padé approximation (at infinity) of f on $\mathbb{T}$ tends to zero by the Markov Theorem. Let $\epsilon > 0$, then there exists a $K > 0$ such that the Kth Padé approximant of f satisfies $||f - f_K||_{L^\infty(\mathbb{T})} < \epsilon$. Notice that $f_K \in \mathcal{R}_K$ has K simple poles, all being elements of $[b, c] \subset \mathbb{D}$, and positive residuals, showing that f_K is a member of the class (1). Applying Lemma 3.1 we find some rational function $f_k \in \mathcal{R}_k$, with

$$||f_K - f_k||_{L^\infty(\mathbb{T})} \leq \sigma_k(A_{f_K}) + \sigma_{k+1}(A_{f_K}) + ... + \sigma_{K-1}(A_{f_K}).$$

Since f_K is of the form (1), we apply for each singular value the estimate of Lemma 2.1, leading to

$$\begin{aligned} ||f_K - f_k||_{L^\infty(\mathbb{T})} &\leq Z_k \cdot (\sigma_0(A_{f_K}) + \sigma_1(A_{f_K}) + ... + \sigma_{K-k-1}(A_{f_K}) \\ &\leq Z_k \cdot \text{trace}(\widetilde{A}_{f_K}) = Z_k \cdot \frac{f_K(1) - f_K(-1)}{2}, \end{aligned}$$

where for the last identities we have used relation (8). It follows that

$$\begin{aligned} ||f - f_k||_{L^\infty(\mathbb{T})} &\leq ||f - f_K||_{L^\infty(\mathbb{T})} + ||f_K - f_k||_{L^\infty(\mathbb{T})} \\ &\leq \epsilon + Z_k \cdot (\frac{|f_K(1) - f(1)| + |f_K(-1) - f(-1)|}{2} + \frac{f(1) - f(-1)}{2}) \\ &\leq \epsilon + Z_k \cdot (\epsilon + \frac{f(1) - f(-1)}{2}). \end{aligned}$$

Using relation (18) and taking into considerations that $\epsilon > 0$ is arbitrary, we obtain $||f - f_k||_{L^\infty(\mathbb{T})} \leq Z_k \cdot dist(f, \mathcal{R}_0)$, as claimed in (4). □

Remark 3.5. *We have seen in the above proof that, for any b, c, for any f of class (1), and for any $k \geq 0$,*

$$\mathrm{dist}(f, H_k^\infty) \leq Z_k \cdot \mathrm{dist}(f, H^\infty),$$

with equality for the function $\widetilde{f}$ of Lemma 2.3. Hence Z_k is the best constant in this inequality. Also, in the second part of the proof of Theorem 1.1 we have shown that, for any b, c, for any f of class (1), and for any $k \geq 0$,

$$\mathrm{dist}(f, \mathcal{R}_k) \leq Z_k \cdot \mathrm{dist}(f, \mathcal{R}_0).$$

We do not know whether Z_k is the best constant in this last inequality. However, for the function $\widetilde{f} \in \mathcal{R}_{k+1}$ as in Lemma 2.3 we know from Lemma 3.2 that its kth AAK approximant is an element of $\mathcal{R}_k$, implying that $\mathrm{dist}(\widetilde{f}, \mathcal{R}_k) = \mathrm{dist}(\widetilde{f}, H_k^\infty) = Z_k \cdot \mathrm{dist}(\widetilde{f}, H^\infty)$. Thus the second estimate can be improved at most by the factor $\mathrm{dist}(\widetilde{f}, \mathcal{R}_0)/\mathrm{dist}(\widetilde{f}, H^\infty) > 1$. Let us give a simple upper bound for this quantity which is invariant under a Moebius transformation of the argument of the form $z' = T(z) = (z - \alpha)/(1 - \alpha z)$, $\alpha \in (-1, 1)$. We find α such that

$$T([b, c]) = [b', c'], \quad c' = -b' = \frac{1}{a} - \sqrt{\frac{1}{a^2} - 1}, \quad a = \frac{c - b}{1 - bc}.$$

Hence using relation (18) we get the following upper bound

$$\begin{aligned} 1 \leq \frac{\mathrm{dist}(\widetilde{f}, \mathcal{R}_0)}{\mathrm{dist}(\widetilde{f}, H^\infty)} &= \frac{trace(\widetilde{A}_{\widetilde{f}})}{||\widetilde{A}_{\widetilde{f}}||} = \frac{\int_{b'}^{c'} \frac{d\widetilde{\mu}}{1 - x^2}}{||\widetilde{A}_{\widetilde{f}}||} \\ &\leq \frac{1}{1 - (c')^2} \cdot \frac{|(1, \widetilde{A}_{\widetilde{f}} 1)|}{||\widetilde{A}_{\widetilde{f}}||} \leq \frac{1}{1 - (c')^2} = \frac{a^2/2}{a^2 - 1 + \sqrt{1 - a^2}}. \end{aligned}$$

□

Remark 3.6. *For functions f as in (1) with μ satisfying a Szegő condition on $[b, c]$, the following asymptotics has been shown in [BPS01, Theorem 4]*

$$\lim_{k \to \infty} \frac{\mathrm{dist}(f, H_k^\infty)}{\gamma^k} = 2\pi \mathcal{G}(w_\mu), \quad w_\mu(x) = \sqrt{\frac{(x - b)(c - x)}{(1 - xb)(1 - xc)}} \cdot \frac{d\mu}{dx}(x),$$

where γ is as in Theorem 1.1, and $\mathcal{G}(v)$ for any positive function v is defined by

$$\log(\mathcal{G}(v)) = \int_b^c \frac{\log(v(x))}{\sqrt{(1 - bx)(1 - cx)(x - b)(c - x)}} \frac{1 - bc}{2K(a)} dx.$$

We claim that

$$\pi \mathcal{G}(w_\mu) \leq \mathrm{dist}(f, H^\infty), \tag{19}$$

with equality for the particular measure

$$\frac{d\nu}{dx}(x) = \sqrt{\frac{(1-xb)(1-xc)}{(x-b)(c-x)}}. \tag{20}$$

It follows from the above asymptotics and (19) that relation (5) is true and may not be improved since there is equality in (5) for the particular measure (20).

Notice that for constants $c > 0$ we have $\mathcal{G}(c) = c$. Since for any unit measure σ and any positive function v we have the relation $\int \log(v)\, d\sigma \leq \log(\int v\, d\sigma)$ (known as link between geometric and arithmetic mean), we may conclude that

$$\mathcal{G}(v) \leq \int_b^c \frac{v(x)}{\sqrt{(1-bx)(1-cx)(x-b)(c-x)}} \frac{1-bc}{2K(a)}\, dx$$

and hence

$$2\pi\mathcal{G}(w_\mu) \leq 2\pi \frac{1-bc}{2K(a)} \int_b^c \frac{d\mu(x)}{(1-bx)(1-cx)}.$$

We notice that, by (7),

$$\begin{aligned}
&\int_b^c \frac{d\mu(x)}{(1-bx)(1-cx)} \leq ||A_f|| \cdot (||\frac{1}{\sqrt{(1-bx)(1-cx)}}||_{L^2(\mathbb{T})})^2 \\
&= \mathrm{dist}(f, H^\infty) \cdot \frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{|1-e^{it}b| \cdot |1-e^{it}c|} \\
&= \mathrm{dist}(f, H^\infty) \cdot \frac{2K(\frac{2\sqrt{a}}{1+a})}{\pi(1-b)(1+c)} = \mathrm{dist}(f, H^\infty) \cdot \frac{2K(a)}{\pi(1-bc)}.
\end{aligned}$$

Thus (19) holds. Let us now consider the particular measure $\mu = \nu$, here $G(w_\nu) = 1$, and hence $\mathrm{dist}(f, H^\infty) \geq \pi$ *by (19). In order to show the opposite inequality, notice that*

$$f(z) - g(z) = \pi\sqrt{\frac{(1-zb)(1-zc)}{(z-b)(z-c)}}, \quad \text{where} \quad g(z) = \int_{1/c}^{1/b} \sqrt{\frac{(1-xb)(1-xc)}{(x-b)(c-x)}} \frac{dx}{z-x},$$

and $g \in H^\infty$. Since $f - g$ is of modulus π on the unit circle, we obtain $\mathrm{dist}(f, H^\infty) \leq \pi$. □

It remains to show that a stable reduced model deviating at least from h on the imaginary axis may be constructed as claimed in Corollary 1.2.

Proof of Corollary 1.2. Consider the variable transformation $z = T(w) = (w + 1/||D^{-1}||)/(w - 1/||D^{-1}||)$, mapping the imaginary axis $i\mathbb{R}$ to $\mathbb{T}$, and the convex hull of the spectrum of D, namely $[-||D||, -1/||D^{-1}||]$, to the interval $[0, c]$, with $c = (\kappa(D)-1)/(\kappa(D)+1) \in [0, 1)$. By passing to the orthogonal basis of eigenvectors if necessary, we may suppose without loss of generality that D is diagonal, with eigenvalues $-||D|| = \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n = -1/||D^{-1}||$. Writing $\lambda_j' := T(\lambda_j) \in [0, a]$, we are left with the function

$$\begin{aligned}
-f(z) \quad &:= \quad h(w) = \beta + \sum_{j=1}^n \frac{e_j d_j}{w - \lambda_j} = \beta + \sum_{j=1}^n \frac{e_j d_j}{w - \lambda_j} \\
&= \quad \beta + \sum_{j=1}^n e_j d_j \cdot \Big(\frac{2\lambda_n}{(\lambda_j + \lambda_n)^2} \frac{1}{z - \lambda_j'} - \frac{1}{\lambda_j + \lambda_n} \Big).
\end{aligned} \tag{21}$$

We first discuss the symmetric case $e = d$ and hence $d_j e_j \geq 0$ for all j. Since $\lambda_n < 0$, we may conclude that $f \in \mathcal{R}_n$ is of the form (1), with $b = 0$, and c as above. Also, using (18) we obtain

$$dist(f, \mathcal{R}_0) = \frac{f(1) - f(-1)}{2} = \frac{h(0) - h(\infty)}{2} = \frac{-d^* D^{-1} d}{2} = \frac{|d^* D^{-1} d|}{2}.$$

With $f_k \in \mathcal{R}_k$ from Lemma 3.1 and $h_k(w) := -f_k(z)$, $h_k \in \mathcal{R}_k$, we may conclude using Lemma 2.1 that

$$\max_{w \in i \cdot \mathbb{R}} |h(w) - h_k(w)| = ||f - f_k||_{L^\infty(\mathbb{T})} \leq 4 \cdot \gamma^k \cdot dist(f, \mathcal{R}_0) = 2|d^* D^{-1} d| \gamma^k$$

where

$$-\log(\gamma) = \frac{2}{cap\,([0,a], \mathbb{T})} = 2\mu(a) = 4\mu(\sqrt{1 - 1/\kappa(B)^2}) = \frac{2\pi K(1/\kappa(B))}{K(\sqrt{1 - 1/\kappa(B)^2})}, \quad a = \frac{\kappa(B) - 1}{\kappa(B) + 1},$$

as claimed in (6).

In the unsymmetric case $d \neq e$ it may happen that that some of the coefficients $d_j e_j$ in (21) are negative. Here we write $f(z) = f^*(z) - f^{**}(z)$, with each of the two functions being of of the form (1). Approximating each of them with a rational functions of degree k', we obtain a rational function of McMillan degree at most $2k' \leq k$, together with the error estimate

$$4 \cdot \gamma^k \cdot dist(f, \mathcal{R}_0) = 2\gamma^{k'} \sum_{j=1}^{n} |\frac{d_j e_j}{\lambda_j}|,$$

where the sum on the right hand side can be estimated by $\sqrt{|e^T D^{-1} e| \cdot |d^T D^{-1} d|}$, as claimed in (6). □

Let us finally notice that, by Remark 3.5, the right hand side of estimate (6) in the symmetric case $d = e$ can be at most be improved by dividing by

$$\frac{a^2/2}{a^2 - 1 + \sqrt{1 - a^2}} = \frac{(\sqrt{\kappa(B)} + 1)^2}{4\sqrt{\kappa(B)}}.$$

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