A note on the convergence of Ritz values for sequences of matrices

by

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Abstract

While discussing the convergence of the Lanczos method, Trefethen and Bau observed a relationship with electric charge distributions, and claimed that the Lanczos iteration tends to converge to eigenvalues in regions of "too little charge" for an equilibrium distribution. Recently, Kuijlaars found a theoretical justification for this phenomenon by considering the Lanczos method applied to a suitable sequence of matrices with similar spectra which may occur for instance in the discretization of PDEs while varying the parameter of discretization. The aim of the present note is to improve the result of Kuijlaars: we obtain a better rate of convergence under weaker regularity assumptions, and show that this new rate of convergence is sharp.

Key words: Lanczos method, Convergence of Ritz values, Logarithmic potential theory.
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1 Introduction

In order to approximate eigenvalues of large real symmetric matrices A of order N via the Lanczos method with starting vector $b \in \mathbb{R}^N$, one successively constructs vectors $v_1, v_2, ...$, with $v_1, ..., v_n$ being an orthonormal basis of the Krylov space $b, Ab, ..., A^{n-1}b$ for all $n \geq 1$, and computes the so-called Ritz values, namely, the eigenvalues $\theta_{1,n} < ... < \theta_{n,n}$ of the (tridiagonal) matrix S_n obtained by orthogonal projection

$$S_n = (v_1, ..., v_n)^T \cdot A \cdot (v_1, ..., v_n).$$

Depending on the eigenvector components $\beta_1, .., \beta_N$ of the starting vector b, some of the eigenvalues $\lambda_1 \leq ... \leq \lambda_N$ of A are well approximated by Ritz values even if n is much smaller than the dimension N. Classical results on convergence and on technical details of the Lanczos method may be found in many textbooks. Let us cite here the well-known Kaniel-Page-Saad estimate for extremal eigenvalues [GoVL93, PPV95, Saa96, TrBa97]

$$|\frac{\theta_{1,n} - \lambda_1}{\lambda_N - \lambda_1}| \leq \frac{\tau}{T_{n-1}(1 + 2\frac{\lambda_2 - \lambda_1}{\lambda_N - \lambda_2})^2}, \quad \tau := \frac{1}{\beta_1^2} \sum_{j=2}^{n} \beta_j^2, \tag{1.1}$$

T_n being the nth Chebyshev polynomial of the first kind, or its modification established by Sleijpen and van der Sluis [SlVS96, Theorem 3.1]

$$|\frac{\theta_{1,n} - \lambda_1}{\theta_{1,1} - \lambda_1}| \leq \frac{\tau + 1}{\tau + T_{n-1}(1 + 2\frac{\lambda_2 - \lambda_1}{\lambda_N - \lambda_2})^2}, \quad \tau = \frac{1}{\beta_1^2} \sum_{j=2}^{n} \beta_j^2. \tag{1.2}$$

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Thus one may expect geometric convergence of the smallest (largest) Ritz value to the smallest (largest) eigenvalue for a fixed matrix A, but the rate of convergence will depend on the size of the eigenvector component β_1, and on the (relative) distance of λ_1 to the other eigenvalues. For an "inner" eigenvalue λ_k lying in the convex hull of the Ritz values, say, $\theta_{\kappa-1,n} < \lambda_k \leq \theta_{\kappa,n}$ for some $\kappa = \kappa(k)$, we have

$$\min_\ell |\lambda_k - \theta_{\ell,n}|^2 \leq |(\lambda_k - \theta_{\kappa-1,n})(\lambda_k - \theta_{\kappa,n})| \leq \frac{|(\lambda_k - \lambda_1)(\lambda_k - \lambda_N)|}{t_{n-2}(\lambda_k)^2} \cdot \sum_{\ell \neq k} \frac{\beta_\ell^2}{\beta_k^2}, \tag{1.3}$$

where t_n denotes the generalized Chebyshev polynomial of degree n of a continuous set E_k which contains all eigenvalues $\lambda_1, ..., \lambda_{k-1}, \lambda_{k+1}, ..., \lambda_N$, for instance the set $E_k = [\lambda_1, \lambda_{k-1}] \cup [\lambda_{k+1}, \lambda_n]$ (see, e.g., [Fi96, Theorem 3.3.18]). Notice again that we may only expect an interesting rate of convergence if λ_k is well separated from the rest of the spectrum, and if $|\beta_k|/||b||$ is sufficiently large.

There exist (worst case) examples A, b with eigenvalue and eigenvector component distribution such that the bounds (1.1), (1.2), or (1.3) are (approximately) sharp. However, for matrices occurring in applications one observes quite often that the above bounds greatly overestimate the actual error, even for a judicious choice of the set E_k (for instance a finite union of intervals representing the parts of the real axis where the spectrum of A is relatively dense). To give an example, consider the 2D Poisson equation on a unit square discretized by finite differences on a uniform mesh with stepsize $1/(m+1)$. As it is well known, the corresponding (scaled) matrix of coefficients has the eigenvalues $4 - 2\cos(\pi j/(m+1)) - 2\cos(\pi k/(m+1))$, $j, k = 1, \ldots, m$. Thus there are about $N \approx m^2/2$ distinct eigenvalues and the spectrum becomes dense everywhere in $[0, 8]$ for $N \to \infty$. However, for $n = n(N)$ and N tending to infinity along ray sequences (i.e., $n/N \to t \in (0,1)$) one may observe geometric convergence of Ritz values outside some interval $[4 - 4\cos(\pi t/2), 4 + 4\cos(\pi t/2)]$. This phenomenon is common for all examples studied in the sequel: an additional information about the (asymptotic) fine structure of the spectrum will enable us to give an improved (asymptotic) counterpart of the classical estimates (1.1) and (1.3).

Trefethen and Bau [TrBa97, p. 279] observed a relationship with electric charge distributions, and claimed that the Lanczos iteration tends to converge to eigenvalues in regions of "too little charge" for an equilibrium distribution. To illustrate this statement, suppose that the N eigenvalues are more or less equally spaced on the interval $[-1, 1]$, except perhaps for a few outliners outside of $[-1, 1]$. In order to approximate the equilibrium measure of $[-1, 1]$ with n points, one requires points with a mutual distance of $\approx 1/n^2$ close to the endpoints of the interval. Choosing such a set from the set of equidistant eigenvalues is no longer possible if n/N becomes larger. Thus, according to the above claim, the Lanczos method should converge outside of $[-r, r]$, with $r = r(n/N) \in (0, 1)$ being decreasing if n/N increases.[2]

Clearly, the above observation of Trefethen and Bau is an asymptotic assertion. In order to be more precise, we need to consider a sequence of real symmetric matrices $(A_N)_N$, with A_N having N distinct eigenvalues $\lambda_{1,N} < ... < \lambda_{N,N}$. We suppose that all eigenvalues are contained in some compact interval, and that the eigenvalue distribution of the matrices A_N is

[2]Indeed, Kuijlaars [Kuij99, Section 4.2] showed that this statement is true with $r(t) = \sqrt{1-t^2}$, see also Example 3.2 below.

asymptotically determined by some (probability) measure σ, i.e.,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} f(\lambda_{j,N}) = \int f(\lambda')\, d\sigma(\lambda') \tag{1.4}$$

for all continuous functions $f : \mathbb{R} \to \mathbb{R}$. Notice that the occurrence of sequences of matrices is natural in the context of a discretization of some continuous operator while varying the discretization parameter.

We suppose that the Lanczos method is applied to matrix A_N with starting vector b_N having eigenvector components $\beta_{1,N}, ..., \beta_{N,N}$, and we are interested in measuring the distance of an eigenvalue $\lambda_{j,N}$ to the set of Ritz values $\theta_{1,n,N} < ... < \theta_{n,n,N}$ obtained in the nth iteration of the Lanczos process. Thus, according to Trefethen and Bau, we will have to compare the measure σ describing the (asymptotic) fine structure of the spectra with an equilibrium distribution. Kuijlaars [Kuij99, Theorem 3.1] showed (under some hypotheses being specified below) that for any set of indices $(k_N)_N$ satisfying

$$\lim_{N\to\infty} \lambda_{k_N,N} = \lambda \tag{1.5}$$

and for any $t \in (0,1)$ there holds

$$\limsup_{\substack{n,N\to\infty \\ n/N\to t}} \min_j |\lambda_{k_N,N} - \theta_{j,n,N}|^{1/n} \leq \exp(-G_{t,\sigma}(\lambda)/2). \tag{1.6}$$

Here the (nonnegative) function $G_{t,\sigma}$ is described by some constrained equilibrium problem in logarithmic potential theory. In fact, complex potential theory is a very useful tool in order to study convergence of Krylov methods in Numerical Linear Algebra, see for instance the excellent survey paper [DTT98] of Driscoll, Toh, and Trefethen. For example, the same function $G_{t,\sigma}$ is used for explaining the superlinear convergence of the conjugate gradient method or other Krylov subspace methods [BeKu99]. We refer the reader to [Ran95, SaTo97] for detailed accounts of logarithmic potential theory.

The above mentioned constrained equilibrium problem has been considered and analyzed recently by Rakhmanov [Rak96] and Dragnev and Saff [DrSa97] in order to describe the nth root asymptotics of discrete orthogonal polynomials. Let $t \in (0,1)$, and suppose that σ has a continuous logarithmic potential

$$U^{\sigma}(\lambda) = \int \log \frac{1}{|\lambda - \lambda'|}\, d\sigma(\lambda'), \qquad \lambda \in \mathbb{C}.$$

According to [Rak96], [DrSa97, Theorem 2.1], under all Borel probability measures μ satisfying the constraint $0 \leq t\mu \leq \sigma$ there exists a unique measure μ_t having minimal logarithmic energy. By [Rak96, Theorem 3] and [DrSa97, Theorem 2.1], the minimizer μ_t is uniquely characterized by the following variational condition: there exists a real constant F_t such that

$$G_{t,\sigma}(\lambda) := F_t - U^{\mu_t}(\lambda) \begin{cases} = 0 & \text{if } \lambda \in S(t) := \operatorname{supp}(\sigma - t\mu_t) \\ \geq 0 & \text{for } \lambda \in \mathbb{R} \setminus S(t). \end{cases} \tag{1.7}$$

Thus, if for instance the equilibrium measure of $\operatorname{supp}(\sigma)$ is less than or equal to σ/t then it will coincide with μ_t (and $G_{t,\sigma}(\lambda) = 0$ for $\lambda \in \operatorname{supp}(\sigma)$). Things become however more involved (and more interesting) if this inequality is no longer true. Here $S(t)$ will be a proper subset of $\operatorname{supp}(\sigma)$, and it is known that $S(t)$ decreases if t increases. In addition, it is shown in [BeKu99, Theorem 3.1, Eqn.(2.22)] using arguments from [BuRa99] that $G_{t,\sigma}$ is known provided that $S(t')$ is given for $0 < t' < t$: denoting by g_E the Green function with logarithmic singularity at infinity of a compact set E, we have

$$G_{t,\sigma}(\lambda) = \frac{1}{t}\int_0^t g_{S(t')}(\lambda)\, dt', \text{ and} \tag{1.8}$$

$$\Lambda(t,\sigma) := \{\lambda \in \mathbb{R} : G_{t,\sigma}(\lambda) > 0\} = \mathbb{R} \setminus \bigcap_{0<t'<t} S(t'). \tag{1.9}$$

Thus, according to (1.6), there will be convergence of the Lanczos process in the set $\Lambda(t,\sigma)$ of "too little charge". Also, the rate of convergence is not determined by a single Green function (as one might expect from (1.3)) but by some mean of Green functions.

In accordance with the observation mentioned above, it was shown in [BeKu99, Section 5] that for the model problem resulting from discretizing the 2D Poisson equation we have $S(t) = [4 - 4\cos(\pi t/2), 4 + 4\cos(\pi t/2)]$, and thus there will be geometric convergence for extremal eigenvalues. The case of Hermitian Toeplitz matrices generated by some real–valued, integrable, and bounded symbol ϕ was examined in [BeKu99, Section 4]. According to a classical result of Szegő [BoSi99, Theorem 5.10 and Corollary 5.11], relation (1.4) holds for the measure σ having the logarithmic potential

$$U^\sigma(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\frac{1}{|x - \phi(\theta)|}\, d\theta$$

(provided that most of the eigenvalues have the multiplicity 1). For instance, for the Toeplitz matrix $A_N = (\gamma^{j-k})_{j,k=1,\ldots,N}$, $\gamma \in (-1,1)$ studied by Kac, Murdock and Szegő [KaMuSz53, p. 783] one finds a symbol with range $\mathrm{supp}\,(\sigma) = [a, 1/a]$, $a = (1-\gamma)/(1+\gamma)$, and here $S(t) = \mathrm{supp}\,(\sigma)$ for $t \in (0,a)$, but $S(t) = [a, a/t^2]$ for $t \in (a,1)$ [BeKu99, Section 4]. Thus, for $t > a$, we may observe geometric convergence around the right endpoint of the spectrum. The determination of a mesure σ as in (1.4) is possible for a number of further examples, e.g., for block-Toeplitz-Toeplitz-block matrices with (bivariate) real-valued symbol of Wiener class. For the determination of the corresponding sets $S(t)$ we refer the reader to a future publication.

We still need to mention the additional technical assumptions imposed by Kuijlaars for establishing (1.6): he assumes that for any sequence of indices $(k_N)_N$ satisfying (1.5) the following two assertions are true

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j\neq k_N} \log|\lambda_{k_N,N} - \lambda_{j,N}| = \int \log|\lambda - \lambda'|\, d\sigma(\lambda'), \tag{1.10}$$

$$\lim_{N\to\infty} \left[\frac{|\beta_{k_N,N}|}{||b_N||}\right]^{1/N} = 1. \tag{1.11}$$

The first condition will be true according to (1.4) if the eigenvalues $\lambda_{k_N+j,N}$ for $j \neq 0$ do not approach "too fast" $\lambda_{k_N,N}$ when $N \to \infty$, see Lemma 2.4 below. This separation condition has been suggested by Dragnev and Saff [DrSa97, Definition 3.1] as a sufficient condition for insuring nth root asymptotics for discrete orthogonal polynomials. It holds if the distance $|\lambda_{k_N+j,N} - \lambda_{k_N,N}|$ is bounded below by a constant times $|j|/N$ [Rak96] or by some positive power of this quantity; however, it is excluded that two neighboring eigenvalues approach exponentially. Condition (1.11) means that the starting vector b_N has a sufficiently large eigencomponent for the eigenvalue $\lambda_{k_N,N}$.

Notice that Kuijlaars' assertion is a global one: he imposes conditions (1.10) and (1.11) for all sequences satisfying (1.5), and obtains the asymptotic bound (1.6) for all such sequences.

In Theorem 2.1 below we will give a local counterpart of this assertion where we only impose (1.10) and (1.11) for the sequence of eigenvalues under consideration. In addition we show that the factor 1/2 on the right-hand side of (1.6) may be dropped, and sometimes even be replaced by a factor 2. Under the assumptions of Kuijlaars, this latter estimate is shown to be sharp. A comparison with estimates for zeros of discrete Chebyshev polynomials concludes Section 2. In Section 3 we will consider some examples.

2 The main result

As in the previous section we suppose that $(A_N)_N$ is a sequence of real symmetric matrices, A_N having the eigenvalues $\lambda_{j,N}$, with asymptotic spectrum given by the measure σ. We are interested in the question how the Ritz values $\theta_{1,n,N}, ..., \theta_{n,n,N}$ obtained in the nth iteration of the Lanczos method applied to the matrix A_N and the starting vector b_N approximate the spectrum of A_N for $n, N \to \infty$, where here and in what follows we will always assume that $n = n(N)$ depends on N in such a way that $n(N)/N \to t \in (0,1)$ as $N \to \infty$. Our main findings are summarized in

Theorem 2.1. *Suppose that the asymptotic distribution of the spectra of $(A_N)_N$ is given by σ, which has a continuous potential, and define $G_{t,\sigma}$ as in (1.8).*

(a) *Let $(k_N)_N$ be a sequence as in (1.5) satisfying (1.10) and (1.11). Then for each $t \in (0,1)$ there holds*

$$\limsup_{\substack{n,N\to\infty \\ n/N\to t}} \min_j |\lambda_{k_N,N} - \theta_{j,n,N}|^{1/n} \leq \exp(-G_{t,\sigma}(\lambda)). \tag{2.1}$$

(b) *Let $I \subset \Lambda(t,\sigma)$ be some closed interval, and suppose that conditions (1.10) and (1.11) hold for any sequence $(\lambda_{k_N,N})_N \subset I$ having a limit λ. Then for each N there exists an "exceptional" index $k^*(I,N)$ (possibly the trivial choice $k^*(I,N) = 0$) such that*

$$\limsup_{\substack{n,N\to\infty \\ n/N\to t}} \min_j |\lambda_{k_N,N} - \theta_{j,n,N}|^{1/n} \leq \exp(-2G_{t,\sigma}(\lambda)) < 1 \tag{2.2}$$

for any $(\lambda_{k_N,N})_N \subset I$ tending to some λ, and satisfying $k_N \neq k^*(I,N)$ for all N. If I is unbounded then there is no such exceptional eigenvalue (i.e., $k(I,N)^* = 0$ for all N).

(c) *Under the assumptions of Kuijlaars [Kuij99, Theorem 3.1], the preceding estimates are sharp. More precisely, suppose that conditions (1.10) and (1.11) hold for any sequence $(\lambda_{k_N,N})_N$ having a limit λ, and let $I \subset \Lambda(t,\sigma)$ be some closed interval. Then for each N there exists an "exceptional" index $k^{**}(I,N)$ (possibly the trivial choice $k^{**}(I,N) = 0$) such that*

$$\liminf_{\substack{n,N\to\infty \\ n/N\to t}} \min_j |\lambda_{k_N,N} - \theta_{j,n,N}|^{1/n} \geq \exp(-2G_{t,\sigma}(\lambda)) \tag{2.3}$$

for any $(\lambda_{k_N,N})_N \subset I$ tending to some λ, and satisfying $k_N \neq k^{**}(I,N)$ for all N. If I is unbounded then there is no such exceptional eigenvalue (i.e., $k^{**}(I,N) = 0$ for all N).

(d) *The exceptional eigenvalues described in parts (b),(c) do not occur simultaneously for fixed N (i.e., $k^*(I,N) \cdot k^{**}(I,N) = 0$ for all N). In addition, there exists an example (see Example 3.4 below) satisfying the regularity assumptions of part (c) where both types of exceptional eigenvalues occur, and where equality in (2.1) occurs for a particular subsequence of eigenvalues.*

Under the assumptions of Kuijlaars, it follows in particular from Theorem 2.1(b),(c) that for sequences of eigenvalues $(\lambda_{k_N,N})_N$ tending to some λ not being an element of the convex hull of $\mathbb{R} \setminus \Lambda(t,\sigma)$ we have

$$\lim_{\substack{n,N\to\infty \\ n/N\to t}} \min_j |\lambda_{k_N,N} - \theta_{j,n,N}|^{1/n} = \exp(-2G_{t,\sigma}(\lambda)).$$

Thus for such "extremal" eigenvalues we may exactly determine the rate of (geometric) convergence. However, things may become more complicated if $\mathbb{R} \setminus \Lambda(t,\sigma)$ consists of several intervals and if λ lies between two of such intervals. Here we may have either a better rate of convergence (e.g., $\lambda_{k_N,N}$ coincides with a Ritz value) or only the weaker convergence rate specified in (2.1). Notice also that in such gaps there may be Ritz values which are not close to the set of eigenvalues.

Some further remarks on Theorem 2.1 and its proof may be found at the end of Section 2.

In [Kuij99, Proof of Theorem 3.1], Kuijlaars used some recent results on nth root asymptotics of discrete orthogonal polynomials, a research activity initiated by Rakhmanov [Rak96] and Dragnev and Saff [DrSa97] followed by a number of authors, see, e.g., [Beck98, KuRa98, KuVA98]. To see how these orthogonal polynomials are involved, we denote by $v_{1,N}, v_{2,N}, ...$ the othonormal basis produced by the Lanczos process applied to A_N and starting vector b_N. By construction, there exist monic polynomials $p_{n,N}$ of degree n such that $v_{n+1,N} = p_{n,N}(A_N)b_N/||p_{n,N}(A_N)b_N||$ for $n \geq 0$, and thus for $m \neq n$

$$0 = (v_{m+1,N}, v_{n+1,N}) = (p_{m,N}(A_N)b_N, p_{n,N}(A_N)b_N) = \sum_{j=1}^{N} \beta_{j,N}^2 \cdot p_{m,N}(\lambda_{j,N}) \cdot p_{n,N}(\lambda_{j,N}),$$

in other words, $p_{n,N}$ is the nth monic orthogonal polynomial with respect to some discrete scalar product depending on N. It is also well known and easily verified that the Ritz values $\theta_{1,n,N}, ..., \theta_{n,n,N}$ are just the (distinct) zeros of $p_{n,N}$, i.e., the abscissa of the corresponding Gaussian quadrature formula of order n. In other words, we are interested in the question how the abscissa of the nth Gaussian quadrature formula approximate the abscissa of the original discrete scalar product (i.e., the Nth Gaussian quadrature formula). Here the following probably well-known assertion is helpful which is also the starting point for establishing estimates like (1.1), (1.2), or (1.3).

Lemma 2.2. *If $\lambda_{k,N} \leq \theta_{1,n,N}$ then*

$$\theta_{1,n,N} - \lambda_{k,N} = \min\Big\{ \frac{\sum_{j=1, j\neq k}^{N} \beta_{j,N}^2 (\lambda_{j,N} - \theta_{1,n,N}) q(\lambda_{j,N})^2}{\beta_{k,N}^2 q(\lambda_{k,N})^2} : \partial q < n, q(\lambda_{k,N}) \neq 0 \Big\}.$$

Here the minimum is attained for the polynomial $q(x) = p_{n,N}(x)/(x - \theta_{1,n,N})$. If $\lambda_{k,N} \in$

$[\theta_{1,n,N}, \theta_{n,n,N}]$, *say,* $\theta_{\kappa(k,N)-1,n,N} \leq \lambda_{k,N} \leq \theta_{\kappa(k,N),n,N}$, *then*

$$(\lambda_{k,N} - \theta_{\kappa(k,N)-1,n,N})(\theta_{\kappa(k,N),n,N} - \lambda_{k,N}) = \min\Big\{ \frac{\sum_{j=1, j\neq k}^{N} \beta_{j,N}^2 (\lambda_{j,N} - \theta_{\kappa(k,N)-1,n,N})(\lambda_{j,N} - \theta_{\kappa(k,N),n,N}) q(\lambda_{j,N})^2}{\beta_{k,N}^2 q(\lambda_{k,N})^2} : \partial q < n-1, q(\lambda_{k,N}) \neq 0 \Big\}.$$

Here the minimum is attained for the polynomial $q(x) = p_{n,N}(x)/[(x - \theta_{\kappa(k,N)-1,n,N})(x - \theta_{\kappa(k,N),n,N})]$.

Proof. We will show here the second part of the assertion; similar arguments may be applied to establish the first part, compare [SlVS96]. From Gaussian quadrature we know that there exist real numbers $\beta_{1,n,N}, ..., \beta_{n,n,N}$ such that

$$\sum_{j=1}^{N} \beta_{j,N}^2 \cdot p(\lambda_{j,N}) = \sum_{j=1}^{n} \beta_{j,n,N}^2 \cdot p(\theta_{j,n,N}) \tag{2.4}$$

for all polynomials p of degree at most $2n-1$. If q is now a polynomial of degree less than $n-1$ with $q(\lambda_{k,N}) \neq 0$ and $p(x) = (x - \theta_{\kappa(k,N)-1,n,N})(x - \theta_{\kappa(k,N),n,N}) \cdot q(x)^2$, then the right hand side of (2.4) is ≥ 0, and thus

$$(\lambda_{k,N} - \theta_{\kappa(k,N)-1,n,N})(\theta_{\kappa(k,N),n,N} - \lambda_{k,N}) \leq \frac{\sum_{j=1, j\neq k}^{N} \beta_{j,N}^2 (\lambda_{j,N} - \theta_{\kappa(k,N)-1,n,N})(\lambda_{j,N} - \theta_{\kappa(k,N),n,N}) q(\lambda_{j,N})^2}{\beta_{k,N}^2 q(\lambda_{k,N})^2}.$$

Finally, notice that for the choice $q(x) = p_{n,N}(x)/[(x - \theta_{\kappa(k,N)-1,n,N})(x - \theta_{\kappa(k,N),n,N})]$ the right hand side of (2.4) equals zero, and thus there is equality in the above estimate. □

For a proof of Theorem 2.1 we require two further lemmas, the first cited from [BeKu99].

Lemma 2.3. *Suppose that (1.4) holds with* σ *being a finite Borel measure on* $\mathbb{R}$ *with compact support. Let* $t \in (0,1)$ *and let* μ *be a Borel probability measure such that* $t\mu \leq \sigma$. *Let* $n = n_N \leq N$ *such that* $n/N \to t$. *Then there exists a sequence of sets* $(Z_N)_N$ *such that*

(a) $\#Z_N = n$,

(b) $Z_N \subset \{\lambda_{1,N}, .., \lambda_{N,N}\}$, *and*

(c) *for all continuous functions* f,

$$\lim_{N\to\infty} \frac{1}{n} \sum_{\lambda \in Z_N} f(\lambda) = \int f(\lambda) d\mu(\lambda).$$

Furthermore, if K *is a closed set such that* $\sigma(\partial K) = 0$ *and* $\sigma(K) = t\mu(K)$, *then the sets* Z_N *can be chosen such that in addition to* (a), (b) *and* (c), *we also have for* N *large enough,*

(d) $\{\lambda_{1,N}, .., \lambda_{N,N}\} \cap K \subset Z_N$.

Proof. See [BeKu99, Lemma 6.1]. □

Lemma 2.4. *Suppose that (1.4) holds, with σ having a finite logarithmic potential.*

(a) *Suppose that (1.5) holds. Then (1.10) is true if and only if, for all $\epsilon > 0$, there exist $\delta, C > 0$ such that*

$$\frac{1}{N} \sum_{\substack{j=1 \\ |\lambda_{j,N}-\lambda_{k_N,N}|<\delta}}^{N} \log \frac{1}{|\lambda_{j,N} - \lambda_{k_N,N}|} < \epsilon$$

for all $N \geq C$.

(b) *Let $I \subset \Lambda(t,\sigma)$ be some compact interval, and suppose that the separation condition (1.10) holds for any sequence $(\lambda_{k_N,N})_N \subset I$ having a limit λ. Then*

$$\lim_{N\to\infty} \min\{|\lambda_{k\pm1,N} - \lambda_{k,N}|^{1/N} : \lambda_{k,N} \in I\} = 1. \tag{2.5}$$

Proof. For establishing part (a), we notice first that $U^\sigma(\lambda) < \infty$ together with Lebesgue's dominated convergence theorem imply that there exists a $\delta > 0$ such that $U^{\sigma|_\delta}(\lambda) < \epsilon/3$, where $\sigma|_\delta$ denotes the restriction of σ to the interval $[\lambda - \delta, \lambda + \delta]$. From (1.4) and (1.5) we may conclude that

$$\left| \frac{1}{N} \sum_{|\lambda_{j,N}-\lambda_{k_N,N}|\geq\delta} \log \frac{1}{|\lambda_{j,N} - \lambda_{k_N,N}|} - U^{\sigma-\sigma|_\delta}(\lambda) \right| < \epsilon/3 \tag{2.6}$$

for all sufficiently large N, say, $N \geq C'$. In other words, the difference between $U^\sigma(\lambda)$ and the sum in (2.6) is smaller than $2\epsilon/3$ for all $N \geq C'$. This leads to the equivalence claimed in part (a).

If the assertion of part (b) is not true, then we may find a sequence $(k_N)_N$ of indices and $\eta_N \in \{\pm1\}$ with

$$\lambda_{k_N,N} \in I, \quad \text{and} \quad \liminf_{N\to\infty} \frac{1}{N} \log|\lambda_{k_N,N} - \lambda_{k_N+\eta_N,N}| =: -\epsilon < 0.$$

Since I is compact, we may assume without loss of generality (by extracting a convergent subsequence) that (1.5) holds. Then also (1.10) is valid by assumption, in contradiction to part (a). □

Proof of Theorem 2.1(a). We first recall from, e.g., [DrSa97, Lemma 5.2] that with U^σ also U^{μ_t} and thus $G_{t,\sigma}$ are continuous. Since all eigenvalues (and hence all Ritz values) are included in some compact set, the assertion is trivial if $G_{t,\sigma}(\lambda) = 0$. Otherwise, let $\epsilon \in (0, G_{t,\sigma}(\lambda))$, and define $K := \{\lambda' \in \mathbb{R} : G_{t,\sigma}(\lambda') \geq \epsilon\}$. Notice that K is closed and contains a neighborhood $(\lambda - \eta, \lambda + \eta)$ of λ by continuity of $G_{t,\sigma}$. Furthermore, $K \cap S(t)$ is empty according to (1.7), and σ has no masspoints, showing that $\sigma(\partial K) = 0$ and $\sigma(K) = t\mu_t(K)$.

We apply Lemma 2.3 with $\mu = \mu_t$ and n being replaced by $n - 1$, and define

$$q_N(x) := \prod_{\lambda_{j,N} \in Z'_N} (x - \lambda_{j,N}), \quad \text{where } Z'_N := Z_N \setminus \{\lambda_{k_N,N}\}.$$

Then q_N is of degree $n-2$ by property (a) of Lemma 2.3. Notice that all eigenvalues $\lambda_{j,N}$ with $0 < |\lambda_{j,N} - \lambda_{k_N,N}| < \delta$ are elements of Z_N' for $\delta \in (0,\eta)$ by properties (b) and (d) of Lemma 2.3. Comparing (1.4) with property (c), and taking into account assumption (1.10) and Lemma 2.4(a), we may conclude that

$$\lim_{N\to\infty} \log |q_N(\lambda_{k_N,N})|^{1/n} = -U^{\mu_t}(\lambda). \tag{2.7}$$

We now estimate the modulus of q_N on the rest of the spectrum of A_N. First from property (d) we get that

$$\max_{j\neq k_N} |q_N(\lambda_{j,N})| = \max_{\lambda_{j,N}\notin K} |q_N(\lambda_{j,N})|.$$

Notice that property (c) remains valid if the sets Z_N are replaced by Z_N', i.e., the sequence of normalized zero counting measures of $(q_N)_N$ has the weak* limit μ_t. Since the complement of K is bounded by construction, it follows from the principle of descent [SaTo97, Theorem I.6.8] that

$$\limsup_{N\to\infty} \max_{j\neq k_N} \log |q_N(\lambda_{j,N})|^{1/n} \leq \limsup_{N\to\infty} \sup_{\lambda\notin K} [-U^{\mu_t}(\lambda)] \leq -F_t + \epsilon, \tag{2.8}$$

the last inequality resulting from the definition of K.

Suppose now that $\lambda_{k_N,N}$ is an element of $[\theta_{1,n,N}, \theta_{n,n,N}]$ for sufficiently large N (otherwise the reasoning is even simpler). Choosing $q = q_N$ in the second part of Lemma 2.2 leads to the estimate

$$\begin{aligned}
&\log \limsup_{\substack{n,N\to\infty \\ n/N\to t}} \min_j |\lambda_{k_N,N} - \theta_{j,n,N}|^{\frac{1}{n}} \\
\leq\ & \limsup_{\substack{n,N\to\infty \\ n/N\to t}} \log |(\lambda_{k_N,N} - \theta_{\kappa(k_N,N)-1,n,N})(\lambda_{k_N,N} - \theta_{\kappa(k_N,N),n,N})|^{\frac{1}{2n}} \\
\leq\ & \limsup_{\substack{n,N\to\infty \\ n/N\to t}} \log \left[\frac{\max_{j\neq k_N} |q_N(\lambda_{j,N})|}{|q_N(\lambda_{k_N,N})|}\right]^{1/n} + \log \left[\frac{\sum_{j\neq k_N} \beta_{j,N}^2}{\beta_{k_N,N}^2}\right]^{1/(2n)}
\end{aligned}$$

which according to (1.11), (2.7), and (2.8) is bounded above by $-G_{t,\sigma}(\lambda) + \epsilon$. Since $\epsilon > 0$ was arbitrary, we have established part (a). □

Proof of Theorem 2.1(b). Define $\alpha(N), \alpha'(N) \in \{1, ..., N\}$ and $\widetilde{\kappa}(k,N) \in \{1, ..., n\}$ by

$$\begin{aligned}
&\{\lambda_{\alpha(N),N}, \lambda_{\alpha(N)+1,N} ..., \lambda_{\alpha'(N),N}\} = I \cap \{\lambda_{j,N} : 1 \leq j \leq N\}, \\
&\min_j |\lambda_{k,N} - \theta_{j,n,N}| = |\lambda_{k,N} - \theta_{\widetilde{\kappa}(k,N),n,N}|.
\end{aligned}$$

Notice that an unbounded interval I containing eigenvalues necessarily contains either $\lambda_{1,N}$ (and then $\alpha(N) = 1$ and $\widetilde{\kappa}(1,N) = 1$) or $\lambda_{N,N}$ (and then $\alpha'(N) = N$ and $\widetilde{\kappa}(N,N) = n$).

Since $I \subset \Lambda(t,\sigma)$ is closed and $G_{t,\sigma}(\lambda) \to +\infty$ for $|\lambda| \to \infty$, we may find an $\epsilon \in (0,1)$ with $\epsilon < G_{t,\sigma}(\lambda)$ for $\lambda \in I$. By assumption, the union of the sets of eigenvalues is bounded, and by using a compactness argument we may conclude from Theorem 2.1(a) that

$$|\lambda_{k,N} - \theta_{\widetilde{\kappa}(k,N),n,N}| \leq \exp(-n\epsilon) \tag{2.9}$$

for all $\alpha(N) \leq k \leq \alpha'(N)$ and all sufficiently large N. On the other hand, we know from Lemma 2.4(b) that

$$|\lambda_{k,N} - \lambda_{k+1,N}| \geq 3\exp(-n\epsilon) \tag{2.10}$$

	Case 1	Case 2
$\lambda_{k-1,N} - \theta_{j-2,n,N}$	??	$\in (0, e^{-n\epsilon}]$ (by separation and (2.9))
$\theta_{j-1,n,N} - \lambda_{k-1,N}$	??	$\geq 2e^{-n\epsilon}$ (by (2.10))
$\lambda_{k,N} - \theta_{j-1,n,N}$	> 0 (by separation)	$\in [0, e^{-n\epsilon}]$ (assumption)
$\theta_{j,n,N} - \lambda_{k,N}$	$\in [0, e^{-n\epsilon}]$ (assumption)	> 0 (by separation)
$\lambda_{k+1,N} - \theta_{j,n,N}$	$\geq 2e^{-n\epsilon}$ (by (2.10))	??
$\theta_{j+1,n,N} - \lambda_{k+1,N}$	$\in (0, e^{-n\epsilon}]$ (by separation and (2.9))	??
Conclusion	$j+1 = \widetilde{\kappa}(k+1, N)$	$j-2 = \widetilde{\kappa}(k-1, N)$

Table 1: *Interlacing property of Ritz values and eigenvalues for adjacent eigenvalues in $\Lambda(\sigma, t)$ for some $\epsilon > 0$ and sufficiently large N. At least one of the cases 1 or 2 is true.*

for all $\alpha(N) \leq k < \alpha'(N)$ and all sufficiently large N. Combining these last two assertions, we may conclude that the quantities $\widetilde{\kappa}(k, N)$ are distinct for $\alpha(N) \leq k \leq \alpha'(N)$ and sufficiently large N. On the other hand, eigenvalues separate the Ritz values[3], and this property will enable us to conclude. Indeed, for some $k \in \{\alpha(N)+1, ..., \alpha'(N)-1\}$ and for fixed (sufficiently large) N we know from (2.9) that there is a j such that $\lambda_{k,N} - \theta_{j-1,n,N} \in [0, e^{-n\epsilon}]$ or $\theta_{j,n,N} - \lambda_{k,N} \in [0, e^{-n\epsilon}]$, and thus at least one of the two cases described in Table 1 is true.

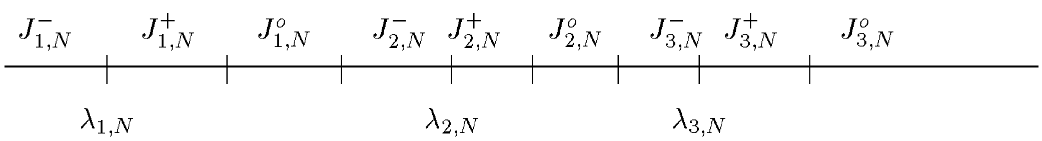
For $\alpha(N) - 1 \leq k \leq \alpha'(N)$, define $\eta_{k,N} := \frac{\lambda_{k+1,N} - \lambda_{k,N}}{3}$ ($\eta_{0,N} = \eta_{N,N} = +\infty$), and consider the intervals

$$J^+_{k,N} = (\lambda_{k,N}, \lambda_{k,N} + \eta_{k,N}), \quad J^-_{k,N} = (\lambda_{k,N}, \lambda_{k,N} - \eta_{k-1,N}), \quad \alpha(N) \leq k \leq \alpha'(N), \tag{2.11}$$

$$J^o_{k,N} = [\lambda_{k,N} + \eta_{k,N}, \lambda_{k+1,N} - \eta_{k,N}], \quad \alpha(N) \leq k < \alpha'(N), \tag{2.12}$$

$$J^o_{\alpha(N)-1,N} := (-\infty, \lambda_{\alpha(N)} - \eta_{\alpha(N)-1,N}], \quad J^o_{\alpha'(N),N} := [\lambda_{\alpha'(N)} + \eta_{\alpha'(N),N}, +\infty). \tag{2.13}$$

Notice that the intervals $J^o_{\alpha(N)-1,N}, ..., J^-_{k,N}, J^+_{k,N}, J^o_{k,N}, ..., J^o_{\alpha'(N),N}$ form a partition of $\mathbb{R} \setminus \{\lambda_{1,N}, ..., \lambda_{N,N}\}$. For instance, in the case $\alpha(N) = 1$ and $\alpha'(N) \geq 3$ we get



By applying recursively the information of the cases of Table 1, we may conclude that one and only one of the cases described in Table 2 is valid. We define $k^*(I, N) = k$ in case (e), and $k^*(I, N) = 0$ else. Notice that $k^*(I, N) = 0$ in the case of an unbounded interval: indeed, the closest Ritz value of $\lambda_{N,N} \in I$, namely $\theta_{n,n,N}$, is smaller than $\lambda_{N,N}$, and thus we have case (b). Similarly, $\lambda_{1,N} \in I$ implies that case (a) is true.

A proof of (2.2) for subsequences $(k_N)_N$ with $k_N = 1$ (or the symmetric counterpart $k_N = N$) for all N follows by adapting the proof of Theorem 2.1(a). For all other $k = k_N \in \{\alpha(N), ..., \alpha'(N)\} \setminus \{1, N, k^*(I, N)\}$ we have by Table 2

$$(\theta_{\kappa(k,N),n,N} - \lambda_{k,N})(\lambda_{k,N} - \theta_{\kappa(k,N)-1,n,N}) \geq \min\{\eta_{k-1,N}, \eta_{k,N}\} \cdot |\lambda_{k,N} - \theta_{\widetilde{\kappa}(k,N),n,N}|$$

for sufficiently large N, where the minimum on the right-hand side can be bounded below using (2.5). Combining this inequality with Lemma 2.2 and (2.7), (2.8), we may conclude as in the proof of Theorem 2.1(a) that (2.2) and thus Theorem 2.1(b) is true. □

[3] More precisely, if $\theta_{j,n,N} \in [\lambda_{k,N}, \lambda_{k+1,N})$ then $\theta_{j+1,n,N} > \lambda_{k+1,N}$. Similarly, if $\theta_{j-1,n,N} \in (\lambda_{k-1,N}, \lambda_{k,N}]$ then $\theta_{j-2,n,N} < \lambda_{k-1,N}$.

Case (a)	$\theta_{1+i,n,N} \in J^+_{\alpha(N)+i,N}$, $i = 0, ..., \alpha'(N) - \alpha(N)$
Case (a')	for $k = \alpha(N)$: $\theta_{j-1,n,N} \in J^o_{k-1,N}$, $\theta_{j+i,n,N} \in J^+_{k+i,N}$, $i = 0, ..., \alpha'(N) - k$
Case (b)	$\theta_{n-i,n,N} \in J^-_{\alpha(N)'-i,N}$, $i = 0, ..., \alpha'(N) - \alpha(N)$
Case (b')	for $k = \alpha'(N)$: $\theta_{j,n,N} \in J^o_{k,N}$, $\theta_{j-i-1,n,N} \in J^-_{k-i,N}$, $i = 0, ..., k - \alpha(N)$
Case (c)	for some $k \in \{\alpha(N) + 1, ..., \alpha'(N)\}$: $\theta_{j-i,n,N} \in J^-_{k-i,N}$, $i = 1, ..., k - \alpha(N)$, and $\theta_{j+i,n,N} \in J^+_{k+i,N}$, $i = 0, ..., \alpha'(N) - k$
Case (d)	for some $k \in \{\alpha(N), ..., \alpha'(N)\}$: $\lambda_{k,N} = \theta_{j,n,N}$ $\theta_{j-i,n,N} \in J^-_{k-i,N}$, $i = 1, ..., k - \alpha(N)$, and $\theta_{j+i,n,N} \in J^+_{k+i,N}$, $i = 1, ..., \alpha'(N) - k$
Case (e)	for some $k \in \{\alpha(N), ..., \alpha'(N)\}$: $\theta_{j-i-1,n,N} \in J^-_{k-i,N}$, $i = 0, ..., k - \alpha(N)$, and $\theta_{j+i,n,N} \in J^+_{k+i,N}$, $i = 0, ..., \alpha'(N) - k$
Case (f)	for some $k \in \{\alpha(N) + 1, ..., \alpha'(N)\}$: $\theta_{j-1,n,N} \in J^o_{k-1,N}$, $\theta_{j-i-1,n,N} \in J^-_{k-i,N}$, $i = 1, ..., k - \alpha(N)$, and $\theta_{j+i,n,N} \in J^+_{k+i,N}$, $i = 0, ..., \alpha'(N) - k$

Table 2: *Interlacing property of Ritz values and eigenvalues in closed subintervals I of $\Lambda(\sigma, t)$. The intervals $J^o_{\alpha(N)-1,N}, ..., J^-_{k,N}, J^+_{k,N}, J^o_{k,N}, ..., J^o_{\alpha'(N),N}$ form a partition of $\mathbb{R} \setminus \{\lambda_{1,N}, ..., \lambda_{N,N}\}$. Exactly one of the above cases is true for any (sufficiently large) N.*

Proof of Theorem 2.1(c). For the proof of the sharpness, we require some recent result of Rakhmanov [Rak96, Theorem 2] and Dragnev and Saff [DrSa97, Theorem 3.3] on nth root asymptotics for the discrete orthogonal polynomials $p_{n,N}$: Under the assumptions of part (c)[4], there holds for the norm of $p_{n,N}$

$$\lim_{\substack{n,N\to\infty \\ n/N\to t}} \log\Big[\sum_{j=1}^{N} \beta_{j,N}^2 \cdot p_{n,N}(\lambda_{j,N})^2\Big]^{\frac{1}{2n}} = -F_t, \tag{2.14}$$

furthermore, we have for the zeros of $p_{n,N}$

$$\lim_{\substack{n,N\to\infty \\ n/N\to t}} \frac{1}{n}\sum_{j=1}^{n} f(\theta_{j,n,N}) = \int f(\lambda')\, d\mu_t(\lambda') \tag{2.15}$$

for all continuous functions $f : \mathbb{R} \to \mathbb{R}$.

We will again restrict our attention to sequences $(k_N)_N$ with $k_N \neq 0, N$ for all N, otherwise the reasoning is even simpler. In this case, according to Table 2 we find Ritz values with $\theta_{j_N-1,n,N} \leq \lambda_{k_N,N} \leq \theta_{j_N,n,N}$, and we need to minorize

$$\frac{\displaystyle\sum_{j=1, j\neq k_N}^{N} \frac{|\beta_{j,N}|^2}{||b_N||^2}(\lambda_{j,N} - \theta_{j_N-1,n,N})(\lambda_{j,N} - \theta_{j_N,n,N})q_N(\lambda_{j,N})^2}{\frac{|\beta_{k_N,N}|^2}{||b_N||^2} q_N(\lambda_{k_N,N})^2} \tag{2.16}$$

for the polynomial $q_N(x) = p_{n,N}(x)/[(x - \theta_{j_N-1,n,N})(x - \theta_{j_N,n,N})]$. This task is possible as long as the terms in the sum are all positive, there may however be exceptional indices k_N where this

[4]We use here the separation condition of [DrSa97] generalizing the one of [Rak96]. Various other sufficient separation conditions have been considered in [Beck98, KuVA98], see also [KuRa98] for a summary. Notice also that the authors in [DrSa97] impose that supp (σ) is an interval, but this is not essential for our setting.

property is not true, see cases (c) and (d) in Table 2. Therefore we define

$$k^{**}(I,N) = \begin{cases} k & \text{if case (d) holds,} \\ k & \text{if case (c) holds and} \\ & |\beta_{k,N}|^2 |p_{n,N}(\lambda_{k,N}) q_N(\lambda_{k,N})| \le |\beta_{k-1,N}|^2 |p_{n,N}(\lambda_{k-1,N}) q_N(\lambda_{k-1,N})|, \\ k-1 & \text{if case (c) holds and} \\ & |\beta_{k,N}|^2 |p_{n,N}(\lambda_{k,N}) q_N(\lambda_{k,N})| > |\beta_{k-1,N}|^2 |p_{n,N}(\lambda_{k-1,N}) q_N(\lambda_{k-1,N})|, \\ 0 & \text{else.} \end{cases}$$

Let $k_N \neq k^{**}(I,N)$, and, provided that case (c) is true, we also suppose that $k_N \notin \{k-1, k\}$. Then we have according to Table 2 the more precise interlacing property

$$\lambda_{k_N-1,N} \le \theta_{j_N-1,n,N} < \lambda_{k_N,N} < \theta_{j_N,n,N} \le \lambda_{k_N+1,N},$$

and thus all terms in the sum of (2.16) are positive. Notice that (2.15) remains valid if we drop the terms for $j \in \{j_N - 1, j_N\}$. Hence, by the principle of descent, the nth root of the denominator of (2.16) may asymptotically be majorized by

$$\limsup_{\substack{n,N\to\infty \\ n/N\to t}} |q_N(\lambda_{k_N,N})|^{2/n} \le \exp(-2U^{\mu_t}(\lambda)).$$

In order to show that the numerator of (2.16) is not essentially smaller than the norm in (2.14), we recall that $S(t) = \operatorname{supp}(\sigma - t\mu)$ is non-empty. Consequently, we find another sequence of indices $(k'_N)_N$ with $(\lambda_{k'_N,N})_N$ converging to some $\lambda' \in S(t)$ such that the intervals $[(\lambda_{k'_N-1,N} + \lambda_{k'_N,N})/2, (\lambda_{k'_N,N} + \lambda_{k'_N+1,N})/2]$ do not contain any Ritz values. Taking into account that Ritz values are separated by eigenvalues, we may conclude that, for any $\delta \in (0,1)$,

$$\begin{aligned} 0 \;&\le\; \frac{1}{n} \sum_{|\theta_{j,n,N} - \lambda_{k'_N,N}| \le \delta} \log \frac{1}{|\theta_{j,n,N} - \lambda_{k'_N,N}|} \\ &\le\; \frac{1}{n} \log \frac{1}{\min(\delta, |\lambda_{k'_N+1,N} - \lambda_{k'_N,N}|/2)} + \frac{1}{n} \log \frac{1}{\min(\delta, |\lambda_{k'_N,N} - \lambda_{k'_N-1,N}|/2)} \\ &\quad + \frac{1}{n} \sum_{0 < |\lambda_{j,N} - \lambda_{k'_N,N}| \le \delta} \log \frac{1}{|\lambda_{j,N} - \lambda_{k'_N,N}|}. \end{aligned}$$

By assumption (1.10) and Lemma 2.4(a), for any $\epsilon > 0$ we find $\delta \in (0,1)$ and $C > 0$ such that the right hand side of the preceding expression is smaller than ϵ for all $N \ge C$. Taking into account (2.15), we may conclude by applying again Lemma 2.4(a) that

$$\lim_{\substack{n,N\to\infty \\ n/N\to t}} |p_{n,N}(\lambda_{k'_N,N})|^{1/n} = \exp(-U^{\mu_t}(\lambda')) = \exp(-F_t), \tag{2.17}$$

where the right-hand equality follows from (1.7). Consequently, the nth root of the numerator of (2.16) may asymptotically be minorized by

$$\begin{aligned} &\liminf_{\substack{n,N\to\infty \\ n/N\to t}} \Big[\frac{|\beta_{k'_N,N}|^2}{||b_N||^2} (\lambda_{k'_N,N} - \theta_{j_N-1,n,N})(\lambda_{k'_N,N} - \theta_{j_N,n,N}) q_N(\lambda_{k'_N,N})^2\Big]^{1/n} \\ \ge\; &\liminf_{\substack{n,N\to\infty \\ n/N\to t}} [p_{n,N}(\lambda_{k'_N,N})^2]^{1/n} = \exp(-2F_t). \end{aligned}$$

where we have used (1.11) and the fact that the eigenvalues are uniformly bounded. A combination with Lemma 2.2 leads to the estimate

$$\liminf_{\substack{n,N\to\infty \\ n/N\to t}} |(\lambda_{k_N,N} - \theta_{j_N-1,n,N})(\theta_{j_N,n,N} - \lambda_{k_N,N})|^{1/n} \ge \exp(-2G_{t,\sigma}(\lambda)),$$

and thus (2.3) follows.

It remains to examine the case (c) where

$$\lambda_{k-2,N} < \theta_{j_N-1,n,N} < \lambda_{k-1,N} < \lambda_{k,N} < \theta_{j_N,n,N} < \lambda_{k+1,N},$$

and $k_N \in \{k-1, k\}$ but $k_N \neq k^{**}(I,N)$. In this case we have by construction

$$\begin{aligned}
& 2|\beta_{k_N,N}|^2 q_N(\lambda_{k_N}, N)^2 \cdot (\lambda_{k_N,N} - \theta_{j_N-1,n,N})(\theta_{j_N,n,N} - \lambda_{k_N,N}) \\
= \; & 2|\beta_{k_N,N}|^2 |p_{n,N}(\lambda_{k_N}, N) q_N(\lambda_{k_N}, N)| \geq \sum_{j=k-1}^{k} |\beta_{j,N}|^2 |p_{n,N}(\lambda_j, N) q_N(\lambda_{k_N}, N)| \\
= \; & -\sum_{j=k-1}^{k} |\beta_{j,N}|^2 q_N(\lambda_j, N)^2 \cdot (\lambda_{j,N} - \theta_{j_N-1,n,N})(\lambda_{j,N} - \theta_{j_N,n,N}) \\
= \; & \sum_{j\neq k-1,k} |\beta_{j,N}|^2 q_N(\lambda_j, N)^2 \cdot (\lambda_{j,N} - \theta_{j_N-1,n,N})(\lambda_{j,N} - \theta_{j_N,n,N}),
\end{aligned}$$

where now all terms in the sum on the right-hand side are ≥ 0. This leads to a lower bound for $(\lambda_{k_N,N} - \theta_{j_N-1,n,N})(\theta_{j_N,n,N} - \lambda_{k_N,N})$ which can be minorized as before. □

Proof of Theorem 2.1(d). By construction, $k^*(I,N) \neq 0$ only in case (e) of Table 2, and $k^{**}(I,N) \neq 0$ only in the cases (c) or (d). Since these cases are mutually exclusive, we have $k^*(I,N) \cdot k^{**}(I,N) = 0$ for all N. □

In order to conclude this section, we compare our findings with similar properties for the zeros of (generalized) Chebyshev polynomials: for some polynomial p of degree $< N$, we consider the two norms (depending on N)

$$||p||_{2,N} = \Big[\sum_{j=1}^{N} \frac{|\beta_{j,N}|^2 |p(\lambda_{j,N})|^2}{||b_N||^2}\Big]^{1/2}, \quad ||p||_{\infty,N} = \max_{1\leq j\leq N} \frac{|\beta_{j,N}||p(\lambda_{j,N})|}{||b||_N}.$$

Furthermore, for some fixed sequence $n = n(N)$ with $n(N)/N \to t \in (0,1)$ we consider monic polynomials $T_{\infty,N}$ of degree n with minimal norm $||\cdot||_{\infty,N}$ (the so–called generalized Chebyshev polynomials) and monic polynomials $T_{2,N}$ of degree n with minimal norm $||\cdot||_{2,N}$ (i.e., $T_{2,N}$ is the corresponding monic orthogonal polynomial $p_{n,N}$).

Suppose that the assumption of Kuijlaars holds. Obviously, our two norms are closely related, namely $||p||_{\infty,N} \leq ||p||_{\infty,N} \leq \sqrt{N} \cdot ||p||_{\infty,N}$ for any polynomial. Indeed, it follows for instance from [Beck98, Theorem 1.3] that $|T_{\infty,N}|$ and $|T_{2,N}|$ have the same nth root asymptotic behavior in $\mathbb{C}\backslash\mathbb{R}$, the same zero distribution (2.15), and their norms have the same nth root limit (2.14). Thus it is perhaps surprising to discover that the zeros of these polynomials approach elements of $\{\lambda_{1,N}, ..., \lambda_{N,N}\} \cap \Lambda(t,\sigma)$ for $N \to \infty$ with different rates of convergence. Indeed, for the zeros $\theta_{1,n,N} < ... < \theta_{n,n,N}$ of $T_{2,n}$ the rate of approximation is specified in Theorem 2.1. For the zeros of $T_{\infty,N}$ we have

Corollary 2.5. *Denote by $\widetilde{\theta}_{1,n,N} < ... < \widetilde{\theta}_{n,n,N}$ the zeros of $T_{\infty,N}$. Under the assumption of Kuijlaars, the parts (a),(b),(c) of Theorem 2.1 remain valid for $\widetilde{\theta}_{j,n,N}$ instead of $\theta_{j,n,N}$ if we multiply the exponents on the right hand side of (2.1), (2.2), and (2.3), respectively, by a factor 1/2.*

Proof. The estimate (2.1) (with a factor 1/2 in the exponent) has been shown in [Kuij99, Theorem 3.1]. For completeness, we give some hints how to show the other two estimates.

Given some close interval $I \subset \Lambda(t,\sigma)$, we may conclude from part (a) that there is some $\epsilon > 0$ such that any interval of the form

$$J_{k,N} := (\lambda_{k,N} - e^{-\epsilon n}, \lambda_{k,N} + e^{-\epsilon n}), \quad \lambda_{k,N} \in I,$$

contains at least one $\widetilde{\theta}_{j,n,N}$ for sufficiently large N. Also, we again have the property (2.10) for all k provided that N is sufficiently large, and thus these $J_{k,N}$ have a non-empty intersection.

It is well known that for each N there exists an alternant $\widetilde{\lambda}_{0,N} < ... < \widetilde{\lambda}_{n,N}$ out of the set $\{\lambda_{j,N}\}$ such that

$$T_{\infty,N}(\widetilde{\lambda}_{j,N}) = (-1)^{n-j} \cdot ||T_{\infty,N}||_{\infty,N}, \quad j = 0, ..., n,$$

and thus $\widetilde{\lambda}_{j-1,N} < \widetilde{\theta}_{j,N} < \widetilde{\lambda}_{j,N}$ for $j = 1, ..., n$. One may conclude similarly to the proof of Theorem 2.1(b) that there is either at most one interval $J_{k',n}$ containing $\widetilde{\theta}_{j-1,n,N}$ and $\widetilde{\theta}_{j,n,N}$ implying that $\lambda_{k',N}$ is part of the alternant, or there is at most one interval $J_{k',n}$ containing exactly one $\widetilde{\theta}_{j,n,N}$ but no point of the alternant. All other intervals $J_{k,n}$ will contain exactly one $\widetilde{\theta}_{j,n,N}$ and the point $\lambda_{k,N}$ is part of the alternant.

Therefore, given $(\lambda_{k_N,N})_N \subset I$ converging to some λ and $J_{k_N,N}$ containing exactly one $\widetilde{\theta}_{j_N,n,N}$, we have

$$\begin{aligned} 1 &\geq \limsup_{N\to\infty} \Big[\frac{|T_{\infty,N}(\lambda_{k_N,N})|}{||T_{\infty,N}||_{\infty,N}}\Big]^{1/n} \\ &= \limsup_{N\to\infty} |\lambda_{k_N,N} - \widetilde{\theta}_{j_N,n,N}|^{1/n} \cdot \lim_{N\to\infty} \Big[\frac{|T_{\infty,N}(\lambda_{k_N,N})|}{|\lambda_{k_N,N} - \widetilde{\theta}_{j_N,n,N}| \cdot ||T_{\infty,N}||_{\infty,N}}\Big]^{1/n} \\ &= \limsup_{N\to\infty} |\lambda_{k_N,N} - \widetilde{\theta}_{j_N,n,N}|^{1/n} \cdot \exp(G_{t,\sigma}(\lambda)), \end{aligned}$$

showing part (b). Similarly, in the case of $\lambda_{k_N,N}$ being part of the alternant and $\widetilde{\theta}_{j_N,n,N} \in J_{k_N,N}$ we get by the principle of descent

$$\begin{aligned} 1 &= \liminf_{N\to\infty} \Big[\frac{|T_{\infty,N}(\lambda_{k_N,N})|}{||T_{\infty,N}||_{\infty,N}}\Big]^{1/n} \\ &\leq \liminf_{N\to\infty} |\lambda_{k_N,N} - \widetilde{\theta}_{j_N,n,N}|^{1/n} \cdot \limsup_{N\to\infty} \Big[\frac{|T_{\infty,N}(\lambda_{k_N,N})|}{|\lambda_{k_N,N} - \widetilde{\theta}_{j_N,n,N}| \cdot ||T_{\infty,N}||_{\infty,N}}\Big]^{1/n} \\ &\leq \liminf_{N\to\infty} |\lambda_{k_N,N} - \widetilde{\theta}_{j_N,n,N}|^{1/n} \cdot \exp(G_{t,\sigma}(\lambda)), \end{aligned}$$

showing part (c). □

3 Some examples

In order to illustrate Theorem 2.1 and in particular the phenomenon of exceptional eigenvalues, we consider some simple examples, namely the diagonal matrices

$$A_N = \operatorname{diag}(\lambda_{j,N})_{j=1,...,N}, \quad \text{and} \quad b_N = (1, ..., 1)^T,$$

with the following choices for the eigenvalues

$$\lambda_{j,N} = \cos(\pi \frac{2j-1}{2N}) \cdot |\cos(\pi \frac{2j-1}{2N})|^{\alpha-1}, \quad \alpha > 0, \tag{3.1}$$

$$\lambda_{j,N} = \frac{2j-1-N}{N-1}. \tag{3.2}$$

In all these cases one easily verifies that condition (1.4) holds with $\mathrm{supp}\,(\sigma) = [-1,1]$.

Notice that, for (3.1) with $\alpha = 1$, the measure σ equals the equilibrium distribution of $[-1,1]$, and thus $\mu_t = \sigma$ for all $0 < t < 1$. Here $\Lambda(t,\sigma) \cap [-1,1]$ is empty, and thus Theorem 2.1 does not give any useful information. Indeed, here $p_{n,N}$ coincides up to some constant with the Chebyshev polynomial T_n. Though for n being a divisor of N some eigenvalues coincide with Ritz values, we may not observe some geometric convergence for $n, N \to \infty$, $n/N \to t \in (0,1)$.

Things become more interesting for (3.2) or for (3.1) with $\alpha = 2$ (more eigenvalues close to zero) or for (3.1) with $\alpha = 1/2$ (more eigenvalues close to the endpoints). In order to apply Theorem 2.1, we need to compute $G_{t,\sigma}$ and $\Lambda(t,\sigma)$ introduced in (1.8), (1.9), or more generally the sets $S(t)$ of (1.7). For the cases described above, we may apply

Lemma 3.1. *Let P be a polynomial of degree m, with $P^{-1}((0,1))$ consisting of m disjoint real intervals, and denote by $\omega_1, ..., \omega_m : [0,1] \to \mathbb{R}$ the different branches of P^{-1}. Furthermore, suppose that ϕ is strictly increasing and Hölder continuous on $[0,1]$, with $\phi([0,1]) = [0,1]$. Define the probability measure σ with* $\mathrm{supp}\,(\sigma) = P^{-1}([0,1])$ *by*[5]

$$\int f(x)\, d\sigma(x) = \frac{1}{m} \sum_{h=1}^{m} \int_0^1 f(\omega_h(\phi^{-1}(\xi)))\, d\xi.$$

(a) *Let $\xi_{1,M}, ..., \xi_{M,M} \in [0,1]$, with $1/(M+1) \le \xi_{\ell+1,M} - \xi_{\ell,M} \le 1/(M-1)$ for all ℓ, M. Then the sequence of sets*

$$\{\lambda_{j,N} : j = 1, ..., N\} = \{\omega_h(\phi^{-1}(\xi_{\ell,M})) : \ell = 1, ..., M, h = 1, ..., m\},$$

(i.e., there are $N \in [m(M-1)+1, mM]$ distinct eigenvalues), satisfies (1.4), and condition (1.10) is true for any sequence as in (1.5).

(b) *The quantity $m \cdot G_{t,\sigma}(\omega_h(z))$ does not depend neither on the choice of the branch h nor on the choice of the polynomial P.*

(c) *Suppose in addition that ϕ' exists in $(0,1)$, and that $\psi(x) := \pi\sqrt{x(1-x)} \cdot \phi'(x)$ is strictly increasing in $(0,1)$. Then $S(t) = P^{-1}([r(t),1])$ for all $0 < t < 1$, where $r(t) = 0$ for $t \le \psi(0+)$, and else $r = r(t)$ is the unique solution in $(0,1)$ of the equation*

$$t = \int_0^r \sqrt{\frac{1-x}{r-x}} \phi'(x)\, dx. \tag{3.3}$$

Proof. The property (1.4) mentioned in part (a) is easily verified. In order to verify (1.10), we apply Lemma 2.4(a). Thus it is sufficient to show that there exists some constants $C_1, C_2 > 0$ such that for all $j \ne j'$ and for all N we have $\log|\lambda_{j,N} - \lambda_{j',N}| \ge C_1 \cdot \log|j/N - j'/N| - C_2$.

[5] Notice that σ has a continuous potential.

Obviously, it is sufficient to consider eigenvalues resulting from the same branch of P^{-1}, i.e., $\lambda_{j,N} = \omega_h(\phi^{-1}(\xi_{\ell,M}))$ and $\lambda_{j',N} = \omega_h(\phi^{-1}(\xi_{\ell',M}))$, with

$$|\xi_{\ell,M} - \xi_{\ell',M}| \geq \frac{|\ell - \ell'|}{M+1} = \frac{|j - j'|}{M+1} \geq \frac{1}{2}\frac{|j - j'|}{N}.$$

Consequently,

$$\log \frac{|j - j'|}{N} \leq \log 2 + \log |\xi_{\ell,M} - \xi_{\ell',M}| = \log 2 + \log |\phi(P(\lambda_{j,N})) - \phi(P(\lambda_{j',N}))|,$$

and the above estimate follows from the fact that, with ϕ, also $\phi \circ P$ is Hölder continuous on $\omega_h([0,1])$.

In order to show part (b), we denote by $\widetilde{\sigma}$ the mesure obtained for the particular choice $P = 1$, and by $\widetilde{\mu}_t$ the solution of the constrained energy problem corresponding to the parameters t and $\widetilde{\sigma}$. Furthermore, let C be the absolute value of the leading coefficient of P, and define the probability measure μ by

$$\int f(\lambda)\, d\mu(\lambda) = \frac{1}{m}\sum_{k=1}^{m} \int f(\omega_k(x))\, d\widetilde{\mu}_t(x).$$

Since also

$$\int f(\lambda)\, d\sigma(\lambda) = \frac{1}{m}\sum_{k=1}^{m} \int f(\omega_k(x))\, d\widetilde{\sigma}(x).$$

we may conclude that μ satisfies the constrain $0 \leq t\mu \leq \sigma$, with $\mathrm{supp}\,(\sigma - t\mu) = P^{-1}(\mathrm{supp}\,(\widetilde{\sigma} - t\widetilde{\mu}_t))$. Furthermore, for any $\lambda \in \mathrm{supp}\,(\sigma)$, say, $\lambda = \omega_{h'}(x')$ with $x' \in \mathrm{supp}\,(\widetilde{\sigma}) = [0,1]$, we obtain for the potential

$$U^{\mu}(\lambda) = \frac{1}{m}\int \log\Big(\prod_{k=1}^{m} \frac{1}{|\lambda - \omega_k(x)|}\Big)\, d\widetilde{\mu}_t(x) = \frac{\log C + U^{\widetilde{\mu}_t}(x')}{m}.$$

Taking into account (1.7) for the parameters $(t, \widetilde{\sigma})$, we may conclude that μ satisfies the equilibrium condition (1.7) for the parameters (t, σ), and thus $\mu = \mu_t$. In particular it follows from the above relation that $G_{t,\sigma}(\lambda) = G_{t,\widetilde{\sigma}}(x')/m$ for $\lambda = \omega_{h'}(x')$, as claimed in part (b).

For establishing part (c), we first notice that, according to (b), it is sufficient to consider the case $P = 1$. However, then the statement can be found in [BeKu99, Lemma 3.1(a)], which again has been shown using results of [KuDr99] (compare also with [Kuij99, Theorems 5.1 and 6.1]). □

Example 3.2. *The choice (3.2) of equidistant eigenvalues have been originally discussed by Rakhmanov in the context of nth root asymptotics of some classical family of orthogonal polynomials. The convergence of Ritz values for these matrices have been estimated in [Kuij99, Section 4.2] where one may also find numerical experiments.*

Notice that here Lemma 3.1 applies with $P(\lambda) = 1 - \lambda^2$ and $\phi(x) = 1 - \sqrt{1-x}$. From (3.3) we find that $r(t) = t^2$ for $0 < t < 1$, and thus

$$S(t) = [-\sqrt{1-t^2}, \sqrt{1-t^2}], \quad \Lambda(t,\sigma) = \mathbb{R} \setminus S(t) = (-\infty, -\sqrt{1-t^2}) \cup (\sqrt{1-t^2}, +\infty).$$

Hence the outer eigenvalues will be well approximated by the Lanczos method. Also, around some $\lambda \in \Lambda(t,\sigma)$ *we have as convergence rate the quantity* $\exp(-2G_{t,\sigma}(\lambda))$ *by Theorem 2.1(b),(c), where, according to [BeKu99, Eqn.(3.10)],*

$$G_{t,\sigma}(\pm 1) = \frac{(1+t)\log(1+t) + (1-t)\log(1-t)}{2t}.$$

We refer the reader to [Kuij99, Section 7] for a discussion of measures σ with an ultra-spherical density including Example 3.2.

Example 3.3. *We now consider the choice (3.1) for* $\alpha = 2$. *Here Lemma 3.1 applies with* $P(x) = 1 - x^2$ *and* $\phi^{-1}(\xi) = 1 - \cos^4(\pi\xi/2)$, *in particular*

$$\phi'(x) = \frac{1}{2\pi} \frac{\sqrt{1+\sqrt{1-x}}}{\sqrt{x}(1-x)^{3/4}}.$$

In particular, ψ *is strictly increasing, and* $\psi(0+) = 1/\sqrt{2}$. *Thus for* $0 < t \leq 1/\sqrt{2}$ *we obtain* $S(t) = [-1,1]$, *and there is no geometric convergence of Ritz values. For* $t \in (1/\sqrt{2}, 1)$, $\Lambda(t,\sigma)$ *will be again of the form* $(-\infty, -\sqrt{1-r(t)}) \cup (\sqrt{1-r(t)}, +\infty)$ *(geometric convergence for extremal eigenvalues), where according to (3.3)* $r = r(t) \in (0,1)$ *is found from*

$$\pi t = \int_0^r \sqrt{\frac{1-x}{r-x}} \frac{\sqrt{1+\sqrt{1-x}}}{2\sqrt{x}(1-x)^{3/4}}\, dx = \int_{\sqrt{1-r}}^1 \frac{y}{\sqrt{(y^2-(1-r))y(1-y)}}\, dy$$

leading to complete elliptic integrals. We omit further details.

Example 3.4. *Finally we consider the choice (3.1) for* $\alpha = 1/2$. *Here Lemma 3.1 applies with* $P(x) = x^2$ *and* $\phi^{-1}(\xi) = \cos(\pi(1-\xi)/2)$, *in particular*

$$\phi'(x) = \frac{2}{\pi} \frac{1}{\sqrt{1-x^2}}.$$

Again, ψ *is strictly increasing, and* $\psi(0+) = 0$. *Thus,*

$$\Lambda(t,\sigma) = (-\infty,-1) \cup (-\sqrt{r(t)}, \sqrt{r(t)}) \cup (1,+\infty), \quad 0 < t < 1$$

(geometric convergence for eigenvalues around the origin), where according to (3.3) $r = r(t) \in (0,1)$ *is found from*

$$\frac{\pi t}{2} = \int_0^r \frac{dx}{\sqrt{(r-x)(1+x)}} = \arccos(\frac{1-r}{1+r}).$$

Consequently,

$$r(t) = \frac{1-\cos(\pi t/2)}{1+\cos(\pi t/2)} = (\frac{\sin(\pi t/2)}{1+\cos(\pi t/2)})^2.$$

Moreover, according to Lemma 3.1(b) and (1.8) we have

$$G_{t,\sigma}(0) = \frac{1}{2t}\int_0^t g_{[r(t'),1]}(0)\, dt' = \frac{-1}{2t}\int_0^t \log(\cos(\frac{\pi t'}{2}))\, dt',$$

which cannot be further simplified. Notice that $G_{t,\sigma}(0)$ *tends for* $t \to 1$ *to* $\log(\sqrt{2})$, *and behaves for small* t *like* $\pi^2 t^2/48$.

For this example we may have a closer look at the exceptional eigenvalues mentioned in Theorem 2.1(d). Notice that the eigenvalues (and the eigenvector components) are symmetric *with respect to the origin. Thus $p_{2n,N}$ is odd, and $p_{2n+1,N}$ is even. Moreover, $\lambda_{N+1,2N+1} = \theta_{n+1,2n+1,2N+1} = 0$ (case (d) of Table 2), and thus*

$$\lim_{\substack{n,N\to\infty \\ n/N\to t}} \min_j |\lambda_{N+1,2N+1} - \theta_{j,2n+1,2N+1}|^{1/n} = 0 \quad (k^{**}(I,2N+1) = N+1).$$

Also, by symmetry, the two Ritz values $\theta_{n,2n,2N+1}$ and $\theta_{n+1,2n,2N+1} = -\theta_{n,2n,2N+1}$ are closest to $\lambda_{N+1,2N+1} = 0$, and thus here necessarily case (e) occurs, with

$$\lim_{\substack{n,N\to\infty \\ n/N\to t}} \min_j |\lambda_{N+1,2N+1} - \theta_{j,2n,2N+1}|^{1/n} = \exp(-G_{t,\sigma}(0)) \quad (k^{*}(I,2N+1) = N+1).$$

Similarly, one may show that case (f) occurs for $p_{2n+1,2N}$ (and thus no exceptional eigenvalues), and case (c) for $p_{2n,2N}$ (again an additional symmetry argument shows that then there are no exceptional eigenvalues in Theorem 2.1(c)). To summarize, it follows from Theorem 2.1 that for any sequence of eigenvalues $(\lambda_{k_N,N})_N$ tending to some $\lambda \in \Lambda(t,\sigma)$ we will have the exact geometric rate $\exp(-2G_{t,\sigma}(\lambda))$*, provided that $\lambda_{k_N,N} \neq 0$ for sufficiently large N.*

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