On a conjecture of E.A. Rakhmanov

by

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Revised version from March 26, 99

Abstract

It is shown that a conjecture of E.A. Rakhmanov is true concerning the zero distribution of orthogonal polynomials with respect to a measure having a discrete real support. We also discuss the case of extremal polynomials with respect to some discrete L_p–norm, $0 < p \leq \infty$, and give an extension to complex supports.

Furthermore, we present properties of weighted Fekete points with respect to discrete complex sets, such as the weighted discrete transfinite diameter and a weighted discrete Bernstein–Walsh–like inequality.

Key words: Discrete orthogonality, Fekete points, Constrained equilibrium problem.
Subject Classifications: AMS(MOS): 33C45, 42C05.

1 Introduction and statement of results

In this paper we shall investigate asymptotic properties of extremal polynomials $T_{n,p}(z) = z^n + \textit{lower powers}$ with regard to some discrete L_p–norm

$$||w_n \cdot T_{n,p}||_{L_p(E_n)} = \min\{||w_n \cdot P||_{L_p(E_n)} : P(z) = z^n + \textit{lower powers}\}, \tag{1}$$

where

$$||f||_{L_\infty(E_n)} := \sup_{z \in E_n} |f(z)|, \quad ||f||_{L_p(E_n)} := \left[\sum_{z \in E_n} |f(z)|^p \right]^{1/p}, \tag{2}$$

$0 < p < \infty$, with E_n being suitable finite or countable subsets of the complex plane, $\#E_n \geq n+1$, and $w_n(z)$, $z \in E_n$, being (sufficiently fast decreasing) positive numbers.

For the case $p = 2$ of discrete orthogonal polynomials, examples include the discrete Chebyshev polynomials [Rak96] (choose $w_n = 1$, $E_n = \{0, 1, ..., n\}$) or other classical families like Krawtchouk or Meixner polynomials [DaSa98, DrSa97, KuVA98]. A study of asymptotics of such polynomials has some important applications, e.g., in coding theory. It was Rakhmanov [Rak96] who first observed that a particular constrained (weighted) energy problem in complex potential theory may furnish a method for calculating the nth root asymptotics of extremal polynomials with respect to so–called *ray sequences* obtained by a suitable renormalization of the sets E_n. Further progress has been made by Dragnev and Saff for real sets E_n being uniformly bounded [DrSa97]; they also obtained asymptotics for discrete L_p–norms with $0 < p \leq \infty$.

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Generalizations for unbounded real sets E_n and exponentially decreasing weights have been discussed by Kuijlaars and Van Assche [KuVA98] ($0 < p \le \infty$) and Kuijlaars and Rakhmanov [KuRa98] ($p = 2$). Damelin and Saff [DaSa98] studied the case $p = \infty$ for more general classes of weights. In this paper we establish nth root asymptotics for the extremal constants of (1), $0 < p \le \infty$, for possibly complex and unbounded sets E_n and for a fairly large class of weights. In particular we show that two conjectures of Rakhmanov [KuRa98] are true concerning some separation assumption for the sets E_n.

The nth root asymptotic of the extremal constants $||w_n \cdot T_{n,p}||_{L_p(E_n)}$ as well as the asymptotic distribution of zeros of $T_{n,p}$ may be expressed in terms of the solution of the constrained weighted energy problem: for some positive Borel measure μ, we define its (logarithmic) potential, and its energy, respectively, by

$$U^\mu(z) := \int \log \frac{1}{|z-t|}\, d\mu(t), \quad I(\mu) := \int\int \log \frac{1}{|z-t|}\, d\mu(t)\, d\mu(z),$$

and write $\mathrm{supp}(\mu) \subset \overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ for its (closed) support. Given some positive Borel measure σ with total mass $||\sigma|| > 1$, the constrained energy problem consists in minimizing $I(\mu)$, where μ is some probability measure satisfying in addition the constraint that $\sigma - \mu$ is some nonnegative measure. The set of such measures will be denoted by $\mathcal{M}^\sigma := \{\mu \ge 0 : ||\mu|| = 1, \sigma - \mu \ge 0\}$. In our context it will be useful to introduce a weighted analogue of this problem. Its unique solution has been characterized by Dragnev and Saff [DrSa97, Theorem 2.1 and Remark 2.3], and further investigated by several other authors. We summarize some of their findings in Theorem 1.1 below, here additional assumptions enable us to obtain a simplified statement.

Theorem 1.1 (see [DrSa97]) *Let Q be a continuous real–valued function on some closed set $E \subset \mathbb{C}$, $w := \exp(-Q)$, and, if E is unbounded, suppose that $Q(z) - \log|z| \to +\infty$ for $|z| \to \infty$. Furthermore, let σ be a positive measure, with connected support $\mathrm{supp}(\sigma) \subset E$. Suppose that $||\sigma|| > 1$ and that, for any compact $K \subset \mathrm{supp}(\sigma)$, the restriction $\sigma|_K$ of σ to K has a continuous potential.*[2] *Then there exists a unique measure $\lambda_w^\sigma \in \mathcal{M}^\sigma$ such that*

$$I_w(\lambda_w^\sigma) = \min_{\mu \in \mathcal{M}^\sigma} I_w(\mu), \quad \textit{where } I_w(\mu) = \int\int \log \frac{1}{|z-t| w(t) w(z)}\, d\mu(t)\, d\mu(z).$$

This extremal measure λ_w^σ has a compact support.[3] *Furthermore, the following equilibrium condition holds*

$$F_w^\sigma := \max_{x \in \mathrm{supp}(\lambda_w^\sigma)} U^{\lambda_w^\sigma}(x) + Q(x) = \min_{x \in \mathrm{supp}(\sigma - \lambda_w^\sigma)} U^{\lambda_w^\sigma}(x) + Q(x).$$

A combination of [DrSa97, Theorem 3.3] (for compact E), [DaSa98, Theorem 2.5] (for $p = \infty$), [KuVA98, Theorem 7.4 and Lemma 8.3] (for $0 < p \le \infty$), and [KuRa98, Theorem 7.1] (for $p = 2$, see also [KuRa99]) gives the following well–known characterization of the asymptotic behavior of the extremal quantities of (1) in terms of a weighted constrained energy problem.

Theorem 1.2 (see [DaSa98, DrSa97, KuRa98, KuVA98]) *Let $0 < p \le \infty$. Furthermore, let σ, Q, $w = \exp(-Q)$ be as in Theorem 1.1, with $\mathrm{supp}(\sigma) = E$ being a real interval. Suppose that the sets $E_n \subset E$, $n \ge 0$, and the weights $w_n(z) \ge 0$, $z \in E_n$, $n \ge 0$, satisfy the following four conditions*

[2] In particular it follows that σ does not have any mass points.

[3] Our assumptions on σ imply (see, e.g., [Rak96] or Lemma 2.1(a) below) that any measure $\mu \in \mathcal{M}^\sigma$ with compact support will have a continuous potential, and thus $I_w(\mu) = I(\mu) + 2\int Q(t)\, d\mu(t)$.

(i) *For every function f continuous on E with compact support there holds*[4]

$$\lim_{n\to\infty} \frac{1}{n} \sum_{z\in E_n} f(z) = \int f(z)\, d\sigma(z).$$

(ii) *For every compact $K \subset \mathbb{C}$ there holds*

$$\lim_{n\to\infty} \Big[\max_{z\in K\cap E_n} |w_n(z)^{1/n} - w(z)| \Big] = 0.$$

Furthermore, $w_n(z)^{1/n}/w(z)$ is bounded above uniformly for $z \in E_n$ and $n \geq 0$.

(iii) *For every compact $K \subset E$ there holds*

$$\lim_{n\to\infty} \max_{y\in K\cap E_n} \Big| \prod_{x\in E_n\cap K, x\neq y} |y-x|^{1/n} - \exp(-U^{\sigma|_K}(y)) \Big| = 0.$$

(iv) *In case of unbounded E and $p < \infty$, suppose in addition that there exist two positive functions a, b, and constants $\alpha_1 > \alpha_2 > 0$, such that*

$$\#(E_n \cap \{|z| \leq R\}) \leq a(R) \cdot b(n) \quad \text{for all } R, n \geq 0, \quad \lim_{n\to\infty} b(n)^{1/n} = 1, \tag{3}$$

$Q(z) \geq |z|^{\alpha_1}$ for sufficiently large $z \in E$, and $\log a(R) \leq R^{\alpha_2}$ for sufficiently large $R > 0$.

Then we have for the extremal constants

$$\lim_{n\to\infty} \Big[||w_n \cdot T_{n,p}||_{L_p(E_n)} \Big]^{1/n} = e^{-F_w^\sigma},$$

and the sequence of normalized zero counting measures[5] *of $T_{n,p}$, $n \geq 0$, has the weak* limit λ_w^σ.*

In many classical cases (e.g., discrete Chebyshev polynomials [Rak96], Krawtchouk polynomials [DrSa97, DaSa98], Meixner polynomials [KuVA98, DaSa98], Charlier polynomials [KuVA98], Stieltjes–Carlitz polynomials [KuVA98, DaSa98], or discrete Freud polynomials [KuVA98, KuRa98]), the corresponding sets E_n and weights w_n will satisfy the hypotheses of Theorem 1.2, but only after a renormalization of the set E_n by dividing by some power of n. Here Theorem 1.2 may be used to determine the nth root limit of scaled counterparts of the extremal constants $||w_n \cdot T_{n,p}||_{L_p(E_n)}$, also, the measure λ_w^σ is obtained as weak* limit of the corresponding "contracted" zero counting measures (compare, e.g., [KuVA98, Theorem 2.2]).

Beside a generalization to complex sets E_n, the main contribution of this paper is to relax both conditions (iii) and (iv) of Theorem 1.2 (see Theorem 1.3 below). Let us shortly discuss why at least similar assumptions are necessary.

Condition (iv) insures the finiteness of $||w_n P||_{L_p(E_n)}$ for a polynomial of degree at most n, at least for sufficiently large n. Such an additional condition is required for $p < \infty$ for controlling the contribution to the L_p norm of in modulus large elements of E_n.

[4] Since σ has no mass points, it follows in particular that $\#(E_n \cap I)/n$ tends to $\sigma(I)$ for any bounded interval I.

[5] For a polynomial P with zeros $z_1, ..., z_n$ counting multiplicities, its normalized zero counting measure $\nu(P)$ is defined by $\int f(t)\nu(P)(t) = (1/n) \cdot [f(z_1) + f(z_2) + ... + f(z_n)]$.

Condition (iii) of Theorem 1.2 was proposed in [DrSa97], following [DaSa98] it is sufficient to impose (iii) for just one suitable compact K. It may be shown [DrSa97, Lemma 3.2] that, e.g., sets of zeros of suitable orthogonal polynomials satisfy condition (iii). Also, (iii) will be true if the mutual distance of elements in E_n is bounded below [Rak96], namely

$$\liminf_{n\to\infty} \inf_{x,y\in E_n\cap K, x\neq y} n\cdot|x-y|>0$$

for all compact sets K. Let us also mention that condition (iii) was relaxed by some authors [DaSa98, KuVA98, KuRa98] by allowing for an exceptional set of capacity zero. However, as it becomes clear from the discussion in [KuRa98, Section 8], we have to impose some separation property for the abscissas.

In [KuRa98, Conjecture 2], Rakhmanov has conjectured that Theorem 1.2 remains valid in the case of a compact real E if condition (iii) is replaced by the requirement:

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{x,y\in E_n, x\neq y}\log\frac{1}{|x-y|}=I(\sigma)<\infty. \tag{4}$$

In the case of unbounded real E, Rakhmanov [KuRa98, Conjecture 3] suggested that Theorem 1.2 remains valid if condition (iii) is replaced by the requirement: there exists some open set V such that

$$\mathrm{supp}(\lambda_w^\sigma)\subset V, \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{n^2}\sum_{x,y\in E_n\cap V, x\neq y}\log\frac{1}{|x-y|}=I(\sigma|_V)<\infty. \tag{5}$$

Of course, in case of a compact set E, condition (5) is weaker than (4). Also, one may show that condition (iii) of Theorem 1.2 implies (5).

In the present paper we propose a solution for the two Rakhmanov conjectures [KuRa98, Conjecture 2 and 3], namely the following generalization of Theorem 1.2:

Theorem 1.3 *Let E, σ, Q, and $w=\exp(-Q)$ be as in Theorem 1.1,* $\mathrm{supp}(\sigma)$ *being connected (but not necessarily real), and E containing* $\mathrm{supp}(\sigma)$ *(but not necessarily identical). Also, let $w_n, E_n\subset E$ satisfy the conditions (i),(ii) of Theorem 1.2, and suppose that (5) holds for a bounded open set V. Furthermore, let either $p=\infty$, or suppose that there exists a $p'\in(0,p)$ satisfying*

$$\limsup_{n\to\infty}[||f^n||_{L_{p'}(E_n)}]^{1/n}<\infty, \quad f(z):=|z|\cdot w(z). \tag{6}$$

Then the following holds.

(a) *We have for the asymptotic constants*

$$\lim_{n\to\infty}\left[||w_n\cdot T_{n,p}||_{L_p(E_n)}\right]^{1/n}=e^{-F_w^\sigma}.$$

(b) *Suppose in addition that the polynomial convex hull of* $\mathrm{supp}(\lambda_w^\sigma)$ *has two–dimensional Lebesgue measure zero.*[6] *Then for every sequence $(P_n)_{n\geq 0}$ of monic polynomials, P_n of degree n, such that*

$$\lim_{n\to\infty}\left[||w_n\cdot P_n||_{L_p(E_n)}\right]^{1/n}=e^{-F_w^\sigma}$$

we have $\nu(P_n)\xrightarrow{*}\lambda_w^\sigma$.

[6] This is the case, e.g., if $\mathrm{supp}(\sigma)$ is real. For the general case confer the remark at the end of Section 2.

A proof of this and the following statement will be given in the next section. In order to see that Theorem 1.2 is contained in Theorem 1.3, notice first that, according to Lemma 2.7 below, condition (iv) of Theorem 1.2 implies (6) for all $p' \in (0, \infty)$. In addition, if condition (iii) holds for some compact set K satisfying $\sigma(\partial K) = 0$, with its interior V containing $\mathrm{supp}(\lambda_w^\sigma)$, then also condition (5) is shown to be true.

One easily constructs examples (e.g., take $w = 1$ and as $E = \mathrm{supp}(\sigma)$ the unit circle) where $(\nu(T_{n,p}))_{n\geq 0}$ does not have the weak* limit λ_w^σ. However, the measure of balayage of any weak* limit onto the outer boundary of $\mathrm{supp}(\lambda_w^\sigma)$ will always coincide with that of λ_w^σ.

Our proof of Theorem 1.3 strongly relies on the following result for *weighted Fekete points* with respect to discrete sets.

Theorem 1.4 *Let E, σ, Q, and $w = \exp(-Q)$ be as in Theorem 1.1, $\mathrm{supp}(\sigma)$ being a connected (but not necessarily real) subset of E, and let $w_n, E_n \subset E$ satisfy the conditions (i) and (ii) of Theorem 1.2. We define for $n \geq 0$*

$$\delta_n(E_n) := \sup_{E_n' \subset E_n, \# E_n' = n+1} \Big[\prod_{x,y \in E_n', x \neq y} w_n(x)^{1/n} \cdot w_n(y)^{1/n} \cdot |x-y| \Big]^{1/n^2},$$

and denote by E_n^* a set where the supremum is attained (the existence of such a set follows from Lemma 2.3 below), referred to as a set of nth weighted Fekete points (with respect to $(E_n, w_n^{1/n})$).

If condition (5) holds for a bounded open set V, then

(a) *$(\delta_n(E_n))_{n\geq 0}$ converges to $\exp(-I_w(\lambda_w^\sigma))$. Furthermore, the sequence of normalized counting measures of E_n^*, $n \geq 0$, has the weak* limit $\lambda := \lambda_w^\sigma$.*

(b) *Following [KuVA98, Section 8], we define for $d \geq 0$ the compact sets $S_d := \{z \in E : U^\lambda(z) + Q(z) \leq d + F_w^\sigma\}$. Then for any $d > 0$ there exists an $N \geq 0$ such that*

$$E_n^* \subset S_d \quad \text{for all } n \geq N.$$

In other words, Fekete points only accumulate in S_0.

(c) *For every $\epsilon > 0$ there exists an $N \geq 0$ such that for all $n \geq N$, for all polynomials P of degree at most n and for all $z \in \mathbb{C}$*

$$|P(z)| \leq ||w_n P||_{L_\infty(E_n^*)} \cdot \exp(n \cdot (\epsilon + F_w^\sigma - U^\lambda(z))). \tag{7}$$

This result is sharp in the sense that there exist monic polynomials ϕ_n of degree n (with zeros in E_n^*), $n \geq 0$, satisfying

$$\begin{aligned} &||w_n \phi_n||_{L_\infty(E_n)} = ||w_n \phi_n||_{L_p(E_n^*)}, \quad n \geq 0, \quad 0 < p \leq \infty, \\ &\lim_{n\to\infty} [||w_n \phi_n||_{L_\infty(E_n)}]^{1/n} = \exp(-F_w^\sigma), \\ &\lim_{n\to\infty} |\phi_n(z)|^{1/n} = \exp(-U^\lambda(z)), \end{aligned}$$

the latter relation holding uniformly on compact sets of $\mathbb{C} \setminus S_0$.

(d) *Let either $p = \infty$, or suppose that there exists a $p' \in (0, p)$ satisfying (6). If $\mathcal{N}$ denotes any open neighborhood of the compact set $S_0 = \{z \in E : U^\lambda(z) + Q(z) \leq F_w^\sigma\}$, and $E_n' := \mathcal{N} \cap E_n$, then there exist positive constants c, N such that*

$$||w_n P||_{L_p(E_n')} \leq ||w_n P||_{L_p(E_n)} \leq (1 + e^{-c \cdot n})^{1/p} \cdot ||w_n P||_{L_p(E_n')}$$

(in the case $0 < p < \infty$) and

$$||w_n P||_{L_\infty(E_n')} = ||w_n P||_{L_\infty(E_n)}$$

for all $n \geq N$ and for all polynomials P of degree at most n.

Remark 1.5 (a) *In the continuous (unconstrained) case $E_n = E$ for $n \geq 0$, the limit of $(\delta_n(E_n))_{n \geq 0}$ is usually referred to as the weighted transfinite diameter (see, e.g., [SaTo97, Chapter III.1]). Thus Theorem 1.4(a) may be understood as the discrete analogue of [SaTo97, Theorem III.1.3]. Also, a localization of weighted Fekete points similar to Theorem 1.4(b) is well–known in the continuous case.*

(b) *Theorem 1.4(c) may be interpreted as some discrete analogue of a weighted Bernstein–Walsh inequality (for the continuous case compare [SaTo97, Theorem III.2.1]). A related result has been given by Kuijlaars and Van Assche [KuVA98, Lemma 8.1 and Corollary 8.2] without using Fekete points.*

The assertion of Theorem 1.4(c) has an important application to the estimation of the condition number of least squares polynomial approximation, for details see [BeSa98].

(c) *The continuous analogue of Theorem 1.4(d) may be found in [SaTo97, Theorem III.6.1] for $p < \infty$, and follows from the weighted Bernstein–Walsh inequality for $p = \infty$. For the special case of exponentially decreasing weights and real E_n, this result has been given before in [KuVA98, Corollary 8.2 and Lemma 8.3].*

In this context, it is also interesting to recall the results [DaSa98, Theorem 2.1, Theorem 2.3, and Theorem 2.5] of Damelin and Saff, who described a continuous set where a discrete L_∞–norm "lives".

(d) *Let K be some compact set containing in its interior the set S_0. It follows from Theorem 1.4(d) that the discrete "Chebyshev" polynomials $T_{n,\infty}$ with respect to the (possibly unbounded) set E_n and the (uniformly bounded) set $E_n \cap K$ coincide for sufficiently large n.*

(e) *Theorem 1.4(d) in combination with [StTo92, Lemma 1.3.2.] allows to prove that zeros of the extremal polynomials $T_{n,p}$, $0 < p \leq \infty$, only accumulate in the convex hull of S_0, and that the number of zeros of $T_{n,p}$ lying in a compact subset of $\mathbb{C} \setminus S_0$ will be bounded uniformly in n.*

For concrete examples of discrete L_p–extremal polynomials and the determination of the corresponding extremal measure we refer the reader to [Rak96, DrSa97, KuVA98, DaSa98, KuRa98]. Possible generalizations of the above results will be discussed in Section 3.

2 Proofs

In what follows we will always suppose that E, σ, Q, and $w = \exp(-Q)$ are as in Theorem 1.1, $\mathrm{supp}(\sigma)$ being a connected (but not necessarily real) subset of E, and that $w_n, E_n \subset E$, $n \geq 0$, satisfy the conditions (i) and (ii) of Theorem 1.2. In contrast, additional requirements like conditions (iii),(iv), (5), or (6) will be mentioned explicitly.

The "normalized" counting measure $\nu_n(A)$ of some finite set A is defined by

$$\int f(z)\,\nu_n(A)(z) = \frac{1}{n}\sum_{z\in A} f(z),$$

with the corresponding discrete energy given by

$$I_n^*(\nu_n(A)) := \frac{1}{n^2}\sum_{x,y\in A, x\neq y} \log\frac{1}{|x-y|}.$$

We start by establishing/recalling some elementary properties.

Lemma 2.1 **(a)** *For any $\mu \in \mathcal{M}^\sigma$ with compact support, the potential U^μ is continuous in $\mathbb{C}$.*

(b) *Let $O \subset \mathbb{C}$ be open, and let F be some compact subset of O. Then there exists some compact set F' with*

$$\sigma(\partial F') = 0, \quad \textit{and} \quad F \subset \mathit{Int}(F') \subset F' \subset O.$$

(c) *If $B \subset \mathbb{C}$ is a bounded Borel set with $\sigma(\partial B) = 0$, then $(\nu_n(E_n \cap B))_{n\geq 0}$ has the weak* limit $\sigma|_B$, in particular $\#(E_n \cap B)/n \to \sigma(B)$.*

(d) *Let $\mu \in \mathcal{M}^\sigma$, with compact support, O be some open set containing* $\mathrm{supp}(\mu)$*, and Λ some infinite set of integers. Then we may find $\Lambda' \subset \Lambda$ and sets $E_n' \subset E_n \cap O$, $\#E_n' = n+1$, $n \in \Lambda'$, such that $(\nu_n(E_n'))_{n\in\Lambda'}$ has the weak* limit μ.*

Proof: Assertion (a) has already been mentioned in [Rak96]: with $K := \mathrm{supp}(\mu)$, notice that $\lambda := \sigma|_K - \mu$ is a nonnegative Borel measure. Since U^μ and U^λ are lower semi–continuous on $\mathbb{C}$, and by assumption $U^{\sigma|_K}$ is continuous on $\mathbb{C}$, the statement follows from the representation $U^\mu = U^{\sigma|_K} - U^\lambda$. For showing (b) we just recall from [SaTo97, Theorem 0.1.5] that for the Euclidean topology in $\mathbb{C}$ there exists a basis $\{H_\tau\}$ of open disks satisfying $\sigma(\partial H_\tau) = 0$. Part (c) follows from condition (i) of Theorem 1.2 together with [SaTo97, Theorem 0.1.4 and Theorem 0.1.5].

A proof of part (d) is slightly more technical: we first choose a compact set K with $\mathrm{supp}(\mu) \subset \mathrm{Int}(K) \subset K \subset O$. By assumption, $\mathrm{supp}(\sigma)$ is connected, $\sigma(\mathbb{C}) > 1$, and $\mu \leq \sigma$. Thus $\sigma(K) > 1$. In the second step let us show that there exist a constant $C > 0$, and integers $n_k \in \Lambda$, m_k, as well as Borel sets $V_{j,k}$, $j = 1, ..., m_k$, such that

$$\mathrm{diam}(V_{j,k}) \leq \frac{C}{\sqrt{m_k}}, \quad \sigma(\partial V_{j,k}) = 0, \quad V_{j,k} \cap V_{j',k} = \{\}$$
$$K \subset \bigcup_{j=1}^{m_k} V_{j,k}, \quad \frac{\#(E_{n_k} \cap V_{j,k})}{n_k + 1} \geq \sigma(V_{j,k}) - \frac{1}{m_k^3},$$

for $1 \leq j, j' \leq m_k$, $j \neq j'$. In fact, for sufficiently small $r_k \leq 1/k$, we may cover K by at most $(\mathrm{diam}(K)/r_k)^2$ closed squares with side length r_k while staying in O. Each of these squares may be covered by an open disk Δ of radius $< 2r_k$ satisfying $\sigma(\partial\Delta) = 0$ (compare [SaTo97, p.5]), which for sufficiently small r_k will be a subset of O. By taking intersections between these disks, we will stay with $m_k \leq (\mathrm{diam}(K)/r_k)^2$ Borel sets $V_{j,k}$ as quoted above. By part (c), $\#(E_n \cap V_{j,k})/(n+1) \to \sigma(V_{j,k})$ for $j = 1, ..., m_k$, and thus we may find an integer $n_k \in \Lambda$ as described above. Notice that by construction

$$\sum_{j=1}^{m_k} \#(E_{n_k} \cap V_{j,k}) \geq (n_k+1) \cdot \sigma(\bigcup_{j=1}^{m_k} V_{j,k}) - \frac{n_k+1}{m_k^2} \geq (n_k+1) \cdot (\sigma(K) - \frac{1}{m_k^2}), \tag{8}$$

the latter being greater than $n_k + 1$ (at least for sufficiently large k).

Imposing in addition that $n_k + 1 \geq m_k^3$, we may construct a set E'_{n_k} by taking exactly $\ell_{j,k}$ elements from $V_{j,k} \cap E_n$, with $|\mu(V_{j,k}) - \ell_{j,k}/(n_k+1)| \leq 1/m_k^3$, $j = 1, 2, ..., m_k$. As above one checks that this set E'_{n_k} contains $(n_k+1) \cdot (1+\epsilon_k)$ elements, with $|\epsilon_k| \leq 1/m_k^2$. From (8) we see that further points are available if necessary, and thus we may modify slightly the construction of E'_{n_k} such that

$$\#E'_{n_k} = n_k + 1, \quad \text{and} \quad \mu(V_{j,k}) + \frac{2}{m_k^2} \geq \frac{\ell_{j,k}}{n_k+1} \geq \mu(V_{j,k}) - \frac{2}{m_k^2}.$$

Let $\mu_n := \nu_n(E'_n)$ for $n \in \Lambda' := \{n_k : k \geq 0\}$. In order to establish $\mu_n \xrightarrow{*} \mu$, let f be continuous. Then f is bounded and uniformly continuous on some neighborhood of K, and hence the expression

$$|\int f(t)\, d\mu(t) - \int f(t)\, d\mu_{n_k}(t)| \leq \sum_{j=1}^{m_k} \Big| \int_{V_{j,k}} f(t)\, d\mu(t) - \frac{1}{n_k} \cdot \sum_{x \in E'_{n_k} \cap V_{j,k}} f(x) \Big|$$

becomes arbitrarily small for $k \to \infty$. □

We continue with analyzing the Rakhmanov condition (5).

Lemma 2.2 **(a)** *Let K be compact, and $E'_n \subset E_n \cap K$, $n \geq 0$, with $\mu_n := \nu_n(E'_n) \xrightarrow{*} \mu$. Then*

$$\liminf_{n \to \infty} I_n^*(\mu_n) \geq I(\mu).$$

(b) *With the notation of part (a), we have $\lim_{n\to\infty} I_n^*(\mu_n) = I(\mu)$ iff for every $\epsilon > 0$ we find an $\delta_0 > 0$ such that for every $0 < \delta \leq \delta_0$ with some $n_0 = n_0(\delta)$ there holds*

$$\frac{1}{n^2} \sum_{x,y \in E'_n, 0<|x-y|<\delta} \log \frac{\delta}{|x-y|} < \epsilon, \quad n \geq n_0. \tag{9}$$

(c) *If (5) holds for a bounded open set V, then condition (9) is true for all $K \subset V$.*

Proof: Writing $f_\delta(z) := \log(1/\max\{\delta, |z|\})$, we first observe that

$$\frac{1}{n^2} \sum_{x,y \in E'_n, 0<|x-y|<\delta} \log \frac{\delta}{|x-y|} - I_n^*(\mu_n) + I(\mu) = -a_1(n,\delta) + a_2(n,\delta) + a_3(\delta), \tag{10}$$

with

$$a_1(n,\delta) := \int\int f_\delta(x-y)\, d\mu_n(x)\, d\mu_n(y) - \int\int f_\delta(x-y)\, d\mu(x)\, d\mu(y),$$
$$a_2(n,\delta) := \frac{\#E'_n}{n^2} \cdot \log\frac{1}{\delta}, \quad a_3(\delta) := I(\mu) - \int\int f_\delta(x-y)\, d\mu(x)\, d\mu(y).$$

For a proof of parts (a),(b), notice that $(a_1(n,\delta))_{n\geq 0}$ tends to zero for all $\delta > 0$ since $\mu_n \xrightarrow{*} \mu$, and $(x,y) \to f_\delta(x-y)$ is continuous in $K \times K$. Also, $(a_2(n,\delta))_{n\geq 0}$ tends to zero for all $\delta > 0$. Furthermore, $a_3(\delta) \to 0$ for $\delta \to 0$ by the monotone convergence theorem.

The assertion of part (a) now follows by observing that the sum on the left–hand side of (10) is nonnegative. Part (b) is a consequence of (10) and of the fact that

$$\lim_{\delta\to 0}\lim_{n\to\infty} -a_1(n,\delta) + a_2(n,\delta) + a_3(\delta) = 0.$$

In order to show (c), notice that the Rakhmanov condition (5) may be equivalently rewritten as $I_n^*(\mu_n) \to I(\mu)$, with $\mu_n := \nu_n(E_n \cap V)$, and $\mu := \sigma|_V$. If $\sigma(\partial V) = 0$, then from Lemma 2.1(c) and part (b) we may conclude that (9) is true for $E_n' = E_n \cap V$, and thus in particular for any subset of $E_n \cap K$, as claimed in part (c). The reasoning is slightly more involved in the case $\sigma(\partial V) > 0$: using Lemma 2.1(b) we find some bounded open neighborhood O of the closure of V with $\sigma(\partial O) = 0$. Writing shorter $\sigma_n := \nu_n(E_n \cap O)$, it follows from Lemma 2.1(c) that $\sigma_n \xrightarrow{*} \sigma|_O$. Denote by χ_V the characteristic function of V. Then the function $g_\delta(x,y) := \chi_V(x)\cdot\chi_V(y)\cdot f_\delta(x-y)$ is lower semi–continuous for each $\delta > 0$, and from [SaTo97, Theorem 0.1.4] we obtain

$$\liminf_{n\to\infty} a_1(n,\delta) = \liminf_{n\to\infty} \int\int g_\delta(x-y)\, d\sigma_n(x)\, d\sigma_n(y) - \int\int g_\delta(x-y)\, d\sigma|_O(x)\, d\sigma|_O(y) \geq 0$$

for all $\delta > 0$. Consequently, using (5) and (10) we get

$$\limsup_{\delta\to 0}\ \limsup_{n\to\infty} \frac{1}{n^2} \sum_{x,y\in E_n\cap V, 0<|x-y|<\delta} \log\frac{\delta}{|x-y|} \leq 0,$$

and thus (9) holds for E_n' being any subset of $V \cap E_n$. □

We now investigate some properties of nth weighted Fekete points with respect to $(E_n, w_n^{1/n})$. Notice that if $\zeta, x_0, ..., x_n \in E_n$ then by construction

$$\frac{\delta_n(\{\zeta, x_1, ..., x_n\})}{\delta_n(\{x_0, x_1, ..., x_n\})} = \Big|\frac{w_n(\zeta)\cdot(\zeta - x_1)\cdot\ldots\cdot(\zeta - x_n)}{w_n(x_0)\cdot(x_0-x_1)\cdot\ldots\cdot(x_0-x_n)}\Big|^{2/n^2}, \tag{11}$$

and this expression is ≤ 1 for any $\zeta \in E_n$ if $\{x_0, x_1, ..., x_n\} = E_n^*$. Thus one may easily determine the L_∞ norm of such weighted polynomials.

The problem of finding Fekete points with respect to $(E_n, w_n^{1/n})$ may be understood as the problem of minimizing some discrete weighted energy, here the set E_n^* plays the role of the "support of the extremal measure". An important feature for the continuous case is that these "supports" are uniformly bounded for $n \geq 0$. A similar fact is true for our discrete problem.

Lemma 2.3 *If (5) holds for a bounded open set V, then there exists some compact set K containing all sets of Fekete points E_n^* for $n \geq 0$, i.e.,*

$$\delta_n(E_n \cap K) = \delta_n(E_n), \quad n \geq 0.$$

Proof: Assume that the assertion of Lemma 2.3 is not true. Then we may construct a strictly increasing sequence $(n_k)_{k\geq 0}$, with $\delta_{n_k}(E_{n_k}) > \delta_{n_k}(E_{n_k} \cap \{|x| \leq k\})$. Consequently, we find sets $E^*_{n_k}$, $k \geq 0$, satisfying[7]

$$\max\{|x| : x \in E^*_{n_k}\} \geq k, \quad \#E^*_{n_k} = n_k + 1, \quad \text{and} \quad \delta_{n_k}(E^*_{n_k}) \geq \delta_{n_k}(E_{n_k})/2^{1/n_k}$$

for all $k \geq 0$. For $n = n_k$, write $E^*_n = \{x_{0,n}, ..., x_{n,n}\}$, $n \geq 0$, with $w(x_{0,n}) = \min_{0 \leq j \leq n} w(x_{j,n})$. Then there exist $0 \leq j_k \leq n_k$ with $|x_{j_k,n_k}| \to \infty$, implying $w(x_{j_k,n_k}) \to 0$, and thus $w(x_{0,n_k}) \to 0$ and $|x_{0,n_k}| \to \infty$ for k tending to infinity. We claim that

$$\lim_{n=n_k, k\to\infty} \max_{x \in E_n} |w_n(x) \cdot f_n(x)|^{1/n} = 0, \quad f_n(x) = \prod_{j=1}^{n} (x - x_{j,n}). \tag{12}$$

In fact, from (11) together with $\delta_{n_k}(E^*_{n_k}) \geq \delta_{n_k}(E_{n_k})/2^{1/n_k}$ it follows that

$$\max_{x \in E_n} |w_n(x) \cdot f_n(x)| \leq 2^{n/2} \cdot \max_{x \in E^*_n} |w_n(x) \cdot f_n(x)| = 2^{n/2} \cdot |w_n(x_{0,n}) \cdot f_n(x_{0,n})|.$$

Therefore, according to the assumption (ii) of Theorem 1.2, for establishing (12) it is sufficient to show that

$$\lim_{n=n_k, k\to\infty} w(x_{0,n}) \cdot |f_n(x_{0,n})|^{1/n} = 0.$$

Define $h(r) := \max\{|x| \cdot w(x) : x \in E, |x| \geq r\}$. Then $\lim_{r\to\infty} h(r) = 0$ by assumption on $w = \exp(-Q)$ (see Theorem 1.1), and

$$w(x_{0,n}) \cdot |x_{0,n} - x_{j,n}| \leq \begin{cases} w(x_{0,n})|x_{0,n}| + w(x_{0,n})|x_{j,n}| \leq 2h(|x_{0,n}|) & \text{if } |x_{j,n}| \leq |x_{0,n}|, \\ w(x_{0,n})|x_{0,n}| + w(x_{j,n})|x_{j,n}| \leq 2h(|x_{0,n}|) & \text{if } |x_{j,n}| \geq |x_{0,n}|, \end{cases}$$

for $j = 1, 2, ..., n$. Consequently, $|w(x_{0,n})^n f_n(x_{0,n})|^{1/n} \leq 2h(|x_{0,n}|) \to 0$, proving our claim (12).

We now choose some compact sets K', K'' with

$$\text{supp}(\lambda^\sigma_w) \subset \text{Int}(K'') \subset K'' \subset \text{Int}(K') \subset K' \subset V,$$

where according to Lemma 2.1(b) we may suppose without loss of generality that $\sigma(\partial K') = 0$. Furthermore, we define $E'_n := E_n \cap K'$, and $\mu_n := \nu_n(E'_n)$, with weak* limit $\sigma|_{K'}$ (see Lemma 2.1(c)). Since $\sigma(K'') > 1$, we may conclude that

$$\liminf_{n\to\infty} \frac{m_n}{n} > 0, \quad m_n := \#((E'_n \setminus E^*_n) \cap K'').$$

Taking into account assumption (ii) and the fact that w is bounded away from zero on K'', it follows from (12) that for every $\delta > 0$

$$\begin{aligned} &\lim_{n=n_k, k\to\infty} \min_{x \in (E'_n \setminus E^*_n) \cap K''} \frac{1}{n} \sum_{1 \leq j \leq n, |x - x_{j,n}| < \delta} \log \frac{\delta}{|x - x_{j,n}|} \\ &\geq \lim_{n=n_k, k\to\infty} \min_{x \in (E'_n \setminus E^*_n) \cap K''} \log \frac{\delta}{|f_n(x)|^{1/n}} = +\infty, \end{aligned}$$

and thus

$$\lim_{n=n_k, k\to\infty} \frac{1}{n \cdot m_n} \sum_{x \in (E'_n \setminus E^*_n) \cap K''} \sum_{1 \leq j \leq n, |x - x_{j,n}| < \delta} \log \frac{\delta}{|x - x_{j,n}|} = +\infty.$$

[7]By assumption on Q (see Theorem 1.1), the function $(x, y) \to w(x) \cdot w(y) \cdot |x - y|$ is bounded above on $E \times E$. Taking into account condition (ii) of Theorem 1.2, we may conclude that $\delta_n(E_n)$ is finite for each $n \geq 0$.

By construction, there exists a $\delta_0 > 0$ such that

$$\{x_{j,n} : 1 \leq j \leq n, |x - x_{j,n}| < \delta\} \subset E_n'$$

for all $\delta \in (0, \delta_0]$ and for all $x \in (E_n' \setminus E_n^*) \cap K''$, implying that

$$\lim_{n=n_k, k\to\infty} \frac{1}{n^2} \sum_{x,y\in E_n', 0<|x-y|<\delta} \log \frac{\delta}{|x-y|} = +\infty, \quad 0 < \delta \leq \delta_0,$$

in contradiction with Lemma 2.2(c). □

Proof of Theorem 1.4(a): Let E_n^* be a set of nth weighted Fekete points as described in Theorem 1.4, and write $\nu_n := \nu_n(E_n^*)$. From Lemma 2.3 we know that $\mathrm{supp}(\nu_n) \subset K'$ for some compact K', and Helly's theorem asserts that from each subsequence of $(\nu_n)_{n\geq 0}$ we may extract a subsequence $(\nu_n)_{n\in\Lambda}$ that converges weak* to some probability measure ν. Notice that $\nu \in \mathcal{M}^\sigma$, since for any nonnegative continuous function f with compact support there holds according to condition (i) of Theorem 1.2

$$0 \leq \lim_{n\to\infty, n\in\Lambda} \sum_{x\in E_n\setminus E_n^*} \frac{f(x)}{n} = \lim_{n\to\infty} \sum_{x\in E_n} \frac{f(x)}{n} - \lim_{n\to\infty, n\in\Lambda} \int f(x)\, d\nu_n(x) = \int f(x)\, d(\sigma - \nu)(x).$$

Choose a compact set $K \subset V$, with its interior O containing $\mathrm{supp}(\lambda_w^\sigma)$. We now apply Lemma 2.1(d) with O, Λ as above for the extremal measure $\mu = \lambda_w^\sigma$, giving some set $\Lambda' \subset \Lambda$ and some sets E_n', $\#E_n' = n+1$, with $(\mu_n)_{n\in\Lambda'}$ having the weak* limit λ_w^σ, where $\mu_n := \nu_n(E_n')$. Then by the construction of Fekete points we have

$$\log \delta_n(E_n) \geq \log \delta_n(E_n') = \frac{1}{n^2} \sum_{x,y\in E_n', x\neq y} \log(w_n(x)^{1/n} \cdot w_n(y)^{1/n} \cdot |x-y|)$$

$$= -I_n^*(\mu_n) - \frac{2}{n} \sum_{x\in E_n'} \log \frac{1}{w_n(x)^{1/n}}.$$

By assumption (ii), the sequence $(\log w_n^{-1/n})_{n\in\Lambda'}$ tends to $\log w^{-1} = Q$ uniformly in the compact set K. Furthermore, since $\mu_n \xrightarrow{*} \lambda_w^\sigma$, with $\mathrm{supp}(\mu_n) \subset K \subset V$, we may conclude from Lemma 2.2(b),(c) that

$$\lim_{n\to\infty, n\in\Lambda'} I_n^*(\mu_n) = I(\lambda_w^\sigma), \qquad \lim_{n\to\infty, n\in\Lambda'} \frac{2}{n} \sum_{x\in E_n'} \log \frac{1}{w_n(x)^{1/n}} = 2 \int Q(z)\, d\lambda_w^\sigma(z).$$

Therefore

$$-I_w(\lambda_w^\sigma) = -I(\lambda_w^\sigma) - 2\int Q(z)\, d\lambda_w^\sigma(z) = \lim_{n\to\infty, n\in\Lambda'} \log \delta_n(E_n') \leq \liminf_{n\to\infty, n\in\Lambda'} \log \delta_n(E_n) \leq -I_w(\nu),$$

where for the last estimate we have applied Lemma 2.2(a). On the other hand, $I_w(\nu) \geq I_w(\lambda_w^\sigma)$ by the extremal property of Theorem 1.1, with equality iff $\nu = \lambda_w^\sigma$. Consequently, any weak* limit of $(\nu_n)_{n\geq 0}$ coincides with λ_w^σ, and $\lim_{n\to\infty} \log \delta_n(E_n) = -I_w(\lambda_w^\sigma)$, as claimed in Theorem 1.4(a). □

Remark 2.4 *From the preceding proof it follows that $\nu_n(E_n^*) \xrightarrow{*} \lambda_w^\sigma$ and $I_n^*(\nu_n(E_n^*)) \to I(\lambda_w^\sigma)$. Hence, by Lemma 2.2(b), condition (9) holds for $E_n' = E_n^*$. Moreover, using Lemma 2.2(c) one verifies that condition (9) is even true for any sequence of sets $(E_n')_{n\geq 0}$ satisfying $E_n' \subset E_n^* \cup (E_n \cap K)$, where K is some compact subset of V.* □

For a proof of Theorem 1.4(b),(c) we need the following two technical lemmas.

Lemma 2.5 *Suppose that (5) holds for a bounded open set V, and write shorter $\lambda = \lambda_w^\sigma$. For $n \geq 0$, $y \in E_n^*$, let*

$$f_{y,n}(z) := \prod_{x \in E_n^*, x \neq y} (z - x), \quad c_{y,n} := |w_n(y) \cdot f_{y,n}(y)|^{1/n}.$$

Then

$$|f_{y,n}(z)| \leq \exp(-n \cdot U^\lambda(z)) \cdot \max_{t \in S} |\exp(n \cdot U^\lambda(t)) \cdot f_{y,n}(t)|, \quad n \geq 0, \quad y \in E_n^*, \tag{13}$$

$$\lim_{n \to \infty} \max_{y \in E_n^*} \max_{t \in S} |\exp(n \cdot U^\lambda(t)) \cdot f_{y,n}(t)|^{1/n} = 1, \tag{14}$$

$$\limsup_{n \to \infty, n \in \Lambda} \log c_{\zeta_n, n} \leq -U^\lambda(\zeta) - Q(\zeta), \quad \text{if } \zeta_n \in E_n^* \text{ and } \lim_{n \to \infty, n \in \Lambda} \zeta_n = \zeta, \tag{15}$$

where S is some compact disk containing $\operatorname{supp}(\lambda)$ *in its interior.*

Proof: Claim (13) is an easy consequence of the maximum principle for subharmonic functions (see, e.g., [SaTo97, Theorem I.2.4]): since $z \to U^\lambda(z) + \log|z|$ is harmonic in some neighborhood of $\overline{\mathbb{C}} \setminus S$, it follows that $h(t) := n \cdot U^\lambda(t) + \log|f_{y,n}(t)|$ is subharmonic in some neighborhood of $\overline{\mathbb{C}} \setminus S$, and thus $h(z) \leq \max_{t \in S} h(t)$ for all $z \in \overline{\mathbb{C}}$.

In order to show (14), let $y_n \in E_n^*$ such that

$$\max_{t \in S} |\exp(n \cdot U^\lambda(t)) \cdot f_{y_n,n}(t)| = \max_{y \in E_n^*} \max_{t \in S} |\exp(n \cdot U^\lambda(t)) \cdot f_{y,n}(t)|, \quad n \geq 0.$$

We first notice that U^λ is continuous in S, and that $-\log|f_{y_n,n}(z)|^{1/n} = U^{\mu_n}(z)$ for some probability measure μ_n with support being uniformly bounded in n, and $\mu_n \xrightarrow{*} \lambda$. Since the set S has the K–property [NiSo88, Section 5.4.3], we may conclude from [NiSo88, Theorem 5.4.3, p.182] that

$$\begin{aligned}
&\lim_{n \to \infty} \log \max_{t \in S} |\exp(n \cdot U^\lambda(t)) \cdot f_{y_n,n}(t)|^{1/n} \\
&= -\lim_{n \to \infty} \min_{t \in S} |U^{\mu_n}(t) - U^\lambda(t)| = -\min_{t \in S} |U^\lambda(t) - U^\lambda(t)| = 0,
\end{aligned}$$

as claimed in (14).

For a proof of (15) we write

$$\log c_{\zeta_n,n} = -U^{\mu_n}(\zeta_n) - \log(1/w_n(\zeta_n)^{1/n}),$$

where we observe that the term on the right tends to $-Q(\zeta)$ for $n \to \infty$. Since $\mu_n \xrightarrow{*} \lambda$, the assertion of (15) now follows from the principle of descent [SaTo97, Theorem I.6.8]. $\Box$

Lemma 2.6 *Suppose that (5) holds for a bounded open set V, and write shorter $\lambda = \lambda_w^\sigma$. Then with $c_{y,n}$ as in Lemma 2.5 there holds*

$$\lim_{n \to \infty} \min_{y \in E_n^*} \log c_{y,n} = -F_w^\sigma.$$

Proof: First notice that any $\zeta \in \mathrm{supp}(\lambda)$ is a limit point of Fekete points according to Theorem 1.4(a), and thus by (15)

$$\limsup_{n\to\infty} \min_{y\in E_n^*} \log c_{y,n} \le \min_{\zeta\in\mathrm{supp}(\lambda)} [-U^\lambda(\zeta) - Q(\zeta)] = -F_w^\sigma,$$

the latter following from Theorem 1.1. For showing the opposite inequality, define $x_n \in E_n^*$ by

$$\min_{y\in E_n^*} \log c_{y,n} = \log c_{x_n,n}. \tag{16}$$

Recall that $U^\lambda + Q$ is continuous in E, and that $\mathrm{supp}(\sigma)$ is supposed to be connected. Given an $\epsilon > 0$, according to Theorem 1.1 we may find some compact $J \subset V$ with $(\sigma-\lambda)(J) > 0$, and

$$Q(z) + U^\lambda(z) \le F_w^\sigma + \epsilon, \quad z \in J \cap E,$$

where in view of Lemma 2.1(b) we may suppose in addition that $\sigma(\partial J) = 0$. We define $E_{0,n} = E_{1,n} \cup E_{2,n}$, with $E_{1,n} = E_n^* \setminus \{x_n\}$, and $E_{2,n} = (J \cap E_n) \setminus E_n^*$. Then we obtain the weak* limits

$$\nu_n(E_{1,n}) \xrightarrow{*} \lambda_1 = \lambda, \quad \nu_n(E_{2,n}) \xrightarrow{*} \lambda_2 = (\sigma-\lambda)|_J, \quad \nu_n(E_{0,n}) \xrightarrow{*} \lambda_0 = \lambda_1 + \lambda_2.$$

Writing shorter $I_n(A) := I_n^*(\nu_n(A))$ for some finite set A, we observe that by Lemma 2.2(b) and Remark 2.4

$$\lim_{n\to\infty} I_n(E_{j,n}) = I(\lambda_j), \quad j = 0,1,2, \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \sum_{x\in E_{2,n}} \log \frac{1}{w_n(x)^{1/n}} = \int Q(t)\, d\lambda_2(t).$$

The decomposition

$$I_n(E_{0,n}) - I_n(E_{1,n}) - I_n(E_{2,n}) = \frac{2}{n^2} \sum_{y\in E_{2,n}} \sum_{x\in E_{1,n}} \log \frac{1}{|x-y|} = \frac{2}{n} \sum_{y\in E_{2,n}} \log \frac{1}{|f_{x_n,n}(y)|^{1/n}}$$

with $f_{y,n}$ as in Lemma 2.5 leads to

$$\lim_{n\to\infty} \frac{1}{n} \sum_{y\in E_{2,n}} \log |w_n(y) f_{x_n,n}(y)|^{1/n} = -\int Q(t)\, d\lambda_2(t) + \frac{1}{2}\cdot(-I(\lambda_0) + I(\lambda_1) + I(\lambda_2))$$
$$= -\int [Q(t) + U^{\lambda_1}(t)]\, d\lambda_2(t) = -\int [Q(t) + U^\lambda(t)]\, d(\sigma-\lambda)|_J(t).$$

Now with $d_n := \#E_{2,n}/n$, $d_n \to (\sigma-\lambda)(J) > 0$, there holds

$$\frac{1}{n} \sum_{y\in E_{2,n}} \log |w_n(y) f_{x_n,n}(y)|^{1/n} \le d_n \cdot \max_{y\in E_{2,n}} \log |w_n(y) f_{x_n,n}(y)|^{1/n}.$$

Taking into account the property (11) of Fekete points, we may conclude that

$$\liminf_{n\to\infty} \log c_{x_n,n} \ge \liminf_{n\to\infty} \max_{y\in E_{2,n}} \log |w_n(y) f_{x_n,n}(y)|^{1/n}$$
$$\ge -\frac{1}{(\sigma-\lambda)(J)} \int [Q(t) + U^\lambda(t)]\, d(\sigma-\lambda)|_J(t) \ge -\max_{t\in J\cap E} [Q(t) + U^\lambda(t)] \ge -F_w^\sigma - \epsilon,$$

the latter inequality following from the definition of J. Since $\epsilon > 0$ was arbitrary, we therefore have established the assertion of Lemma 2.6. □

We are now prepared to proceed with the proof of Theorem 1.4.

Proof of Theorem 1.4(b),(c): For a proof of part (b), suppose in contrast that there is a sequence of Fekete points accumulating outside of S_0. Then we may construct a set Λ and $\zeta_n \in E_n^*$, with $(\zeta_n)_{n\in\Lambda}$ converging to some $\zeta \in E$, and $U^\lambda(\zeta) + Q(\zeta) = F_w^\sigma + d > F_w^\sigma$. With the notation of Lemma 2.5, it follows from (15) that

$$\limsup_{n\to\infty, n\in\Lambda} \log c_{\zeta_n,n} \leq -U^\lambda(\zeta) - Q(\zeta) < -F_w^\sigma,$$

in contradiction to Lemma 2.6.

We proceed by showing (7): By the Lagrange interpolation formula, we have for any polynomial P of degree at most n

$$|P(z)| = |\sum_{y\in E_n^*} \frac{f_{y,n}(z)}{f_{y,n}(y)} \cdot P(y)| \leq (n+1) \cdot ||w_n P||_{L_\infty(E_n^*)} \cdot \max_{y\in E_n^*} \frac{|f_{y,n}(z)|}{(c_{y,n})^n},$$

where $f_{y,n}, c_{y,n}$ are as in Lemma 2.5. Thus (7) is an immediate consequence of the assertions (13), (14) of Lemma 2.5 together with Lemma 2.6.

In order to show the second part of Theorem 1.4(c), we follow (16) and define for $n \geq 0$

$$\phi_n(z) = f_{x_n,n}(z), \quad \text{with } x_n \in E_n^* \text{ satisfying } \min_{y\in E_n^*} \log c_{y,n} = \log c_{x_n,n}.$$

The relation $||w_n\phi_n||_{L_\infty(E_n)} = ||w_n\phi_n||_{L_\infty(E_n^*)}$ is a trivial consequence of the construction of Fekete points (see (11)), and $||w_n\phi_n||_{L_\infty(E_n^*)} = ||w_n\phi_n||_{L_p(E_n^*)}$ follows from the fact that ϕ_n is different from zero at exactly one element of E_n^*. The claimed limit relation for $c_{x_n,n} = [||w_n\phi_n||_{L_\infty(E_n)}]^{1/n}$ has been established in Lemma 2.6. Furthermore, since $-\log|\phi_n(z)|^{1/n} = U^{\mu_n}(z)$ for some measure μ_n, with $(\mu_n)_{n\geq 0}$ asymptotically supported in S_0 and $\mu_n \xrightarrow{*} \lambda$ (see Theorem 1.4(a)(b)), we get $-\log|\phi_n(z)|^{1/n} \to U^\lambda(z)$ uniformly on compact subsets of $\mathbb{C} \setminus S_0$. $\Box$

Proof of Theorem 1.4(d): The left–hand inequalities claimed in part (d) are trivial since $E_n' \subset E_n$. In order to show the right–hand ones, we choose a $d > 0$ such that $S_{3d} \subset \mathcal{N}$. Outside of S_{3d}, Theorem 1.4(c) will enable us to control the size of the quantity $w_n(z)|P(z)|$; however, in the case of unbounded E, we first have to describe the behavior for large $|z|$.

If E is bounded, let $\Delta = E$. Otherwise, a compact set Δ is constructed as follows: let

$$C_0 := \sup_{n\geq 0} \sup_{z\in E_n} w_n(z)^{1/n}/w(z),$$

being finite by assumption (ii). Recall that $U^\lambda(z) + \log|z|$ is continuous in $\overline{\mathbb{C}} \setminus \{0\}$, and vanishes at infinity. Also, $f(z) = |z| \cdot w(z) = \exp(-Q(z) + \log|z|)$ tends to zero for $|z| \to \infty$, $z \in E$. Thus we may find some compact set Δ such that for all $z \in E \setminus \Delta$ there holds

$$\exp(-U^\lambda(z)) \leq 2 \cdot |z|, \tag{17}$$

$$2 \cdot C_0 \cdot \exp(d + F_w^\sigma) \cdot f(z) \leq \exp(-d). \tag{18}$$

Moreover, in the case $p < \infty$ we may suppose in addition that with $p' \in (0,p)$ as in (6) there holds

$$2 \cdot C_0 \cdot \exp(d + F_w^\sigma) \cdot \Big[2 \cdot \limsup_{n\to\infty} [||f^n||_{L_{p'}(E_n)}]^{1/n}\Big]^{p'/p} \cdot f(z)^{1-p'/p} \leq \exp(-d), \quad z \in E \setminus \Delta. \tag{19}$$

By Theorem 1.4(b), the set S_{3d} (and thus E_n') will contain the set of Fekete points E_n^* for all sufficiently large n. From Theorem 1.4(c) with $\epsilon = d$ we may conclude that there exists some $N_1 \geq 0$ such that for every $n \geq N_1$, for any polynomial P of degree at most n, and for all $z \in \mathbb{C}$ we have

$$\Big[\frac{|P(z)|}{||w_n P||_{L_\infty(E_n')}}\Big]^{1/n} \leq \exp(d + F_w^\sigma - U^\lambda(z)). \tag{20}$$

We now estimate separately the contribution to the L_p–norm of the sets $(E_n \cap \Delta) \setminus E_n'$ and $E_n \setminus \Delta$. By assumption (ii) we may find an $N_2 \geq N_1$ such that

$$w_n(z) \leq w(z)^n \cdot e^{nd}, \quad z \in E_n \cap \Delta, \quad n \geq N_2.$$

Taking into account (20) and the definition of the set S_{3d}, we may conclude that for all $n \geq N_2$ and for all $z \in (E_n \cap \Delta) \setminus E_n' \subset \Delta \setminus S_{3d}$

$$\Big[\frac{|w_n(z)P(z)|}{||w_n P||_{L_\infty(E_n')}}\Big]^{1/n} \leq \exp(2d + F_w^\sigma - U^\lambda(z) - Q(z)) \leq \exp(-d).$$

It follows from assumption (i) that there exists an $N_3 \geq N_2$ such that $\#(E_n \cap \Delta) \leq C_1 \cdot n$ for all $n \geq N_3$ with some constant C_1. Consequently,

$$\Big[\frac{||w_n P||_{L_p((E_n \cap \Delta)\setminus E_n')}}{||w_n P||_{L_\infty(E_n')}}\Big]^{1/n} \leq [C_1 \cdot n]^{1/(np)} \cdot \exp(-d), \quad n \geq N_3. \tag{21}$$

In the case $p = \infty$, combining (17), (18), and (20) we obtain also

$$\Big[\frac{||w_n P||_{L_\infty(E_n\setminus\Delta)}}{||w_n P||_{L_\infty(E_n')}}\Big]^{1/n} \leq \frac{\sup_{z \in E_n\setminus\Delta} C_0 \cdot |w(z)| \cdot |P(z)|^{1/n}}{[||w_n P||_{L_\infty(E_n')}]^{1/n}}$$
$$\leq \sup_{z \in E_n\setminus\Delta} C_0 \cdot \exp(d + F_w^\sigma - U^\lambda(z)) \cdot w(z) \leq \sup_{z \in E_n\setminus\Delta} 2C_0 \cdot \exp(d + F_w^\sigma) \cdot f(z) \leq \exp(-d)$$

for all $n \geq N_1$ and for all polynomials P of degree at most n, which together with (21) is sufficient for the assertion of Theorem 1.4(d).

It remains to discuss the case $0 < p < \infty$, here we have according to (17), (19), and (20)

$$\Big[\frac{||w_n P||_{L_p(E_n\setminus\Delta)}}{||w_n P||_{L_\infty(E_n')}}\Big]^{1/n} \leq 2C_0 \cdot \exp(d + F_w^\sigma) \cdot [||f^n||_{L_p(E_n\setminus\Delta)}]^{1/n}$$
$$\leq 2C_0 \cdot \exp(d + F_w^\sigma) \cdot \Big[[||f^n||_{L_{p'}(E_n\setminus\Delta)}]^{1/n}\Big]^{p'/p} \cdot \sup_{z\in E\setminus\Delta} f(z)^{1-p'/p} \leq \exp(-d)$$

for all sufficiently large n and for all polynomials P of degree at most n. Thus the assertion follows by recalling (21) and by observing that $||w_n P||_{L_\infty(E_n')} \leq ||w_n P||_{L_p(E_n')}$. □

In the context of Theorem 1.4(d) and Theorem 1.3 it is of interest to know some simple sufficient conditions for (6). Here we propose the following

Lemma 2.7 *The relation*

$$\limsup_{n\to\infty} [||f^n||_{L_p(E_n)}]^{1/n} < \infty, \quad f(z) = |z| \cdot w(z),$$

holds for some $p \in (1, \infty)$ if one of the following conditions is satisfied.

(a) *There exist sets K_n satisfying*

$$\limsup_{n\to\infty}[\#(E_n \cap K_n)]^{1/n} < \infty, \quad \textit{and} \quad \limsup_{n\to\infty}[||f^n||_{L_p(E_n\setminus K_n)}]^{1/n} < \infty.$$

(b) *There exists an $\alpha > 0$ such that*

$$\limsup_{n\to\infty}[||f^\alpha||_{L_p(E_n)}]^{1/n} < \infty.$$

(c) *Condition (iv) of Theorem 1.2 holds.*

(d) *Condition (3) of Theorem 1.2 holds; furthermore, there exist constants $\alpha_1, \alpha_2 > 0$ such that $Q(z) \geq (1+\alpha_1)\cdot \log|z|$ for sufficiently large $z \in E$, and $a(R) \leq R^{\alpha_2}$ for sufficiently large $R > 0$.*

Proof: We first recall that f is bounded above on E, and tends to zero for $|z| \to \infty$. Thus part (a) follows from the simple observation that

$$[||f^n||_{L_p(E_n\cap K_n)}]^{1/n} \leq [\#(E_n \cap K_n)]^{1/(p\cdot n)} \cdot ||f||_{L_\infty(E_n\cap K_n)}.$$

Let K be some compact set with $f(z) < 1$ for $z \in E \setminus K$. Choosing $K = K_n$ we have according to assumption (i)

$$\lim_{n\to\infty}[\#(E_n \cap K_n)]^{1/n} = 1, \quad \text{and trivially} \quad ||f^n||_{L_p(E_n\setminus K_n)} \leq ||f^\alpha||_{L_p(E_n\setminus K_n)}$$

for sufficiently large n. Therefore (b) implies (a). For a proof of (c), define the decreasing function $c(x) = \exp(-x^{\alpha_1})$. Then with a suitable compact set K we have by assumption (iv)

$$\left[||f^\alpha||_{L_p(E_n\setminus K)}\right]^p \leq \sum_{k=0}^{\infty} \sum_{z\in E_n, k\leq|z|<k+1} c(|z|)^{\alpha\cdot p} \leq b(n)\cdot\sum_{k=0}^{\infty} a(k+1)\cdot c(k)^{\alpha\cdot p},$$

where the right–hand sum is convergent for all $\alpha, p > 0$ by assumption on a, c. Thus (c) implies (b) for all $\alpha > 0$. A proof of (d) is similar, here we choose $c(z) = |z|^{-\alpha_1}$, and observe again that the above sum is convergent for $p > 0$ and sufficiently large $\alpha > 0$. □

We still have to show the Rakhmanov conjecture.

Proof of Theorem 1.3(a): The assertion will follow by deriving lower and upper bounds for the extremal constants in terms of the Fekete points E_n^* and the associated Fekete polynomials ϕ_n. Let K be some compact set, with its interior containing S_0. According to assumption (i), we may find some constant C such that $\#(E_n \cap K) \leq C\cdot n$ for all $n \geq 0$. The extremal constant is clearly majorized by the $L_p(E_n)$–norm of the Fekete polynomial ϕ_n of Theorem 1.4(c): by Theorem 1.4(d), there exist constants $c, N > 0$ such that for all $n \geq N$ we have

$$\begin{aligned}||w_nT_{n,p}||_{L_p(E_n)} &\leq ||w_n\phi_n||_{L_p(E_n)} \leq (1+e^{-cn})\cdot||w_n\phi_n||_{L_p(E_n\cap K)} \\ &\leq (1+e^{-cn})^{1/p}\cdot[\#(E_n\cap K)]^{1/p}\cdot||w_n\phi_n||_{L_\infty(E_n)}.\end{aligned}$$

On the other hand,

$$||w_nP||_{L_p(E_n)} \geq ||w_nP||_{L_\infty(E_n^*)} \geq ||w_nP||_{L_1(E_n^*)}/(n+1)$$

for any polynomial P. Thus by definition of $T_{n,p}$ we have

$$(n+1) \cdot ||w_n T_{n,p}||_{L_p(E_n)} \geq \min\{||w_n P||_{L_1(E_n^*)} : P(z) = z^n + \textit{lower powers}\} =: \Gamma_n.$$

Here Γ_n may be computed explicitly in terms of the quantities $f_{y,n}$ of Lemma 2.5: we first notice that every monic polynomial P of degree n may be written as

$$P(z) = \sum_{y \in E_n^*} a_y \cdot f_{y,n}(z), \quad \text{with} \quad a_y \in \mathbb{C}, \quad \sum_{y \in E_n^*} a_y = 1,$$

and therefore

$$||w_n P||_{L_1(E_n^*)} = \sum_{y \in E_n^*} |a_y| \cdot ||w_n f_{y,n}||_{L_\infty(E_n^*)} \geq \min_{y \in E_n^*} ||w_n f_{y,n}||_{L_\infty(E_n^*)} \cdot \sum_{y \in E_n^*} |a_y|.$$

Consequently,

$$\Gamma_n = \min_{y \in E_n^*} ||w_n f_{y,n}||_{L_\infty(E_n^*)} = ||w_n \phi_n||_{L_\infty(E_n^*)} = ||w_n \phi_n||_{L_\infty(E_n)},$$

the latter equalities following from the definition of ϕ_n (see the proof of Theorem 1.4(c)). Since the sequence $([||w_n \phi_n||_{L_\infty(E_n^*)}]^{1/n})_{n \geq 0}$ tends to $\exp(-F_w^\sigma)$ by Theorem 1.4(c), the assertion of Theorem 1.3(a) becomes immediate. $\square$

Proof of Theorem 1.3(b): In the first step of the proof, let us (roughly) estimate the number of zeros of P_n of large modulus: for $R \geq 0$, denote by $m_n(R)$ the number of zeros of P_n counting multiplicities of modulus $\geq R$, and by Δ_R the closed disk centred at zero with radius R. First, according to Theorem 1.1 and Lemma 2.3 we may find some $r > 0$ such that the interior of $\Delta := \Delta_r$ contains both $\operatorname{supp}(\lambda_w^\sigma)$ and all Fekete sets E_n^* for $n \geq 0$. We now construct a polynomial Q_n of degree n via

$$Q_n(z) = \prod_{P_n(\zeta)=0, |\zeta|<4r} (z - \zeta) \cdot \prod_{P_n(\zeta)=0, |\zeta|\geq 4r} (z - \frac{r\zeta}{|\zeta|}),$$

and hence $|Q_n(z)| \leq |P_n(z)| \cdot (2/3)^{m_n(4r)}$ for all $z \in \Delta$ and for all $n \geq 0$. Consequently

$$\begin{aligned} e^{-F_w^\sigma} &= \lim_{n\to\infty} \Big[||w_n \cdot P_n||_{L_p(E_n)}\Big]^{1/n} \geq \limsup_{n\to\infty} \Big[||w_n \cdot P_n||_{L_\infty(E_n \cap \Delta)}\Big]^{1/n} \\ &\geq \limsup_{n\to\infty} (3/2)^{m_n(4r)/n} \cdot \liminf_{n\to\infty} \Big[||w_n \cdot Q_n||_{L_\infty(E_n \cap \Delta)}\Big]^{1/n}. \end{aligned}$$

One easily verifies that $\tilde{\sigma} := \sigma|_\Delta$ and $\tilde{E}_n = E_n \cap \Delta$, satisfy the assumptions of Theorem 1.3(a) (at least for sufficiently large r), with

$$\lambda_w^{\tilde{\sigma}} = \lambda_w^\sigma, \quad F_w^{\tilde{\sigma}} = F_w^\sigma.$$

Therefore,

$$\liminf_{n\to\infty} \Big[||w_n \cdot Q_n||_{L_\infty(E_n \cap \Delta)}\Big]^{1/n} \geq e^{-F_w^\sigma},$$

implying that $m_n(4r)/n$ tends to 0 for $n \to \infty$. This shows that the two sequences $(\nu(P_n))_{n \geq 0}$ and $(\nu(Q_n))_{n \geq 0}$ of normalized zero counting measures will have the same weak* accumulation points (in the sense of Theorem 1.2(i)).

We may therefore suppose without loss of generality that the measures $\mu_n = \nu(P_n)$ are supported in the compact set $K := \Delta_{4r}$. By Helly's theorem, we find a subsequence $(\mu_n)_{n\in\Lambda}$ with weak* limit μ, where μ is a some probability measure with support being a subset of K. Let $\epsilon > 0$. Taking into account that $E_n^* \subset K$ for all $n \geq 0$, we obtain from Theorem 1.4(c) for each $z \in \mathbb{C}$

$$\begin{aligned} 0 = F_w^\sigma + \lim_{n\to\infty} \log\Big[||w_n \cdot P_n||_{L_p(E_n)}\Big]^{1/n} &\geq F_w^\sigma + \limsup_{n\to\infty, n\in\Lambda} \log\Big[||w_n \cdot P_n||_{L_\infty(E_n\cap K)}\Big]^{1/n} \\ \geq -\epsilon + U^\lambda(z) + \limsup_{n\to\infty, n\in\Lambda} \log|P_n(z)|^{1/n} &= -\epsilon + U^\lambda(z) - \liminf_{n\to\infty, n\in\Lambda} U^{\mu_n}(z), \end{aligned}$$

where $\lambda = \lambda_w^\sigma$. According to the lower envelope theorem [SaTo97, Theorem I.6.9], we may conclude that

$$0 \geq -\epsilon + U^\lambda(z) - U^\mu(z), \quad z \in \mathbb{C} \setminus M,$$

where M is a (compact) set of capacity zero. It follows from the continuity of U^λ that $\lambda(M) = 0$, and thus $U^\lambda(z) \leq \epsilon + U^\mu(z)$ for all $z \in \mathbb{C}$ by the principle of domination [SaTo97, Theorem II.3.2]. Thus for $\epsilon \to 0$ we obtain $U^\lambda(z) \leq U^\mu(z)$ for all $z \in \mathbb{C}$.

From the minimum principle, applied to the function $U^\mu - U^\lambda$ being superharmonic in $\mathbb{C} \setminus \mathrm{supp}(\lambda)$, we may conclude that $U^\mu = U^\lambda$ in the unbounded connected component Ω of $\overline{\mathbb{C}} \setminus \mathrm{supp}(\lambda)$. If now the two–dimensional Lebesgue measure of the polynomial convex hull of $\mathrm{supp}(\lambda)$ (i.e., the complement of Ω) is zero, then $\mu = \lambda$ for any weak* limit of $(\mu_n)_{n\geq 0}$ by the Unicity Theorem [SaTo97, Corollary II.2.2], and hence $\mu_n \xrightarrow{*} \lambda$. $\square$

If the two–dimensional Lebesgue measure of the polynomial convex hull S of $\mathrm{supp}(\lambda)$ is greater than zero, then we may conclude from [SaTo97, Theorem II.4.1] that the measure of balayage $\hat{\mu}$ of any weak* limit of $(\nu(P_n))_{n\geq 0}$ from $\mathrm{Int}(S)$ onto ∂S will coincide with the corresponding measure of balayage of λ_w^σ. In all cases, we know from the lower envelope theorem that

$$\limsup_{n\to\infty} |P_n(z)|^{1/n} = \exp(-U^{\lambda_w^\sigma}(z))$$

quasi everywhere in Ω (i.e., up to a set of capacity zero), and $|P_n|^{1/n} \to \exp(-U^{\lambda_w^\sigma})$ uniformly in compact subsets of (asymptotically) zero–free regions. For instance, taking into account Remark 1.5(e), we may conclude that $(|T_{n,p}|^{1/n})_{n\geq 0}$ tends to $\exp(-U^{\lambda_w^\sigma})$ uniformly in compact subsets of the complement of the convex hull of $S_0 = \{z \in E : U^{\lambda_w^\sigma}(z) + Q(z) \leq F_w^\sigma\}$.

3 Conclusions

We have established nth root asymptotics for extremal polynomials (1) with respect to some discrete L_p–norm, generalizing work of Rakhmanov, Dragnev, Saff, Damelin, Kuijlaars, and Van Assche, and thereby proving two conjectures of Rakhmanov. As a main tool we investigated properties of weighted Fekete points formed from discrete sets.

We conclude by commenting on possible generalizations of the results presented in this paper: In all assertions, the condition of $\mathrm{supp}(\sigma)$ being connected may be replaced by the weaker requirement that $\mathrm{supp}(\lambda_w^\sigma)$ has a non–empty intersection with $\mathrm{supp}(\sigma - \lambda_w^\sigma)$. However, one may not allow arbitrary supports as it becomes clear from [DrSa97, Example 2.4].

Also, following [KuVA98, DaSa98] we may allow for a weight w being non–negative and continuous on E, provided that $w_n = w^n$, and provided that $\sigma(\{z \in E : w(z) > 0\}) > 1$.

Finally, condition (5) of Rakhmanov is still quite sensitive with respect to single elements of E_n: for instance, (5) fails to hold if there exist distinct $x_n, x_n' \in E_n \cap \mathrm{supp}(\lambda_w^\sigma)$ with $\liminf \exp(n^3) \cdot |x_n - x_n'| < \infty$. It seems to be therefore interesting to find a separation condition replacing (5) where a certain undesired part of E_n may be neglected. By slightly modifying the above proofs one shows that in fact (5) may be replaced by the following: there exist uniformly bounded sets $\widetilde{E}_n \subset E_n$ with

$$\sigma_n := \nu_n(\widetilde{E}_n) \xrightarrow{*} \widetilde{\sigma} \quad \text{and} \quad \lim_{n\to\infty} I_n^*(\sigma_n) = I(\widetilde{\sigma}), \tag{22}$$

where $\widetilde{\sigma} \geq \lambda_w^\sigma$, and $\mathrm{supp}(\lambda_w^\sigma) \cap \mathrm{supp}(\widetilde{\sigma} - \lambda_w^\sigma)$ is nonempty.

Acknowledgements: The author is grateful to Prof. Vilmos Totik for providing a proof of Lemma 2.2(b). I also would like to thank Prof. Ed Saff for helpful discussions.

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