The Condition Number of real Vandermonde, Krylov and positive definite Hankel matrices

by

Bernhard Beckermann [1]

Abstract

We show that the Euclidean condition number of any positive definite Hankel matrix of order $n \geq 3$ may be bounded from below by $\gamma^{n-1}/(16n)$ with $\gamma = \exp(4 \cdot Catalan/\pi) \approx 3.210$, and that this bound may be improved at most by a factor $8\gamma n$. Similar estimates are given for the class of real Vandermonde matrices, the class of row scaled real Vandermonde matrices, and the class of Krylov matrices with Hermitian argument. Improved bounds are derived for the case where the abscissa or eigenvalues are included in a given real interval. Our findings confirm that all such matrices — including for instance the famous Hilbert matrix — are ill–conditioned already for "moderate" order.

As application, we describe implications of our results for the numerical condition of various tasks in Numerical Analysis such as polynomial and rational interpolation at real nodes, determination of real roots of polynomials, computation of coefficients of orthogonal polynomials, or the iterative solution of linear Hermitian systems of equations.

Key words: Condition number, Vandermonde matrix, Krylov matrix, Hankel matrix, Hilbert matrix.
Subject Classifications: AMS(MOS): 15A12, 65F35; CR: G1.3, G1.6.

1 Introduction

In the sequel of the paper, we will consider (square or rectangular) matrices A having $n+1$ columns and m rows, with $m > n$. We denote (unless explicitly stated) by $||\cdot||$ the Euclidean vector and matrix norm. Then the (Euclidean) condition number $\kappa(A)$ of A is defined to be the square root of the ratio of the largest and the smallest eigenvalue of the Hermitian positive definite matrix $A^H \cdot A$, with A^H denoting the Hermitian counterpart of A. It is well–known that the condition number serves as a measure for the linearly independence of the columns of A [GoVL93, p.80] since it gives a measure for the relative distance to the set of matrices which do not have full rank

$$\frac{1}{\kappa(A)} = \min\Big\{\frac{||A-B||}{||A||} : rank(B) < n\Big\}.$$

Also, according to [GoVL93, Subsection 2.7.2] we have

$$\kappa(A) = \max_{y\neq 0} \frac{||y||}{||A\cdot y||} \cdot \max_{x\neq 0} \frac{||A\cdot x||}{||x||}. \tag{1}$$

Writing $x(b)$ for the solution of the system of linear equations $Ax = b$ in the square case $m = n+1$, it follows from (1) that $\kappa(A)$ is the maximum factor of magnification of relative errors, if the relative error in the right hand side data is measured by $||\Delta b||/||b||$, and the relative error of the solution by $||x(b+\Delta b) - x(b)||/||x(b)||$. Also, in the case $m > n+1$ we

[1] Laboratoire d'Analyse Numérique et d'Optimisation, UFR IEEA – M3, UST Lille, F-59655 Villeneuve d'Ascq CEDEX, France, e-mail: bbecker@ano.univ-lille1.fr

have a similar interpretation for the sensitivity of the minimal norm solution of the least square problem $\min ||Ax - b||$ with respect to perturbations of the right hand side.

Hankel matrices $H_n := (h_{j+k})_{j,k=0,1,...,n}$ have constant entries along antidiagonals. It is well–known (see for instance the proof of Theorem 2.1) that the set of real positive semidefinite Hankel matrices may be identified with the set of Gram matrices

$$H_n(\mu) = (h_{j+k})_{j,k=0,1,...,n}, \quad h_k = \int x^k \, d\mu(x)$$

where μ is some positive (possibly discrete) Borel measure, with support $supp\,(\mu)$ being some subset of the real line $\mathbb{R}$. Positive definite Hankel matrices are suspected to be very ill–conditioned even for not very large order. Here the perhaps most famous example is the *Hilbert matrix* $\tilde{H}_n = (1/(j+k+1))_{j,k=0,1,...,n}$, with its condition number growing like $(1+\sqrt{2})^{4n}/\sqrt{n} \approx 34^n/\sqrt{n}$ (see [Tod54] and [Wif70, Equation (3.35)]). We will be interested in determining sharp lower and upper bounds for the quantity

$$\Gamma_n(I)^2 := \inf\{\kappa(H_n(\mu)) : supp\,(\mu) \subset I\},$$

with the interval $I = [a, b]$, $-\infty \leq a < b \leq +\infty$. Recently, Tyrtyshnikov showed that $\Gamma_n(\mathbb{R})$ is at least exponentially increasing ([Tyr94b], the reasoning in [Tyr94a] is incomplete). By refining [Bec96, Corollary 5.14] we will here give lower and upper bounds for any n which only differ by n times some constant, showing that already positive definite Hankel matrices of moderate size are very ill–conditioned. In particular (see Example 3.3 below), it follows that the Hilbert matrix is still not the "most ill–conditioned" positive definite Hankel matrix since here the value of $\Gamma_n([0, 1])$ is approximately attained.

Let us mention some other structured matrices occurring in different fields of Numerical Analysis which are closely connected to positive definite Hankel matrices. Square or rectangular real *Vandermonde matrices*

$$V_n = V_n(z_1, ..., z_m) = \begin{pmatrix} 1 & z_1 & z_1^2 & \cdots & z_1^n \\ 1 & z_2 & z_2^2 & \cdots & z_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & z_m & z_m^2 & \cdots & z_m^n \end{pmatrix}.$$

occur beside polynomial interpolation also in other applications, e.g., the determination of Christoffel numbers for a Gaussian quadrature rule, the interpolatory approximation of linear functionals, or the discretization of differential equations. Their condition number for real abscissa $z_1, .., z_m$ has been investigated by Gautschi in a number of papers [Gau75a, Gau75b, Gau90, GaIn88], it is a measure, e.g., for the magnification of relative errors in polynomial interpolation. Sometimes it is more appropriate to allow for an additional weight function, i.e., for a preconditioning by multiplication on the left by some diagonal matrix. Notice that $V_n^H \cdot V_n$ is a positive definite Hankel matrix of order $n+1$, and a similar result holds if we add some row scaling. Thus the quantities

$$\Gamma_n^V(I) := \inf\{\kappa(V_n(z_0, ..., z_n)) : z_0, ..., z_n \in I\},$$
$$\Gamma_n^D(I) := \inf\{\kappa(D \cdot V_n(z_1, ..., z_m)) : z_1, ..., z_m \in I, \, D \text{ diagonal }\}$$

will be very much related to the quantity $\Gamma_n(I)$ introduced above. It is well-known that real Vandermonde matrices are ill-conditioned. Wilkinson studied three different configurations of nodes [Wil65, p.372f] and obtained $\kappa(V_{19}(1/20, 2/20, .., 20/20)) \geq 2^{42}$, $\kappa(V_{19}(-9/10, -8/10, .., 10/10)) \geq 2^{24}$, and $\kappa(V_{19}(2^{-1}, 2^{-2}, .., 2^{-20})) \geq 2^{171}$. Taylor [Tay78, Section 4], Gautschi and Inglese

[GaIn88, Theorem 2.1 and Theorem 3.1] and Tyrtyshnikov [Tyr94a, Theorem 4.1] proposed lower bounds for $\Gamma_n^V(\mathbb{R})$ of the form γ^n, with the best estimate $\gamma \geq 2$. We will show in Theorem 4.1 below that the optimal value for γ is given by $1+\sqrt{2}$.

For solving a large linear system $B \cdot x = c$ with the square matrix B of order m being sparse, one usually prefers iterative methods such as the method of conjugate gradients, or a Lanczos–type method (see, e.g., [GoOL89, BrSa93]). Also, a Lanczos–type method may be applied in order to obtain the approximate spectrum of B (see, e.g., [GoVL93, Chapter 9]). Here, with a suitable vector b, one successively determines an orthogonal basis of the so–called nth *Krylov space* spanned by the columns of the Krylov matrix

$$K_n(B;b) := (b, B\cdot b, B^2\cdot b, ..., B^n \cdot b).$$

Again, the quality of the generating system of Krylov vectors may be measured by studying the condition number of $K_n(B;b)$. Depending on the spectrum $\sigma(B)$ of B, one observes quite often that the condition number grows at least exponentially in n, see for instance [Wil65, p.374] or the numerical results reported in [Car94, Chapitre 4.6]. Here we will discuss the case of Hermitian matrices B (and thus $\sigma(B) \subset \mathbb{R}$), and consider the quantity

$$\Gamma_n^K(I) := \inf\{\kappa(K_n(B;b)) : \; B \text{ of order } m > n \text{ Hermitian}, \, \sigma(B) \subset I, \, b \in \mathbb{C}^m \}.$$

The paper is organized as follows: The exact relationship between $\Gamma_n^V, \Gamma_n^D, \Gamma_n^K$ and Γ_n is given in Theorem 2.1 below. In the sequel of Section 2 we relate the problem of determining $\Gamma_n(I)$ to a constrained weighted Chebyshev approximation problem, which enables us to describe approximately $\Gamma_n(I)$ in terms of some products. Lower and upper bounds for $\Gamma_n(I)$ of the form $f(n)\cdot\Gamma(I)^n$ with at most polynomial functions f are obtained in Section 3. In Section 4 we study the quantity Γ_n^V related to square real Vandermonde matrices. Applications to the problems of determining real polynomial roots, and of rational interpolation at real nodes, respectively, are described in Section 5 and Section 6. The closing section gives a summary along with topics for future research.

2 Estimates in terms of a polynomial extremal problem

Theorem 2.1 *There holds for each interval $I \subset \mathbb{R}$ and for all $n \geq 0$*

$$\Gamma_n(\mathbb{R}) \leq \Gamma_n(I) = \Gamma_n^K(I) = \Gamma_n^D(I) \leq \Gamma_n^V(I).$$

Proof: Let us first show that $\Gamma_n(I) \geq \Gamma_n^D(I)$. First of all, if H_n is some Hankel matrix of order $n+1$ which is positive semidefinite but not positive definite then H_n is singular and thus $\kappa(H_n) = \infty$. Let therefore $H_n = (h_{j+k})_{j,k=0,..,n}$ be some positive definite real Hankel matrix. Then, we may choose h_{2n+1} and h_{2n+2} such that the extended Hankel matrix H_{n+1} is also positive semidefinite (e.g., take $h_{2n+1} = 0$ and h_{2n+2} being sufficiently large). Thus the linear functional c acting on the space of polynomials of degree at most $2n+2$ by $c(x^j) = h_j$ is positive (see, e.g., [NiSo88, Chapter II.5,p.49ff]), and we may associate orthonormal polynomials $p_0, .., p_{n+1}$. Furthermore, we have a Gaussian quadrature rule

$$h_j = \int x^j \, d\mu_n(x), \quad j = 0, 1, ..., 2n+1, \qquad \int f(x)\, d\mu_n(x) := \sum_{k=0}^n d_k^2 \cdot f(z_k),$$

with real numbers d_k, and with $z_0, ..., z_n$ being the real and distinct zeros of p_{n+1}. Consequently, we obtain a representation $H_n = H_n(\mu_n)$ as a Gram matrix with respect to some positive measure supported on the real line, as claimed in the introduction. Since

$$H_n = H_n(\mu_n) = W_n^T \cdot W_n, \quad W_n = diag\,(d_0, ..., d_n) \cdot V_n(z_0, ..., z_n),$$

it follows that $\kappa(H_n) = \kappa(W_n)^2$, and thus $\kappa(H_n)^{1/2} \geq \Gamma_n(\mathbb{R}) \geq \Gamma_n^D(\mathbb{R})$. If we know in addition that H_n is some Gram matrix $H_n(\mu)$ with $supp\,(\mu) \subset I$, then all zeros of q_{n+1} lie in I, and hence $\Gamma_n(I) \geq \Gamma_n^D(I)$.

Let now B be some Hermitian matrix of order $m > n$, and $b \in \mathbb{C}^m$. Furthermore, denote by $z_1, ..., z_m \in I$ the eigenvalues of B, and by U the (unitary) matrix consisting of the corresponding basis of eigenvectors. Then $U^H \cdot B \cdot U = diag\,(z_1, ..., z_m)$, and thus

$$K_n(B;b)^H \cdot K_n(B;b) = (b^H B^{j+k} b)_{j,k=0,...,n} = H_n(\nu), \quad \int f(x)\, d\nu(x) := \sum_{j=1}^{m} d_j^2 \cdot f(z_j),$$

where d_j is the absolute value of the jth component of $U^H \cdot b$, $j = 1, ..., m$. Consequently, $\kappa(K_n(B;b))^2 = \kappa(H_n(\nu))$, showing that $\Gamma_n^K(I) \geq \Gamma_n(I)$.

The inequality $\Gamma_n^D(I) \geq \Gamma_n^K(I)$ now follows from the fact that any row scaled Vandermonde matrix $diag\,(d_1, ..., d_m) \cdot V_n(z_1, ..., z_m)$ may be rewritten as a Krylov matrix $K_n(B;b)$ with $B = diag\,(z_1, ..., z_m)$ and $b = (d_1, ..., d_m)^T$. Finally, the inequality $\Gamma_n^D(I) \leq \Gamma_n^V(I)$ is trivial since we may choose as diagonal matrix the identity. □

As a complement of Theorem 2.1, notice the trivial inclusion properties

$$\Gamma_n(I) \geq \Gamma_n(J) \text{ for } I \subset J, \quad \Gamma_n(\mathbb{R}) = \lim_{b\to\infty} \Gamma_n([-b,b]), \quad \Gamma_n([0,+\infty)) = \lim_{b\to\infty} \Gamma_n([0,b]). \tag{2}$$

Similar results hold for Γ_n^V. We will see in Corollary 3.2 and Theorem 3.6 that $\Gamma_n(\mathbb{R})$ is much smaller than $\Gamma_n([-1,1])$. This is no longer true for Vandermonde matrices, as shown in Theorem 4.1.

The aim of the sequel of this section is to give lower and upper bounds for $\Gamma_n(I)$ for finite intervals $I = [a,b]$ in terms of solutions of a constrained weighted Chebyshev approximation problem, and to provide bounds for the solution of the latter problem. More explicit bounds will be given in Section 3.

Denote by $\mathbb{P}_n$ the set of polynomials with complex coefficients of degree at most n. For compact sets $H, G \subset \mathbb{C}$ and for a weight function q continuous and positive on G we introduce the quantities

$$\Delta_n(H;G,q) := \max_{z\in H} \Delta_n(z;G,q), \quad \Delta_n(z;G,q) := \max_{P\in\mathbb{P}_n} \frac{|P(z)|}{\max_{x\in G} \frac{|P(x)|}{\sqrt{q(x)}}}. \tag{3}$$

Recall that the set $\{P \in \mathbb{P}_n : P(z) = 1\}$ verifies the Haar condition on $I = [a,b]$ for any $z \notin (a,b)$, and thus up to normalization there exists a unique polynomial where the maximum in $\Delta_n(z;I,q)$ or $\Delta_n(H;I,q)$ is attained. A such polynomial with normalization $\max_{x\in I} |P(x)|/\sqrt{q(x)} = 1$ will be referred to as *extremal polynomial*. Notice also that, by the maximum principle for analytic functions, $\Delta_n(z;I,q) = \Delta_n(H;I,q)$ for some z on the (outer) boundary of H. In the sequel we will be only interested in the case where q is a polynomial of degree $2n$. Here, according to Bernstein and Szegö (see, e.g., [MMR94, Theorem 1.2.12, p.394] or [Sze67, Section 2.6]), the extremal polynomial is known for real $z \notin I$. Moreover, as seen in Lemma 2.4 below, also for complex z we may give a good approximation of $\Delta_n(z;q,I)$. We will be manly interested in the case $H = \mathbb{D}$, the closed unit disk.

Theorem 2.2 *Let $I=[a,b]$ with $a<b$, and define*

$$q_n^L(z) := 1+z^{2n}, \quad q_n^R(z) := 1+z^2+z^4+...+z^{2n}.$$

Then $\Delta_n(\mathbb{D};I,q_n^L)/\sqrt{2n+2} \le \Gamma_n(I) \le (n+1)\cdot\Delta_n(\mathbb{D};I,q_n^R)$ for all $n\ge 1$. Moreover, for the particular cases $a\ge 0$ and $a=-b$ we have the sharper upper bound $\Gamma_n(I)\le \Delta_n(\mathbb{D};I,q_n^R)$.

For a proof of Theorem 2.2 we require some further characterizations of the constrained weighted Chebyshev approximation problem (3) for the particular cases $a\ge 0$ and $a=-b$. In what follows, we write shorter $\vec{P}=(a_0,...,a_n)^T$ for a polynomial $P(z)=\sum_{j=0}^n a_j z^j$ of degree n. Also, we denote by $||\cdot||_1$ the 1–Hölder norm. Notice that for any polynomial P with r nontrivial coefficients there holds

$$||\vec{P}||^2 = \frac{1}{2\pi}\int |P(e^{it})|^2\,dt \le \max_{|z|=1}|P(z)|^2 \le ||\vec{P}||_1^2 \le r\cdot||\vec{P}||^2. \tag{4}$$

Lemma 2.3 *Let $I=[a,b]$, and let q be an even polynomial of degree $\le 2n$ being positive on I. Furthermore, define $\zeta\in\mathbb{D}$ by $\Delta_n(\mathbb{D};I,q)=\Delta_n(\zeta;I,q)$. Then $\zeta=-1$ in the case $a\ge 0$, and $\zeta=i$ if $a=-b$.*

Also, in these two cases there exists a set $H=\{z_0,...,z_n\}$ with $\Delta_n(\mathbb{D};I,q)=\Delta_n(\mathbb{D};H,q)=\Delta_n(\zeta;H,q)$. Moreover, for the corresponding Lagrange polynomials $\ell_0,...,\ell_n$ there holds

$$\max_{z\in\mathbb{D}}\sum_{j=0}^n d_j\cdot|\ell_j(z)|^d = \sum_{j=0}^n d_j\cdot|\ell_j(\zeta)|^d \tag{5}$$

for all $d,d_0,...,d_n\ge 0$ in the case $a\ge 0$, and also in the case $a=-b$ if we impose the additional symmetry condition $d_j=d_{n-j}$ for $j=0,..,n$.

Proof: Notice first that for any $H\subset I$ and for all $z\in\mathbb{C}$ there holds trivially $\Delta_n(z;H,q)\ge\Delta_n(z;I,q)$. In the sequel of the proof we will require some additional characterizations of extremal solutions P of $\Delta_n(\zeta;I,q)$. Here we will follow Freund and Ruscheweyh [FrRu86, Section 3] who treated the case $q=1$. Consider the set of extremal points

$$H := \{z\in\mathbb{R} : \frac{|P(z)|}{\sqrt{q(z)}} = \max_{x\in I}\frac{|P(x)|}{\sqrt{q(x)}}\} = \{z\in\mathbb{R} : \frac{|P(z)|}{\sqrt{q(z)}} = 1\}.$$

Notice that $\rho(x):=q(x)-|P(x)|^2$, restricted on the real axis, is a real polynomial of degree $\le 2n$. On counting the number of zeros of ρ and ρ' one verifies (compare [FrRu86, Proposition (3.3)]) that H contains exactly $n+1$ elements $z_0,...,z_n\in I$, with $b=z_n>z_{n-1}>...>z_1>z_0=a$. Moreover, denoting by $\ell_0,...,\ell_n$ the corresponding Lagrange polynomials one shows as in [FrRu86, Theorem (3.5)] that there exists a $\sigma\in\mathbb{C}$ of absolute value 1 with

$$|P(z_j)|\cdot|\ell_j(\zeta)| = \sigma\cdot P(z_j)\cdot\ell_j(\zeta), \quad j=0,1,...,n.$$

Therefore

$$\sigma\cdot P(\zeta) = \sigma\cdot\sum_{j=0}^n P(z_j)\ell_j(\zeta) = \sum_{j=0}^n\sqrt{q(z_j)}|\ell_j(\zeta)| = \Delta_n(\zeta;H,q),$$

and we may conclude that $\Delta_n(\zeta;I,q)=|P(\zeta)|=\sigma\cdot P(\zeta)=\Delta_n(\zeta;H,q)$.

Consider now the case $0 \le a < b$ and $\zeta = -1$. Then all extremal points $z_0, ..., z_n$ are nonnegative, and thus $||\vec{\ell_j}||_1 = |\ell_j(-1)|$ for $j = 0, 1, ..., n$. We may conclude from (4) that $|\ell_j(z)| \le |\ell_j(\zeta)|$ for all j and for all $|z| = 1$, showing (5). In particular,

$$\Delta_n(z; I, q) \le \Delta_n(z; H, q) = \sum_{j=0}^{n} \sqrt{q(z_j)}|\ell_j(z)| \le \sum_{j=0}^{n} \sqrt{q(z_j)}||\vec{\ell_j}||_1 = \Delta_n(\zeta; H, q) = \Delta_n(\zeta; I, q)$$

for any $|z| \le 1$, and therefore $\Delta_n(\mathbb{D}; I, q) = \Delta_n(\mathbb{D}; H, q) = \Delta_n(\zeta; H, q) = \Delta_n(\zeta; I, q)$ as claimed in the assertion of Lemma 2.3.

The case of a symmetric interval $I = [-b, b]$ is slightly more involved, here we choose $\zeta = i$ and P, H as above. Recall that $q(z) = q(-z)$ for all $z \in I$. Thus for the polynomial $Q(z) := (\sigma \cdot P(z) + \overline{\sigma} \cdot \overline{P(-\overline{z})})/2$ we obtain $Q(\zeta) = |P(\zeta)| = \sigma \cdot P(\zeta)$, and $\max_{x \in I} |Q(x)|/\sqrt{q(x)} \le 1$. By definition and uniqueness (up to a constant) of the extremal polynomial P it follows that $Q(z) = \sigma \cdot P(z)$. In particular, the set H has to be symmetric, that is, $z_j = -z_{n-j}$ for $j = 0, 1, ..., n$. Hence, for $|z| = 1$, $d > 0$ and for $j \ne n/2$

$$\begin{aligned} |\ell_j(z)|^d + |\ell_{n-j}(z)|^d &= \frac{|z - z_j|^d + |z + z_j|^d}{(2|z_j|)^d} \cdot \prod_{k \notin \{j, n-j\}} \frac{|z^2 - z_j^2|^{d/2}}{|z_k^2 - z_j^2|^{d/2}} \\ &\le \frac{|i - z_j|^d + |i + z_j|^d}{(2|z_j|)^d} \cdot \prod_{k \notin \{j, n-j\}} \frac{(1 + z_j^2)^{d/2}}{|z_k^2 - z_j^2|^{d/2}} \le |\ell_j(i)|^d + |\ell_{n-j}(i)|^d, \end{aligned}$$

whereas for even n there holds $|\ell_{n/2}(z)|^d \le |\ell_{n/2}(i)|^d$. This shows (5). By taking into account the symmetry of q, we may conclude that $\Delta_n(z; I, q) \le \Delta_n(z; H, q) \le \Delta_n(\zeta; H, q) = \Delta_n(\zeta; I, q)$. □

Proof of Theorem 2.2: In order to show the lower bound, let $z_1, ..., z_m \in I$, $m > n$, and let $D = diag(d_1, ..., d_m)$. Writing $W_n := D \cdot V_n(z_1, ..., z_m)$, we have

$$\kappa(W_n) = \max_{P \in \mathbb{P}_n} \frac{||\vec{P}||}{||(d_j \cdot P(z_j))_{j=1,...,m}||} \cdot ||W_n||.$$

Here $||W_n|| \ge ||W_n'||$, with W_n' obtained by taking the first and the last column of W_n. Therefore

$$||W_n||^2 \ge ||(W_n')^T W_n'|| \ge \frac{1}{2} trace[(W_n')^T W_n'] = \frac{1}{2} ||(d_j \cdot \sqrt{q_n^L(z_j)})_{j=1,...,m}||^2.$$

Let $P \in \mathbb{P}_n$ denote an extremal polynomial of $\Delta := \Delta_n(\mathbb{D}; I, q_n^L)$. According to its normalization we have $|P(z_j)| \le \sqrt{q_n^L(z_j)}$ for $j = 0, 1, ..., n$, and therefore by (4)

$$\kappa(W_n) \ge \frac{1}{\sqrt{2n+2}} \cdot \frac{\max_{|z|=1} |P(z)|}{||(d_j \cdot P(z_j))_{j=1,...,m}||} \cdot ||(d_j \cdot \sqrt{q_n^L(z_j)})_{j=1,...,m}||^2 \ge \frac{1}{\sqrt{2n+2}} \cdot \max_{|z|=1} |P(z)|,$$

the latter being equal to $\Delta/\sqrt{2n+2}$ by definition of P. Consequently, $\Gamma_n(I) = \Gamma_n^D(I) \ge \Delta/\sqrt{2n+2}$ by Theorem 2.1.

The upper bound for $\Gamma_n(I)$ claimed in Theorem 2.2 will follow by estimating $\kappa(H_n(\mu_n))$ for some discrete measure μ_n with mass points $z_0, ..., z_n \in I$. In the case of a general interval I we choose as mass points the set of *weighted Fekete points*, namely, the point $(z_0, z_1, ..., z_n)$ where

$$F(u_0, ..., u_n) := \prod_{\substack{j,k=0 \\ j \ne k}}^{n} \frac{(u_j - u_k)^2}{q_n^R(u_j)}$$

takes its maximum in I^{n+1}. Denoting by $\ell_0, ..., \ell_n$ the corresponding Lagrange polynomials, we have by definition for all $j = 0, 1, ..., n$ and for all $z \in I$

$$1 \geq \frac{F(z_0, ..., z_{j-1}, z, z_{j+1}, ..., z_n)}{F(z_0, ..., z_{j-1}, z_j, z_{j+1}, ..., z_n)} = \ell_j(z)^2 \frac{q_n^R(z_j)}{q_n^R(z)}.$$

We now consider the discrete measure μ_n defined by

$$\int f(x)\, d\mu_n(x) := \sum_{j=0}^{n} \frac{f(z_j)}{q_n^R(z_j)}.$$

Then $||H_n(\mu_n)|| \leq trace(H_n(\mu_n)) = n + 1$. Moreover, for any polynomial $P \in \mathbb{P}_n$

$$\max_{x\in I} \frac{|P(x)|^2}{q_n^R(x)} \leq \max_{x\in I} \frac{(\sum_{j=0}^n |P(z_j)| \cdot |\ell_j(x)|)^2}{q_n^R(x)}$$
$$\leq \int |P(t)|^2\, d\mu_n(t) \cdot \max_{x\in I} \frac{\sum_{j=0}^n q_n^R(z_j) \cdot |\ell_j(x)|^2}{q_n^R(x)} \leq (n+1) \cdot \int |P(t)|^2\, d\mu_n(t),$$

and therefore

$$||H_n(\mu_n)^{-1}|| = \max_{P\in\mathbb{P}_n} \frac{||\vec{P}||^2}{\int |P(t)|^2\, d\mu_n(t)} \leq (n+1) \cdot \max_{P\in\mathbb{P}_n} \frac{||\vec{P}||^2}{\max_{x\in I} \frac{|P(x)|^2}{q_n^R(x)}},$$

with the right hand side being $\leq (n+1) \cdot \Delta_n(\mathbb{D}; I, q_n^R)^2$ by (4), as required for the upper bound of $\Gamma_n(I)$ for a general interval I.

In the cases $a \geq 0$ or $a = -b$ we choose the quantities ζ, $z_0, .., z_n$ as in Lemma 2.3 with $q = q_n^R$. Consider the discrete measure μ_n defined by

$$\int f(x)\, d\mu_n(x) := \sum_{j=0}^{n} \frac{|\ell_j(\zeta)|}{\sqrt{q_n^R(z_j)}} \cdot f(z_j).$$

Then

$$||H_n(\mu_n)|| \leq trace(H_n(\mu_n)) = \sum_{j=0}^{n} \frac{|\ell_j(\zeta)|}{\sqrt{q_n^R(z_j)}} \cdot q_n^R(z_j) = \Delta(\mathbb{D}; I, q_n^R).$$

Also, for any $z \in \mathbb{C}$ we have by the Cauchy–Schwarz inequality

$$\max_{P\in\mathbb{P}_n} \frac{|P(z)|^2}{\int |P(x)|^2\, d\mu_n(x)} = \max_{P\in\mathbb{P}_n} \frac{|\sum_{j=0}^n P(z_j)\ell_j(z)|^2}{|\sum_{j=0}^n \frac{|\ell_j(\zeta)|}{\sqrt{q_n^R(z_j)}} \cdot |P(z_j)|^2} = \sum_{j=0}^{n} |\ell_j(z)|^2 \cdot \frac{\sqrt{q_n^R(z_j)}}{|\ell_j(\zeta)|}.$$

Taking into account (4) and (5), it follows that

$$||H_n(\mu_n)^{-1}|| \leq \max_{|z|=1} \sum_{j=0}^{n} |\ell_j(z)|^2 \cdot \frac{\sqrt{q_n^R(z_j)}}{|\ell_j(\zeta)|} = \sum_{j=0}^{n} |\ell_j(\zeta)|^2 \cdot \frac{\sqrt{q_n^R(z_j)}}{|\ell_j(\zeta)|} = \Delta_n(\mathbb{D}; I, q_n^R),$$

and thus $\Gamma_n(I) \leq \kappa(H_n(\mu_n))^{1/2} \leq \Delta_n(\mathbb{D}; I, q_n^R)$. □

By using classical techniques, we will now solve approximately the constrained Chebyshev approximation problems occurring in Theorem 2.2 which will enable us to give more explicit bounds in Theorem 2.5 below. In what follows, for a fixed interval $I = [a, b]$ we denote by Φ the Riemann function mapping the exterior of I conformally on the exterior of the closed unit disk

$\mathbb{D}$, and by Ψ its inverse. Notice both functions have a continuous extension to the boundary, and that

$$\Psi(w) = \frac{a+b}{2} + \frac{b-a}{2} \cdot \frac{1}{2}(w + \frac{1}{w}),$$

a shifted *Joukowski function.*

Lemma 2.4 *Let $I = [a, b]$, and let q be an even polynomial of degree $\leq 2n$ being positive on I. We consider the (unique) factorisation $q(\Psi(w)) = Q(w) \cdot Q(1/w)$ where Q is a real polynomial of degree $\leq 2n$ veryfing $Q(0) > 0$ and $Q(w) \neq 0$ for $w \in \mathbb{D}$. Then for all $z \in \mathbb{C}$*

$$(|w^n \cdot Q(1/w) + \frac{q(z)}{w^n \cdot Q(1/w)}|)/2 \leq \Delta_n(z; I, q) \leq |w^n \cdot Q(1/w)|, \quad w = \Phi(z).$$

Proof: The upper bound for $\Delta_n(z; I, q)$ usually is referred to as a weighted Bernstein–Walsh inequality [SaTo97, Theorem III.2.1, p.153]: for any polynomial $P \in \mathbb{P}_n$, the function $w^n \cdot P(\Psi(w))$ is a polynomial in w of degree at most $2n$. Also, $w^{2n} \cdot Q(1/w)$ is a polynomial in w of degree $2n$, and therefore $f(w) := P(\Psi(w))^2/Q(1/w)^2$ is analytic in some neighborhood of the complement of $\mathbb{D}$, including infinity. From the maximum principle for analytic functions we may conclude that

$$|f(\zeta)| \leq \max_{w=1} |f(w)| = \max_{w=1} \frac{|P(\Psi(w))|^2}{|Q(w)| \cdot |Q(1/w)|} = \max_{w=1} \frac{|P(\Psi(w))|^2}{q(\Psi(w))} = \max_{x \in I} \frac{|P(x)|^2}{q(x)}$$

for all $|\zeta| \geq 1$, showing that $\Delta_n(z; I, q) \leq |w^n \cdot Q(1/w)|$ for all $z \in \mathbb{C}$. For the other estimate, following Szegö and Bernstein we consider T defined by

$$T(\Psi(w)) := \frac{1}{2} \cdot (w^n Q(1/w) + w^{-n} Q(w)). \tag{6}$$

T coincides with the weighted Chebyshev polynomial of degree n on I, (see, e.g., [MMR94, Theorem 1.2.12, p.394] or [Sze67, Section 2.6]), in particular there holds $\max_{x \in I} |T(x)|/\sqrt{q(x)} = 1$. Thus $\Delta_n(z; I, q) \geq |T(z)|$ for all $z \in \mathbb{C}$, as claimed in the assertion. □

It is well-known that in the case $z \in \mathbb{R} \setminus I$ the lower bound of Lemma 2.4 is attained. On the other hand, it seems that there are no general results for non–real z in the case where q has exact degree $2n$ (the case $q = 1$ and imaginary z is explicitly solved in [FrRu86, Theorem (2.6)]).

The results of Theorem 2.2, Lemma 2.3, and Lemma 2.4 may now be combined in order to give explicit bounds for arbitrary intervals I. For simplicity, in the following Theorem we will only discuss the symmetric cases $a = 0$ and $a = -b$, and refer the reader to Remark 3.4 and Remark 3.5 for the general case.

Theorem 2.5 *Let $b > 0$, and denote as before by Φ the Riemann map of $[-b, b]$. Furthermore, let*

$$B_n^R(b) := \prod_{j=1}^{n} \frac{|1 + |\Phi(i)|^2 \cdot \Phi(e^{\pi i j/(n+1)})^2|}{|\Phi(e^{\pi i j/(n+1)})| \cdot |\Phi(i)| \cdot (2/b)},$$

and $B_n^L(b) := B_{2n-1}^R(b)/B_{n-1}^R(b)$ for $n \geq 1$, where $B_0^L(b) := B_0^R(b) := 1$. Then there holds for $n \geq 1$

$$\frac{B_n^L(b)}{2\sqrt{2n+2}} \leq \Gamma_n([-b, b]) \leq B_n^R(b), \quad \frac{B_{2n}^L(\sqrt{b})}{2\sqrt{2n+2}} \leq \Gamma_n([0, b]) \leq B_{2n}^R(\sqrt{b}).$$

Proof: For studying the interval $I = [-b, b]$, we first determine explicitly the canonical factorization of Lemma 2.4 for a polynomial q with symmetric and conjugate zeros (as it is the case for the polynomials q_n^L and q_n^R)

$$q(z) := \prod_{j=1}^{2n} (z - z_j), \quad \overline{z_j} = z_{2n+1-j} = -z_{n+1-j} \text{ for } j = 1, ..., n.$$

Here

$$q(\Psi(w)) = (b/2)^{2n} w^{2n} + \text{ lower powers } = (b/2)^{2n} \cdot w^{-2n} \cdot \prod_{j=1}^{2n} (w - \Phi(z_j)) \cdot (w - \frac{1}{\Phi(z_j)})$$

$$= Q(w) \cdot Q(1/w), \quad Q(w) = (b/2)^n \cdot \prod_{j=1}^{2n} \frac{(w - \Phi(z_j))}{\sqrt{|\Phi(z_j)|}}.$$

Taking into account the symmetries $\Phi(z) = -\Phi(-z) = \overline{\Phi(\overline{z})}$ for $z \in \mathbb{C}$, we obtain for $w \in \mathbb{C}$ with $w = i \cdot |w|$

$$\Theta_n(w) := w^n \cdot Q(1/w) = \prod_{j=1}^{n} \frac{(1 - w\Phi(z_j))(1 - w\overline{\Phi(z_j)})}{w \cdot (2/b) \cdot |\Phi(z_j)|} = (-i)^n \prod_{j=1}^{n} \frac{|1 + |w|^2 \Phi(z_j)^2|}{|w| \cdot (2/b) \cdot |\Phi(z_j)|}.$$

Consider now the case $q = q_n^R$, having as zeros the $2n + 2$th roots of unity except ± 1. Then $i^n \cdot \Theta_n(\Phi(i)) = B_n^R(b)$, and a combination of Theorem 2.2 with Lemma 2.3 and Lemma 2.4 leads to the upper bound for $\Gamma_n([-b, b])$ as claimed in Theorem 2.5.

We now consider the case $q = q_n^L$, and write $B_n^L(b) := i^n \cdot \Theta_n(\Phi(i))$. Notice that the zeros of q_n^L are those $(4n)$th roots of unity which are not simultaneously $2n$th roots of unity, and therefore $B_n^L(b) = B_{2n-1}^R(b)/B_{n-1}^R(b)$ for $n \geq 1$, as claimed in the assertion. Moreover, since $q_{2n}^L(i) = 2$ and $q_{2n-1}^L(i) = 0$, we obtain for $w = \Phi(i)$

$$|w^n \cdot Q(1/w) + \frac{q_n^L(i)}{w^n \cdot Q(1/w)}| = \begin{cases} B_n^L(b) & \text{if } n \text{ is odd,} \\ |B_n^L(b) + 2/B_n^L(b)| \geq B_n^L(b) & \text{if } n \text{ is even.} \end{cases}$$

Therefore, Theorem 2.2, Lemma 2.3 and Lemma 2.4 yield the lower bound for $\Gamma_n([-b, b])$ as claimed in Theorem 2.5.

It remains to examine the case $a = 0$. First notice that $q_n^R(x^2) \leq q_{2n}^R(x)$ for all $x \in \mathbb{R}$, and thus

$$\Delta_n(\mathbb{D}; [0, b], q_n^R) = \max_{|z|=1} \max_{P \in \mathbb{P}_n} \frac{|P(z^2)|}{\max\limits_{x \in [-\sqrt{b}, \sqrt{b}]} |P(x^2)| / \sqrt{q_n^R(x^2)}}$$

$$\leq \max_{|z|=1} \max_{P \in \mathbb{P}_n} \frac{|P(z^2)|}{\max\limits_{x \in [-\sqrt{b}, \sqrt{b}]} |P(x^2)| / \sqrt{q_{2n}^R(x)}} \leq \Delta_{2n}(\mathbb{D}; [-\sqrt{b}, +\sqrt{b}], q_{2n}^R).$$

Consequently, the upper bound for $\Gamma_n([0, b])$ follows as above from Theorem 2.2, Lemma 2.3, and Lemma 2.4. In order to establish the lower bound, recall from above that $B_{2n}^L(\sqrt{b})/2 \leq |T(i)|$, where T is the weighted Chebyshev polynomial of degree $2n$ of the weight $\sqrt{q_{2n}^L}$ on the interval $[-\sqrt{b}, \sqrt{b}]$. It follows from (6) that $T(x) = T(-x)$, i.e., we may write $T(x) = P(x^2)$, with P a polynomial of degree n. Since $q_{2n}^L(x) = q_n^L(x^2)$, we may conclude that

$$\frac{B_{2n}^L(\sqrt{b})}{2} \leq \frac{|T(i)|}{\max\limits_{x \in [-\sqrt{b}, \sqrt{b}]} |T(x)| / \sqrt{q_{2n}^L(x)}} = \frac{|P(-1)|}{\max\limits_{x \in [0, b]} |P(x)| / \sqrt{q_n^L(x)}} \leq \Delta_n(-1; [0, b], q_n^L),$$

the right hand side being equal to $\Delta_n(\mathbb{D};[0,b],q_n^L)$ by Lemma 2.3. Thus $\Gamma_n([0,b]) \geq B_{2n}^L(\sqrt{b})/(2\sqrt{2n+2})$ by Theorem 2.2. □

Example 2.6 *In order to illustrate the assertion of Theorem 2.5, let us compute for some small n explicitly the proposed bounds. First notice that in the case $I=[-b,b]$ we have*

$$\Phi(z) = \frac{z}{b} \cdot (1 + \sqrt{1 - \frac{b^2}{z^2}}), \quad z \notin [-b,b],$$

where the branch of the square root is chosen such that $\sqrt{e^{it}} = e^{it/2}$ for $-\pi < t < \pi$. Consequently, $|\Phi(i)| = (1+\sqrt{b^2+1})/b = \Phi(i)/i$, and

$$B_1^L(b) = B_1^R(b) = \frac{|1-\Phi(i)^4|}{|\Phi(i)|^2 \cdot (2/b)} = \frac{2}{b} \cdot \sqrt{1+b^2}.$$

Introducing the substitution $b = \tan(\beta)$, $\beta \in (0, \pi/2)$, we obtain the simpler expressions

$$|\Phi(i)| = \frac{1}{\tan(\beta/2)}, \quad B_1^R(b) = \frac{2}{\sin(\beta)}.$$

Moreover, for $0 < t < \pi/2$ we get with $\zeta := \sqrt{1 - b^2 e^{-2it}}$

$$h_b(t) := \frac{|1 + |\Phi(i)|^2 \cdot \Phi(e^{it})^2|^2}{|\Phi(e^{it})|^2 \cdot |\Phi(i)|^2 \cdot (2/b)^2} = |\frac{(1+\zeta)/\tan(\beta/2) + (1-\zeta)\cdot\tan(\beta/2)}{2}|^2$$

$$= |\frac{1+\cos(\beta)\cdot\zeta}{\sin(\beta)}|^2 = |\frac{1+\sqrt{\cos^2(\beta) - \sin^2(\beta)e^{-2it}}}{\sin(\beta)}|^2$$

$$= \frac{1 + |\cos^2(\beta) - \sin^2(\beta)e^{-2it}| + \sqrt{2}\sqrt{|\cos^2(\beta) - \sin^2(\beta)e^{-2it}| + \cos^2(\beta) - \sin^2(\beta)\cos(2t)}}{\sin^2(\beta)},$$

in particular for $\beta = \pi/4$ and for $\beta = \pi/2$.

$$h_1(t) = \frac{1 + \sqrt{\sin(t)} + \sqrt{2\sin(t)(1+\sin(t))}}{2}, \quad h_\infty(t) = 2 \cdot (1 + \sin(t)).$$

Thus for instance $B_2^R(\infty) = h_\infty(\pi/3) = 2 + \sqrt{3}$ and $B_3^R(\infty) = B_1^R(\infty) \cdot h_\infty(\pi/4) = 4 + 2\sqrt{2}$. Similarly, $B_5^R(\infty) = 12 + 6\sqrt{3}$, $B_2^L(\infty) = 2 + \sqrt{2}$, and $B_3^L(\infty) = 6$. The lower and upper bounds for $\Gamma_n([0,b])$ given in Theorem 2.5 have been computed numerically for $n = 1, 2, ..., 20$ and $b = 1/2$, $b = 1$ and $b = 10$, as displayed in Figure 1. □

3 Exponential estimates

The bounds proposed in Theorem 2.2, and Theorem 2.5, for arbitrary intervals, and for intervals of the form $[0,b]$ or $[-b,b]$, respectively, are sharp up to a factor $\approx (n+1)^{3/2}$, and $\approx (n+1)^{1/2}$, respectively. However, their representation as a product does not show clearly that the underlying matrices — positive definite Hankel matrices or Krylov matrices with Hermitian argument or real row scaled Vandermonde matrices — are ill–conditioned already for moderate order. The aim of this Section is to give more explicit bounds of the form $f^L(n) \cdot \Gamma(I)^n \leq \Gamma_n(I) \leq f^R(n) \cdot \Gamma(I)^n$, where $f^L(n), f^R(n)$ behave like powers of $n+1$, and thus the choice of $\Gamma(I)$ is optimal. We start with treating the case of intervals $[-b,b]$ and $[0,b]$ in Theorem 3.1.

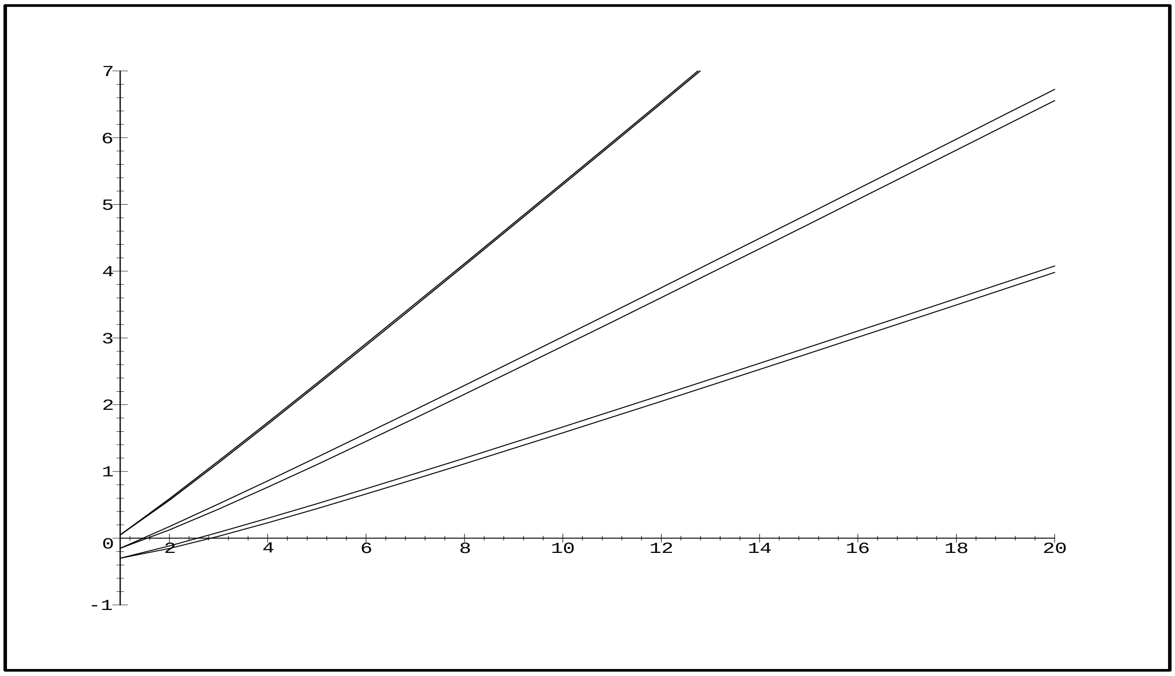

Figure 1: THE LOGARITHM TO THE BASIS 10 OF THE BOUNDS FOR $\Gamma_n([0,b])$ OF THEOREM 2.5, VALUES FOR $n=1,2,...,20$ AND FOR $b=1/2$, $b=1$ AND $b=10$ (FROM ABOVE TO BELOW).

The particular case $b=1$ is subject of Corollary 3.2 and Example 3.3, where we compare our findings with the well–known growth rate of the condition number of the Hilbert matrix. An extension to arbitrary intervals is given in Remark 3.4, and the case of intervals in or outside the unit disk is studied in Remark 3.5. Finally we discuss the case of an arbitrary positive definite Hankel matrix in Theorem 3.6 where a suitable combination of the mathematical constants e,π and *Catalan* plays an important role.

Theorem 3.1 *For $b>0$ define*

$$\Gamma([-b,b]) := (b^{-1}+\sqrt{1+b^{-2}})\cdot\begin{cases} 1 & \text{if } b\le 1,\\ \exp\Big(\dfrac{2}{\pi}\displaystyle\int_1^b \log(x)\cdot\frac{\sqrt{1+b^2}}{(1+x^2)\cdot\sqrt{b^2-x^2}}\,dx\Big) & \text{if } b>1,\end{cases}$$

and $\Gamma([0,b]) := \Gamma([-\sqrt{b},\sqrt{b}])^2$. *Then there holds for $n\ge 1$*

$$\frac{\Gamma([-b,b])^n}{2\sqrt{2n+2}} \le \Gamma_n([-b,b]) \le \sqrt{n+1}\cdot\Gamma([-b,b])^n, \quad \frac{\Gamma([0,b])^n}{2\sqrt{2n+2}} \le \Gamma_n([0,b]) \le \sqrt{2n+1}\cdot\Gamma([0,b])^n.$$

Proof: In view of Theorem 2.5, for establishing the above assertion it is sufficient to show that for $n\ge 1$ there holds

$$\Gamma([-b,b])^n \le B_n^L(b) \le B_n^R(b) \le \sqrt{n+1}\cdot\Gamma([-b,b])^n. \tag{7}$$

Let q be an even real polynomial positive on $I=[-b,b]$. Recall that the real polynomial Q in the canonical factorization $Q(1/w)\cdot Q(w) = q(\Psi(w))$ verifies $Q(0)>0$, $Q(w)\neq 0$ for $|w|\le 1$,

b	$1/3$	$1/2$	1	2	10
$\Gamma([-b,b])$	$3+\sqrt{10}\approx 6.162$	$2+\sqrt{5}\approx 4.236$	$1+\sqrt{2}\approx 2.414$	1.935	1.797

Table 1: THE QUANTITY $\Gamma([-b,b])$ OF THEOREM 3.1 FOR SOME PARTICULAR VALUES OF b.

and $|Q(e^{it})|^2 = q(b\cdot\cos(t))$ for $t\in[0,2\pi]$. By the Poisson formula, any function analytic and different form zero in some neighborhood of the unit disk can be recovered in the unit disk up to an imaginary constant from its absolute value on the boundary. Here we have for $|w|<1$

$$\log Q(w) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log(|Q(e^{it})|)\cdot\frac{1+we^{-it}}{1-we^{-it}}\,dt + i\cdot c = \frac{1}{4\pi}\int_{-\pi}^{\pi}\log(q(b\cdot\cos(t))\cdot\frac{1+we^{-it}}{1-we^{-it}}\,dt + i\cdot c$$

with a real constant c. Comparing the values at zero of both sides, we may conclude that $c=0$. In particular, for $w=1/\Phi(i)$ with $w=|w|/i$ we obtain

$$F[q] := \log|Q(1/\Phi(i))| = \frac{1}{4\pi}\int_{-\pi}^{\pi}\log(q(b\cdot\cos(t))\cdot Re\Big(\frac{\Phi(i)+e^{-it}}{\Phi(i)-e^{-it}}\Big)\,dt,$$

with

$$Re\Big(\frac{\Phi(i)+e^{-it}}{\Phi(i)-e^{-it}}\Big) = \frac{(\Phi(i)^2+1)/(2\cdot\Phi(i))}{(\Phi(i)^2-1)/(2\cdot\Phi(i))+i\sin(t)} = \frac{1}{\sqrt{1+b^2}+b\sin(t)}.$$

Consequently,

$$\begin{aligned}F[q] &= \frac{1}{4\pi}\int_{-\pi}^{\pi}\log(q(b\cdot\cos(t))\cdot\frac{dt}{\sqrt{1+b^2}+b\sin(t)}\\ &= \frac{1}{4\pi}\int_{0}^{\pi}\log(q(b\cdot\cos(t))\cdot\Big(\frac{1}{\sqrt{1+b^2}+b\sin(t)}+\frac{1}{\sqrt{1+b^2}-b\sin(t)}\Big)\,dt\\ &= \frac{1}{2\pi}\int_{0}^{\pi}\log(q(b\cdot\cos(t))\cdot\frac{\sqrt{1+b^2}}{1+b^2\cos^2(t)}\,dt = \frac{1}{\pi}\int_0^b\log(q(x))\cdot\frac{\sqrt{1+b^2}}{(1+x^2)\cdot\sqrt{b^2-x^2}}\,dx,\end{aligned}$$

where for the last transformation we have used the fact that q is even. The right hand side of the preceding formula may now be used to define $F[q]$ for any function q continuous and positive on $[0,b]$. Notice that $q(x)\le\tilde q(x)$ for $x\in[0,b]$ implies $F[q]\le F[\tilde q]$. Also, for $\tilde q(x):=\max(1,|x|)$ there holds $\tilde q(x)^{2n}\le q_n^L(x)\le q_n^R(x)\le(n+1)\tilde q(x)^{2n}$, and thus

$$\begin{aligned}&2n\cdot F[\tilde q] = F[q^{2n}] \le F[q_n^L]\le F[q_n^R]\\ &\le F[(n+1)\tilde q^{2n}] = 2n\cdot F[\tilde q]+\log(n+1)\cdot F[e^0] = 2n\cdot F[\tilde q]+\log(\sqrt{n+1}).\end{aligned}$$

Taking into account that $|\Phi(i)| = b^{-1}+(1+b^{-2})^{1/2}$ and that $F[q]=0$ for $b\le 1$, the inequalities (7) now follow by recalling from the proof of Theorem 2.5 that $B_n^L(b) = |\Phi(i)|^n\cdot\exp(F[q_n^L])$, and $B_n^R(b) = |\Phi(i)|^n\cdot\exp(F[q_n^R])$. □

In Table 1 we have computed numerically the quantity $\Gamma([-b,b])$ for some values of b. We observe that the entries for $b=1/2$, $b=1$ and $b=10$ describe the asymptotic behavior of the curves in Figure 1. Also, $\Gamma([-1,1]) = 1+\sqrt{2}$, and thus Theorem 3.1 implies

Corollary 3.2 *Let $n\ge 1$. Then for measures supported in $[-1,1]$ or for Krylov matrices with scaled hermitian B (i.e., $||B||\le 1$) there holds*

$$\frac{(1+\sqrt{2})^n}{2\sqrt{2n+2}} \le \Gamma_n([-1,1]) \le \sqrt{n+1}\cdot(1+\sqrt{2})^n,$$

whereas for measures supported in $[0,1]$ *or for Krylov matrices with scaled hermitian positive semidefinite* B *there holds*

$$\frac{(1+\sqrt{2})^{2n}}{2\sqrt{2n+2}} \le \Gamma_n([0,1]) \le \sqrt{2n+1}\cdot(1+\sqrt{2})^{2n}.$$

□

Other bounds for $\Gamma_n(I)$ for the intervals $I=[0,1]$ and $I=[-1,1]$ have been given already in [Bec96, Theorem 5.11], they are obtained by stressing the link to ordinary Vandermonde matrices. However, both bounds of Corollary 3.2 are tighter roughly by a factor $\sqrt{n}$.

Example 3.3 *The* Hilbert matrix, *defined by* $\tilde{H}_n = (1/(j+k+1))_{j,k=0,\ldots,n}$, *is a well-known example for an extremely ill-conditioned matrix. It seems to be Todd [Tod54] who gave a first asymptotic expression for the Turing condition number of* $\tilde{H}_n$. *Let us here mention a result of Wilf [Wif70, Equation (3.35)] who showed that*

$$||\tilde{H}_n^{-1}|| = \frac{1}{\sqrt{\pi}\cdot 2^{15/4}} \cdot \frac{(1+\sqrt{2})^{4n+4}}{\sqrt{n}} \cdot (1+o(1)_{n\to\infty}).$$

Also, it is well-known that $||\tilde{H}_n|| \in [1,\pi]$, *and thus* $\kappa(\tilde{H}_n)$ *behaves like a constant times* $(1+\sqrt{2})^{4n}/\sqrt{n} \approx 33.97^n/\sqrt{n}$ *for large* n. *Notice that the Hilbert matrix is a Gram matrix with respect to the measure* $d\mu(x) = dx$ *on* $[0,1]$. *This shows that the lower bound of Corollary 3.2 for* $\Gamma_n([0,1])$ *may be at most improved by a constant times* $n^{1/4}$, *and that the upper bound may be asymptotically replaced by the square root of the condition number of the Hilbert matrix.* □

Remark 3.4 *By changing slightly the arguments used in the proof of Theorem 3.1 we may also obtain exponential bounds for arbitrary intervals* $I=[a,b]$. *For some polynomial* q *positive on* I *let* $Q(w)$ *denote the polynomial in the canonical factorization of* $q(\Psi(w))$, *and define* $F[q](z) := \log(|Q(1/\Phi(z))|)$. *Then as above*

$$F[q](z) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \log(q(\Psi(e^{it}))\cdot \mathrm{Re}(\frac{\Phi(z)+e^{-it}}{\Phi(z)-e^{-it}})\, dt,$$

which is also defined for more general functions q. *Taking* $\tilde{q}(z) = \max(0,\log(|z|))$, *one shows as before that* $2n\cdot F[\tilde{q}](z) \le F[q_n^L](z) \le F[q_n^R](z) \le 2n\cdot F[\tilde{q}](z) + \log(\sqrt{n+1})$, *and*

$$\log(|\Phi(z)|) + 2F[\tilde{q}](z) - \tilde{q}(z) = \frac{1}{2\pi}\int_0^{2\pi} g_I(z,e^{it})\, dt =: G_I(z),$$

where $g_I(\cdot,\zeta)$ *denotes the Green function with pole at infinity of* $\mathbb{C}\setminus I$. *The function* G_I *usually is referred to as the* Green potential *of the equilibrium distribution of* $\mathbb{D}$ *[SaTo97, Chapter II.5, p.124]. Notice that in the case* $I\subset\mathbb{D}$ *there holds* $G_I(z) = \log(|\Phi(z)|) = g_I(z,\infty)$ *for* $z\in\mathbb{D}$. *In correspondence with Theorem 3.1 define*

$$\log(\Gamma(I)) := \max_{z\in\mathbb{D}} \frac{1}{2\pi}\int_0^{2\pi} g_I(z,e^{it})\, dt$$

considered already in [Bec96, Theorem 4.13(c)] for more general sets I. *From Lemma 2.4 we may conclude that*

$$\frac{1}{2}(\Gamma(I)^n - 2\Gamma(I)^{-n}) \le \Delta_n(\mathbb{D};I,q_n^L) \le \Delta_n(\mathbb{D};I,q_n^R) \le \sqrt{n+1}\cdot\Gamma(I)^n,$$

and thus by Theorem 2.2 for all intervals $I = [a,b]$ and for all $n \geq 1$

$$\frac{1}{2\sqrt{2n+2}}(\Gamma(I)^n - 2\Gamma(I)^{-n}) \leq \Gamma_n(I) \leq \begin{cases} \sqrt{n+1}\cdot\Gamma(I)^n & \text{if } a \geq 0, \\ (n+1)^{3/2}\cdot\Gamma(I)^n & \text{otherwise.} \end{cases}$$

□

Remark 3.5 *Using the definition of, e.g., Γ_n^K one easily verifies that $\Gamma_n(I) = \Gamma_n(-I) = \Gamma_n(I^{-1})$, $I^{-1} = \{1/x : x \in I\}$. Thus, using the results of Remark 3.4, we may give explicit exponential bounds for intervals lying in or outside the unit disk. Let $I = [a,b]$, where we suppose (without loss of generality) that $a+b \geq 0$, and $b \leq 1$. Then $\Gamma(I) = \max_{|z|=1}|\Phi(z)|$ by Remark 3.4. The right hand side has been given explicitly in [Bec96, Lemma 2.18], namely,*

$$\Gamma(I) = \begin{cases} \Phi(-1) = \frac{2+a+b}{b-a} + ((\frac{2+a+b}{b-a})^2 - 1)^{1/2} & \text{if } -2ab \leq a+b, \\ (-\frac{1}{ab})^{1/2} + (-\frac{1}{ab}+1)^{1/2} & \text{if } -2ab \geq a+b. \end{cases}$$

□

We now turn to the quantities $\Gamma_n(\mathbb{R})$ and $\Gamma_n([0,\infty))$ defined by (2). It is not difficult to verify that the function B of Theorem 3.1 is strictly decreasing, and thus bounds could be obtained by taking the limit in Theorem 3.1 (or in Remark 3.4). However, better bounds are obtained directly from Theorem 2.5, again refining [Bec96, Theorem 5.11 and Corollary 5.14].

Theorem 3.6 *Let $\Gamma(\mathbb{R}) := \exp(2\cdot \text{Catalan}/\pi) \approx 1.792$. Then*

$$\frac{\Gamma(\mathbb{R})^n}{4\sqrt{n+1}} \leq \Gamma_n(\mathbb{R}) \leq \frac{\Gamma(\mathbb{R})^{n+1}}{\sqrt{2}}, \quad n \geq 2,$$

i.e., a positive definite Hankel matrix of order $n+1 \geq 3$ has a (Euclidean) condition number bounded below by $\Gamma(\mathbb{R})^{2n}/(16n+16) \approx 3.210^n/(16n+16)$, and this bound is tight at most up to a factor $\Gamma(\mathbb{R})^2\cdot(8n+8)$. Moreover, for a Krylov matrix with Hermitian positive semidefinite argument

$$\frac{\Gamma(\mathbb{R})^{2n}}{4\sqrt{n+1}} \leq \Gamma_n([0,\infty)) \leq \frac{\Gamma(\mathbb{R})^{2n+1}}{\sqrt{2}}, \quad n \geq 1.$$

Proof: We know from Theorem 2.5 and Example 2.6 that

$$\log B_n^R(\infty) := \lim_{b\to\infty} \log B_n^R(b) = \sum_{j=1}^{n} h(\frac{j}{n+1}) + \frac{h(0)+h(1)}{2} - \log(\sqrt{2}),$$

where $h(t) := \log(2+2\sin(\pi\cdot t)))/2$. Up to a factor $1/(n+1)$, the sum of h–values is the composite trapezoidale quadrature rule with stepsize $1/(n+1)$ approximating the integral

$$J = \int_0^1 h(t)\,dt = \frac{1}{\pi}\int_0^{\pi/2} \log(2+2\sin(t)))\,dt = \frac{2}{\pi}\int_0^{\pi/2}\log(\cot(t/2))\,dt = \frac{2}{\pi}\cdot \textit{Catalan}.$$

Hence

$$J - \frac{1}{n+1}\log(\sqrt{2}\cdot B_n^R(\infty)) = -\frac{h''(\xi)}{12\cdot(n+1)^2}$$

for some $\xi \in (0,1)$ by the well–known quadrature error formula. Now

$$h'(t) = \frac{\pi}{2}\frac{\cos(\pi\cdot t)}{1+\sin(\pi\cdot t)}, \quad h''(t) = -\frac{\pi^2}{2}\frac{1}{1+\sin(\pi\cdot t)}, \quad \frac{-h''(\xi)}{12} \in (\frac{\pi^2}{48}, \frac{\pi^2}{24}),$$

and therefore with $\eta_n := \exp(\frac{-\pi^2/48}{n})$

$$\frac{B_n^R(\infty)}{\exp(J)^{n+1}/\sqrt{2}} \in [\eta_{n+1}^2, \eta_{n+1}], \quad \frac{B_n^L(\infty)}{\exp(J)^n} = \frac{B_{2n-1}^R(\infty)}{B_{n-1}^R(\infty)\exp(J)^n} \in [\eta_n^3, \eta_n].$$

Also, for all $n \geq 2$ there holds $1 \geq \eta_n^3 \geq \eta_2^3 = \exp(-\pi^2/32) > 1/\sqrt{2}$. The assertion now follows from Theorem 2.5 by combining the above elements. □

4 Real Vandermonde matrices

By adapting the reasoning of the preceding sections we are now able to show the following result for real Vandermonde matrices

Theorem 4.1 *There holds for $n \geq 1$*

$$\frac{\Delta_n(\mathbb{D}; [-1,1], 1)}{\sqrt{n+1}} \leq \Gamma_n^V(\mathbb{R}) \leq \Gamma_n^V([-1,1]) \leq (n+1)\cdot\Delta_n(\mathbb{D}; [-1,1], 1),$$

$$\frac{\Delta_n(\mathbb{D}; [0,1], 1)}{\sqrt{n+1}} \leq \Gamma_n^V([0,+\infty)) \leq \Gamma_n^V([0,1]) \leq (n+1)\cdot\Delta_n(\mathbb{D}; [0,1], 1),$$

where $\Delta_n(\mathbb{D}; [-1,1], 1) = \sqrt{2}\cdot(1+\sqrt{2})^{n-1}$, *and* $2\cdot\Delta_n(\mathbb{D}; [0,1], 1) = (1+\sqrt{2})^{2n} + (1+\sqrt{2})^{-2n}$.

Proof: Let $x_0, ..., x_n \in \mathbb{R}$. By possibly changing the sign leaving invariant the condition number of $V := V_n(x_0, ..., x_n)$ we may suppose that $b := \max_j x_j = \max_j |x_j|$. In the case of nonnegative nodes let $I = [0,1]$, and otherwise $I = [-1,1]$. First notice that $||V||$ is greater than the norm of any of its rows,

$$||V|| \geq \sqrt{1 + |b|^2 + ... + |b|^{2n}} = ||D \cdot e||, \quad e = (1, ..., 1)^T, \quad D = diag(1, b, ..., b^n).$$

Let P be an extremal polynomial of $\Delta_n(\mathbb{D}; I, 1)$. Then $||V \cdot D^{-1} \cdot \vec{P}|| \leq \sqrt{n+1}\max_j |P(x_j/b)| \leq \sqrt{n+1}$, and thus by (4)

$$\sqrt{n+1}\cdot\kappa(V) \geq ||D^{-1}\vec{P}|| \cdot ||D \cdot e|| \geq ||P||_1 \geq \max_{z\in\mathbb{D}} |P(z)| = \Delta_n(\mathbb{D}; I, 1),$$

leading to the first inequalities. We now choose $\zeta \in \mathbb{D}$, $z_0, ..., z_n \in I$ as in Lemma 2.3 with $q = 1$. Then

$$||V_n(z_0, ..., z_n)^{-1}|| \leq \max_{|z|\leq 1} \max_{a_0,...,a_n \in \mathbb{C}} \frac{|\sum_{j=0}^n a_j \ell_j(z)|}{||(a_0, ..., a_n)^T||} \leq \max_{|z|\leq 1} \sum_{j=0}^n |\ell_j(z)| = \Delta_n(\mathbb{D}; I, 1)$$

by Lemma 2.3. Using the fact that $|z_j| \leq 1$, we also have the trivial estimate $||V_n(z_0, ..., z_n)|| \leq n+1$, as required for the upper bound of $\Gamma_n^V(I)$. It remains to determine $\Delta_n(\mathbb{D}; I, 1)$. We know from Lemma 2.3 (or from [FrRu86, Theorem (6.12)]) that $\Delta_n(\mathbb{D}; [-1,1], 1) = \Delta_n(i; [-1,1], 1)$, with the latter being explicitly determined for $n \geq 1$ in [FrRu86, Theorem (2.6)]. Also, it is well–known that an extremal polynomial of $\Delta_n(\mathbb{D}; [0,1], 1) = \Delta_n(-1; [0,1], 1)$ is given by the shifted Chebyshev polynomial $T_n(-1+2x)$, and thus $\Delta_n(\mathbb{D}; [0,1], 1) = |T_n(-3)| = T_n(3)$, as claimed in the assertion. □

By a similar reasoning one may obtain bounds for the p–condition number of real square Vandermonde matrices being sharp up to a factor $(n+1)^{n+1/p}$ (see [Bec96, Theorem 5.8]). Gautschi and Inglese [GaIn88, Theorem 2.1 and Theorem 3.1] proposed lower bounds for the 1–condition number, which are weaker at least for $n \geq 6$. The Euclidean condition number of real square Vandermonde matrices has been estimated by Tyrtyshnikov [Tyr94a, Theorem 4.1]; again the bounds of Theorem 4.1 are tighter for $n \geq 2$.

5 An application to the sensitivity of polynomial roots

A very impressive and well–known example for the limitation of finite precision arithmetic was given by Wilkinson (see, e.g., [StBu96, Chapter 5.8]): try to expand in a floating point environment the polynomial $p(z) = (z-1)\cdot(z-2)\cdot..\cdot(z-20)$ in its monomial basis, add possibly a "very small" perturbation, and then compute its zeros. Depending on the precision of the machine, one finds answers which have no significant digits.

To understand better this phenomenon one has to study the non-linear map M : $(x_1,...,x_n) \to (a_0,...,a_{n-1})$ defined by

$$P(z) = (z-x_1)\cdot(z-x_2)\cdot ...\cdot(z-x_n) = z^n + \sum_{j=0}^{n-1} a_j z^j.$$

The map M is locally invertible at a point $X = (x_1,...,x_n)$ if its Jacobian $J_M := (\frac{\partial M}{\partial X})$ is non-singular at X, which again is true iff P has simple zeros.

As a mathematical model for the numerical condition of the above procedure, suppose that a vector of zeros $X \in \mathbb{R}^n$ with distinct elements is given. We add a "small" perturbation ΔX, compute an approximation Y^* of $M(X+\Delta X)$, and then X^* as an approximation of $M^{-1}(Y^*)$. From the definition of M one easily sees that

$$\frac{\partial a_j}{\partial x_k}(X) = c_{j,k}, \quad -\prod_{s\neq k}(z-x_s) = \sum_{j=0}^{n-1} z^j c_{j,k} = -P'(x_k)\cdot \ell_k(z), \quad k = 1,..,n,$$

with the associated Lagrange polynomials $\ell_1,...,\ell_n$. The inverse of $J_M(X)$ is a row scaled Vandermonde matrix, more precisely,

$$J_M(X)^{-1} = J_{M^{-1}}(M(X)) = D\cdot V_{n-1}(x_1,...,x_n), \quad D = diag\Big(\frac{-1}{P'(x_1)},...,\frac{-1}{P'(x_n)}\Big).$$

In order to measure the sensitivity of a non-linear map at a given point (with regard to absolute perturbations) one usually uses the norm of its Jacobian. Therefore, a measure for the magnification $||X^*-X||/||\Delta X||$ of errors is given by $||J_M(X)||\cdot||J_{M^{-1}}(M(X))|| = \kappa(J_M(X))$, which is as least as large as $1.792^{n-1}/(4\sqrt{n})$ by Theorem 3.6 (and much larger for Wilkinson's configuration of nodes). We may conclude that the procedure of passing from the roots to the coefficients in the basis of monomials and vice versa is quite sensitive for a polynomial having real zeros.

6 Rational interpolation at real nodes

Let us finally mention another application of our results, namely the numerical condition of the problem of rational interpolation at real distinct nodes. First recall from (1) that $\kappa(A) \geq \kappa(A')$ provided that A' results from A by dropping some columns. In particular, if A contains $n+1$ columns building up a Krylov matrix $K_n(B,b)$ with Hermitian B then the lower bounds of, e.g., Theorem 3.6 for the condition number are still valid.

"Striped" Krylov matrices containing two column blocks of Krylov matrices occur naturally in the context of the (linearized) rational interpolation problem: Given two positive integers μ,ν, real nodes $x_0,..,x_{\mu+\nu}$, and data $(f_0,g_0),..,(f_{\mu+\nu},g_{\mu+\nu})$, we look for polynomials P and Q

(represented using the basis of monomials) with degrees bounded by μ, and ν, respectively, such that

$$f_j \cdot P(x_j) - g_j \cdot Q(x_j) = 0, \quad j = 0, .., \mu + \nu.$$

With $f := (f_0, .., f_{\mu+\nu})^T$, $g = (g_0, .., g_{\mu+\nu})$, and $X = diag(x_0, .., x_{\mu+\nu})$, this problem leads to a homogeneous system of $(\mu+\nu+1)$ linear equations and $(\mu+\nu+2)$ unknowns, where the matrix of coefficients is given by $(K_\mu(X; f)|K_\nu(X; g))$. For the purpose of scaling, one usually fixes either the leading coefficient of Q or the value of Q at zero, leading to the square matrices of coefficients

$$A = (K_\mu(X; f)|K_{\nu-1}(X; g)) \text{ or } A = (K_\mu(X; f)|K_{\nu-1}(X; X \cdot g)).$$

One could imagine that in the case of clustered data, i.e., of nearly coinciding nodes, this matrix is ill-conditioned. However, the condition number of A already is very large independent of the location of the (real) nodes, namely at least $1.79^n/(4\sqrt{n+1})$ with $n = \min\{\mu, \nu - 1\}$ by Theorem 3.6, and this lower bound certainly may be improved.

7 Conclusions

In this paper we have given lower bounds for the Euclidean condition number of various classes of real matrices, improving results given earlier. Our bounds are of the form $f(n) \cdot \gamma^n$, where the given γ optimal, and the function $f(n)$ may be at most improved by some small power of n. Similarly, one may obtain bounds for other Hölder norms [Bec96].

Our results show clearly that the use of the basis of monomials is numerically very delicate for several tasks in Numerical Analysis, namely for: polynomial interpolation at real nodes, rational interpolation at real nodes, iterative methods for solving linear Hermitian systems, the determination of roots of polynomials having only real roots, or the explicit computation of polynomials being orthogonal with respect to a measure supported on the real line.

It remains the open question how the basis of monomials behaves for other configuration of abscissa or eigenvalues or supports (for instance, a Vandermonde matrix at roots of unity has the "perfect" Euclidean condition number 1). Also, it would be interesting to have similar characterizations for other bases of polynomials. Here results have been given by a large number of authors (see, e.g., [Bec96] and the references cited therein). The nth root behavior for optimal lower bounds for rather large classes of such matrices is investigated in [BeSt98] by using the tool of complex potential theory. The numerical condition of such matrices for a given complex configuration of nodes is studied in [BeSa98].

References

[Bec96] B. Beckermann, On the numerical condition of polynomial bases: Estimates for the Condition Number of Vandermonde, Krylov and Hankel matrices, Habilitationsschrift, Universität Hannover, April 1996.

[BeSa98] B. Beckermann & E.B. Saff, The numerical condition of (weighted) Vandermonde–like matrices for prescribed complex abscissa, in preparation (1998).

[BeSt98] B. Beckermann & H. Stahl, Some n–th root estimates for the condition number of Krylov, Krylov–like and Gram matrices, in preparation (1998).

[BrSa93] C. Brezinski & H. Sadok, Lanczos–type algorithms for solving systems of linear equations, *Appl. Numer. Math.* **11** (1993) 443-473.

[Car94] J.-F. Carpraux, Quelques outils d'aide au contrôle de précision en calcul scientifique, Thèse, Université de Rennes I (1994).

[FrRu86] R. Freund & S. Ruscheweyh, On a class of Chebyshev approximation problems which arise in connection with a conjugate gradient type method, *Numer. Math.* **48** (1986) 525-542.

[Gau75a] V. Gautschi, Norm estimates for inverses of Vandermonde matrices, *Numer. Math.* **23** (1975) 337-347.

[Gau75b] W. Gautschi, Optimally Conditioned Vandermonde Matrices, *Numer. Math.* **24** (1975) 1-12.

[Gau90] W. Gautschi, How (un)stable are Vandermonde systems?, in: Asymptotic and Computational Analysis, R. Wong Ed., Lecture Notes in Pure and Appl. Math. **124**, Decker, New York (1990) 193-210.

[GaIn88] W. Gautschi & G. Inglese, Lower Bounds for the Condition Number of Vandermonde Matrices, *Numer. Math.* **52** (1988) 241-250.

[GoOL89] G.H. Golub & D.P O'Leary, Some history of the conjugate and Lanczos algorithms, *SIAM Rev.* **31**(1989) 50-102.

[GoVL93] G.H. Golub, C.F. Van Loan, Matrix Computations, Second Edition, Johns Hopkins University Press, Baltimore, London (1993).

[MMR94] G.V. Milovanovic, D.S. Mitrinovic & Th.M. Rassias, Topics in polynomials: Extremal problems, Inequalities, Zeros, World Scientific, Singapure (1994).

[NiSo88] E.M. Nikishin & V.N. Sorokin, Rational Approximations and Orthogonality, *Translations of Mathematical Monographs* **92**, Am. Math. Soc., Providence, R.I. (1991).

[SaTo97] E.B. Saff & V. Totik, Logarithmic potentials with external fields, Springer, Berlin (1997).

[StBu96] J. Stoer & R. Bulirsch, Introduction to Numerical Analysis, Second Edition, Springer (1996).

[Sze67] G. Szegö, Orthogonal polynomials, Third Edition, AMS Colloquium Publications Volume 23, Providence, Rhode Island (1967).

[Tay78] J.M. Taylor, The condition of Gram matrices and related problems, *Proc. of the Royal Soc. of Edinburgh* **80A** (1978) 45-56.

[Tod54] J. Todd, The Condition Number of the Finite Segment of the Hilbert Matrix, *Nat. Bur. of Standards Appl. Math. Series* **39** (1954) 109-116.

[Tyr94a] E.E. Tyrtyshnikov, How bad are Hankel matrices? *Numer. Math.* **67** (1994) 261-269.

[Tyr94b] E.E. Tyrtyshnikov, Personal communication, June 1994.

[Wif70] H. S. Wilf, Finite Sections of Some Classical Inequalities, Springer, Heidelberg (1970).

[Wil65] J.H. Wilkinson, The algebraic Eigenvalue problem, Claredon Press, Oxford (1965).