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On the Classification of the Spectrum of Second Order Difference Operators

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Abstract. We study nonsymmetric second order difference operators acting in the Hilbert space ℓ^2 and describe the resolvent set and the essential spectrum of such operators in terms of related formal orthogonal polynomials. As an application, we obtain new results on the growth of orthonormal polynomials outside and inside the support of the underlying measure of orthogonality.

1. Introduction

We denote by ℓ^2 the Hilbert space of complex quadratic summable sequences and by $(e_n)_{n\geq 0}$ its usual orthonormal basis. Furthermore, for a linear operator T in ℓ^2, we denote by $\mathcal{D}(T)$, $\mathcal{R}(T)$, $\mathcal{N}(T)$, and $\sigma(T)$, its domain of definition, its range, its kernel, and its spectrum, respectively.

Given complex numbers a_n, b_n, $n \geq 0$, with $a_n \neq 0$ for all n, we associate the infinite tridiagonal *complex Jacobi matrix*

$$\mathcal{A} = \begin{pmatrix} b_0 & a_0 & 0 & \cdots & \cdots \\ a_0 & b_1 & a_1 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

In the *symmetric case* $b_n, a_n \in \mathbb{R}$ for all n one recovers the classical Jacobi matrix. Denoting by $\mathcal{C}_0 \subset \ell^2$ the linear space of finite linear combinations of the basis elements $e_0, e_1, \ldots$, we may identify via the usual matrix product a complex Jacobi matrix $\mathcal{A}$ with an operator acting on $\mathcal{C}_0$, which in the sequel is also denoted by $\mathcal{A}$. As in [K, Section III.5.3] one verifies that $\mathcal{A}$ is closable. Thus there exist a unique minimal

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closed extension of $\mathcal{A}$, the *closure* A of $\mathcal{A}$, referred to as the second order difference operator of $\mathcal{A}$.

The aim of the present paper is to characterize the spectrum of A in terms of particular solutions of the second order difference equation

$$z \cdot y_n = a_n y_{n+1} + b_n y_n + a_{n-1} y_{n-1}, \quad n \geq 0, \tag{1.1}$$

where $a_{-1} := 1$; these particular solutions $(p_n(z))_{n\geq -1}$, $(q_n(z))_{n\geq -1}$ are obtained from the initializations

$$q_0(z) = 1, \quad q_{-1}(z) = 0, \quad p_0(z) = 0, \quad p_{-1}(z) = -1. \tag{1.2}$$

By the Shohat–Favard Theorem, any $(q_n(z))_{n\geq 0}$ verifying (1.1) and (1.2) is a sequence of formal orthogonal polynomials with respect to some *regular* linear functional c acting on the space of polynomials (and vice versa), that is, $c(q_j \cdot q_k)$ vanishes if $j \neq k$, and is equal to 1 otherwise. Such a linear functional may be given explicitly in terms of the second order difference operator A defined above: by recurrence one easily verifies that $q_n(A)e_0 = q_n(A)^* e_0 = e_n$, $n \geq 0$, and thus $c(p) = (e_0, p(A)e_0)$ for any polynomial p. The moments of this linear functional c may be obtained by a formal expansion at infinity of its symbol, defined by

$$\phi(z) = c(\frac{1}{z - \cdot}) = (e_0, (zI - A)^{-1} e_0), \quad z \in \Omega(A) := \mathbb{C} \setminus \sigma(A),$$

usually referred to as the *Weyl function* of the difference operator A. It is also well–known (see, e.g., [W] or [BK]) that the formal expansion of ϕ at infinity admits a representation as a J–fraction

$$\frac{1 \rfloor}{\lfloor z - b_0} + \frac{-a_0^2 \rfloor}{\lfloor z - b_1} + \frac{-a_1^2 \rfloor}{\lfloor z - b_2} + \frac{-a_2^2 \rfloor}{\lfloor z - b_3} + \dots,$$

with its nth convergent being equal to p_n/q_n, the Padé approximant of order n (at infinity).

Our interest in characterizing the resolvent set $\Omega(A)$ and particular parts of the spectrum of A is also motivated by results partly well–known [W] and partly obtained recently [AKA, B, BK] concerning the convergence of J–fractions for bounded A (that is, $\sup |a_n| + \sup |b_n| < \infty$). For instance, there is convergence of subsequences of $(p_n(z)/q_n(z))_{n\geq 0}$ to $\phi(z)$ pointwise for each $z \in \Omega(A)$ [AKA, Theorem 2], and uniformly in some neighborhood of $z \in \Omega(A)$ [B, Theorem 4.5]. Also, $(p_n/q_n)_{n\geq 0}$ converges locally uniformly to ϕ outside the closure of the numerical range of A [BK, Theorem 3.10], or in any other subdomain of the unbounded connected component of $\Omega(A)$ which does not contain poles [B, Theorem 4.1]. In addition, one may establish general convergence in the sense of [LW, Chapter II.1.5] outside the essential spectrum. Finally [B, Theorem 3.1], $(p_n/q_n)_{n\geq 0}$ converges in capacity in the unbounded connected component of $\Omega(A)$.

The paper is organized as follows: we generalize in Theorem 2.1 a criterion for the resolvent set given by Aptekarev, Kaliaguine and Van Assche [AKA, Theorem 1]. For bounded operators, we establish in Theorem 2.3 a second criterion based only on the asymptotic behavior of the sequence of formal orthogonal polynomials $(q_n)_{n\geq 0}$. A variant of this criterion enables us in Theorem 3.1 to characterize the essential spectrum of bounded second order difference operators. The case of unbounded operators is studied in Proposition 3.3 and Proposition 3.4. In particular, we study in Proposition 3.2 closed extensions of A preserving the matrix product, and the link to the indeterminate case as introduced by WALL [W, Definition 22.1]. Finally, we apply our new characterizations for determining the (essential) spectrum of an asymptotically periodic complex Jacobi matrix.

2. Two criteria for the resolvent set

As our first result, we extend a result of APTEKAREV, KALIAGUINE AND VAN ASSCHE [AKA, Theorem 1] (see also the slight improvement given in [BK, Theorem 2.1]) who characterized the resolvent set of a bounded difference operator A in terms of two solutions of (1.1).

Theorem 2.1. *Suppose that there exists a strictly increasing sequence $(n_j)_{j\geq 0}$ of nonnegative integers, with both $(n_{j+1}-n_j)_{j\geq 0}$ and $(a_{n_j})_{j\geq 0}$ being bounded. Then $z\in\Omega(A)$ iff there exist constants $\gamma,\lambda,C\in\mathbb{C}$ depending on z with $0<\lambda<1$ such that*

$$|q_k(z)|\cdot|\gamma\cdot q_n(z)-p_n(z)|\leq C\cdot\lambda^{n-k},\quad 0\leq k\leq n. \tag{2.1}$$

In this case, we have for the resolvent $R=(zI-A)^{-1}$

$$(e_k,Re_n)=(e_n,Re_k)=q_k(z)\cdot[\gamma\cdot q_n(z)-p_n(z)],\quad 0\leq k\leq n, \tag{2.2}$$

in particular $\gamma=\phi(z)=(e_0,(zI-A)^{-1}e_0)$.

Proof. First suppose that $z\in\Omega(A)$, and write $R:=(zI-A)^{-1}$. Then $(zI-A)R$ is the identity on ℓ^2, and $R(zI-A)$ is the identity on $\mathcal{D}(A)$. In particular,

$$(e_j,(zI-A)Re_k)=(e_j,R(zI-A)e_k)=\delta_{j,k}$$

for $j,k=0,1,2,...$, leading to inhomogeneous second order difference equations for the sequences $((e_k,Re_j))_{j\geq 0}$ and $((e_j,Re_k))_{j\geq 0}$ for each $k\geq 0$. As in the proof of [W, Theorem 60.1] or [AKA, Theorem 1] one deduces the representation (2.2). It remains to show (2.1) which in the case of bounded A (or bounded $(a_j)_{j\geq 0}$) is the classical decay rate for inverses of banded matrices [DMS]. For our more general case, we write shorter $r_n(z):=q_n(z)\phi(z)-p_n(z)$, and notice that

$$(zI-A)y_n=(\underbrace{0,..,0}_{n-1},a_nq_{n+1}(z),-a_nq_n(z),0,0,...)\in\mathcal{C}_0, \tag{2.3}$$

$$\text{with}\quad y_n:=(q_0(z),q_1(z),...,q_n(z),0,0,...)\in\mathcal{C}_0.$$

Consequently, we have for all $n\geq 0$

$$\sum_{j=0}^{n}|q_j(z)|^2=||R(zI-A)y_n||^2\leq||R||^2\cdot|a_n|^2\cdot(|q_n(z)|^2+|q_{n+1}(z)|^2). \tag{2.4}$$

Thus the estimate of Lemma 2.2 below holds for $\beta_n := |q_n(z)|^2 + |q_{n+1}(z)|^2$, $n \geq -1$. Let α, p be as in Lemma 2.2, and $\lambda^2 := \tilde{\lambda}$. Then (2.1) is true for all indices $\max\{0, n-p\} \leq k \leq n$ with $C \geq C_0 := ||R|| \cdot \lambda^{-p}$ according to (2.2). In the case $k < n-p$, we may find a j with $n-p \leq j < n$, and $|a_j| \leq \alpha$. Applying Lemma 2.2 we obtain

$$\begin{aligned}&|q_k(z)|^2 \cdot |r_n(z)|^2 \leq \beta_{k-1} \cdot |r_n(z)|^2 \leq \tilde{C} \cdot \lambda^{2j+2-2k} \cdot |\alpha|^2 \cdot \beta_j \cdot |r_n(z)|^2 \\ &\leq \tilde{C} \cdot \lambda^{2j+2-2k} \cdot |\alpha|^2 \cdot C_0^2 \cdot (\lambda^{2n-2j} + \lambda^{2n-2j-2}),\end{aligned}$$

showing (2.1).

We now suppose that (2.1) holds, and define an operator R with $\mathcal{D}(R) = \mathcal{C}_0$ by (2.2). First as in [AKA, Proof of Theorem 1] one easily verifies using (2.1) that R is bounded. Consequently, its closure (also denoted by R) is a bounded linear operator defined on the whole space ℓ^2. Using the recurrence (1.1) one shows that $R(zI - A)y = y$ for all $y \in \mathcal{C}_0$. Since A is the closure of $\mathcal{A}$, its domain of definition is given by

$$\mathcal{D}(A) = \Big\{ y \in \ell^2 : \exists\, (y_n)_{n\geq 0} \subset \mathcal{C}_0, y = \lim_{n\to\infty} y_n, (\mathcal{A}y_n)_{n\geq 0} \text{ converges (with limit } Ay) \Big\}. \tag{2.5}$$

Consequently, $R(zI - A)y = y$ for all $y \in \mathcal{D}(A)$, showing that $(zI - A)$ has a bounded left inverse. For the inclusion $z \in \Omega(A)$ it remains to show that the image of $zI - A$ equals ℓ^2. First it follows from [K, Theorem IV.5.2] that $\mathcal{R}(zI - A)$ is closed. Also, $e_k - q_k(z)e_0 = [q_k(A) - q_k(zI)]e_0 \in \mathcal{R}(zI - A)$ for all $k \geq 1$, and $\mathcal{C}_0$ is dense in ℓ^2. Consequently, it is sufficient to show that $e_0 \in \mathcal{R}(zI - A)$. Let $y_n := ((e_0, Re_0), (e_1, Re_0), ..., (e_n, Re_0), 0, 0, ...) \in \mathcal{C}_0$, then $(y_n)_{n\geq 0}$ tends to $y := Re_0$. Furthermore, using (1.1) we get

$$(zI - A)y_n = e_0 + a_n \cdot (\underbrace{0, .., 0}_{n-1}, (e_{n+1}, Re_0), -(e_n, Re_0), 0, 0, ...) \in \mathcal{C}_0.$$

By assumption on $(a_n)_{n\geq 0}$, the sequence $((zI - A)y_{n_j})_{j\geq 0}$ tends to e_0, and thus $y \in \mathcal{D}(A)$, with $(zI - A)y = e_0$. □

Lemma 2.2. *Let p, α be some constants and suppose that there exist a strictly increasing sequence of nonnegative integers $(n_j)_{j\geq -1}$ with $n_{-1} = -1$ verifying $n_j - n_{j-1} \leq p$ and $|a_{n_j}| \leq \alpha$ for all $j \geq 0$. Provided that*

$$0 < \sum_{j=-1}^{n-1} \beta_j \leq \rho \cdot |a_n|^2 \cdot \beta_n, \quad n \geq 0,$$

with some constant ρ, there exist constants $\tilde{\lambda} \in (0,1)$ and $\tilde{C} > 0$ such that

$$\beta_k \leq \tilde{C} \cdot \tilde{\lambda}^{n-k} \cdot |a_n^2| \cdot \beta_n, \quad -1 \leq k < n.$$

Proof. Let us first verify be recurrence on $n - k$ that, for $-1 \leq k < n$,

$$\beta_k \cdot \prod_{\ell=k+1}^{n-1} (1 + \frac{1}{\rho \cdot |a_\ell|^2}) \leq \rho \cdot |a_n^2| \cdot \beta_n. \tag{2.6}$$

Here the case $n = k+1$ is trivial. In the case $n > k+1$ we have

$$\rho \cdot |a_n|^2 \cdot \beta_n \geq \beta_k + \sum_{j=k+1}^{n-1} \beta_j \geq \beta_k \cdot \left(1 + \sum_{j=k+1}^{n-1} \frac{1}{\rho \cdot |a_j|^2} \cdot \prod_{\ell=k+1}^{j-1} (1 + \frac{1}{\rho \cdot |a_\ell|^2}) \right),$$

where the sum on the right–hand side simplifies. This shows (2.6). By assumption on $(a_n)_{n\geq 0}$, at least j of the $n-k-1$ numbers $|a_{k+1}|, ..., |a_{n-1}|$ are bounded by α, with the integer j being bounded below by $(n-k)/p - 1$. Consequently,

$$\prod_{\ell=k+1}^{n-1} (1 + \frac{1}{\rho \cdot |a_\ell|^2}) \geq (1 + \frac{1}{\rho \cdot \alpha^2})^j \geq (1 + \frac{1}{\rho \cdot \alpha^2})^{(n-k)/p-1},$$

and a combination with (2.6) gives the assertion of Lemma 2.2. □

For bounded operators A, we may now give a new criterion for the resolvent set based only on asymptotic properties of one of the solutions of (1.1), namely, the sequence of formal orthonormal polynomials.

Theorem 2.3. *If $z \in \Omega(A)$ then*

$$\sup_{n\geq 0} \frac{\sum_{j=0}^n |q_j(z)|^2}{|a_n|^2[|q_n(z)|^2 + |q_{n+1}(z)|^2]} =: \rho < \infty. \tag{2.7}$$

Provided that A is bounded, also the converse is true.

Proof. First with the help of (2.3) and (2.4) we see that $z \in \Omega(A)$ implies (2.7) with $\rho \leq ||(zI-A)^{-1}||^2$. Now, let A be bounded, and suppose that (2.7) holds. In the first part of the proof we want to show that $(T_n(w_n'))_{n\geq 0}$ is a Cauchy sequence (in the complex plane $\mathbb{C}$), where

$$T_n(w) = T_n(w,z) = \frac{p_{n+1}(z) + wp_n(z)}{q_{n+1}(z) + wq_n(z)}, \quad w_n' := \overline{q_n(z)/q_{n+1}(z)} \in \overline{\mathbb{C}}.$$

First notice that the conclusion of Lemma 2.2 holds for $\beta_n := |q_n(z)|^2 + |q_{n+1}(z)|^2$ according to (2.7). Also, in the case of finite w_n', w_{n-1}' we have $|q_{n+1}(z) + w_n' q_n(z)|^2 = ||(1, w_n')^T||^2 \cdot \beta_n$, and

$$\begin{aligned} &||(1, -w_n') \cdot \begin{pmatrix} p_{n+1}(z) & -q_{n+1}(z) \\ -p_n(z) & q_n(z) \end{pmatrix} \cdot \begin{pmatrix} q_{n-1}(z) & q_n(z) \\ p_{n-1}(z) & p_n(z) \end{pmatrix} \cdot (w_{n-1}', 1)^T|| \\ &= \frac{1}{|a_{n-1} \cdot a_n|} \cdot ||(1, -w_n') \cdot \begin{pmatrix} z - b_n & a_{n-1} \\ -a_n & 0 \end{pmatrix} \cdot (w_{n-1}', 1)^T|| \\ &\leq \frac{||(1, w_n')^T|| \cdot ||(1, w_{n-1}')^T|| \cdot ||(zI - A)e_k||}{|a_{n-1} \cdot a_n|}. \end{aligned}$$

Combining these elements, we may conclude that there exist $C_3, C_4 > 0$ such that for all $k < n$

$$\beta_k \cdot |T_n(w_n') - T_{n-1}(w_{n-1}')| \leq \frac{C_3 \cdot \beta_k}{|a_{n-1} \cdot a_n| \cdot \sqrt{\beta_{n-1} \cdot \beta_n}} \leq C_4 \cdot \tilde{\lambda}^{n-k}.$$

By a slightly different reasoning, the same conclusion is obtained in the cases $w_n' = \infty$ or $w_{n-1}' = \infty$. It follows that $(T_n(w_n'))_{n\geq 0}$ is a Cauchy sequence. Denoting the limit of $(T_n(w_n'))_{n\geq 0}$ by $\gamma \in \mathbb{C}$, and $r_n(z) := \gamma \cdot q_n(z) - p_n(z)$, we have $|T_n(w_n) - \gamma| \leq C/|a_n| \cdot \beta_n$ for some constant $C > 0$, and

$$\begin{aligned} & |T_n(w_n') - \gamma|^2 \cdot \beta_n^2 = |(r_n(z), r_{n+1}(z)) \cdot (\overline{q_n(z)}, \overline{q_{n+1}(z)})^T|^2 \\ = \quad & -|(r_n(z), r_{n+1}(z)) \cdot (-q_{n+1}(z), q_n(z))^T|^2 \\ & +||(r_n(z), r_{n+1}(z)) \cdot \begin{pmatrix} \overline{q_n(z)} & -q_{n+1}(z) \\ \overline{q_{n+1}(z)} & q_n(z) \end{pmatrix} ||^2 \\ = \quad & -\frac{1}{|a_n|^2} + ||(r_n(z), r_{n+1}(z))||^2 \cdot \beta_n, \end{aligned}$$

and thus for $0 \leq k \leq n$

$$\begin{aligned} & |q_k(z)|^2 |r_n(z)|^2 \leq \beta_{k-1} \cdot ||(r_n(z), r_{n+1}(z))||^2 \\ & \leq [1 + C] \cdot \frac{\beta_{k-1}}{|a_n|^2 \beta_n} \leq [1 + C] \cdot \tilde{C} \cdot \tilde{\lambda}^{n-k+1}. \end{aligned}$$

Therefore, $z \in \Omega(A)$ by Theorem 2.1. □

Remark 2.4. Let A be bounded. It is easy to see that we may reformulate Theorem 2.3 as follows: we have $z \in \sigma(A)$ iff the sequence $(y_n/||y_n||)_{n\geq 0}$ with y_n as in (2.3) contains a subsequence of approximate eigenvectors. Also, it follows from (2.3) that

$$\frac{\sum_{j=0}^n |q_j(z)|^2}{|a_n|^2 [|q_n(z)|^2 + |q_{n+1}(z)|^2]} \geq \frac{1}{||zI - A||^2}, \quad z \in \mathbb{C}, n \geq 0.$$

Thus we have $z \in \Omega(A)$ iff the sequence of numerators in (2.7), and denominators in (2.7), respectively, have the same asymptotic behavior (namely at least exponential growth by Lemma 2.2).

Remark 2.5. It seems that the statement of Theorem 2.3 was unnoticed even for the classical case of real bounded Jacobi matrices. Here the spectrum of A coincides with the support of the measure of orthogonality of $(q_n)_{n\geq 0}$. The exponential growth of $(|q_n|^2 + |q_{n+1}|^2)_{n\geq 0}$ outside the support was already established by SZWARC [Szw1, Corollary 1], who showed by examples [Szw2] that there may be also exponential growth inside the support.

Remark 2.6. For bounded A, it follows from (2.4) and is shown in [B, Proposition 2.2] that the sequence $(a_n q_{n+1}/q_n)_{n\geq 0}$ of ratios of monic formal orthogonal polynomials — referred to as a *tail sequence* of the corresponding J–fraction [LW, Section II.1.2] — is normal in $\Omega(A)$ with respect to the chordal metric. At least in the case of real a_j, b_j, $\Omega(A)$ is the maximal set with this property: define $\delta_n(z)$ as the distance from $z \in \mathbb{C}$ to the set of zeros of q_n, then it is well–known [Sze, Theorem 6.1.1] that $\delta_n(z) \to 0$ for z being a non–isolated element of $\sigma(A)$. The same property may be shown to be valid for isolated elements of $\sigma(A)$. In contrast, according to normality we have $\delta_n(z) \not\to 0$ for $z \in \Omega(A)$, more precisely, $\liminf \max\{\delta_n(z), \delta_{n+1}(z)\} > 0$ for any $z \in \Omega(A)$.

3. A criterion for the essential spectrum

In the theory of second order difference equations, one often is interested in describing the perturbation of solutions of (1.1) while perturbing the coefficients. In terms of operators, if the entries of the difference of two (complex) Jacobi matrices tend to zero along diagonals, then the difference of the corresponding difference operators is known to be *compact*. Following KATO [K, Chapter IV.5.6], here it is of interest to consider some closed non–empty subset of $\sigma(A)$, the *essential spectrum* $\sigma_{ess}(A)$; this part of the spectrum remains invariant under compact perturbations [K, Theorem IV.5.35]. We will show in the first part of the proof of Proposition 3.2 below that for second order difference operators we have the simple characterization

$$\sigma_{ess}(A) = \{ z \in \mathbb{C} : \mathcal{R}(zI - A) \text{ is not closed } \}. \tag{3.1}$$

Denote by $\mathcal{A}^{(k)}$ the "shifted" (complex) Jacobi matrix obtained by replacing (a_j, b_j) in $\mathcal{A}$ by (a_{j+k}, b_{j+k}), $j \geq 0$. As in [K, Chapter IV.6.1] one shows that the difference operators corresponding to $\mathcal{A} = \mathcal{A}^{(0)}$, and to $\mathcal{A}^{(k)}$, respectively, have the same essential spectrum for any $k \geq 0$, and thus $\sigma_{ess}(A) \subset \sigma(A) \cap \sigma(A^{(1)})$. In fact, as a corollary of the characterization given below we obtain equality

Theorem 3.1. *Let A be bounded. Then $z \notin \sigma_{ess}(A)$ iff*

$$\sup_{n \geq 0} \frac{\sum_{j=0}^{n} [|p_j(z)|^2 + |q_j(z)|^2]}{|a_n|^2 [|p_n(z)|^2 + |p_{n+1}(z)|^2 + |q_n(z)|^2 + |q_{n+1}(z)|^2]} =: \rho' < \infty. \tag{3.2}$$

In this case, there exists a nontrivial minimal *solution* $(s_n(z))_{n \geq 0} \in \ell^2$ *of (1.1) verifying*

$$\lim_{n \to \infty} \frac{|s_n(z)|^2 + |s_{n+1}(z)|^2}{|p_n(z)|^2 + |p_{n+1}(z)|^2 + |q_n(z)|^2 + |q_{n+1}(z)|^2} = 0, \tag{3.3}$$

with $z \in \Omega(A)$ iff $s_{-1}(z) \neq 0$, and $z \in \Omega(A^{(1)})$ iff $s_0(z) \neq 0$.

Proof. We will prove in Proposition 3.3 below in a more general setting that $z \notin \sigma_{ess}(A)$ implies (3.2). In the first part of the present proof, let us show that (3.2) implies (3.3). Let $\delta \in (0, 1/\sqrt{2})$. For fixed z verifying (3.2) we write

$$M_n := \begin{pmatrix} q_n(z) & q_{n+1}(z) \\ p_n(z) & p_{n+1}(z) \end{pmatrix},$$

denote by $|| \cdot ||_F$ the Froebenius matrix norm, and by $\chi(\cdot, \cdot)$ the chordal metric. We may find $w_n \in \overline{\mathbb{C}}$ such that

$$||M_n \cdot (w_n, 1)^T|| / ||(w_n, 1)^T|| \geq \delta \cdot ||M_n||_F, \quad n \geq 0.$$

Then as in the proof of Theorem 2.3 we have

$$\begin{aligned} \chi(T_n(w_n), T_{n-1}(w_{n-1})) &\leq \frac{||(1, w_n)^T|| \cdot ||(1, w_{n-1})^T|| \cdot ||(zI - A)e_k||}{|a_{n-1} \cdot a_n| \cdot ||M_{n-1}(w_{n-1}, 1)^T|| \cdot ||M_n(w_n, 1)^T||} \\ &\leq \frac{||(zI - A)e_k||}{\delta^2 |a_{n-1} \cdot a_n| \cdot ||M_{n-1}||_F \cdot ||M_n||_F}. \end{aligned}$$

Again by using Lemma 2.2 we may conclude that $(T_n(w_n))_{n\geq 0}$ converges to some $\phi(z) \in \overline{\mathbb{C}}$, with error estimate

$$\chi(T_n(w_n), \phi) \leq \frac{C}{|a_n| \cdot ||M_n||_F^2}, \quad n \geq 0, \tag{3.4}$$

$C > 0$ being a suitable constant. We write $\phi(z) = s_0(z)/s_{-1}(z)$, with $||(s_{-1}(z), s_0(z))^T|| = 1$, and define $(s_n(z))_{n\geq 1}$ by (1.1), i.e., $s_n(z) = s_0(z)q_n(z) - s_{-1}(z)p_n(z)$, $n \geq -1$. Then

$$\chi(T_n(w_n), \phi)^2 \cdot \frac{||M_n(w_n, 1)^T||^2}{||(w_n, 1)^T||^2} = \frac{|(s_n(z), s_{n+1}(z)) \cdot (w_n, 1)^T|^2}{||(w_n, 1)^T||^2}$$
$$= ||(s_n(z), s_{n+1}(z))^T||^2 - \frac{|(s_n(z), s_{n+1}(z)) \cdot (1, -\overline{w_n})^T|^2}{||(w_n, 1)^T||^2}.$$

In the case $||(q_n(z), q_{n+1}(z))^T|| \geq ||(p_n(z), p_{n+1}(z))^T||$, the choice $w_n = \overline{q_n(z)}/\overline{q_{n+1}(z)}$ is admissible, otherwise we may take $w_n = \overline{p_n(z)}/\overline{p_{n+1}(z)}$. In both cases we conclude that

$$||(s_n(z), s_{n+1}(z))^T||^2 - \frac{2}{|a_n|^2 \cdot ||M_n||_F^2} \leq \chi(T_n(w_n), \phi)^2 \cdot ||M_n||_F^2, \quad n \geq 0,$$

and thus by (3.4)

$$\frac{||(s_n(z), s_{n+1}(z))^T||}{||M_n||_F} \leq \frac{\tilde{C}}{|a_n| \cdot ||M_n||_F^2}, \quad n \geq 0,$$

with some constant $\tilde{C}$. Here the right–hand side tends geometrically to zero by Lemma 2.2, showing (3.3) and the inclusion $(s_n(z))_{n\geq 0} \in \ell^2$.

Now, suppose that $\phi(z) \neq \infty$, that is, $s_{-1}(z) \neq 0$. Then

$$|1 - \sqrt{1 + |\phi(z)|^2} \cdot \frac{||(q_n(z), q_{n+1}(z))^T||}{||M_n||_F}|$$
$$\leq \frac{||M_n - (1, \phi(z))^T \cdot (q_n(z), q_{n+1}(z))||_F}{||M_n||_F} = \frac{||(s_n(z), s_{n+1}(z))^T||}{|s_{-1}(z)| \cdot ||M_n||_F},$$

with the right–hand side tending to zero. Consequently, (3.2) implies (2.7), and thus $z \in \Omega(A)$ by Theorem 2.3. On the other hand, $s_{-1}(z) = 0$ implies that $(q_n(z))_{n\geq 0}$ is a multiple of $(s_n(z))_{n\geq 0}$, and hence an element of ℓ^2. Again from Theorem 2.3 it follows that $z \notin \Omega(A)$. Similarly, in the case $s_0(z) \neq 0$ one shows that $(||M_n||)_{n\geq 0}$ behaves asymptotically like $(||(p_n(z), p_{n+1}(z))^T||)_{n\geq 0}$. Taking into account that $(a_0 \cdot p_{n+1}(z))_{n\geq -1}$ is the sequence of formal orthogonal polynomials associated to the shifted difference operator $A^{(1)}$, the equivalence between $s_0(z) \neq 0$ and $z \in \Omega(A^{(1)})$ follows again from Theorem 2.3. As a consequence, for a $z \in \mathbb{C}$ verifying (3.2) there holds $z \in \Omega(A) \cup \Omega(A^{(1)}) \subset \mathbb{C} \setminus \sigma_{ess}(A)$. □

In what follows we will consider more in detail unbounded second difference operators. Here there are several closed operators which may be associated with a complex

Jacobi matrix. To be more precise, let us consider the adjoint A^* of the closure A of $\mathcal{A}$. We write $\mathcal{A}^H$ for the infinite matrix obtained by taking complex conjugates in $\mathcal{A}$, i.e., by taking the Hermitian counterpart of $\mathcal{A}$. As usual [K, Section III.5.5], $\mathcal{D}(A^*)$ consists of all $y \in \ell^2$ such that there exists an $y^* \in \ell^2$ with the property

$$(3.5) \qquad (y, Av) = (y^*, v) \quad \text{for all} \quad v \in \mathcal{D}(A),$$

and then $A^* y = y^*$. According to (2.5), it is sufficient to verify that (3.5) holds for $v \in \mathcal{C}_0$. Thus, with $y^* = (y_n^*)_{n\geq 0}$, $y = (y_n)_{n\geq 0}$, equation (3.5) may be rewritten as

$$y_k^* = \overline{(y, Ae_k)} = \overline{a_{k-1}} y_{k-1} + \overline{b_k} y_k + \overline{a_k} y_{k+1} = (e_k, \mathcal{A}^H \cdot y), \quad k \geq 0$$

(here the matrix product formally is defined for any sequence $y \in \ell^2$). Thus we obtain the shorter characterisation

$$(3.6) \qquad \mathcal{D}(A^*) = \{y \in \ell^2 : \mathcal{A}^H \cdot y \in \ell^2\},$$

with $A^* y = \mathcal{A}^H \cdot y$ for $y \in \mathcal{D}(A^*)$. Also, we know from [K, Section III.5.5 and Problem III.5.24] that $A^* = (\mathcal{A})^*$ is closed.

In the case of $a_n, b_n \in \mathbb{R}$, it is known that A^* is an extension of A, and there exist classical characterizations in terms of solutions of (1.1) for the case $A^* = A$ of a self-adjoint operator. This again is known to be equivalent to the existence of a unique solution of the moment problem, that is, to the existence of a unique positive measure μ satisfying $c(f) = \int f(x)\, d\mu(x)$ for all functions f continuous in $\mathbb{R}$ (see, e.g., [NS, Chapter II.7]). Notice that here $\sigma_{ess}(A)$ is a subset of the real line (see [K, Problem V.3.7]), and that $\dim \mathcal{N}(zI - A) = 0$ for all $z \notin \mathbb{R}$. In the general case we propose

Proposition 3.2. *The operator $B := (\mathcal{A}^H)^*$ is the maximal closed extension of A preserving the matrix product, that is, $By = \mathcal{A} \cdot y$ for all $y \in \mathcal{D}(B)$. Moreover, provided that $\sigma_{ess}(A) \neq \mathbb{C}$, there is equivalent*

- (a) *Recurrence (1.1) has two lin. indep. ℓ^2–solutions for one $z \in \mathbb{C}$.*
- (b) *Recurrence (1.1) has two lin. indep. ℓ^2–solutions for all $z \in \mathbb{C}$.*
- (c) *There holds $\dim \mathcal{N}(zI - B) > \dim \mathcal{N}(zI - A)$ for all $z \in \mathbb{C}$.*
- (d) *There holds $\dim \mathcal{N}(zI - B) > \dim \mathcal{N}(zI - A)$ for one $z \in \mathbb{C}$.*
- (e) *$B \neq A$, that is, $\mathcal{D}(A)$ is a proper subset of $\mathcal{D}(B)$.*

In any of these cases, $\Omega(A)$ is empty.

Proof. It follows from (3.6) that B is the maximal closed extension of $\mathcal{A}$ preserving the matrix product. Thus the same is true for A, the minimal closed extension of $\mathcal{A}$. For a proof of the second part, one easily verifies using (3.6) that

$$(3.7) \quad 0 \leq \dim \mathcal{N}(zI - A) \leq \dim \mathcal{N}(zI - B) = \dim \mathcal{N}((zI - A)^*) \leq 1, \quad z \in \mathbb{C},$$

with $\dim \mathcal{N}(zI - B) = 1$ iff $(q_n(z))_{n\geq 0} \in \ell^2$. Moreover, in the case of closed $\mathcal{R}(zI - A)$, the codimension of $\mathcal{R}(zI - A)$ in ℓ^2 coincides with the dimension of $(\mathcal{R}(zI - A))^\perp =$

$\mathcal{N}((zI-A)^*)$ [K, Lemma III.1.40], and thus is bounded by one. Comparing with [K, Section IV.5.6], we obtain the simple characterization of $\sigma_{ess}(A)$ as stated in (3.1).

We are now prepared to show the second part. The equivalence between (a) and (b) has been established by WALL [W, Theorem 22.1] (referred to as the *indeterminate case*). Also, by taking suitable linear combinations we see that property (b) is equivalent to $\dim \mathcal{N}((zI-A)^*) = 1$, and $e_0 \in \mathcal{R}((zI-A)^*)$ for all $z \in \mathbb{C}$. If $\dim \mathcal{N}(zI-A) = 1$, then $y' := (q_n(z))_{n\geq 0} \in \mathcal{D}(A)$, and with $y \in \mathcal{D}((zI-A)^*)$ verifying $(zI-A)^* y = e_0$ we get $(e_0, y') = ((zI-A)^* y, y') = (y, (zI-A)y') = 0$, in contradiction to the fact that $(e_0, y') = q_0(z) = 1$. Consequently, (b) implies (c), and the implications $(c) \Rightarrow (d)$ and $(d) \Rightarrow (e)$ are trivially true. Let finally (e) hold, and consider a $z \notin \sigma_{ess}(A)$. Then both $\mathcal{R}(zI-A)$ and $\mathcal{R}((zI-A)^*)$ are closed by (3.1) and [K, Theorem IV.5.13]. Notice that $\mathcal{R}(zI-B) \supset \mathcal{R}(zI-A) = (\mathcal{N}((zI-A)^*))^\perp$, and that $\mathcal{R}(zI-B)$ is obtained by taking complex conjugates of elements of $\mathcal{R}((zI-A)^*) = (\mathcal{N}(zI-A))^\perp$. Consequently, $A \neq B$ implies $\dim \mathcal{N}((zI-A)^*) \neq \dim \mathcal{N}(zI-A)$; more precisely, $1 = \dim \mathcal{N}((zI-A)^*)$ and $0 = \dim \mathcal{N}(zI-A)$ by (3.7). Thus $\mathcal{R}((zI-A)^*) = \ell^2$, in particular, $e_0 \in \mathcal{R}((zI-A)^*)$, which together with $1 = \dim \mathcal{N}((zI-A)^*)$ implies property (a). □

Let us recall from [W, Exercise V.5.1] that the determinate case holds for instance if $\sum_{n=0}^\infty 1/|a_n|$ diverges, as it is the case for the operator discussed in Theorem 2.1.

We are now prepared to discuss extensions of Theorem 3.1 to the case of unbounded second order difference operators

Proposition 3.3. *For an arbitrary second order difference operator A, the inclusion $z \in \mathbb{C} \setminus \sigma_{ess}(A)$ implies that (3.2) holds. In the indeterminate case, the converse is not true.*

Proof. In the first part of the proof let us examine the indeterminate case, that is

$$\sum_{j=0}^\infty |p_j(z)|^2 + |q_j(z)|^2 < \infty$$

for all $z \in \mathbb{C}$. One easily verifies by recurrence that $a_n \cdot (q_n(z) \cdot p_{n+1}(z) - q_{n+1}(z) \cdot p_n(z)) = 1$ for all $n \geq -1$ and for all $z \in \mathbb{C}$. From the Cauchy–Schwarz inequality it follows that

$$\begin{aligned} &\frac{1}{|a_n|^2[|p_n(z)|^2 + |p_{n+1}(z)|^2 + |q_n(z)|^2 + |q_{n+1}(z)|^2]} \\ &\leq |p_n(z)|^2 + |p_{n+1}(z)|^2 + |q_n(z)|^2 + |q_{n+1}(z)|^2, \end{aligned}$$

with the right–hand side tending to zero. Consequently, here (3.2) holds for all $z \in \mathbb{C}$.

We now suppose that (3.2) does not hold for some $z \in \mathbb{C} \setminus \sigma_{ess}(A)$. Let $\Lambda \subset \mathbb{N}$ be infinite such that

$$\lim_{n\to\infty, n\in\Lambda} \frac{\sum_{j=0}^n [|p_j(z)|^2 + |q_j(z)|^2]}{|a_n|^2[|p_n(z)|^2 + |p_{n+1}(z)|^2 + |q_n(z)|^2 + |q_{n+1}(z)|^2]} = \infty,$$

and consider $\Lambda' := \{n \in \Lambda : ||y_n|| > ||y_n'||\}$, with

$$(zI - A)y_n' = (-1, \underbrace{0, .., 0}_{n-2}, a_n p_{n+1}(z), -a_n p_n(z), 0, 0, ...) \in \mathcal{C}_0,$$
$$\text{with} \quad y_n' := (p_0(z), p_1(z), ..., p_n(z), 0, 0, ...) \in \mathcal{C}_0,$$

and $y_n \in \mathcal{C}_0$ as defined by (2.3). As shown above, the indeterminate case is excluded, and thus $(||y_n|| + ||y_n'||)_{n \geq 0}$ tends to infinity by Proposition 3.2(b).

If Λ' is not finite, then $(||y_n||)_{n \in \Lambda'}$ tends to infinity by construction. Consequently, the sequence $(v_n)_{n \in \Lambda'}$, $v_n := y_n/||y_n|| \in \mathcal{D}(A)$, is a sequence of approximate eigenvectors, namely,

$$||v_n|| = 1, \quad n \in \Lambda', \quad \lim_{n\to\infty, n \in \Lambda'} (zI - A)v_n = 0, \quad \lim_{n\to\infty, n\in\Lambda'} (e_j, v_n) = 0, \quad j \geq 0,$$

and thus $(v_n)_{n \in \Lambda'}$ does not contain a convergent subsequence. From [K, Theorems IV.5.10 and IV.5.11] it follows that $z \in \sigma_{ess}(A)$, a contradiction. It remains to discuss the case of finite Λ', but here using the sequence $(y_n'/||y_n'||)_{n \in \Lambda \setminus \Lambda'}$ we obtain the same conclusion. □

With the same argument as in the first part of the preceding proof one shows that (2.7) holds in the indeterminate case for all $z \in \mathbb{C}$ though here $\Omega(A)$ is empty.

If A is bounded, then of course $\Omega(A)$ contains $|z| > ||A||$. We may conclude from Proposition 3.2(a) that the sequence $(s_n(z))_{n \geq -1}$ of Theorem 3.1 is unique up to multiplication with a constant. In the general case we have the following generalization of Theorem 3.1

Proposition 3.4. *Suppose that the determinate case holds, and let* $k \geq 0$*. Then* $\sigma_{ess}(A) = \sigma(A^{(k)}) \cap \sigma(A^{(k+1)})$*. More precisely, for any* $z \in \mathbb{C} \setminus \sigma_{ess}(A)$ *there exist a non–trivial* ℓ^2*–solution* $(s_n(z))_{n \geq -1}$ *of (1.1), with*

$$\Omega(A^{(k)}) = \{z \in \mathbb{C} \setminus \sigma_{ess}(A) : s_{k-1}(z) \neq 0\}. \tag{3.8}$$

Furthermore, $s_n(z) = a_{k-1} \cdot s_{k-1}(z) \cdot (e_{n-k}, (zI - A^{(k)})^{-1} e_0)$ *for* $n \geq k$ *and* $z \in \Omega(A^{(k)})$*. In particular,* $\phi^{(k)} := s_k/(a_{k-1} \cdot s_{k-1})$ *is a meromorphic continuation of the Weyl function of* $A^{(k)}$ *in* $\mathbb{C} \setminus \sigma_{ess}(A)$*.*

Proof. If $\sigma_{ess}(A) = \mathbb{C}$ then the assertion of Proposition 3.4 is trivially true. We therefore may suppose for the remainder of the proof that a fixed $z \in \mathbb{C} \setminus \sigma_{ess}(A)$ is given. By Proposition 3.2 and (3.7) there holds $\dim \mathcal{N}(zI - A) = \dim \mathcal{N}((zI - A)^*) \leq 1$. Consequently [K, Chapter IV.5.6], z is either an eigenvalue of A, or an element of $\Omega(A)$. In the first case we find a nontrivial $y \in \ell^2$ satisfying $(zI - A)y = 0$, and otherwise there exist a $y \in \ell^2$ with $(zI - A)y = e_0$. Thus in both cases we have found an ℓ^2–solution $(s_n(z))_{n \geq -1}$ of (1.1). Furthermore, we have formally

$$(z\mathcal{I} - \mathcal{A}^{(k)}) \cdot (s_{n+k}(z))_{n \geq 0} = a_{k-1} s_{k-1}(z) e_0, \tag{3.9}$$

and therefore $(s_{n+k}(z))_{n \geq 0} \in \mathcal{D}(A^{(k)})$ by Proposition 3.2. If $s_{k-1}(z) = 0$, then z is an eigenvalue of $A^{(k)}$ and thus $z \notin \Omega(A^{(k)})$. On the other hand, $s_{k-1}(z) \neq 0$

implies that $e_0 \in \mathcal{R}(zI - A^{(k)})$, and thus $z \in \Omega(A^{(k)})$, as claimed in (3.8). Since the quantities $s_k(z)$ and $s_{k-1}(z)$ may not vanish simultaneously according to (1.1), the identity $\sigma_{ess}(A) = \sigma(A^{(k)}) \cap \sigma(A^{(k+1)})$ follows from (3.8). Notice also that (3.9) implies $(s_{n+k}(z))_{n\geq 0} = a_{k-1} \cdot s_{k-1}(z) \cdot (zI - A^{(k)})^{-1} e_0$, as claimed in the second part of the assertion.

Suppose finally that $s_{k-1}(z) \neq 0$. Since $\Omega(A^{(k)})$ is open, s_{k-1} will not vanish in some neighborhood of z, and $\phi^{(k)}$ coincides there with the Weyl function of $A^{(k)}$ according to the above representation. On the other hand, $s_{k-1}(z) = 0$ implies $s_k(z) \neq 0$, here $\phi^{(k+1)}$ is analytic in some neighborhood of z. Since $1/\phi^{(k)}(z) = z - b_k - a_k^2 \cdot \phi^{(k+1)}(z)$, it follows that $1/\phi^{(k)}$ is analytic in some neighborhood of z. □

Remark 3.5. With the assumptions of Proposition 3.4, let ζ be an isolated element of $\sigma(A^{(k)})$. Following [B, Proof of Theorem 5.3] one shows that $\phi^{(k)}$ has an essential singularity at ζ iff $\zeta \in \sigma_{ess}(A)$; otherwise ζ is eigenvalue of $A^{(k)}$, with algebraic multiplicity given by the multiplicity of the pole of $\phi^{(k)}$. This extends a well–known property of real Jacobi–matrices: under the above assumptions, the operator $A^{(k)}$ is self-adjoint for real a_n, b_n. It follows (see, e.g., [K, Section V.3.5]) that $\sigma(A^{(k)}) \subset \mathbb{R}$, and any isolated point of $\sigma(A^{(k)})$ is necessarily an eigenvalue, with algebraic multiplicity coinciding with the geometric multiplicity, i.e., being equal to one. In fact [NS, Chapter II.8.5], here we have for $\phi^{(k)}$ the well-known representation as a Markov function, with the support of the corresponding measure being equal to $\sigma(A^{(k)})$.

Let us finally illustrate our findings by discussing (asymptotically) periodic difference operators.

Example 3.6. Suppose that the complex Jacobi–matrix is m–periodic, that is, $a_{jm+k} = a_k, b_{jm+k} = b_k$, $k = 0, 1, ..., m-1$, $j \geq 0$. It is well–known (see, e.g., [BK, Section 2.2]) that here the sequences $(p_n(z))_{n\geq 0}$ and $(q_n(z))_{n\geq -1}$ verify the recurrence relation

$$y_{(j+1)m+k} = h(z) \cdot y_{jm+k} - y_{(j-1)m+k}, \quad j \geq 0, \quad 0 \leq k < m, \tag{3.10}$$

with some polynomial h for which we have several representations

$$h(z) = \frac{q_{2m-1}(z)}{q_{m-1}(z)} = \frac{p_{2m}(z)}{p_m(z)} = q_m(z) - a_{m-1} p_{m-1}(z).$$

Also, the Weyl function of A is given by

$$\phi(z) = (e_0, (zI - A)^{-1} e_0) = \frac{q_m(z) + a_{m-1} p_{m-1}(z) + \sqrt{h(z)^2 - 1}}{2 a_{m-1} q_{m-1}(z)},$$

with the branch of the root chosen such that $\phi(\infty) = 0$. In [BK, Section 2.3], the authors claimed that $E := \{z \in \mathbb{C} : h(z) \in [-2, 2]\}$ is the essential spectrum of A. Let us here present a different proof based on Theorem 3.1. In fact, according to (3.10) we obtain the representation

$$\begin{bmatrix} p_{jm+k}(z) \\ q_{jm+k}(z) \end{bmatrix} = A_k(z) \cdot \begin{bmatrix} T_j(h(z)/2) \\ U_j(h(z)/2) \end{bmatrix}, \quad j = 0, 1, 2, ..., \quad 0 \leq k < m,$$

with some matrix $A_k(z)$, where T_j and U_j denotes the Chebyshev polynomial of fist and of second kind. Now T_j and U_j are bounded on $[-1,1]$ by $j+1$, and hence $||M_n(z)||$ grows for $z \in E$ at most linearly in n (and not exponentially). It follows from Theorem 3.1 that $E \subset \sigma_{ess}(A)$. On the other hand, for $z \notin E$, the characteristic equation $y^2 = h(z)y - 1$ has roots of different modulus, namely $w(z)$ with $|w(z)| > 1$, and $1/w(z)$. It follows from (3.10) that there exist matrices $B_k(z)$ with

$$\begin{bmatrix} q_{jm+k}(z) \\ p_{jm+k}(z) \end{bmatrix} = B_k(z) \cdot \begin{bmatrix} w(z)^j \\ w(z)^{-j} \end{bmatrix}, \quad j,k = 0,1,2,....$$

Denoting by $adj\, B$ the adjoint of B, we have for $0 \le k < m$ (we omit the argument z)

$$\frac{1}{a_k} = \begin{bmatrix} -p_{jm+k} \\ q_{jm+k} \end{bmatrix}^T \begin{bmatrix} q_{jm+k+1} \\ p_{jm+k+1} \end{bmatrix} = \begin{bmatrix} -w^{-j} \\ w^j \end{bmatrix}^T adj\, B_k \cdot B_{k+1} \begin{bmatrix} w^j \\ w^{-j} \end{bmatrix},$$

with the right–hand side being independent of j and z. Thus $adj\, B_k(z) \cdot B_{k+1}(z)$ is diagonal and of the form $diag\,(d_k(z), 1/a_k + d_k(z))$. It follows that $B_k(z) = C_k(z) diag\,(d_{1,k}(z), d_{2,k}(z))$, $B_{k+1}(z) = C_k(z) diag\,(d_{3,k}(z), d_{4,k}(z))$, with $\det C_k(z) diag\,(d_{2,k}(z), d_{1,k}(z)) diag\,(d_{3,k}(z), d_{4,k}(z)) = diag\,(d_k(z), 1/a_k + d_k(z))$; in particular,

$$M_{jm+k}(z) = C_k(z) \cdot diag\,(w(z)^j, w(z)^{-j}) \cdot D_k(z), \quad D_k(z) = \begin{bmatrix} d_{1,k}(z) & d_{3,k}(z) \\ d_{2,k}(z) & d_{4,k}(z) \end{bmatrix},$$

with $\det D_k(z) \cdot \det C_k(z) = 1/a_k \neq 0$. Consequently, $||M_{jm+k}(z)||$ may be bounded below and above by some positive constant depending on k times $|w(z)|^j$, and $z \notin \sigma_{ess}(A)$ follows from Theorem 3.1. Thus we have shown that $E = \sigma_{ess}(A)$.

Notice that ϕ is meromorphic in the connected and unbounded set $\mathbb{C} \setminus E$ (and in no larger set). It follows from Proposition 3.4 that $\Omega(A)$ is obtained by dropping from $\mathbb{C} \setminus E$ the poles of the Weyl function of A (see [BK, Theorem 2.7]), and this property is true not only for periodic but even for asymptotically periodic difference operators (that is, compact perturbations of periodic operators). This specifies a result of Geronimus [G] and Máté, Nevai and Van Assche [MNA, Theorems 13 and 14] who studied the special case of real coefficients.

References

[AKA] A.I. Aptekarev, V. Kaliaguine & W. Van Assche, Criterion for the resolvent set of non-symmetric tridiagonal operators, *Proc. Amer. Math. Soc.* **123** (1995) 2423-2430.

[B] B. Beckermann, On the convergence of bounded J–fractions on the resolvent set of the corresponding second order difference operator, Publication ANO 376, Université de Lille (1997).

[BK] B. Beckermann & V.A. Kaliaguine, The diagonal of the Padé table and the approximation of the Weyl function of second order difference operators, *Constructive Approximation* **13** (1997) 481-510.

[DMS] S. Demko, W.F. Moss & P.W. Smith, Decay Rates for Inverses of Band Matrices, *Math. Comp.* **43** (1984) 491-499.

[G] J.L. GERONIMUS, On some difference equations and associated systems of orthogonal polynomials, *Zapiski Math. Otdel. Phys.-Math. Facul. Kharkov Universiteta and Kharkov Mathem. Obchestva*, **XXV** (1957), 87-100 (in Russian).

[K] T. KATO, Perturbation theory for linear operators, Springer Verlag (1980).

[LW] L. LORENTZEN & H. WAADELAND, Continued fractions with applications, North–Holland, Amsterdam (1992).

[MNA] A. MÁTÉ, P. NEVAI & W. VAN ASSCHE, The supports of measures associated with orthogonal polynomials and the spectra of the related self adjoint operators, *Rocky Mountain J. of Math.*, **21** (1991), 501-527.

[NS] E.M. NIKISHIN & V.N. SOROKIN, Rational Approximations and Orthogonality, *Translations of Mathematical Monographs* **92**, Am. Math. Soc., Providence, R.I. (1991).

[Szw1] R. SZWARC, A Lower Bound for Orthogonal Polynomials with an Application to Polynomial Hypergroups, *J. Approx. Theory* **81** (1995) 145-150.

[Szw2] ———, A Counterexample to Subexponential Growth of Orthogonal Polynomials, *Constructive Approximation* **11** (1995) 381-389.

[Sze] G. SZEGÖ, Orthogonal polynomials, AMS, Providence (1975).

[W] H.S. WALL, Analytic Theory of Continued Fractions, Chelsea, Bronx NY (1973).

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