Three-fold symmetric polynomials with an algebraic classical behaviour

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Definition
A monic Orthogonal Polynomial Sequence (OPS) \( \{ P_n \}_{n \geq 0} \) is defined by
\[
\langle u_0, P_n P_k \rangle = N_n \delta_{n,k}, \text{ with } N_n \neq 0.
\]
where \( u_0 \) is the first element of the corresponding dual sequence.

▶ Equivalently, \( \{ P_n \}_{n \geq 0} \) is an OPS for \( u_0 \) iff
\[
\langle u_0, x^m P_n \rangle = \begin{cases} 
0 & \text{if } n > m, \\
N_n & \text{if } n = m, \text{ for } n \geq 0.
\end{cases}
\]

▶ It always satisfies the second order recurrence relation
\[
P_{n+1}(x) = (x - \beta_n) P_n(x) - \gamma_n P_{n-1}(x)
\]
with \( P_0 = 1 \) and \( P_{-1} = 0 \) and
\[
\beta_n = \frac{\langle u_0, xP_n^2 \rangle}{\langle u_0, P_n^2 \rangle} \quad \text{and} \quad \gamma_{n+1} = \frac{\langle u_0, P_{n+1}^2 \rangle}{\langle u_0, P_n^2 \rangle} \neq 0, \ n \in \mathbb{N}
\]
Consider a sequence \( \{ \tilde{P}_{\vec{n}} \} \) with \( \vec{n} = (n_1, n_2) \) and \( \deg \tilde{P}_{\vec{n}}(x) = n_1 + n_2 \) such that

\[
\langle u_0, x^k \tilde{P}_{\vec{n}}(x) \rangle = \int_{\Delta_1} x^k \tilde{P}_{\vec{n}}(x) W_0(x) \, dx = 0 \quad k = 0, 1, \ldots, n_1 - 1
\]

\[
\langle u_1, x^k \tilde{P}_{\vec{n}}(x) \rangle = \int_{\Delta_2} x^k \tilde{P}_{\vec{n}}(x) W_1(x) \, dx = 0 \quad k = 0, 1, \ldots, n_2 - 1
\]
Consider a sequence $\{\tilde{P}_{\vec{n}}\}$ with $\vec{n} = (n_1, n_2)$ and deg $\tilde{P}_{\vec{n}}(x) = n_1 + n_2$ such that

$$\langle u_0, x^k \tilde{P}_{\vec{n}}(x) \rangle = \int_{\Delta_1} x^k \tilde{P}_{\vec{n}}(x) W_0(x) \, dx = 0 \quad k = 0, 1, \ldots, n_1 - 1$$

$$\langle u_1, x^k \tilde{P}_{\vec{n}}(x) \rangle = \int_{\Delta_2} x^k \tilde{P}_{\vec{n}}(x) W_1(x) \, dx = 0 \quad k = 0, 1, \ldots, n_2 - 1$$

Now, if we construct a sequence $\{\tilde{P}_n\}_{n \geq 0}$ such that

$$P_{2n}(x) = \tilde{P}_{n,n}(x)$$

$$P_{2n+1}(x) = \tilde{P}_{n,n+1}(x)$$

then $\{P_n\}_{n \geq 0}$ is a 2-OPS.
Definition
Consider a vector linear functional \( u = (u_0, u_1) \) defined on \( \mathcal{P} \) in \( \mathbb{C} \). The sequence of polynomials \( \{P_n\}_{n \geq 0} \), where \( \deg P_n = n \), is said to be 2-orthogonal to \( u = (u_0, u_1) \) if

\[
\langle u_0, x^m P_n \rangle = \begin{cases} 
0 & \text{for } n \geq 2m + 1 \\
N_{2m} \neq 0 & \text{for } n = 2m 
\end{cases} \tag{1}
\]

\[
\langle u_1, x^m P_n \rangle = \begin{cases} 
0 & \text{for } n \geq 2m + 2 \\
N_{2m+1} \neq 0 & \text{for } n = 2m + 1 
\end{cases} \tag{2}
\]
The monic 2-OPS \( \{P_n\}_{n \geq 0} \) for \( \mathbf{u} = (u_0, u_1) \) satisfies a third order recurrence relation (see Van Iseghem’88, Maroni’89)

\[
P_{n+1}(x) = (x - \beta_n)P_n(x) - \alpha_n P_{n-1}(x) - \gamma_{n-1} P_{n-2}(x)
\]  

(3)

with \( P_0(x) = 1 \), \( P_1(x) = x - \beta_0 \) and \( P_2(x) = (x - \beta_1)P_1(x) - \alpha_1 \).
The monic 2-OPS \( \{P_n\}_{n \geq 0} \) for \( \mathbf{u} = (u_0, u_1) \) satisfies a third order recurrence relation (see Van Iseghem’88, Maroni’89)

\[
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\]

(3)

with \( P_0(x) = 1, \ P_1(x) = x - \beta_0 \) and \( P_2(x) = (x - \beta_1)P_1(x) - \alpha_1 \).

Expressions for the recurrence coefficients follow immediately from the definition. For instance,

\[
\gamma_{2n+1} = \frac{\langle u_0, x^{n+1}P_{2n+2} \rangle}{\langle u_0, x^nP_{2n} \rangle}, \quad \gamma_{2n+2} = \frac{\langle u_1, x^{n+1}P_{2n+3} \rangle}{\langle u_1, x^nP_{2n+1} \rangle}, \quad n \geq 0.
\]

Conversely, we also have

\[
N_{2n} := \langle u_0, x^{n+1}P_{2n+2} \rangle = \prod_{k=0}^{n} \gamma_{2k+1}
\]

and

\[
N_{2n+1} := \langle u_1, x^{n+1}P_{2n+3} \rangle = \prod_{k=0}^{n} \gamma_{2k+2}, \quad \text{for} \quad n \geq 0.
\]
Example 1: 2-orthogonal polynomials for Bessel weights

\[ P_{n+1}(x) = (x - \beta_n)P_n(x) - \alpha_n P_{n-1}(x) - \gamma_{n-1} P_{n-2}(x) \]

with

\[ \beta_n = 3n^2 + (2\alpha + 2\beta + 3)n + (1 + \alpha)(1 + \beta) \]
\[ \alpha_n = n(3n + \alpha + \beta)(n + \alpha)(n + \beta), \quad n \geq 1, \]
\[ \gamma_n = n(n + 1)(n + \alpha + 1)(n + \alpha)(n + \beta + 1)(n + \beta), \quad n \geq 2, \]

They satisfy the 3rd order recurrence relation

\[ x^2 P_n''' + (3 + \alpha + \beta)xP_n'' + ((\alpha + 1)(\beta + 1) - x)P'_n = -nP_n \]

and are 2-OPS for \( U = (u_0, u_1) \) satisfying

\[ x^2 u_0'' - (\alpha + \beta - 1)xu_0' - (x - \alpha \beta)u_0 = 0, \quad (\alpha + 1)(\beta + 1)u_1 = -(xu_0)' \]

Such vector functional \( U = (u_0, u_1) \) admits the following integral representation

\[ < u_0, f(x) > = \frac{2}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_0^{+\infty} f(x)x^{(\alpha+\beta)/2} K_{\alpha-\beta}(2\sqrt{x})dx, \]
\[ < u_1, f(x) > = \frac{2}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_0^{+\infty} f(x) \left( x^{(\alpha+\beta)/2} K_{\alpha-\beta}(2\sqrt{x}) \right)' dx, \]

(See Ben Cheikh&Douak’00 and Van Assche&Yakubovich’00.)
Example 2: 2-orthogonal polynomials with constant rec coef

The sequence of polynomials \( \{ P_n(x) \}_{n \geq 0} \) satisfying the recurrence relation

\[
P_{n+1}(x) = xP_n(x) - \frac{4}{27} P_{n-2}(x)
\]

is 2-orthogonal with respect to \( U = (u_0, u_1) \) such that

\[
\begin{align*}
(x^3 - 1) u_0'' + \frac{3}{2} x^2 u_0' - \frac{1}{2} x u_0 &= 0 \\
u_1 &= 3(x^3 - 1) u_0' - \frac{3}{2} x^2 u_0
\end{align*}
\]

Such vector functional admits an integral representation on the real line as follows

\[
\begin{align*}
< u_0, f(x) > &= \int_0^1 f(x) \frac{9\sqrt{3}}{4\pi} \left[ (1 + \sqrt{1-x^3})^{1/3} - (1 - \sqrt{1-x^3})^{1/3} \right] dx \\
&\quad + \int_0^{+\infty} f(x) 3e^{-x} \left[ \lambda_1 \sqrt{x} \cos(\sqrt{3}x) + \lambda_2 x^2 \sin(\sqrt{3}x) \right] dx, \\
< u_1, f(x) > &= \int f(x) U_1(x) dx,
\end{align*}
\]

(See Douak & Maroni'97 for further details.)
Example 3: multiple orthogonal polynomials with exponential weights

Consider the monic polynomials $P_{n,m}$ of degree $n + m$ for which

$$\int_{\Gamma_0 \cup \Gamma_1} x^j P_{n,m}(x) \exp(-x^3 + tx) dx = 0, \ j = 0, \ldots, n - 1,$$

$$\int_{\Gamma_0 \cup \Gamma_2} x^j P_{n,m}(x) \exp(-x^3 + tx) dx = 0, \ j = 0, \ldots, m - 1,$$

with $\Gamma_k = \{z \in \mathbb{C} : \arg z = e^{2k\pi i/3}\}, \ k = 0, 1, 2.$

(see Van Assche & Filipuk & Zhang (2015))

Rodrigues’ formula:

$$e^{-x^3 + tx} P_{n,n+m}(x) = \frac{(-1)^n}{3^n} \frac{d^n}{dx^n} \left( e^{-x^3 + tx} P_{0,m}(x) \right)$$

$$e^{-x^3 + tx} P_{n+m,n}(x) = \frac{(-1)^n}{3^n} \frac{d^n}{dx^n} \left( e^{-x^3 + tx} P_{m,0}(x) \right)$$

where $P_{m,0}$ and $P_{0,m}$ are orthogonal polynomials...

and $\{P_{k,k}\}_k$ is 2-OPS. (Case $t = 0$ already in Pólya and Szegő (1925).

Special case of Gould-Hopper polynomials (1962).)
3-fold symmetric (not necessarily 2-orthogonal) polynomials

Definition

A monic polynomial sequence \( \{B_n\}_{n \geq 0} \) is 3-fold symmetric if and only if

\[
B_n(e^{\frac{2i\pi}{3}} x) = e^{\frac{2in\pi}{3}} B_n(x)
\]

and

\[
B_n(e^{\frac{4i\pi}{3}} x) = e^{\frac{4in\pi}{3}} B_n(x), \quad n \geq 0.
\]

In other words, this is to say that there exist three sequences \( \{B_n^{[j]}\}_{n \geq 0} \) with \( j \in \{0, 1, 2\} \) such that

\[
B_{3n}(x) = B_n^{[0]}(x^3),
\]

\[
B_{3n+1}(x) = xB_n^{[1]}(x^3),
\]

\[
B_{3n+2}(x) = x^2 B_n^{[2]}(x^3),
\]

(The sequences \( \{B_n^{[j]}\}_{n \geq 0} \) are the components of the cubic decomposition of the 3-fold symmetric sequence \( \{B_n\}_{n \geq 0} \).)

(see Barrucand&Dickinson'66)
Whilst we are dealing with 3-fold symmetric and 2-orthogonal sequences, we recall the following result.

**Theorem (Douak & Maroni'92)**

Let \( \{P_n\}_{n \geq 0} \) be a 2-orthogonal polynomial sequence for \( U = (u_0, u_1) \). Then, \( \{P_n\}_{n \geq 0} \) is 3-fold symmetric iff if satisfies the third order recurrence relation

\[
P_{n+1}(x) = xP_n(x) - \gamma_{n-1}P_{n-2}(x), \quad n \geq 2,
\]

with \( P_0(x) = 1 \), \( P_1(x) = x \) and \( P_2(x) = x^2 \).

Moreover, we have
Lemma (Douak & Maroni’92)

If the a 3-fold symmetric sequence \( \{P_n\}_{n \geq 0} \) is 2-orthogonal, then the three components in the cubic decomposition of \( \{P_n\}_{n \geq 0} \) are also 2-orthogonal fulfilling the recurrence relations:

\[
P^{[k]}_{n+1}(x) = (x - \beta^{[k]}_n)P^{[k]}_n(x) - \alpha^{[k]}_n P^{[k]}_{n-1}(x) - \gamma^{[k]}_{n-1} P^{[k]}_{n-2}(x),
\]

where

\[
\beta^{[k]}_n = \gamma_{3n-1+k} + \gamma_{3n+k} + \gamma_{3n+1+k}, \quad n \geq 0,
\]

\[
\alpha^{[k]}_n = \gamma_{3n-2+k} \gamma_{3n+k} + \gamma_{3n-1+k} \gamma_{3n-3+k} + \gamma_{3n-2+k} \gamma_{3n-1+k}, \quad n \geq 1,
\]

\[
\gamma^{[k]}_n = \gamma_{3n-2+k} \gamma_{3n+k} \gamma_{3n+2+k} \neq 0, \quad n \geq 2,
\]

for each \( k = 0, 1, 2 \).
Theorem. (Aptekarev et al.'00)
If $\gamma_n > 0$ for $n \geq 1$ in

$$P_{n+1}(x) = xP_n(x) - \gamma_{n-1}P_{n-2}(x),$$

then $\{P_n\}_{n\geq0}$ is a 2-OPS w.r.t. the vector of linear functionals $(u_0, u_1)$ and

$$\langle u_0, f(x) \rangle = \int_S f(x) d\mu_0(x) \quad (4)$$

$$\langle u_1, f(x) \rangle = \int_S f(x) d\mu_1(x) \quad (5)$$

where $S$ represents the starlike set

$$S := \bigcup_{k=0}^2 \Gamma_k \quad \text{with} \quad \Gamma_k = [0, e^{2\pi ik/3}_\infty),$$

and the measures have a common support which is a subset of $S$ and are
invariant under rotations of $2\pi/3$. 
Theorem. (Ben Romdhane'08)
Let \( \{P_n\}_{n \geq 0} \) be a 2-OPS satisfying
\[
P_{n+1}(x) = xP_n(x) - \gamma_{n-1}P_{n-2}(x).
\]
If \( \gamma_n > 0 \), then the following statements hold
(a) If \( x \) is a zero of \( P_{3n+j} \), then \( \omega^k x \) are also zeros of \( P_{3n+j} \) with \( \omega = e^{2\pi i/3} \)
(b) 0 is a zero of \( P_{3n+j} \) of multiplicity \( j \) when \( j = 1, 2 \)
(c) \( P_{3n+j} \) has \( n \) distinct positive real zeros
\[
0 < x_{n,1}^{(j)} < \ldots < x_{n,n}^{(j)}
\]
(d) Between two real zeros of \( P_{3n+j+3} \) there exist only one zero of \( P_{3n+j+2} \) and only one zero of \( P_{3n+j+1} \), ie,
\[
x_{n,k}^{(j+2)} < x_{n,k+1}^{(j)} < x_{n,k+1}^{(j+1)} < x_{n,k+1}^{(j+2)}
\]
Theorem. (AL & Van Assche’18)
Let \( \{P_n\}_{n \geq 0} \) be a 2-OPS satisfying

\[
P_{n+1}(x) = xP_n(x) - \gamma_{n-1}P_{n-2}(x).
\]

If \( \gamma_n > 0 \) and, additionally,

\[
\gamma_{2n} = c_0 n^\alpha + o(n^\alpha) \quad \text{and} \quad \gamma_{2n+1} = c_1 n^\alpha + o(n^\alpha)
\]

for large \( n \), with \( c_0, c_1 > 0 \) and \( \alpha \geq 0 \), then the largest zero in absolute value \( |x_{n,n}| \) behaves as

\[
|x_{n,n}| \leq \frac{3}{2^{2/3}} c^{1/3} n^{\alpha/3} + o(n^{\alpha/3}), \quad n \geq 1,
\]

(6)

where \( c = \max\{c_0, c_1\} \).
**Proof.** Consider the Hessenberg matrix

\[
H_n = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
\gamma_1 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
0 & \gamma_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\
& & & & & & \ddots & \\
0 & 0 & 0 & 0 & \cdots & \gamma_{n-2} & 0 & 0
\end{pmatrix}
\]

Hence,

\[
H_n \begin{pmatrix}
P_0(x) \\
P_1(x) \\
\vdots \\
P_{n-1}(x)
\end{pmatrix} = x \begin{pmatrix}
P_0(x) \\
P_1(x) \\
\vdots \\
P_{n-1}(x)
\end{pmatrix} - P_n(x) \begin{pmatrix}
0 \\
0 \\
\vdots \\
1
\end{pmatrix}
\]

and each zero of \( P_n(x) \) is an eigenvalue of the matrix \( H_n \).
The spectral radius of the matrix $H_n$,

$$\rho(H_n) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } H_n\},$$

is bounded from above by $\|H_n\|$ where $\| \cdot \|$ denotes a matrix norm. In particular

$$\|H_n\|_S = \|S^{-1}H_nS\|_\infty = \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} |(S^{-1}H_nS)_{i,j}| \right\},$$

where $S = \text{diag}(d_1, \ldots, d_k, \ldots, d_n)$ is non-singular matrix and $(S^{-1}H_nS)_{i,j}$ if the $i$th row and $j$th column entry of the product matrix $S^{-1}H_nS$ we obtain

$$\|H_n\|_S = \max \left\{ \frac{d_2}{d_1}, \frac{d_3}{d_2}, \frac{d_4 + d_1\gamma_1}{d_3}, \ldots, \frac{d_k + d_{k-3}\gamma_{k-3}}{d_{k-1}}, \ldots, \frac{d_{n-2}\gamma_{n-2}}{d_n} \right\}.$$

Setting $d_k = d^k(k!)^{\alpha/3} \neq 0$, for some $d > 0$, brings

$$\|H_n\|_S \leq 2^{\alpha/3} \left( d + \frac{c}{d^2} \right) n^{\alpha/3} + o(n^{\alpha/3}) \quad \text{as } n \to +\infty.$$

The choice of $d = (2c)^{1/3}$ provides a minimum to $\left( d + \frac{c}{d^2} \right)$ and this gives

$$\|H_n\|_S \leq \frac{3}{41/3} (c n^\alpha)^{1/3} + o(n^{\alpha/3}) \quad \text{as } n \to +\infty.$$
**Definition**

A monic 2-OPS \( \{P_n\}_{n \geq 0} \) is "**classical**" in Hahn’s sense when the sequence of its derivatives \( \{Q_n\}_{n \geq 0} \), with

\[
Q_n(x) = \frac{1}{n+1} P'_{n+1}(x)
\]

is also a 2-OPS.

Hence, as a monic 2-OPS, the sequence \( \{Q_n\}_{n \geq 0} \) satisfies a third order recurrence relation:

\[
Q_{n+1}(x) = (x - \tilde{\beta}_n) Q_n(x) - \tilde{\alpha}_n Q_{n-1}(x) - \tilde{\gamma}_{n-1} Q_{n-2}(x), \quad n \geq 2,
\]

(7)

with \( Q_0 = 1 \), \( Q_1(x) = x - \tilde{\beta}_0 \) and \( Q_2(x) = (x - \tilde{\beta}_1) Q_1(x) - \tilde{\alpha}_1 \).
"Classical" 2-orthogonal polynomials

Between the two recurrence relations

\[ P_{n+1}(x) = (x - \beta_n)P_n(x) - \alpha_n P_{n-1}(x) - \gamma_{n-1}P_{n-2}(x) \]
\[ Q_{n+1}(x) = (x - \tilde{\beta}_n)Q_n(x) - \tilde{\alpha}_n Q_{n-1}(x) - \tilde{\gamma}_{n-1}Q_{n-2}(x), \quad n \geq 2, \]

it follows a nonlinear system of equations

\[
\begin{align*}
(n+2)\tilde{\beta}_n - n\tilde{\beta}_{n-1} &= (n+1)\beta_{n+1} - (n-1)\beta_n \\
(n+3)\tilde{\alpha}_{n+1} - (n+1)\tilde{\alpha}_n &= (n+2)\alpha_{n+2} - (n-1)\alpha_{n+2} + (n+1)(\beta_{n+1} - \tilde{\beta}_n)^2 \\
n\alpha_{n+1}\alpha_{n+2} + (n+2)\tilde{\alpha}_n\tilde{\alpha}_{n+1} - 2(n+1)\tilde{\alpha}_n\alpha_{n+2} &\quad = (n+2)\tilde{\gamma}_n(2\beta_{n+2} - \beta_{n+1} - \beta_{n-1}) - n\gamma_{n+1}(\beta_{n+2} + \beta_n - 2\tilde{\beta}_{n-1}) \\
n(\alpha_{n+1}\gamma_{n+2} + \alpha_{n+3}\gamma_{n+1}) &\quad = \tilde{\gamma}_n \left( 2(n+2)\alpha_{n+3} - (n+3)\tilde{\alpha}_{n+2} \right) + \tilde{\alpha}_n \left( 2(n+1)\gamma_{n+2} - (n+3)\tilde{\gamma}_{n+1} \right) \\
n\gamma_{n+1}\gamma_{n+3} &\quad = \tilde{\gamma}_n \left( 2(n+2)\gamma_{n+3} - (n+4)\tilde{\gamma}_{n+2} \right)
\end{align*}
\]
On the other hand, the 2-orthogonality of $\{P_n\}_{n \geq 0}$ for $U = (u_0, u_1)$ and the 2-orthogonality of $\{Q_n\}_{n \geq 0}$ for $V = (v_0, v_1)$ implies

$$
\begin{bmatrix}
v_0 \\
v_1
\end{bmatrix}
= \Phi
\begin{bmatrix}
u_0 \\
u_1
\end{bmatrix}
$$

(8)

and also that

$$
\begin{bmatrix}
v'_0 \\
v'_1
\end{bmatrix}
= -\Psi
\begin{bmatrix}
u_0 \\
u_1
\end{bmatrix}.
$$

(9)

with

$$
\Phi = \begin{bmatrix}
\phi_{0,0} & \phi_{0,1} \\
\phi_{1,0} & \phi_{1,1}
\end{bmatrix}
$$
and
$$
\Psi = \begin{bmatrix}
0 & 1 \\
\psi(x) & \zeta
\end{bmatrix}
$$

where $\psi(x) = \frac{2}{\gamma_1} P_1(x)$ and $\zeta = -\frac{2\alpha_1}{\gamma_1}$,

whilst $\deg\{\phi_{0,0}, \phi_{0,1}, \phi_{1,1}\} \leq 1$ and $\deg \phi_{1,0} \leq 2$. 
Theorem. (Maroni & Douak’92, Maroni’99)

The monic 2-OPS \( \{P_n\}_{n \geq 0} \) for \( U = (u_0, u_1) \) is "classical" iff there are polynomials \( \psi \) and \( \phi_{i,j} \), with \( i, j \in \{0, 1\} \), and a constant \( \zeta \) such that

\[
\left( \begin{bmatrix} \phi_{0,0} & \phi_{0,1} \\ \phi_{1,0} & \phi_{1,1} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \right)' + \begin{bmatrix} 0 & 1 \\ \psi(x) & \zeta \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

where \( \deg \{\phi_{0,0}, \phi_{0,1}, \phi_{1,1}\} \leq 1 \), \( \deg \phi_{1,0} \leq 2 \) and \( \deg \psi = 1 \).

Relation (11a) reads as follows

\[
\left( \Phi \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \right)' + \Psi \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
If \( \{P_n\}_{n \geq 0} \) is three-fold symmetric, then so is \( \{Q_n\}_{n \geq 0} \) where

\[
Q_n(x) := \frac{1}{n + 1} P'_{n+1}(x), \quad n \geq 0.
\]

This means that for a three-fold symmetric Hahn-classical polynomial sequence \( \{P_n\}_{n \geq 0} \) then \( \{Q_n\}_{n \geq 0} \) is three-fold and satisfies

\[
Q_{n+1}(x) = xQ_n(x) - \tilde{\gamma}_{n-1} Q_{n-2}, \quad \text{for} \quad n \geq 2,
\]

with initial conditions \( Q_k(x) = x^k \) for \( k = 0, 1, 2 \).

in this case we have
Theorem. (AL&Van Assche’18) Let \( \{P_n(x)\}_{n \geq 0} \) be a three-fold symmetric 2-OPS for \((u_0, u_1)\). The following are equivalent:

(a) \( \{P_n(x)\}_{n \geq 0} \) is a three-fold symmetric "classical" 2-orthogonal polynomial sequence.

(b) The vector functional \((u_0, u_1)\) satisfies the matrix differential equation

\[
\left( \Phi \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \right)' + \Psi \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

(11a)

where

\[
\Phi = \begin{bmatrix} \vartheta_1 & (1 - \vartheta_1)x \\ \frac{2}{\gamma_1} (1 - \vartheta_2)x^2 & 2\vartheta_2 - 1 \end{bmatrix}
\]

and

\[
\Psi = \begin{bmatrix} 0 & 1 \\ \frac{2}{\gamma_1}x & 0 \end{bmatrix}
\]

(11b)

for some constants \( \vartheta_1 = \frac{3\gamma_1}{\gamma_2} \) and \( \vartheta_2 = \frac{2\gamma_2}{\gamma_3} \) such that \( \vartheta_1, \vartheta_2 \neq \frac{n-1}{n} \).

(c) There exists a sequence of numbers \( \{\tilde{\gamma}_{n+1}\}_{n \geq 0} \) such that

\[
P_{n+3}(x) = Q_{n+3}(x) + \left((n + 1)\gamma_{n+2} - (n + 3)\tilde{\gamma}_{n+1}\right)Q_n(x)
\]

(12)

with initial conditions \( P_k(x) = Q_k(x) = x^k \) for \( k = 0, 1, 2 \).
Proof. (a) ⇒ (c): consequence of the rec. rel. of \( \{P_n\}_{n \geq 0} \) and \( \{Q_n\}_{n \geq 0} \).
(c) ⇒ (b): If \( \{u_n\}_{n \geq 0} \) and \( \{v_n\}_{n \geq 0} \) are the dual sequences of \( \{P_n\}_{n \geq 0} \) and \( \{Q_n\}_{n \geq 0} \), resp., then

\[
\begin{align*}
v'_n &= -(n + 1)u_{n+1} \\
v_n &= u_n + \left( (n + 1)\gamma_{n+2} - (n + 3)\tilde{\gamma}_{n+1} \right)u_{n+3}.
\end{align*}
\] (13)

The 2-orthogonality of \( \{P_n\}_{n \geq 0} \) implies

\[
\begin{align*}
u_2 &= \frac{x}{\gamma_1}u_0, \quad u_3 = -\frac{1}{\gamma_2}u_0 + \frac{x}{\gamma_2}u_1, \quad u_4 = \frac{x^2}{\gamma_1\gamma_3}u_0 - \frac{1}{\gamma_3}u_1
\end{align*}
\]

If we take \( n = 0 \) and \( n = 1 \) in (13) we obtain

\[
\begin{bmatrix}
v'_0 \\
v'_1
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
\frac{2}{\gamma_1}x & 0
\end{bmatrix} \begin{bmatrix}
u_0 \\
u_1
\end{bmatrix}
\]

With \( n = 0 \) and \( n = 1 \) in (14) leads to

\[
\begin{bmatrix}
v_0 \\
v_1
\end{bmatrix} = \begin{bmatrix}
\vartheta_1 & (1 - \vartheta_1)x \\
\frac{2}{\gamma_1}(1 - \vartheta_2)x^2 & 2\vartheta_2 - 1
\end{bmatrix} \begin{bmatrix}
u_0 \\
u_1
\end{bmatrix}
\]
Proof. (cont.)

The proof of \((b) \Rightarrow (a)\) is essentially about showing that \(\{Q_n\}_{n \geq 0}\) is 2-orthogonal with respect to

\[
\begin{bmatrix}
    v_0 \\
    v_1
\end{bmatrix} = \begin{bmatrix}
    \vartheta_1 & (1 - \vartheta_1)x \\
    \frac{2}{\gamma_1} (1 - \vartheta_2) x^2 & 2\vartheta_2 - 1
\end{bmatrix} \begin{bmatrix}
    u_0 \\
    u_1
\end{bmatrix}
\]

\[\square\]
Proof. (cont.)
The proof of \((b) \Rightarrow (a)\) is essentially about showing that \(\{Q_n\}_{n \geq 0}\) is 2-orthogonal with respect to

\[
\begin{bmatrix}
\nu_0 \\
\nu_1
\end{bmatrix} = \begin{bmatrix}
\vartheta_1 & (1 - \vartheta_1)x \\
\frac{2}{\gamma_1}(1 - \vartheta_2)x^2 & 2\vartheta_2 - 1
\end{bmatrix} \begin{bmatrix}
u_0 \\
u_1
\end{bmatrix}
\]

\(\blacksquare\)

The Pearson equation

\[
\left( \boldsymbol{\Phi} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \right)' + \boldsymbol{\Psi} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

gives

\[
\tilde{\gamma}_n = \frac{n}{n + 2} \vartheta_n \gamma_{n+1}
\]

with

\[
\vartheta_{2n+1} = \left( \frac{1 - (n + 1)(1 - \vartheta_1)}{1 - n(1 - \vartheta_1)} \right) \quad \text{and} \quad \vartheta_{2n+2} = \left( \frac{1 - (n + 1)(1 - \vartheta_2)}{1 - n(1 - \vartheta_2)} \right).
\]

(15)
If we replace each $P$ in
\[ xP_n = P_{n+1} + \gamma_{n-1}P_{n-2} \]
by the corresponding expression given in
\[ P_{n+3}(x) = Q_{n+3}(x) + \left((n + 1)\gamma_{n+2} - (n + 3)\tilde{\gamma}_{n+1}\right)Q_n(x) \]
to then use the recurrence relation
\[ xQ_n = Q_{n+1} + \tilde{\gamma}_{n-1}Q_{n-2} \quad \text{where} \quad \tilde{\gamma}_{n-1} = \frac{n - 1}{n + 1} \vartheta_{n-1}\gamma_n \]
we obtain
\[ \vartheta_{n+2} + \frac{1}{\vartheta_n} = 2, \quad n \geq 1, \]
and
\[ \gamma_{n+2} = \frac{n + 3}{n + 1} \frac{\left(n(\vartheta_n - 1) + 1\right)}{((n + 4)(\vartheta_{n+1} - 1) + 1)}\gamma_{n+1} \neq 0 \]
Lemma (Douak&Maroni’97)

If a 2-symmetric 2-OPS \( \{ P_n \}_{n \geq 0} \) is "classical", then each polynomial is a solution of the third order differential equation

\[
(a_n x^3 - b_n) P''_{n+1} + c_n x^2 P''_{n+1} + d_n x P'_{n+1} = e_n P_{n+1}
\]

where

\[
\begin{align*}
 a_n &= (\vartheta_n - 1)(\vartheta_{n+1} - 1) \\
b_n &= \frac{\gamma_{n+3}((n+3)\vartheta_{n+2}-(n+2))((n+4)\vartheta_{n+1}-(n+3))((n+5)\vartheta_{n+2}-(n+4))}{(n+3)(n+4)} \\
c_n &= \vartheta_n \vartheta_{n+1} - 1 - (n-3)(\vartheta_n - 1)(\vartheta_{n+1} - 1) \\
d_n &= n\vartheta_{n+1} - (n-1)\vartheta_n(2\vartheta_{n+1} - 1) \\
e_n &= n\vartheta_{n+1}, \quad \text{for any } n \geq 1,
\end{align*}
\]

with \( a_0 = b_0 = c_0 = d_0 = e_0 = 0 \).

Here

\[
\vartheta_{2n+1} = \left( \frac{1 - (n + 1)(1 - \vartheta_1)}{1 - n(1 - \vartheta_1)} \right) \quad \text{and} \quad \vartheta_{2n+2} = \left( \frac{1 - (n + 1)(1 - \vartheta_2)}{1 - n(1 - \vartheta_2)} \right).
\]
Proposition. (AL & Van Assche’18) The 2-OPS \( \{P_n(x)\}_{n \geq 0} \) with respect to the vector linear functional \( U = (u_0, u_1) \) satisfy the Hahn’s property if and only if there are coefficients \( \vartheta_1, \vartheta_2 \neq \frac{n-1}{n} \), such that \( U = (u_0, u_1) \) satisfies

\[
\left( \phi(x) u_0 \right)''' + \left( \frac{2}{\gamma_1} (\vartheta_2 + \vartheta_1 - 2) x^2 u_0 \right)' + \frac{2}{\gamma_1} \left( \vartheta_1 - 2 \right) x u_0 = 0 \quad (16)
\]

and

\[
\begin{cases}
(\vartheta_1 - 2) (2\vartheta_2 - 1) u_1 = \phi(x) u'_0 - \frac{2}{\gamma_1} (\vartheta_1 - 1) (2\vartheta_2 - 3) x^2 u_0, & \text{if} \quad \vartheta_1 \neq 2, \\
x u'_1 = 2 u'_0, & \text{if} \quad \vartheta_1 = 2,
\end{cases}
\]

where

\[
\phi(x) = \left( \vartheta_1 (2\vartheta_2 - 1) - \frac{2}{\gamma_1} (\vartheta_1 - 1) (\vartheta_2 - 1) x^3 \right). \quad (17)
\]

and from this we have
Three-fold symmetric "classical" 2-orthogonal polynomials

**Theorem.** (AL & Van Assche’18) For a "classical" threefold symmetric \( \{P_n\}_{n \geq 0} \) 2-orthogonal with respect to \((u_0, u_1)\) and satisfying the rec. rel. with \( \gamma_{n+1} > 0 \):

\[
\langle u_k, f(x) \rangle = \frac{1}{3} \left( \int_0^b f(x) U_k(x) dx + \omega^{2k-1} \int_0^{b\omega} f(x) U_k(\omega^2 x) dx + \omega^{1-2k} \int_0^{b\omega^2} f(x) U_k(\omega x) dx \right),
\]

with \( \omega = e^{2\pi i/3} \) and \( b = \lim_{n \to \infty} \left( \frac{27}{4} \gamma_n \right) \), provided that \( U_0(x) \) and \( U_1(x) \)
**Theorem.** (AL & Van Assche’18) For a "classical" threefold symmetric 
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\[
\langle u_k, f(x) \rangle = \frac{1}{3} \left( \int_0^b f(x)U_k(x)dx + \omega^{2k-1} \int_0^{b\omega} f(x)U_k(\omega^2x)dx + \omega^{1-2k} \int_0^{b\omega^2} f(x)U_k(\omega x)dx \right),
\]

with \( \omega = e^{2\pi i/3} \) and \( b = \lim_{n \to \infty} \left( \frac{27}{4} \gamma_n \right) \), provided that \( U_0(x) \) and \( U_1(x) \)

\[
\begin{align*}
\left( \phi(x)U_0(x) \right)'' + \left( \frac{2(\vartheta_2+\vartheta_1-2)}{\gamma_1} x^2U_0(x) \right)' + \frac{2(\vartheta_1-2)}{\gamma_1} xU_0(x) &= \lambda_0 g_0(x) \\
(\vartheta_1 - 2)(2\vartheta_2 - 1) U_1(x) &= \phi(x)U_0'(x) - \frac{2(\vartheta_1-1)(2\vartheta_2-3)}{\gamma_1} x^2U_0(x) + \lambda_1 g_1(x) \\
xU_1'(x) &= 2U_0'(x) \quad \text{if} \quad \vartheta_1 = 2
\end{align*}
\]

with \( \phi(x) = \left( \vartheta_1 (2\vartheta_2 - 1) - \frac{2(\vartheta_1-1)(\vartheta_2-1)}{\gamma_1} x^3 \right) \), satisfying

\[
\lim_{x \to b} f(x) \frac{d^k}{dx^k} U_0(x) = 0, \quad \text{and} \quad \int_0^b U_0(x)dx = 1
\]

\( \lambda_k \in \mathbb{C} \) and \( \int_{\Gamma} x^n g_k(x)dx = 0 \).
There are four cases to single out:

**Case A:** $\vartheta_1 = \vartheta_2 = 1$. This implies that $\vartheta_n = 1$ for all $n \geq 0$.

**Case B**

**Case B1:** $\vartheta_1 \neq 1$ but $\vartheta_2 = 1$ so that by setting $\vartheta_1 = \frac{\mu+2}{\mu+1}$ it follows

$$\vartheta_{2n-1} = \frac{n + \mu + 1}{n + \mu} \quad \text{and} \quad \vartheta_{2n} = 1 , \quad n \geq 1.$$

**Case B2:** $\vartheta_1 = 1$ but $\vartheta_2 \neq 1$ so that by setting $\vartheta_2 = \frac{\rho+2}{\rho+1}$ it follows

$$\vartheta_{2n-1} = 1 \quad \text{and} \quad \vartheta_{2n} = \frac{n + \rho + 1}{n + \rho} , \quad n \geq 1.$$

**Case C:** $\vartheta_1 \neq 1$ and $\vartheta_2 \neq 1$ and hence by setting $\vartheta_1 = \frac{\mu+2}{\mu+1}$ and $\vartheta_2 = \frac{\rho+2}{\rho+1}$ it follows

$$\vartheta_{2n-1} = \frac{n + \mu + 1}{n + \mu} \quad \text{and} \quad \vartheta_{2n} = \frac{n + \rho + 1}{n + \rho} , \quad n \geq 1.$$
Case A: Appell polynomials

In this case we have \( Q_n(x) := \frac{1}{n+1} P'_{n+1}(x) = P_n(x) \). Additionally

\[
\gamma_{n+1} = (n + 1)(n + 2) \frac{\gamma_1}{2}, \quad \text{and} \quad \begin{cases} 
  u_0'' - \frac{2}{\gamma_1} x u_0 = 0 \\
  u_1 = -u_0'
\end{cases}
\]

With the choice \( \gamma_1 = 2 \), it follows that

\[
\gamma_{n+1} = (n + 1)(n + 2), \quad \text{and} \quad \begin{cases} 
  u_0'' - x u_0 = 0 \\
  u_1 = -u_0'
\end{cases}
\]

and

\[-P'''_{n+1}(x) + xP'_{n+1}(x) = nP_{n+1}(x), \quad n \geq 0.\]

**Remark.** The polynomials appear in the Vorob’ev-Yablonski polynomials associated with rational solutions of Painlevé II equations (Clarkson & Mansfield’03)
Integral representation (AL&Van Assche)

\[
\langle u_0, f \rangle = \int_{\Gamma} f(x)W_0(x)\,dx, \text{ for all } f \in \mathcal{P},
\]

\[
\langle u_1, f \rangle = \int_{\Gamma} f(x)W_1(x)\,dx, \text{ for all } f \in \mathcal{P},
\]

where \( W_0 : \Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \longrightarrow \mathbb{R} \) defined by

\[
W_0(x) = \text{Ai}(x)\mathbb{1}_{\Gamma_0} - e^{-2\pi i/3}\text{Ai}(e^{-2\pi i/3}x)\mathbb{1}_{\Gamma_1} - e^{2\pi i/3}\text{Ai}(e^{2\pi i/3}x)\mathbb{1}_{\Gamma_2}
\]

with \( \Gamma_k = \{ w : \text{arg}(w) = \frac{2k\pi}{3} \} \), with \( k = 0, 1, 2 \),

where the orientations of \( \Gamma_k \) are all taken from left to right.
Remarks.
- All the zeros of $P_n(x)$ are located on $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$
- In each $\Gamma_k$, between two zeros of $P_{n+2}$ there is one zero of $P_n$ and $P_{n+1}$. 
Here we have
\[
\gamma_{2n} = \frac{n(2n+1)(n+\mu)(\mu+2)}{(3n+\mu-1)(3n+\mu+2)}
\gamma_1 = \frac{2\gamma_1(\mu+2)}{9} n + o(n), \quad n \geq 1,
\]
\[
\gamma_{2n+1} = \frac{(n+1)(2n+1)(\mu+2)}{(3n+\mu+2)}
\gamma_1 = \frac{2\gamma_1(\mu+2)}{3} n + o(n), \quad n \geq 0,
\]
• For \( \mu > 0 \), then \( \gamma_n > 0 \) for all \( n \geq 1 \).
• The largest real zero \( x_{n,n}^{(j)} \) of \( P_{3n+j} \) is bounded from above by

\[
x_{n,n}^{(j)} \leq \frac{3^{2/3}}{2^{1/3}} (\gamma_1(\mu+2))^{1/3} n^{1/3} + o(n^{1/3})
\]
Case $B_1$: the zeros of $P_n$

Figure: Zeros of $P_{34}(x; \mu)$ (circle), $P_{35}(x; \mu)$ (star) and $P_{36}(x; \mu)$ (square) with $\mu = 3$, where $P_n(x; \mu)$ is the 2-OPS studied in case $B1$. 
With the choice of $\gamma_1 = 2$, when $\mu > 0$

\[
\begin{aligned}
\frac{1}{3} u_0'' + x^2 u_0' - (\mu - 2)xu_0 &= 0 \\
\end{aligned}
\]

\[
\begin{aligned}
u_1 = -\frac{(\mu+2)}{\mu} \left( u_0' + 3x^2 u_0 \right). \\
\end{aligned}
\]

and for $\mu = 0$:

\[
\begin{aligned}
u_0' + 3x^2 u_0 &= 0 \\
xu_1' &= 2u_0' \\
\end{aligned}
\]
Theorem (AL & Van Assche)

The linear 3-fold symmetric 2-orthogonal vector functional \((u_0, u_1)\) admit the following integral representation:

\[
\langle u_k, f(x) \rangle = \frac{1}{3} \left( \int_0^\infty f(x) U_k(x) dx + \omega^{2k-1} \int_0^\infty f(x) U_k(\omega x) dx + \omega^{1-2k} \int_0^\infty f(x) U_k(\omega^2 x) dx \right),
\]

with \(k = 0, 1\) and

\[
U_0(x) := U_0(x; \mu) = \frac{3\Gamma\left(\frac{\mu+2}{3}\right)}{\Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{2}{3}\right)} e^{-x^3} U\left(\frac{\mu}{3}, \frac{2}{3}; x^3\right),
\]

\[
U_1(x) := U_1(x; \mu) = \frac{9\Gamma\left(\frac{\mu+5}{3}\right)}{\Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{2}{3}\right)} x^2 e^{-x^3} U\left(\frac{\mu}{3} + 1, \frac{5}{3}, x^3\right), \quad \text{for} \quad \mu \neq 0
\]

\[
U_1(x; 0) = 3\sqrt{3} \Gamma\left(\frac{2}{3}\right) \Gamma\left(\frac{2}{3}, x^3\right)
\]

Here

\[
U(a, b; x) = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1}(t+1)^{-a+b-1} e^{-tx} dt \quad \text{and} \quad U(0, b; x) = 1
\]
Proof (idea). We seek an integral representation for $u_0$, that is, we seek a weight function $U_0(x)$ and a path $C$ so that

$$< u_0, f(x) > = \int_C f(x) U_0(x) \, dx,$$

is valid for any polynomial $f$. In particular, we must have

$$< u_0, x^n > = \int_C x^n U_0(x) \, dx, \quad n \geq 0.$$

The functional equation $(\mu + 2) u_0'' + x^2 u_0' - (\mu - 2)x u_0 = 0$ implies that $U_0$ must be a solution of the differential equation

$$(\mu + 2) U_0'' + x^2 U_0' - (\mu - 2)x U_0 = \lambda g(x)$$

where $\lambda$ is a complex constant and $g(x)$ is a function such that

$$\int_C x^n g(x) \, dx = 0, \quad n \geq 0.$$

With $\lambda = 0$, it follows that

$$U_0(x) = c_1 \, _1F_1 \left( \frac{2 - \mu}{3}, \frac{2}{3}; t \right) + c_2 t^{1/3} \, _1F_1 \left( 1 - \frac{\mu}{3}, \frac{4}{3}; t \right)$$
The choice of the constants $c_1$ and $c_2$ as well as the path of integration is dictated by the conditions

$$< u_0, x^n > = \int_C x^n u_0(x) dx, \quad n \geq 0,$$

and

$$\left. \left[ (\mu + 2) (f'(x) - f(x)) u'_0(x) - x^2 f(x) u_0(x) \right]\right|_C = 0, \quad \text{for any } f \in \mathcal{P}.$$

From DLMF (relations (13.2.39) and (13.2.41)) we deduce

$$e^{-z} U(a, b, z) = \frac{\Gamma(1 - b)}{\Gamma(a - b + 1)} \binom{1}{b - a, b; -z} + \frac{\Gamma(b - 1)}{\Gamma(a)} z^{1-b} \binom{1}{1-a, 2-b; -z}$$

which are valid when $b$ is not an integer.

Thus, with $c_1 = \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{1}{3} + \mu)} K$ and $c_2 = \frac{-\mu}{\Gamma(\frac{2}{3} + 1)} K$ and $C = \Gamma$, the result follows. □
The particular choice of $\mu = 1$ produces

$$\gamma_{2n} = \frac{2}{9}(2n + 1)(\mu + 2), \quad n \geq 1,$$

$$\gamma_{2n+1} = \frac{2}{3}(2n + 1)(\mu + 2), \quad n \geq 0,$$

whilst the weight functions become

$$U_0(x; 1) = \frac{\sqrt{x}}{2\sqrt{3}\pi^{3/2}} e^{-\frac{x^3}{18}} K_{\frac{1}{6}} \left(\frac{x^3}{18}\right)$$

$$U_1(x; 1) = \frac{x^2}{4\sqrt{3}\pi^{3/2} (x^3)^{5/6}} e^{-\frac{x^3}{18}} \left(\left(x^3 + 6\right) K_{\frac{1}{6}} \left(\frac{x^3}{18}\right) - x^3 K_{\frac{7}{6}} \left(\frac{x^3}{18}\right)\right)$$

where $K_{\nu}(z)$ represents the modified Bessel function of second kind.
With $\mu = 2$, we have

\[
\gamma_{2n} = \frac{4n(2n + 1)(n + 2)}{(3n + 1)(3n + 4)} \gamma_1, \quad n \geq 1,
\]

\[
\gamma_{2n+1} = \frac{4(n + 1)(2n + 1)}{(3n + 4)} \gamma_1, \quad n \geq 0,
\]

whilst the integral representation becomes

\[
U_0(x; 2) = \frac{\sqrt{3}}{2\pi} \frac{\Gamma\left(\frac{4}{3}\right)}{3^{\frac{1}{3}} 4^{\frac{1}{3}}} \Gamma\left(\frac{1}{3}, \frac{1}{12}x^3\right)
\]

\[
U_1(x; 2) = \frac{6\sqrt{3}\Gamma\left(\frac{4}{3}\right)}{\sqrt[3]{4\pi}} \left(\frac{1}{2} x^2 \Gamma\left(\frac{1}{3}, \frac{1}{12}x^3\right) - 3\sqrt{18e^{-x^3/12}}\right)
\]

where $\Gamma(\alpha, z)$ represents the incomplete Gamma function:

\[
\Gamma(\alpha, z) = \int_z^{+\infty} t^{\alpha-1} e^{-t} \, dt \quad \text{provided that } \alpha > 0.
\]
3rd order differential equation:

\[-\gamma_1(\mu + 2)P_n'''(x) + 2x^2 P_n''(x) + 2x \left( \mu + \frac{3}{4} \left( (-1)^n + 3 \right) - \frac{n}{2} \right) P_n'(x)\]

\[= 2n \left( \mu + \frac{n}{2} + \frac{3(-1)^n}{4} + \frac{5}{4} \right) P_n(x)\]

from which we deduce

\[P_n^{[0]}(x; \mu) = \frac{(-1)^n(3\mu + 6)^n \left( \frac{1}{3} \right)_n \left( \frac{2}{3} \right)_n}{\left( \frac{n}{2} + \frac{(-1)^n}{4} + \frac{\mu}{3} + \frac{5}{12} \right)_n} {}_2F_2 \left( -n, \frac{2\mu + 3n}{6} + \frac{(-1)^n}{4} + \frac{5}{12}; \frac{1}{3}, \frac{2}{3} \right) \frac{x}{3(\mu + 2)} \]

\[P_n^{[1]}(x; \mu) = \frac{(-1)^n(3\mu + 6)^n \left( \frac{2}{3} \right)_n \left( \frac{4}{3} \right)_n}{\left( \frac{n}{2} + \frac{(-1)^n+1}{4} + \frac{\mu}{3} + \frac{11}{12} \right)_n} {}_2F_2 \left( -n, \frac{2\mu + 3n}{6} + \frac{(-1)^{n+1}}{4} + \frac{11}{12}; \frac{2}{3}, \frac{4}{3} \right) \frac{x}{3(\mu + 2)} \]

\[P_n^{[2]}(x; \mu) = \frac{(-1)^n(3\mu + 6)^n \left( \frac{4}{3} \right)_n \left( \frac{5}{3} \right)_n}{\left( \frac{n}{2} + \frac{(-1)^n}{4} + \frac{\mu}{3} + \frac{17}{12} \right)_n} {}_2F_2 \left( -n, \frac{2\mu + 3n}{6} + \frac{(-1)^n}{4} + \frac{17}{12}; \frac{4}{3}, \frac{5}{3} \right) \frac{x}{3(\mu + 2)} \]
In this case we have

\[
\gamma_{2n} = \frac{n(2n + 1)(\rho + 3)}{(3n + \rho)} \gamma_1, \quad n \geq 1,
\]

\[
\gamma_{2n+1} = \frac{(n + 1)(2n + 1)(n + \rho)(\rho + 3)}{(3n + \rho + 3)(3n + \rho)} \gamma_1, \quad n \geq 0.
\]

With the choice of \( \gamma_1 = \frac{2}{3(\rho+3)} \), we obtain
In this case we have

\[
\gamma_{2n} = \frac{n(2n + 1)(\rho + 3)}{3n + \rho} \gamma_1, \quad n \geq 1,
\]

\[
\gamma_{2n+1} = \frac{(n + 1)(2n + 1)(n + \rho)(\rho + 3)}{(3n + \rho + 3)(3n + \rho)} \gamma_1, \quad n \geq 0.
\]

With the choice of \( \gamma_1 = \frac{2}{3(\rho+3)} \), we obtain

\[
Q_{n}^{\text{case } B_2}(x; \mu) = P_{n}^{\text{case } B_1}(x; \mu + 1), \quad \text{for all } n \geq 0,
\]

while

\[
Q_{n}^{\text{case } B_1}(x; \mu) = P_{n}^{\text{case } B_2}(x; \mu + 2), \quad \text{for all } n \geq 0,
\]

which brings

\[
\frac{1}{(n + 2)(n + 1)} \frac{d^2}{dx^2} P_{n+2}(x; \mu) = P_{n}(x; \mu + 3)
\]
Case C

We set

\[ \gamma_1 = \frac{2}{(\mu + 2)(\rho + 3)} \]

so that

\[ \gamma_{2n} \equiv \gamma_{2n}(\mu, \rho) = \frac{2n(2n + 1)(n + \mu)}{(3n + \mu - 1)(3n + \mu + 2)(3n + \rho)}, \quad n \geq 1, \]

\[ \gamma_{2n+1} \equiv \gamma_{2n}(\mu, \rho) = \frac{2(n + 1)(2n + 1)(n + \rho)}{(3n + \mu + 2)(3n + \rho)(3n + \rho + 3)}, \quad n \geq 0. \]

Besides, we have

\[
\begin{cases}
(1 - x^3) u_0'' + x^2(\mu + \rho - 4)u_0' - (\mu - 2)(\rho - 1)xu_0 = 0, \\
\frac{\mu}{(\mu + 2)} u_1 = (x^3 - 1) u_0' - (\rho - 1)x^2 u_0, \quad \text{for } \mu > -1, \\
xu_1' = 2u_0', \quad \text{for } \mu = 0.
\end{cases}
\]

Some of these are related to polynomials introduced by Pincherle (1890) and later extended by Humbert (1920), which were also related to \( _3F_2 \) functions by Baker (1920).
Case C: the weights

Here we have

$$\langle u_k, f(x) \rangle = \frac{1}{3} \left( \int_0^1 f(x) \mathcal{U}_k(x) dx + \omega^{2k-1} \int_0^\omega f(x) \mathcal{U}_k(\omega^2 x) dx + \omega^{1-2k} \int_0^\omega f(x) \mathcal{U}_k(\omega x) dx \right)$$

with

$$\mathcal{U}_0(x) := \mathcal{U}_0(x; \mu, \rho)$$

$$= \frac{3 \Gamma \left( \frac{\mu+2}{3} \right) \Gamma \left( \frac{\rho + 1}{3} \right)}{\Gamma \left( \frac{1}{3} \right) \Gamma \left( \frac{2}{3} \right) \Gamma \left( \frac{\mu + \rho + 2}{3} \right)} (1 - x^3)^{\frac{\mu + \rho - 1}{3}} 2 F_1 \left( \frac{\mu}{3}, \frac{\rho + 1}{3}; 1 - x^3 \right),$$

$$\mathcal{U}_1(x) := \mathcal{U}_1(x; \mu, \rho)$$

$$= \frac{3 \Gamma \left( \frac{\mu+5}{3} \right) \Gamma \left( \frac{\rho + 1}{3} \right)}{\Gamma \left( \frac{2}{3} \right) \Gamma \left( \frac{4}{3} \right) \Gamma \left( \frac{\mu + \rho + 2}{3} \right)} x^2 (1 - x^3)^{\frac{\mu + \rho - 1}{3}} 2 F_1 \left( \frac{\mu}{3} + 1, \frac{\rho + 1}{3}; 1 - x^3 \right).$$
Humbert polynomials: when $\mu = \frac{3\nu-1}{2}$ and $\rho = \frac{3\nu}{2}$, this 2-OPS satisfies

$$P_{n+2}(x; \frac{3\nu-1}{2}, \frac{3\nu}{2}) = xP_{n+1}(x; \frac{3\nu-1}{2}, \frac{3\nu}{2}) - \frac{4}{27} \frac{n(n+1)(3\nu+n-1)}{(\nu+n-1)(\nu+n)(\nu+n+1)} P_{n-1}(x; \frac{3\nu-1}{2}, \frac{3\nu}{2})$$

"Chebyshev"-type polynomials: when $\nu = 1 \Rightarrow (\mu, \rho) = (1, 3/2)$:

$$P_{n+2}(x; 1, \frac{3}{2}) = xP_{n+1}(x; 1, \frac{3}{2}) - \frac{4}{27} P_{n-1}(x; 1, \frac{3}{2})$$

and here

$$U_0(x) = \frac{9\sqrt{3}}{4\pi} \left( \left( 1 + \sqrt{1 - x^3} \right)^{1/3} - \left( 1 - \sqrt{1 - x^3} \right)^{1/3} \right)$$

$$U_1(x) = \frac{27\sqrt{3}}{8\pi} \left( \sqrt{1-x^3} \left[ \left( 1 + \sqrt{1 - x^3} \right)^{2/3} - \left( 1 - \sqrt{1 - x^3} \right)^{2/3} \right] \right)$$
Case C: explicit expressions

\[ P_{3n}(x; \mu, \rho) \] = \[
\frac{(-1)^n \left( \frac{1}{3} \right)_n \left( \frac{2}{3} \right)_n}{\left( \frac{n}{2} + \frac{1}{4}(-1)^{3n} + \frac{\mu}{3} + \frac{5}{12} \right) n \left( \frac{n}{2} - \frac{1}{4}(-1)^{3n} + \frac{\rho}{3} + \frac{1}{4} \right) n} \]

\[ \left( -n, \frac{n}{2} + \frac{1}{4}(-1)^{3n} + \frac{\mu}{3} + \frac{5}{12}, \frac{n}{2} - \frac{1}{4}(-1)^{3n} + \frac{\rho}{3} + \frac{1}{4} \right) ; x^3 \]

\[ \_3F_2 \]

\[ P_{3n+1}(x; \mu, \rho) \] = \[
\frac{(-1)^n \left( \frac{2}{3} \right)_n \left( \frac{4}{3} \right)_n}{\left( \frac{n}{2} - \frac{1}{4}(-1)^{3n} + \frac{\mu}{3} + \frac{11}{12} \right) n \left( \frac{n}{2} + \frac{1}{4}(-1)^{3n} + \frac{\rho}{3} + \frac{3}{4} \right) n} \]

\[ \left( -n, \frac{n}{2} - \frac{1}{4}(-1)^{3n} + \frac{\mu}{3} + \frac{11}{12}, \frac{n}{2} + \frac{1}{4}(-1)^{3n} + \frac{\rho}{3} + \frac{3}{4} \right) ; x^3 \]

\[ \_3F_2 \]

\[ P_{3n+2}(x; \mu, \rho) \] = \[
\frac{(-1)^n \left( \frac{4}{3} \right)_n \left( \frac{5}{3} \right)_n}{\left( \frac{n}{2} + \frac{1}{4}(-1)^{3n} + \frac{\mu}{3} + \frac{17}{12} \right) n \left( \frac{n}{2} - \frac{1}{4}(-1)^{3n} + \frac{\rho}{3} + \frac{5}{4} \right) n} \]

\[ \left( -n, \frac{n}{2} + \frac{1}{4}(-1)^{3n} + \frac{\mu}{3} + \frac{17}{12}, \frac{n}{2} - \frac{1}{4}(-1)^{3n} + \frac{\rho}{3} + \frac{5}{4} \right) ; x^3 \]

\[ \_3F_2 \]
Case C: zeros of $P_n(x; 3, 2)$

zeros of $P_n(x; 3, 2)$ when
- $n=27$
- $n=28$
- $n=29$
THANK YOU!