# Reduction of the Gibbs phenomenon for smooth functions with jumps by the $\epsilon$-algorithm[1]

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Dedicated to our friend Claude Brezinski on the occasion of his retirement

**Abstract**

Recently, Brezinski has proposed to use Wynn's $\epsilon$-algorithm in order to reduce the Gibbs phenomenon for partial Fourier sums of smooth functions with jumps, by displaying very convincing numerical experiments. In the present paper we derive analytic estimates for the error corresponding to a particular class of hypergeometric functions, and obtain the rate of column convergence for such functions, possibly perturbed by another sufficiently differentiable function. We also analyze the connection to Padé-Fourier and Padé-Chebyshev approximants, including those studied recently by Kaber and Maday.

**Key words:** Fourier series, Gibbs phenomenon, convergence acceleration, $\epsilon$-algorithm, Padé-Fourier approximants, Padé-Chebyshev approximants.
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## 1 Introduction

For smoothing the Gibbs phenomenon for partial sums $S_n(f)$ of Fourier series of functions $f : [-\pi, \pi] \mapsto \mathbb{R}$ with jumps, the following procedure, called the complex $\epsilon$-algorithm [Br02], has been proposed by Brezinski. If

$$S_n(f)(t) = \frac{a_0}{2} + \sum_{j=1}^{n} [a_j \cos(jt) + b_j \sin(jt)],$$

add $i$ times the conjugate part

$$\widetilde{S}_n(f)(t) = \sum_{j=1}^{n} [a_j \sin(jt) - b_j \cos(jt)],$$

in order to get

$$S_n(f)(t) + i\widetilde{S}_n(f)(t) = G_n(f)(e^{it}),$$

with $G_n(f)$ the $n$th Taylor series of the (formal) series

$$G(f)(z) = \sum_{j=0}^{\infty} c_j(f) z^j, \quad c_0(f) = \frac{a_0}{2}, \quad \text{and for } j > 1, \quad c_j(f) = a_j - ib_j.$$

Then apply the $\epsilon$-algorithm to the sequence of partial sums $(G_n(f)(e^{it}))_n$ for fixed $t$, and use the real part of the resulting quantities $\epsilon_{2k}^{(n)}(t)$ for approaching $f(t) = \mathrm{Re}\,(G(f)(e^{it})$. The

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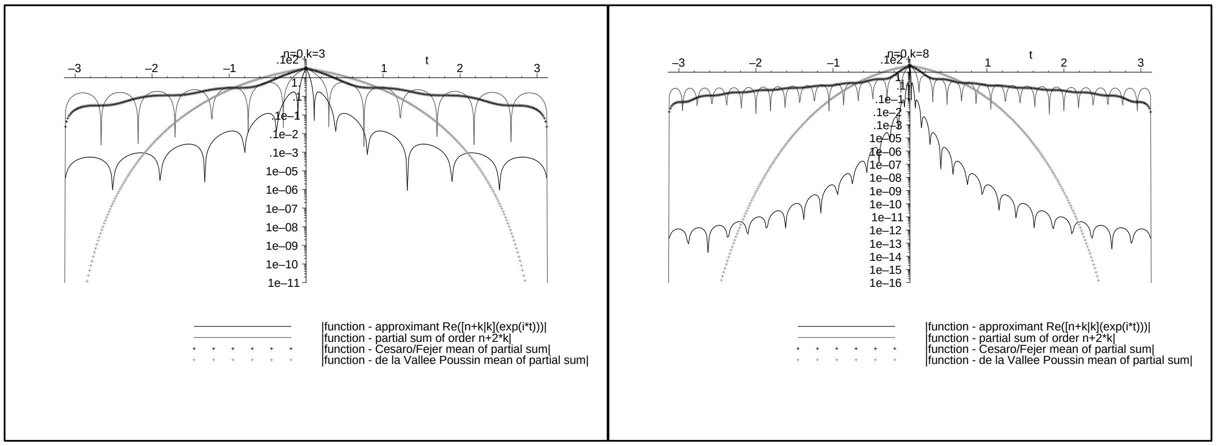


Figure 1: *The modulus of the error of approximation on a logarithmic scale of the Fourier series of the sew tooth function (1.2). On the left we use the first 7 and on the right the first 17 coefficients of the Fourier series. The error for the partial sums is strongly oscillating, and, according to the Gibbs phenomenon, remains quite large (about* $1/10$*) even for higher order Fourier sums. The error for Cesaro means is smoother, but about of the same size, even for arguments far from* $0$*, the singularity of our function, whereas the de la Vallée-Poussin mean gives better approximants only far from* $0$*. In contrast, the errors for* $\epsilon_6^{(0)}$ *and* $\epsilon_{16}^{(0)}$ *are much smaller, even for arguments closer to the singularity.*

acceleration of Fourier series via the $\epsilon$-algorithm applied to the partial sums of $G_n(f)$ has been already proposed by P. Wynn [Wy67], without discussing the link with the Gibbs phenomenon. Wynn gives several examples where classical linear acceleration procedures for Fourier series like Cesaro means or de la Vallée–Poussin means have convergence behavior clearly weaker than the one discussed here, see Figure 1 below. In addition to the fast rate of convergence, Brezinski observed that the real part of the sequence $(\epsilon_{2k}^{(n)}(t))_n$ gives rise to approximations of $f$ with strongly reduced Gibbs oscillations. Though numerical evidence strongly supports Brezinski's smoothing approach, up to now, no theoretical error estimates have been known.

The aim of the present paper is to provide such error estimates for functions $f$ which (either themselves or their derivatives) have one jump on $[-\pi, \pi)$, but are sufficiently regular elsewhere, a typical case of solutions of PDEs where spectral methods have a priori a poor convergence behavior, and there is a need for an acceleration procedure. In [KM05], the authors discuss the simple model function $f(t) = \text{sign}(\cos(t))$, and give interesting acceleration properties for $k$ fixed and $n \to \infty$, see [KM05, Theorem 4.10]. In the present paper we will consider more general functions of the form $f = f_1 + f_2$ where $f_1$ has prescribed discontinuities and is smooth elsewhere while $f_2$ has sufficiently fast decreasing Fourier coefficients. For such type of functions, we will see that the partial Fourier series converges slowly and presents in particular the so-called Gibbs phenomenon of oscillation close to the singularities of $f$ (or $f_1$), and the acceleration properties of the $\epsilon$-algorithm will essentially depend only on $f_1$.

A typical jump function $f_1$ considered in this paper is given by (a multiple of) the $2\pi$ periodic saw tooth function

$$s(t) = \pi + t \quad \text{for } t \in (-\pi, 0], \qquad s(t) = -\pi + t \quad \text{for } t \in (0, \pi], \tag{1.1}$$

having one jump of absolute value $2\pi$ at $t = 0$ in $[-\pi, \pi)$. We have for the saw tooth function the Fourier expansion

$$S_n(s)(t) = -2\sum_{j=1}^{n} \frac{\sin(jt)}{j}, \quad \text{and thus} \quad G(s)(z) = 2i\sum_{j=1}^{\infty} \frac{z^j}{j} = -2i\log(1-z). \tag{1.2}$$

The error obtained by approximating $s(t)$ via partial sums and via the $\epsilon$-algorithm are displayed in Figure 1.

In the present paper we will consider for $G(f_1)$ more general hypergeometric functions of the form

$$G^{(\alpha,\beta)}(z) = {}_2F_1\left( \begin{array}{c} \alpha+1,1 \\ \alpha+\beta+2 \end{array} \middle| z \right), \quad \text{where} \quad {}_2F_1\left( \begin{array}{c} a,b \\ c \end{array} \middle| z \right) = \sum_{j=0}^{\infty} \frac{(a)_j\,(b)_j}{(c)_j\,j!}\, z^j, \tag{1.3}$$

$\alpha, \beta > -1$, and $(a)_0 = 1$, $(a)_j = a(a+1)\cdots(a+j-1)$ for $j > 0$, is the usual Pochhammer symbol. Throughout, we denote by $P_k^{(\alpha,\beta)}$ the Jacobi polynomial of degree $k$, orthogonal with respect to the measure $(1-x)^\alpha(1+x)^\beta dx$, such that

$$\int_{-1}^{1} (P_k^{(\alpha,\beta)}(x))^2(1-x)^\alpha(1+x)^\beta dx = \frac{2^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)k!}, \tag{1.4}$$

see for instance [Chi78, Chapter 5, eqn 2.18]. As seen in Table $1(d)_1-(d)_4, (e)_2, (f)$, and verified by elementary computations, such a class of hypergeometric series allows us to include other functions $f_1(t)$ with a particular jump behavior, like $\mathrm{sign}(\cos(t))$ (compare with [KM05]), having two discontinuities, or like $|\sin(\frac{t}{2})|$, and $(1-\cos(t))\, s(t)$, respectively, with first (and second) order derivative having a discontinuity at 0. Moreover, combined with Table $1(a), (b), (c), (e)_1$, we may easily construct other examples where the argument is shifted, or where $f_1 \in \mathcal{C}^{\ell-1}([-\pi,\pi])$, with its $\ell$th derivative having one discontinuity.

Our main tool in deriving error estimates for the complex $\epsilon$-algorithm will be the connection to Padé approximation of perturbations of the (shifted) logarithm $z \mapsto \log(1-z)$ (or more generally of Stieltjes functions with respect to measures related to the Jacobi orthogonal polynomials) on the unit circle: indeed, as mentioned already in [Br02], it is well-known that

$$\epsilon_{2k}^{(n)}(t) = [n+k|k]_{G(f)}(e^{it}) \tag{1.5}$$

and hence we will have to estimate the modulus of

$$f(t) - \mathrm{Re}\,(\epsilon_{2k}^{(n)}(t)) = \mathrm{Re}\left(G(f)(e^{it}) - [n+k|k]_{G(f)}(e^{it})\right). \tag{1.6}$$

In particular, for $n \to \infty$ and $k$ fixed, we will have to find a Montessus de Ballore type convergence theorem for perturbed Stieltjes functions.

For even $f$ (and hence $b_j = 0$ for all $j$), $S_n(f)(\arccos(x))$ is the partial Chebyshev series of $x \mapsto F(x) := f(\arccos(x))$. Here, according to the well-known formula $T_j(x) = \cos(j\,\arccos(x))$ for the Chebyshev polynomials, it is not difficult to see that $\mathrm{Re}\,(\epsilon_{2k}^{(n)}(\arccos(x)))$ is a rational function in $x$, the so-called Padé-Chebyshev approximant of type $(n+k, k)$ of $g$ due to Gragg [BGM96, pp.383-387]. Following the nomenclature of Baker and Graves-Morris [BGM96, Section 7.4], there are other approaches to rational approximation of Chebyshev series, and these latter methods can also be adapted to Fourier series (see Section 2) or to series of general orthogonal polynomials (see, e.g., [GRS92]): for the so-called *linear* Padé-Chebyshev approximant $R_{m,n}^{LC} = P_{m,n}^{LC}/Q_{m,n}^{LC}$, and the *nonlinear* Padé-Chebyshev approximant $R_{m,n}^{NC} = P_{m,n}^{NC}/Q_{m,n}^{NC}$, respectively, we look for polynomials $P_{m,n}^{LC}, P_{m,n}^{NC}$ of degree $\leq m$ and $Q_{m,n}^{LC}, Q_{m,n}^{NC}$ of degree $\leq n$ such that either the linearized error $F\,Q_{m,n}^{LC} - P_{m,n}^{LC}$ or the error $F - R_{m,n}^{NC}$ itself is orthogonal to $T_0, T_1, ..., T_{m+n}$. Linear Padé-Chebyshev approximants are easy to compute (a solution of a linear system of equations with matrix of coefficients being Toeplitz plus Hankel) but require more coefficients of the Chebyshev series; the acceleration properties of $R_{m,n}^{LC}(x)$ for the sign function and $n$ fixed, $m \to \infty$ have been discussed by Kaber and Maday [KM05]. It was observed numerically by Fleischer [Fl73] and proved rigorously by Gonchar, Rakhmanov and Suetin in [GRS92] for Markov functions that nonlinear Padé-Chebyshev diagonal approximants ($m = n \to \infty$) have

| | $f(t)$ | $G(f)(z)$ | Padé approximant/reversed denominator |
|---|---|---|---|
| $(a)$ | $\check{f}(t-t_0)$ | $G(\check{f})(e^{-it_0}z)$ | $[n+k\|k]_{G(f)}(z) = [n+k\|k]_{G(\check{f})}(e^{-it_0}z)$ |
| $(b)$ | $\check{f}(2t)$ | $G(\check{f})(z^2)$ | $[2n+2k\|2k]_{G(f)}(z) = [n+k\|k]_{G(\check{f})}(z^2)$ |
| $(c)$ | | $p(z) + cz^{n+1}\, G(\check{f})(z)$<br>$\deg p \leq n, c \in \mathbb{C}$.<br>Notice: $G(f) = z^\ell G^{(\alpha,\beta)}$<br>$\Longrightarrow G(\check{f}) = G^{(\alpha+n+1-\ell,\beta)}$ | $[n+k\|k]_{G(f)}(z)$<br>$= p(z) + c\, z^{n+1}\, [k-1\|k]_{G(\check{f})}(z)$ |
| $(d)$ | | $G^{(\alpha,\beta)}(z)$ | $q_{n+k-1,k}(z) = P_k^{(\alpha+n,\beta)}(1-2z)$ |
| $(d)_1$ | $s(t)$, see (1.1) | $-2i\log(1-z)$<br>$= 2i\, z\, G^{(0,0)}(z)$ | $q_{n+k,k}(z) = P_k^{(n,0)}(1-2z)$ |
| $(d)_2$ | $\mathrm{sign}(\cos(t)) =$<br>$(s(t-\frac{\pi}{2}) - s(t+\frac{\pi}{2}))/\pi$ | $\frac{4z}{\pi} G^{(-\frac{1}{2},0)}(-z^2)$ | $q_{2n+2k,2k}(z) = P_k^{(n-\frac{1}{2},0)}(1+2z^2)$ |
| $(d)_3$ | $\|\sin(\frac{t}{2})\|$ | $\frac{2}{\pi} - \frac{4z}{3\pi} G^{(-\frac{1}{2},1)}(z)$ | $q_{n+k,k}(z) = P_k^{(n-\frac{1}{2},1)}(1-2z)$ |
| $(d)_4$ | $\|\sin(t)\| + \sin(t)$ | $\frac{2}{\pi} - iz - \frac{4z^2}{3\pi} G^{(-\frac{1}{2},1)}(z^2)$ | $q_{2n+2k,2k}(z) = P_k^{(n-\frac{1}{2},1)}(1-2z^2)$ |
| $(e)$ | $(\cos(t)-1)^\ell \check{f}(t)$,<br>$a_{j+\ell} = 2^{-\ell}\Delta^{2\ell}\check{a}_j$,<br>$b_{j+\ell} = 2^{-\ell}\Delta^{2\ell}\check{b}_j$ | $p(z) + \frac{(z-1)^{2\ell}}{2^\ell z^\ell}\,[G(\check{f})(z) - G_{\ell-1}(\check{f})(z)]$<br>$\deg p \leq \ell - 1$, | |
| $(e)_1$ | $(\cos(t)-1)^\ell \check{f}(t)$<br>$G(\check{f}) = G^{(\alpha,\beta)}$ | $p + cz^\ell G^{(\alpha,\beta+2\ell)}$<br>$\deg p \leq \ell-1, c = \frac{2^{-\ell}(\beta+1)_{2\ell}}{(\alpha+\beta+2)_{2\ell}}$ | $q_{n+2\ell+k-1,k}(z) = P_k^{(\alpha+n,\beta+2\ell)}(1-2z)$ |
| $(e)_2$ | $(\cos(t)-1)s(t)$ | $-\frac{3}{2}iz + \frac{iz^2}{3} G^{(0,2)}(z)$ | $q_{n+k+1,k}(z) = P_k^{(n,2)}(1-2z)$ |

Table 1: *Some examples for Fourier series $f$, their associated power series $G(f)$ and explicit formulas for Padé approximants $[n+k|k]$ or reversed Padé denominators $q_{n+k,k}$ (up to a normalization constant). The quantities $c, t_0, t_1, t_2, \alpha, \beta$ occurring in the table are real numbers ($\alpha, \beta > -1$), also $n, k, \ell$ are nonnegative integers ($n \geq -k$ for cases (a) and (b)), and $p$ are suitable polynomials, not necessarily the same for different rows of the table.*

better approximation properties than the linear ones. However, the nonlinear approximants are in general difficult to compute, which limits their impact in practical applications. We will show in Theorem 2.1 below that, provided that $f$ is even and the Padé approximant $[n+k|k]_{G(f)}$ of $G(f)$ has no poles in the closed unit disk $|z| \leq 1$, the nonlinear Padé-Chebyshev approximant $R^{NC}_{n+k,k}$ of $F = f \circ \arccos$ is given by

$$R^{NC}_{n+k,k}(\cos(t)) = \mathrm{Re}\,(\epsilon^{(n)}_{2k}(t)) = \mathrm{Re}\,([n+k|k]_{G(f)}(e^{it})). \tag{1.7}$$

This observation, which to our knowledge is original, may help to understand the convergence properties of $\mathrm{Re}\,(\epsilon^{(n)}_{2k}(t))$. A formal link between the denominators of both rational approximants has been given before by Paszkowski [Pa63], without mentioning the necessary hypothesis on the Padé approximant.

The information given in Table 1 requires additional explanations and proofs. Concerning Table 1$(a)$, we notice that a translation of the argument $t$ of $f$ is equivalent to a multiplication of the variable $z$ of $G(f)$ with a constant of modulus one

$$\check{f}(t) = f(t-t_0) \quad\Longrightarrow\quad S_n(\check{f})(t) = S_n(f)(t-t_0) \quad\Longrightarrow\quad G(\check{f})(z) = G(f)(e^{-it_0}z), \tag{1.8}$$

since indeed $\check{a}_j = a_j\cos(jt_0) - b_j\sin(jt_0)$, $\check{b}_j = a_j\sin(jt_0) + b_j\cos(jt_0)$, and thus $c_j(\check{f}) = \check{a}_j - i\check{b}_j = e^{-ijt_0}c_j(f)$. In the last column of rows (a),(b), and (c) we recall some well-known properties of Padé approximation [BGM96]. One may express the Padé approximants of $G^{(\alpha,\beta)}$ by means

| | | |
|---|---|---|
| Input: | integers $n, k \geq 0$, partial sum $G_{n+2k}(z) = \sum_{j=0}^{n+2k} g_j z^j$, a fixed argument $z = e^{it}$ | |
| Initialization: | for $\ell = 0, ..., n+2k$: $\epsilon_0^{(\ell)} = G_\ell(z)$, $\epsilon_{-1}^{(\ell)} = 0$. | |
| Recurrence: | for $j = 0, 2, ..., 2k-1$, for $\ell = 0, ..., n+2k-j-1$: $\epsilon_{j+1}^{(\ell)} = \epsilon_{j-1}^{(\ell+1)} + \left(\epsilon_j^{(\ell+1)} - \epsilon_j^{(\ell)}\right)^{-1}$ | |
| Output: | $[n+k\|k]_G(z) = \epsilon_{2k}^{(n)}$ | |

Table 2: *Evaluating the Padé approximant $[n+k|k]_G(z)$ via the $\epsilon$-algorithm.*

of the Gauss continued fraction (see, e.g., [Ba75, Chapter 5]), in particular, there exist explicit formulas for the Padé denominator [Ba75, Eqn. (5.11)] which will enable us to estimate quite precisely the Padé error on the unit circle of such functions. For the sake of completeness, this connection between Jacobi orthogonal polynomials and the (reversed) Padé denominators of $G^{(\alpha,\beta)}$ claimed in the last column of Table 1($d$) will be shown in Lemma 3.1 in Section 3. The claims in rows $(d)_1 - (d)_4$ are obtained by combining $(d)$ with the statements of $(a)-(c)$, we leave the details for the reader. The claims in rows $(e), (e)_1$ and $(e)_2$ are again not too difficult to verify and left to the reader. From the information in Table 1, we see that, in order to study the error (1.6) for $f$ being equal to one of the functions $s(t)$, $s(t-t_1) - s(t-t_2)$, $\operatorname{sign}(\cos(t))$, $|\sin(\frac{t}{2})|$, or $(1-\cos(t))s(t)$, it is sufficient to estimate the modulus of

$$G^{(\alpha,\beta)}(e^{it}) - [n+k|k]_{G^{(\alpha,\beta)}}(e^{it})$$

(up to some explicitly known constant not depending on $n, k$) in terms of the distance of $e^{it}$ to the singularities of $f$. This will be done in Section 3 below. We will also show that similar bounds hold true for $f = f_1 + f_2$ with $f_1$ as before and $f_2$ sufficiently smooth.

The paper is organized as follows. In Section 2, we exhibit a link between the Padé-Chebyshev approximants of a Chebyshev series (and more general rational approximants of Fourier series) and the ordinary Padé approximants of the corresponding Taylor series. We also relate our analysis to the results of [KM05]. In Section 3, we study the rate of convergence of Padé approximants (in a column of the Padé table) to some specific hypergeometric functions, the real parts of which correspond to functions with prescribed discontinuities, possibly occuring in higher order derivatives. In Section 4, we extend the previous estimates by adding functions with continuous derivatives (up to some order depending on the degree of approximation) to the previous ones. Finally, in Section 5, we present numerical results.

## 2 Rational approximants of Fourier series

The most efficient way of evaluating the value at $z = e^{it}$ of a Padé approximant is known to be Wynn's $\epsilon$-algorithm, as described in Table 2. In this section, we relate the approximant (1.5) to other rational approximants such as linear or nonlinear Padé-Chebyshev and Padé-Fourier approximants.

In what follows we denote as usual by $L_2([-\pi,\pi])$ the set of square integrable functions on $[-\pi,\pi]$, with norm

$$\|f\|_2 := \Big(\frac{1}{\pi}\int_{-\pi}^{\pi} |f(t)|^2\, dt\Big)^{1/2}.$$

For a Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} \left(a_j \cos(jt) + b_j \sin(jt)\right),$$

recall that, by Parceval formula, $\|f\|_2^2 = \frac{|a_0|^2}{2} + \sum_{j=1}^{\infty} \left(|a_j|^2 + |b_j|^2\right)$. We also deal with the Hardy space $H_2$ of functions $G$ being analytic in the unit disk $\mathbb{D}$, with

$$\|G\|_2 := \lim_{r\to 1_-} \Big(\frac{1}{2\pi} \int_{-\pi}^{\pi} |G(re^{it})|^2 \, dt\Big)^{1/2} < \infty.$$

One may show, see e.g. [NS91, Chapter 3.3], that for $G(z) = \sum_{j=0}^{\infty} G_j z^j$, one has

$$\|G\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |G(e^{it})|^2 \, dt = \sum_{j=0}^{\infty} |G_j|^2.$$

In particular we may represent such functions by means of the Cauchy formula with contour being the unit circle $\mathbb{T}$. As a consequence, we have for our real-valued Fourier series $f$ that $f \in L_2([-\pi,\pi])$ if and only if $G(f) \in H_2$, with

$$\|G(f)\|_2^2 = \|f\|_2^2 - \frac{|a_0|^2}{4}. \tag{2.1}$$

We will show in the proof of Theorem 2.1 below that $\mathrm{Re}\,(\epsilon_{2k}^{(n)}(t))$ is a trigonometric rational function in $t \in [-\pi,\pi]$, with the numerator of degree $n+k$, and the denominator of degree $k$. Following the nomenclature of Baker and Graves-Morris [BGM96, Section 7.4], the quantity $\mathrm{Re}\,(\epsilon_{2k}^{(n)}(t))$ equals the Padé-Fourier approximant of type $(n+k,k)$ of $f$. There are other approaches to trigonometric rational approximation of Fourier series, and these latter methods can be also adapted to series of general orthogonal polynomials, see for instance Section 1 for Chebyshev series. For the so-called *linear* Padé-Fourier approximant $R_{m,n}^{LF} = P_{m,n}^{LF}/Q_{m,n}^{LF}$, and the *nonlinear* Padé-Fourier approximant $R_{m,n}^{NF} = P_{m,n}^{NF}/Q_{m,n}^{NF}$, respectively, we look for trigonometric polynomials $P_{m,n}^{LF}, P_{m,n}^{NF}$ of degree $\le m$ and $Q_{m,n}^{LF}, Q_{m,n}^{NF}$ of degree $\le n$ such that either the linearized error $f\,Q_{m,n}^{LF} - P_{m,n}^{LF}$ or the error $f - R_{m,n}^{NF}$ itself is orthogonal to the functions $\sin(jt)$ and $\cos(jt)$ for $j = 0, 1, ...., m+n$.

We have the following link between these rational approximants.

**Theorem 2.1** *Let $n, k \ge 0$, and consider the real–valued Fourier series $f(t) := \frac{a_0}{2} + \sum_{j=1}^{\infty}[a_j \cos(jt) + b_j \sin(jt)] \in L_2([-\pi,\pi])$ together with the associated series $G(z) = G(f)(z) = \frac{a_0}{2} + \sum_{j=1}^{\infty}[a_j - ib_j]z^j$. Suppose that the linear Padé approximant $[n+k|k]_G = P/Q$ of $G$ has no poles in the closed unit disk, i.e.,*

$$\deg P \le n+k, \quad \deg Q \le k, \quad \frac{G(z)Q(z) - P(z)}{z^{n+2k+1}} \text{ is analytic around } 0, \quad \forall |z| \le 1 : Q(z) \ne 0.$$

*Then the nonlinear Padé-Fourier approximant $R_{n+k,k}^{NF}$ of $f$ exists, and*

$$R_{n+k,k}^{NF}(t) = \mathrm{Re}\,(\epsilon_{2k}^{(n)}(t)) = \mathrm{Re}\,([n+k|k]_G(e^{it})), \qquad t \in [-\pi,\pi].$$

*If moreover $f(t) = F(\cos(t))$ is even (and thus $b_j = 0$ for all $j$), then the nonlinear Padé-Chebyshev approximant $R_{n+k,k}^{NC}$ of the Chebyshev series $F(x) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j T_j(x)$ exists, and*

$$R_{n+k,k}^{NC}(\cos(t)) = \mathrm{Re}\,(\epsilon_{2k}^{(n)}(t)) = \mathrm{Re}\,([n+k|k]_G(e^{it})), \qquad t \in [-\pi,\pi].$$

**Proof.** Denote by $P^*$, and $Q^*$, respectively, the polynomials obtained by taking the complex conjugate of the coefficients of $P$, and $Q$, then

$$\mathrm{Re}\,([n+k|k]_G)(e^{it}) = \frac{P(e^{it})Q^*(e^{-it}) + P^*(e^{-it})Q(e^{it})}{2Q(e^{it})Q^*(e^{-it})} =: R(t)$$

Here, the numerator and the denominator are trigonometric polynomials in $t$ of degree $n+k$ and $k$, respectively, showing that $R$ is indeed a candidate for the nonlinear Padé-Fourier approximant of type $(n+k,n)$ of $f$. If in addition $f$ is even, then, with the coefficients of $G$, also the coefficients of $P$ and $Q$ can be chosen to be real. In this latter case, $P=P^*$ and $Q=Q^*$, implying that both numerators and denominator of $R(t)$ are even, and thus cosine polynomials. Using the relation $T_j(x)=\cos(j\ \arccos(x))$ it follows that $R(\arccos(x))$ is indeed a rational function in $x$, and thus a candidate for the nonlinear Padé-Chebyshev approximant of type $(n+k,n)$ of $F=f\circ\arccos$.

In order to conclude, we only need to show that the real-valued function $f-R$ is orthogonal to the functions $\cos(jt)$ and $\sin(jt)$ for $j=0,1,...,n+2k$. We have for $j\in\{0,1,...,n+2k\}$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ijt}(f(t)-R(t))\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ijt}\Big[G(e^{it})-\frac{P(e^{it})}{Q(e^{it})}+\overline{G(e^{it})-\frac{P(e^{it})}{Q(e^{it})}}\Big]\,dt$$

$$=\frac{1}{2\pi i}\int_{|\zeta|=1}\Big[G(\zeta)-\frac{P(\zeta)}{Q(\zeta)}\Big]\frac{d\zeta}{\zeta^{j+1}}+\overline{\frac{1}{2\pi i}\int_{|\zeta|=1}\Big[G(\zeta)-\frac{P(\zeta)}{Q(\zeta)}\Big]\zeta^j\frac{d\zeta}{\zeta}}.$$

By construction, $G$ is an element of the Hardy space $H_2$, and so is $G-P/Q$ by assumption on $Q$. In particular, $G-P/Q$ is analytic in the unit disk and vanishes at zero, and therefore the second integral on the right-hand side vanishes. The first integral equals the $(j+1)$th coefficient of the Taylor expansion of $G-P/Q$ at zero. By assumption, we have that $1/Q$ is analytic in the unit disk, and that the first $n+2k+1$ coefficients in the Taylor expansion of $GQ-P$ do vanish. Hence also the first integral equals zero, and the above claim follows by taking real and imaginary parts. □

Let us illustrate the previous result with the sign function $F(x)=\mathrm{sign}(x)$. We have

$$f(t)=\mathrm{sign}(\cos(t))=\frac{1}{\pi}(s(t-\frac{\pi}{2})-s(t+\frac{\pi}{2})),$$

and, according to (1.8), we get the Chebyshev series expansion

$$F(x)=\frac{4}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}T_{2j+1}(x).$$

The convergence properties of the linear Padé-Chebyshev approximants $R^{LC}_{n+k,k}$ of the sign function $F(x)=\mathrm{sign}(x)$ for fixed $k$ and $n\to\infty$ has been discussed in detail in [KM05]. Paszkowski [Pa63] gave an explicit expression for the nonlinear Padé-Chebyshev approximants $R^{NC}_{n+k,k}$ of the sign function. Let us recover its denominator via Theorem 2.1: we have to compute the Padé approximant of $G=G(f)$, an odd function. In this case, it is well-known and easy to verify that the Padé table of $G(f)$ has a $2\times 2$ block structure

$$[2n+2k-1|2k]_G=[2n+2k|2k]_G=[2n+2k-1|2k+1]_G=[2n+2k|2k+1]_G. \tag{2.2}$$

In particular, the denominator of the linear Padé approximant of $G$ of degree $[2n+2k|2k+1]_G$ are vanishing at zero, and the hypothesis of Theorem 2.1 fails to hold. However, for the other three members of the block (2.2), the denominator is the same, and its reversed counterpart has been given in Table $1(d)_2$. In particular, all zeros of the denominator lie in $(-i\infty,-i)\cup(i,i\infty)$. Thus Theorem 2.1 gives us the following formula for the denominator of the nonlinear Padé–Chebyshev approximant of the sign function

$$Q^{NC}_{2n+2k-1,2k}(\cos(t))=|P_k^{(n-1/2,0)}(1+2e^{2it})|^2$$

in terms of a Jacobi orthogonal polynomial.

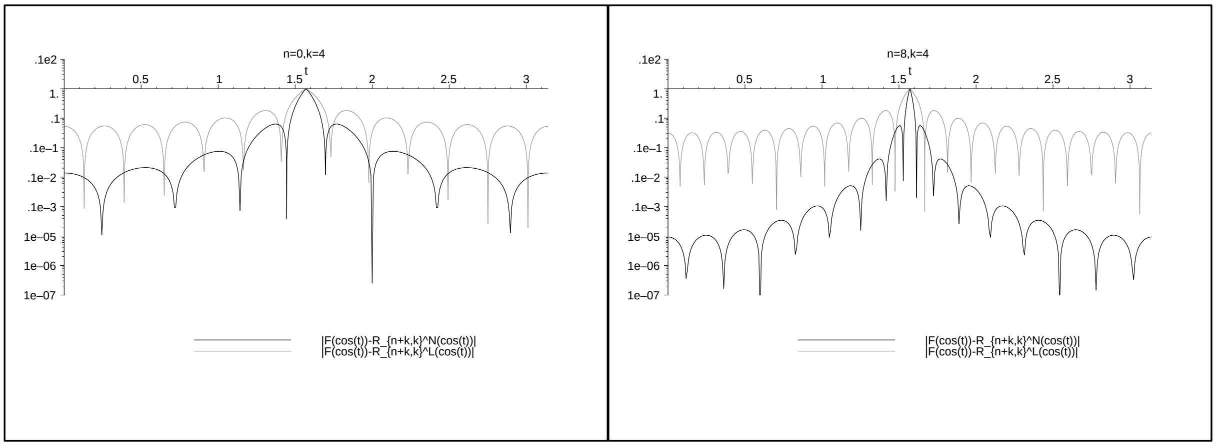


Figure 2: *The modulus of the error of approximation on a logarithmic scale at $x = \cos(t)$ of the linear and the nonlinear Padé-Chebyshev approximant of $F(x) = sign(x)$.*

In contrast, the approximants of index $(\mathcal{N}, \mathcal{M})$ for the sign function used in [KM05] for $\mathcal{N} \geq \mathcal{M} - 1$ are rational functions of numerator degree $2\mathcal{N} + 1$ and denominator degree $2\mathcal{M}$, which coincide with the linear Padé–Chebyshev approximants $R^{LC}_{2n+2k-1,2k} = R^{LC}_{2n+2k,2k}$ for $n \geq 0$. The authors in [KM05] use an explicit formula for the denominator given by Németh and Páris in [NP91]. Our numerical experiments reported in Figure 2 for $k = 2$ and $n \in \{0, 4\}$ seem to indicate that the nonlinear Padé-Chebyshev approximants have better approximation properties.

## 3 Error estimates for hypergeometric functions of type $G^{(\alpha,\beta)}$

In this section we study the Padé approximants to the functions

$$G^{(\alpha,\beta)}(z) = {}_2F_1\left( \begin{array}{c} \alpha+1, 1 \\ \alpha+\beta+2 \end{array} \middle| z \right) = \int \frac{d\mu^{(\alpha,\beta)}(y)}{1 - yz}, \tag{3.1}$$

where $\alpha, \beta > -1$ and the measure $d\mu^{(\alpha,\beta)}$ has the support $[0, 1]$ and the density

$$d\mu^{(\alpha,\beta)}(y) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} y^\alpha (1-y)^\beta dy.$$

We notice that the diagonal Padé approximants $[k|k]_{G^{(\alpha,\beta)}}$ (and, by Table 1(c), also the off-diagonal Padé approximants) are the even convergents of the Gauss continued fraction [Ba75, Chapter 5], from which one may conclude uniform convergence for $k \to \infty$ on compact subsets of $\mathbb{C} \setminus [1, +\infty)$. In the present context we are more interested in convergence on the unit circle including in particular points close to $z = 1$. Also, we are interested in convergence of columns $[n + k|k]_{G^{(\alpha,\beta)}}$ for fixed $k$ and $n \to \infty$.

Here it is useful to recall the well-known explicit representation of the Padé denominator of $G^{(\alpha,\beta)}$ in terms of Jacobi polynomials [Ba75, Eqn. (5.11)]. Indeed, we just need to use the representation of $G^{(\alpha,\beta)}$ as a Stieltjes function, see (3.1), to relate the reversed Padé denominators to the polynomials orthonormal with respect to the measure $d\mu^{(\alpha,\beta)}$ on $[0, 1]$, see [Ba75, Eqn. (7.7)].

**Lemma 3.1** *Let $\alpha, \beta > -1$, and $k \geq 0, n \geq -1$ be two integers. The $(n + k - 1, k)$ Padé denominator $Q_{n+k-1,k}$ of $G^{(\alpha,\beta)}$ is unique (up to multiplication with a scalar). More precisely, we may normalize such that for the reversed denominator we get the formula*

$$\widetilde{Q}_{n+k-1,k}(z) = z^k Q_{n+k-1,k}(1/z) = \gamma^{(\alpha,\beta)}_{k,n} P^{(\alpha+n,\beta)}_k (1 - 2z),$$

*where $P_k^{(\alpha+n,\beta)}$ denotes the classical Jacobi polynomial, see (1.4), and*

$$\gamma_{k,n}^{(\alpha,\beta)} = \sqrt{\frac{(2k+\alpha+n+\beta+1)k!(\alpha+\beta+2)_{n+k-1}}{(\alpha+1)_{n+k}(\beta+1)_k}}. \tag{3.2}$$

*Moreover, for $|z|=1$, the following upper bound holds true,*

$$\left|Q_{n+k-1,k}(z)^2\Big(G^{(\alpha,\beta)}(z) - [n+k-1|k]_{G^{(\alpha,\beta)}}(z)\Big)\right| \le \frac{1}{\operatorname{dist}(z,[0,1])}. \tag{3.3}$$

**Proof.** We have

$$G^{(\alpha,\beta)}(z) = \int \left[1+yz+\cdots+(yz)^{n-1}+\frac{(yz)^n}{1-yz}\right] d\mu^{(\alpha,\beta)}(y) = c_{n-1}(z) + z^n G_n^{(\alpha,\beta)}(z),$$

where $c_{n-1}(z)$ is a polynomial of degree $n-1$ in $z$ and

$$G_n^{(\alpha,\beta)}(z) = \int_0^1 \frac{y^n d\mu^{(\alpha,\beta)}(y)}{1-yz}.$$

The Padé approximant $[n+k-1|k]_{G^{(\alpha,\beta)}}$ is obtained from the Padé approximant $[k-1|k]_{G_n^{(\alpha,\beta)}}$ in the following way

$$[n+k-1|k]_{G^{(\alpha,\beta)}}(z) = c_{n-1}(z) + z^n [k-1|k]_{G_n^{(\alpha,\beta)}}(z).$$

Since $G_n^{(\alpha,\beta)}$ is a Stieltjes function, the reversed Padé denominator $\widetilde{Q}_{n+k-1,k}(z)$ equals the orthonormal polynomial of degree $k$ with respect to the measure $y^n d\mu^{(\alpha,\beta)}(y)$ supported on the interval $[0,1]$ (up to normalization with a scalar), see, e.g., [Ba75, Eqn. (7.7)]. For the Jacobi polynomials of indices $(\alpha+n,\beta)$, we have

$$\int_0^1 P_i^{(\alpha+n,\beta)}(1-2y) P_j^{(\alpha+n,\beta)}(1-2y) y^n d\mu^{(\alpha,\beta)}(y) = 0 \qquad \text{for } i \neq j, \tag{3.4}$$

and it is easily checked from (1.4) that

$$\int_0^1 \Big(P_k^{(\alpha+n,\beta)}(1-2y)\Big)^2 y^n d\mu^{(\alpha,\beta)}(y) = \frac{(\alpha+1)_{n+k}(\beta+1)_k}{(2k+\alpha+n+\beta+1)k!(\alpha+\beta+2)_{n+k-1}} = (\gamma_{k,n}^{(\alpha,\beta)})^{-2}. \tag{3.5}$$

Hence, $\widetilde{Q}_{n+k-1,k}(z) = \gamma_{k,n}^{(\alpha,\beta)} P_k^{(\alpha+n,\beta)}(1-2z)$, and these polynomials are orthonormal with respect to the measure $y^n d\mu^{(\alpha,\beta)}(y)$.

It is well-known, see e.g. [Br80, Chapter 1, Thm 1.17], that the error of the Padé approximant $[k-1|k]$ of a function $f(z) = c\left(\frac{1}{1-xz}\right)$, where $c$ is a linear form acting on the variable $x$, is given by

$$f(z) - [k-1|k]_f(z) = \frac{z^{2k}}{\widetilde{P}_k(z)^2} c\left(\frac{P_k(x)^2}{1-xz}\right), \tag{3.6}$$

where $P_k$ is the orthogonal polynomial of degree $k$ with respect to the linear form $c$. Therefore, the linearized error of the Padé approximant $[k-1|k]_{G_n^{(\alpha,\beta)}}$ of the Stieltjes function $G_n^{(\alpha,\beta)}$ is given by

$$Q_{n+k-1,k}(z)(G_n^{(\alpha,\beta)}(z) - [k-1|k]_{G_n^{(\alpha,\beta)}}(z)) = \frac{z^{2k}}{Q_{n+k-1,k}(z)} \int \frac{\widetilde{Q}_{n+k-1,k}(y)^2}{1-zy} y^n d\mu^{(\alpha,\beta)}(y)$$

which leads for $G^{(\alpha,\beta)}$ to the linearized error

$$Q_{n+k-1,k}(z)G^{(\alpha,\beta)} - P_{n+k-1,k}(z) = \frac{z^{2k+n}}{Q_{n+k-1,k}(z)} \int \frac{\widetilde{Q}_{n+k-1,k}(y)^2}{1-zy} y^n d\mu^{(\alpha,\beta)}(y). \tag{3.7}$$

From (3.7) and the orthonormality properties of $\widetilde{Q}_{n+k-1,k}(z)$, we obtain for $|z|=1$, $z \neq 1$ the following upper bound

$$\left| Q_{n+k-1,k}(z) \Big( Q_{n+k-1,k}(z) G^{(\alpha,\beta)} - P_{n+k-1,k}(z) \Big) \right| \leq \frac{1}{\operatorname{dist}(z,[0,1])}.$$

□

The next lemma gives estimates on the modulus of $Q_{n+k-1,k}(e^{it})$.

**Lemma 3.2** *Set*

$$\nu_{k,n} = |Q_{n+k-1,k}(-1)| = \gamma_{k,n}^{(\alpha,\beta)} P_k^{(\alpha+n,\beta)}(3). \tag{3.8}$$

*Then,*

$$|Q_{n+k-1,k}(e^{it})|^2 \leq \nu_{k,n}^2 \leq \frac{2^{2k}}{k!} \frac{(\alpha+\beta+2)_{n+2k}(n+k+1+\alpha+\beta)_k}{(\alpha+1)_{n+k}(\beta+1)_k}, \tag{3.9}$$

*and for* $0 < \delta \leq |t| \leq \pi$, *we have*

$$|Q_{n+k-1,k}(e^{it})|^2 \geq |Q_{n+k-1,k}(e^{i\delta})|^2 \geq \frac{2^k}{k!} \frac{(\alpha+\beta+2)_{n+2k}}{(\alpha+1)_n(\beta+1)_k} (1-\cos\delta)^k. \tag{3.10}$$

**Proof.** We recall the following representation of the Jacobi polynomials, see [Sze75, Chap.4, Eqn 4.21.2, p. 62],

$$P_k^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_k}{k!} {}_2F_1\left( \begin{array}{c} -k, k+\alpha+\beta+1 \\ \alpha+1 \end{array} \middle| \frac{1-x}{2} \right). \tag{3.11}$$

Thus we get for the reversed denominators

$$\widetilde{Q}_{n+k-1,k}(z) = \gamma_{k,n}^{(\alpha,\beta)} \frac{(\alpha+n+1)_k}{k!} {}_2F_1\left( \begin{array}{c} -k, \alpha+n+1+\beta+k \\ \alpha+n+1 \end{array} \middle| z \right). \tag{3.12}$$

For $\alpha, \beta > -1$, we denote by $a_i$ the coefficient of $x^i$ of the ${}_2F_1$ polynomial. Then

$$\operatorname{sign}(a_i) = \operatorname{sign}((-k)(-k+1)\cdots(-k+i-1)) = (-1)^i,$$

and the coefficients of $\widetilde{Q}_{n+k-1,k}(z)$ have alternating signs, which implies that

$$\max_{|z|=1} \left| \widetilde{Q}_{n+k-1,k}(z) \right| = \left| \widetilde{Q}_{n+k-1,k}(-1) \right|.$$

Denote by $x_{j,k} \in (0,1)$, $j=1,\ldots,k$, the zeros of $\widetilde{Q}_{n+k-1,k}$. We have

$$\begin{aligned} \left| \widetilde{Q}_{n+k-1,k}(-1) \right| &= \gamma_{k,n}^{(\alpha,\beta)} \frac{(\alpha+n+1)_k}{k!} \frac{|(-k)_k|}{k!} \frac{(\alpha+n+\beta+k+1)_k}{(\alpha+n+1)_k} \left| \prod_{j=1}^k (-1-x_{jk}) \right| \\ &\leq 2^k \gamma_{k,n}^{(\alpha,\beta)} \frac{(\alpha+n+\beta+k+1)_k}{k!}. \end{aligned}$$

Plugging the expression (3.2) of $\gamma_{k,n}^{(\alpha,\beta)}$ into the square of the last upper bound leads, after some computations to (3.9). Let us now obtain the lower bound. We can write

$$\widetilde{Q}_{n+k-1,k}(z) = \widetilde{Q}_{n+k-1,k}(0) \prod_{j=1}^k \left( 1 - \frac{z}{x_{jk}} \right).$$

Then we obtain

$$\left|\widetilde{Q}_{n+k-1,k}(e^{it})\right|^2 = \left|\widetilde{Q}_{n+k-1,k}(0)\right|^2 \prod_{j=1}^{k} \frac{2}{x_{jk}} \left( \frac{x_{jk} + 1/x_{jk}}{2} - \cos(t) \right).$$

This shows that $\left|\widetilde{Q}_{n+k-1,k}(e^{it})\right|^2$ is increasing with $t$ and so, for $0 < \delta \leq |t| < \pi$,

$$\left|\widetilde{Q}_{n+k-1,k}(e^{it})\right|^2 \geq \left|\widetilde{Q}_{n+k-1,k}(e^{i\delta})\right|^2$$

$$\geq \left|\widetilde{Q}_{n+k-1,k}(0)\right|^2 \prod_{j=1}^{k} \left[ \frac{2}{x_{jk}} (1 - \cos(\delta)) \right] = \frac{2^k \left|\widetilde{Q}_{n+k-1,k}(0)\right|^2}{\prod_{j=1}^{k} x_{jk}} (1 - \cos(\delta))^k. \tag{3.13}$$

The quotient $\widetilde{Q}_{n+k-1,k}(0)/\prod_{j=1}^{k}(-x_{jk})$ equals the leading coefficient of $\widetilde{Q}_{n+k-1,k}(x)$, which, in view of (3.12), equals

$$\gamma_{k,n}^{(\alpha,\beta)} \frac{(\alpha+n+1)_k}{k!} \frac{(-k)_k(\alpha+n+\beta+k+1)_k}{(\alpha+n+1)_k k!} = (-1)^k \gamma_{k,n}^{(\alpha,\beta)} \frac{(\alpha+n+\beta+k+1)_k}{k!}.$$

On the other hand,

$$\widetilde{Q}_{n+k-1,k}(0) = \gamma_{k,n}^{(\alpha,\beta)} P_k^{(\alpha+n,\beta)}(1) = \gamma_{k,n}^{(\alpha,\beta)} \frac{(\alpha+n+1)_k}{k!}.$$

Making use of these two expressions in (3.13) leads to (3.10). $\square$

**Corollary 3.3** *Assume $f$ is such that $G(f) = G^{(\alpha,\beta)}$. Then, for $0 < \delta \leq |t| \leq \pi$ and for all integers $k \geq 0, n \geq -1$, we have that*

$$|f(t) - \mathrm{Re}\,(\epsilon_{2k}^{(n)}(t))| \leq \frac{k!(\beta+1)_k(\alpha+1)_{n+1}}{2^k \sin\delta(1-\cos\delta)^k(\alpha+\beta+2)_{n+1+2k}}, \tag{3.14}$$

*which implies that, for any $0 < \delta \leq \pi$, $0 \leq \tau \leq 1$ and $k \geq 0$,*

$$\max_{\delta/n^\tau \leq |t| \leq \pi} |f(t) - \mathrm{Re}\,(\epsilon_{2k}^{(n)}(t))| = \mathcal{O}\left(n^{-(1-\tau)(2k+1)-\beta}\right) \quad \textit{as} \quad n \to \infty. \tag{3.15}$$

**Proof.** We know that

$$f(t) - \mathrm{Re}\,(\epsilon_{2k}^{(n)}(t)) = \mathrm{Re}\,(G(f) - [n+k|k]_{G(f)})(e^{it}),$$

and, by (3.3), the modulus of the last expression is less than

$$\left(\mathrm{dist}\,(e^{it},[0,1])\left|Q_{n+k,k}(e^{it})\right|^2\right)^{-1} \leq \frac{k!\,(\alpha+1)_{n+1}(\beta+1)_k}{2^k \sin\delta(1-\cos\delta)^k(\alpha+\beta+2)_{n+1+2k}},$$

for $0 < \delta \leq |t| \leq \pi$, where in the last inequality we have used (3.10). This proves (3.14) from which (3.15) is immediate. $\square$

This result shows the quite impressive convergence properties of the columns of the Padé table for the Stieltjes function $G^{(\alpha,\beta)}$. If we fix the parameters $\delta > 0, \tau = 0$ and the column $k$ of the table, then the error corresponding to the sequence of approximants $[n+k|k](e^{it})$ is of order $\mathcal{O}(n^{-2k-1-\beta})$ as $n$ tends to infinity. This fact explains for the fast convergence of the approximants observed when applying the $\epsilon$-algorithm to the Fourier series of functions like $G^{(\alpha,\beta)}$.

## 4 Error estimates for the sum of a $G^{(\alpha,\beta)}$ function and a smooth function

In this section, we show that the results of Corollary 3.3 remain valid when adding a smooth perturbation to a function $f$ as in Section 3.

**Theorem 4.1** *Let $f = f_1 + f_2$ with $G(f_1) = G^{(\alpha,\beta)}$, $\alpha, \beta > -1$, and $f_2 \in \mathcal{C}^{m-1}(\mathbb{R})$ a $2\pi$–periodic function such that $f_2^{(m)}$ exists almost everywhere, with $f_2^{(m)} \in L_1([0, 2\pi])$. Let $0 < \delta \leq \pi$, $0 \leq \tau < 1$, and $k \geq 0$ an integer such that $m \geq 5/2 + \beta + 2k - \tau$. Then the following estimate holds true*

$$\max_{\delta/n^\tau \leq |t| \leq \pi} |f(t) - \mathrm{Re}\,(\epsilon_{2k}^{(n)}(t))| = \mathcal{O}\left(n^{-(1-\tau)(2k+1)-\beta}\right) \quad \text{as} \quad n \to \infty.$$

In the sequel, we set

$$p_{k,n}(x) = \gamma_{k,n}^{(\alpha,\beta)} P_k^{(\alpha+n,\beta)}(1-2x),$$

that is, $p_{k,n}$ is the orthonormal polynomial of degree $k$ with respect to the measure $y^n d\mu^{(\alpha,\beta)}(y)$ (and thus the reversed $(n+k-1,k)$ Padé denominator of $G(f_1)$, see the preceding section). We also set

$$E_n(f) = f - S_{n-1}(f)$$

for the remainder of order $n$ in the Fourier expansion of $f$.

Before proving Theorem 4.1, we establish three preliminary lemmas.

**Lemma 4.2** *Let $f = f_1 + f_2$ be a function on $[-\pi, \pi]$ with $G(f_1) = G^{(\alpha,\beta)}$ and assume that the Fourier coefficients of $f_2$ decrease sufficiently fast. Namely, we suppose that there exists an $N_k \geq 0$ such that, for all $n \geq N_k$,*

$$\|E_n(f_2)\|_2 \left(\sum_{j=0}^{k} \nu_{j,n}^2\right) < \frac{1}{2}, \tag{4.1}$$

*where the numbers $\nu_{j,n} = |p_{j,n}(-1)|$ have been defined in (3.8). Then, for all $n \geq N_k$, the $(n+k-1,k)$ Padé denominator $Q_{n+k-1,k}$ of $G(f)$ is unique (up to multiplication with a scalar), and its reversed counterpart admits the decomposition*

$$\widetilde{Q}_{n+k-1,k}(z) = p_{k,n}(z) + \sum_{j=0}^{k-1} a_{j,k,n} p_{j,n}(z), \tag{4.2}$$

*with coefficients $a_{j,k,n}$ satisfying*

$$|a_{j,k,n}| \leq 2\nu_{j,n}\nu_{k,n}\|E_n(f_2)\|_2 < 1, \quad j = 0, \ldots, k-1. \tag{4.3}$$

**Proof.** Let $c^{(n)}$ be the linear form acting on the space of polynomials such that $c^{(n)}(z^j)$ is the coefficient of $z^{n+j}$ in the power series of $G(f)$. In view of the integral representation (3.1) of $G^{(\alpha,\beta)}$ and the Cauchy formula for $G(f_2) \in H_2$,

$$G(f_2)(z) = \frac{1}{2\pi i} \int_{|\xi|=1} \frac{G(f_2)(\xi)}{\xi - z} d\xi,$$

we have that for any polynomial $P$,

$$c^{(n)}(P) = \int y^n P(y) d\mu^{(\alpha,\beta)}(y) + \frac{1}{2\pi i} \int_{|\xi|=1} \xi^{-n-1} P\left(\frac{1}{\xi}\right) G(f_2)(\xi) d\xi. \tag{4.4}$$

The second integral equals

$$\frac{1}{2\pi i}\int_{|\xi|=1}\xi^{n-1}P\left(\xi\right)G(f_2)\left(1/\xi\right)d\xi=\frac{1}{2\pi i}\int_{|\xi|=1}\xi^{n-1}P\left(\xi\right)g_n(1/\xi)d\xi,$$

where in the last inequality we have set $g_n(\xi)=G(E_n(f_2))(\xi)$, and have used the fact that $\xi^{n-1}(f_2(1/\xi)-E_n(f_2)(1/\xi))=\xi^{n-1}S_{n-1}(f_2)(1/\xi)$ is analytic in the closed unit disk.

Let us suppose for a moment that there is a reversed denominator $\widetilde{Q}_{n+k-1,k}$ of the Padé approximant $[n+k-1|k]_{G(f)}$ which is of degree exactly $k$. Then, after possibly multiplying with a scalar, there exists coefficients $a_{j,k,n}$ such that (4.2) holds. The order condition for the linearized Padé error yields the orthogonality conditions

$$c^{(n)}(\widetilde{Q}_{n+k-1,k}p_{j,n})=0,\quad j=0,\ldots,k-1. \tag{4.5}$$

Setting

$$A_{l,j}=-\frac{1}{2\pi i}\int_{|\xi|=1}\xi^{n-1}p_{j,n}(\xi)p_{l,n}(\xi)g_n(\frac{1}{\xi})d\xi,\quad l,j=0,\cdots k-1,$$

these orthogonality relations rewrite as

$$(I-A)a=b,$$

where $A$ denotes the matrix $(A_{l,j})_{l,j=0\ldots k-1}$, $a=(a_{0,k,n},\ldots,a_{k-1,k,n})^T$ and $b=(A_{0,k},\ldots,A_{k-1,k})^T$. From the Cauchy-Schwarz inequality together with (2.1) and (3.9) we obtain that

$$|A_{l,j}|^2\le\left(\frac{1}{2\pi}\int_{|\xi|=1}|p_{j,n}(\xi)p_{l,n}(\xi)|^2\,|d\xi|\right)\left(\frac{1}{2\pi}\int_{|\xi|=1}|g_n(1/\xi)|^2\,|d\xi|\right)\le\nu_{j,n}^2\nu_{l,n}^2\|E_n(f_2)\|_2^2, \tag{4.6}$$

so that, in view of (4.1), we have the following upper bound for the Frobenius norm of $A$,

$$\|A\|_F^2=\sum_{l,j=0}^{k-1}A_{j,l}^2\le\|E_n(f_2)\|_2^2\left(\sum_{j=0}^{k-1}\nu_{j,n}^2\right)^2<\frac{1}{4}.$$

As a consequence, the matrix $(I-A)$ is invertible and the vector $a=(a_{0,k,n},\ldots,a_{k-1,k,n})^T$ is given by

$$a=(I-A)^{-1}b=\sum_{m=0}^{\infty}A^mb.$$

Let us show by induction on $m$ that

$$\forall m\ge0,\quad|(A^mb)_j|\le\frac{\nu_{j,n}\nu_{k,n}}{2^m}\|E_n(f_2)\|_2,\quad j=0,\ldots,k-1. \tag{4.7}$$

When $m=0$, this is true, see (4.6). Assume we have

$$|(A^{m-1}b)_j|\le\frac{\nu_{j,n}\nu_{k,n}}{2^{m-1}}\|E_n(f_2)\|_2,\quad j=0,\ldots,k-1.$$

Then,

$$\begin{aligned}|(A^mb)_j|^2&\le\left(\sum_{l=0}^{k-1}A_{j,l}^2\right)\sum_{l=0}^{k-1}|(A^{m-1}b)_l|^2\\&\le(\nu_{j,n}^2(\sum_{l=0}^{k-1}\nu_{l,n}^2)\|E_n(f_2)\|_2^2)(\frac{\nu_{k,n}^2}{4^{m-1}}(\sum_{l=0}^{k-1}\nu_{l,n}^2)\|E_n(f_2)\|_2^2)\\&\le4^{-m}\nu_{k,n}^2\nu_{j,n}^2\|E_n(f_2)\|_2^2.\end{aligned}$$

where in the last inequality we have used (4.1). Hence, (4.7) holds true for any $m \in \mathbb{N}$. It implies that

$$|a_{j,k,n}| \le \sum_{m=0}^{\infty} |(A^m b)_j| \le 2\nu_{j,n}\nu_{k,n}\|E_n(f_2)\|_2,$$

where the last upper bound is less than 1 in view of (4.1).

It finally remains to show that $\widetilde{Q}_{n+k-1,k}$ is necessarily of degree $k$ (which also implies that this reversed Padé denominator is unique up to multiplication with a constant). By contradiction, suppose that $\kappa := \deg \widetilde{Q}_{n+k-1,k} < k$. Then, after possibly multiplying with a scalar, we may write

$$\widetilde{Q}_{n+k-1,k}(z) = p_{\kappa,n}(z) + \sum_{j=0}^{\kappa-1} a_{j,\kappa,n} p_{j,n}(z),$$

and get the same estimates for the coefficients $A_{l,j}$ and $a_{j,\kappa,n}$. In particular, relation (4.5) for $j = \kappa$ leads to

$$\begin{aligned}
0 = |c^{(n)}(p_{\kappa,n}\widetilde{Q}_{n+k-1,k})| &= |1 - A_{\kappa,\kappa} - \sum_{j=0}^{\kappa-1} a_{j,\kappa,n} A_{\kappa,j}| \\
\ge 1 - |A_{\kappa,\kappa}| - \sum_{j=0}^{\kappa-1} |a_{j,\kappa,n}|\,|A_{\kappa,j}| &\ge 1 - \nu_{\kappa,n}^2 \,\|E_n(f_2)\| - \frac{1}{2}\nu_{\kappa,n}^2 \,\|E_n(f_2)\| \ge \frac{1}{4},
\end{aligned}$$

a contradiction. Hence $\deg \widetilde{Q}_{n+k-1,k} = k$. □

We now give a lower bound for the modulus of $\widetilde{Q}_{n+k-1,k}(z)$.

**Lemma 4.3** *Let $f = f_1 + f_2$ satisfy the assumptions of Lemma 4.2. Then, for all $n \ge N_k$, we have*

$$|\widetilde{Q}_{n+k-1,k}(e^{it})| \ge \frac{1}{2}|p_{k,n}(e^{it})|, \quad \textit{provided that} \quad \frac{\pi}{3}\frac{\sqrt{k(\beta+k)}}{n-1} \le |t| \le \pi. \tag{4.8}$$

**Proof.** By the interlacing property of the (simple) zeros $x_{l,j} \in (0,1)$, $l = 1, \dots, j$, of the orthonormal polynomials $p_{j,n}$, we can write

$$\frac{p_{j-1,n}(z)}{p_{j,n}(z)} = \sum_{l=1}^{j} \frac{\beta_{lj}}{z - x_{l,j}}, \quad j = 1, \dots, k,$$

with $\beta_{lj} > 0$. Then,

$$\begin{aligned}
\left|\frac{p_{j-1,n}(z)}{p_{j,n}(z)}\right| &\le \sum_{l=1}^{j} \frac{|x_{l,j}|}{|z - x_{l,j}|}\frac{\beta_{lj}}{|x_{l,j}|} \\
&\le \max_{1\le l\le j}\left|\frac{x_{l,j}}{z - x_{l,j}}\right| \left|\frac{p_{j-1,n}(0)}{p_{j,n}(0)}\right| \le \frac{1}{\operatorname{dist}(z,[0,1])}\left|\frac{p_{j-1,n}(0)}{p_{j,n}(0)}\right|.
\end{aligned}$$

Since $p_{j,n}(0) = \gamma_{j,n}^{(\alpha,\beta)} P_j^{(n+\alpha,\beta)}(1)$, we obtain using (3.11) that, for $j = 1, ..., k$,

$$\begin{aligned}
\left|\frac{p_{j-1,n}(0)}{p_{j,n}(0)}\right| &= \frac{\gamma_{j-1,n}^{(\alpha,\beta)}}{\gamma_{j,n}^{(\alpha,\beta)}}\left|\frac{P_{j-1}^{(\alpha+n,\beta)}(1)}{P_j^{(\alpha+n,\beta)}(1)}\right| \\
&= \sqrt{\frac{2j-1+\alpha+\beta+n}{2j+1+\alpha+\beta+n}\frac{j(\beta+j)}{(\alpha+\beta+n+j)(\alpha+j+n)}} \le \frac{\sqrt{k(\beta+k)}}{n-1}.
\end{aligned}$$

Thus, by assumption on $t$,

$$\left|\frac{p_{j-1,n}(e^{it})}{p_{j,n}(e^{it})}\right| \le \frac{1}{\mathrm{dist}(e^{it},[0,1])}\frac{\sqrt{k(\beta+k)}}{n-1} \le \frac{1}{\sin|t/2|}\frac{\sqrt{k(\beta+k)}}{n-1} \le \frac{1}{3}.$$

Since from (4.3) we know that $|a_{j,k,n}| < 1$, we obtain

$$\left|\frac{\widetilde{Q}_{n+k-1,k}(e^{it})}{p_{k,n}(e^{it})}\right| \ge 1 - \left|\sum_{j=0}^{k-1} a_{j,k,n}\frac{p_{j,n}(e^{it})}{p_{k,n}(e^{it})}\right| \ge 1 - \sum_{j=0}^{k-1}\left|\frac{p_{j,n}(e^{it})}{p_{k,n}(e^{it})}\right| \ge 1 - \sum_{j=0}^{k-1}\frac{1}{3^{k-j}} = \frac{1}{2}.$$

□

**Lemma 4.4** *Let $f = f_1 + f_2$ satisfy the assumptions of Lemma 4.2, and let*

$$e_{n,k}(z) = G(f)(z) - [n+k-1|k]_{G(f)}(z),$$

*be the error corresponding to the Padé approximant $[n+k-1|k]_{G(f)}$. Then for all $n \ge N_k$ and $|z| = 1$ we have*

$$|Q_{n+k-1,k}(z)^2 e_{n,k}(z)| \le \frac{2}{\mathrm{dist}\,(z,[0,1])} + 4\nu_{k,n}^2 \sum_{j=0}^{\infty} \|E_{n+j}(f_2)\|_2. \tag{4.9}$$

**Proof.** By adapting the reasoning leading to (3.7), the error $e_{n,k}(z)$ can be written in the following way

$$e_{n,k}(z) = \frac{z^{n+2k}}{Q_{n+k-1,k}(z)^2} c^{(n)}\left(\frac{\widetilde{Q}_{n+k-1,k}(x)^2}{1-xz}\right),$$

where $c^{(n)}$ has been defined at the beginning of the proof of Lemma 4.2. Replacing $c^{(n)}$ by the expression obtained there, we get

$$\begin{aligned} &Q_{n+k-1,k}(z)^2 e_{n,k}(z) \\ &= z^{n+2k}\int \frac{\widetilde{Q}_{n+k-1,k}(y)^2}{1-yz} y^n d\mu^{(\alpha,\beta)}(y) + \frac{z^{n+2k}}{2\pi i}\int_{|\xi|=1} \frac{\xi^{n-1}\widetilde{Q}_{n+k-1,k}(\xi)^2}{1-\xi z} g_n(1/\xi)d\xi. \end{aligned}$$

Let us denote by $I_1$ and $I_2$ the two terms in the previous sum. We first bound the modulus of $I_1$. Using the decomposition (4.2), we have

$$\widetilde{Q}_{n+k-1,k}(y)^2 = \sum_{j,l=0}^{k} a_{j,k,n}a_{l,k,n}p_{j,n}(y)p_{l,n}(y),$$

where $a_{k,k,n} = 1$. From the orthonormality of the $p_{j,n}$ with respect to the measure $y^n d\mu^{(\alpha,\beta)}(y)$ and the fact that $|z| = 1$, we obtain

$$|I_1| \le \mathrm{dist}\,(z,[0,1])^{-1}\int \widetilde{Q}_{n+k-1,k}(y)^2 y^n d\mu^{(\alpha,\beta)}(y) = \mathrm{dist}(z,[0,1])^{-1}\sum_{j=0}^{k} a_{j,k,n}^2. \tag{4.10}$$

Moreover, from (4.3) and assumption (4.1), we derive that

$$\sum_{j=0}^{k-1} a_{j,k,n}^2 \le 4\nu_{k,n}^2\|E_n(f_2)\|_2^2\left(\sum_{j=0}^{k-1}\nu_{j,n}^2\right) \le 1,$$

which, together with (4.10) and the fact that $a_{k,k,n} = 1$, shows that

$$|I_1| \le 2/\mathrm{dist}\,(z, [0,1]). \tag{4.11}$$

For the second term, we have

$$\begin{aligned} I_2 &= \frac{z^{n+2k}}{2\pi i}\sum_{j=0}^{\infty} z^j \int_{|\xi|=1} \xi^{n-1+j}\widetilde{Q}_{n+k,k}(\xi)^2 g_n(1/\xi)d\xi \\ &= \frac{z^{n+2k}}{2\pi i}\sum_{j=0}^{\infty} z^j \int_{|\xi|=1} \xi^{n-1+j}\widetilde{Q}_{n+k-1,k}(\xi)^2 g_{n+j}(1/\xi)d\xi, \end{aligned}$$

where in the second equality, we have used the fact that $\xi^{n+j-1}(g_{n+j} - g_n)(1/\xi)$ is analytic in the unit disk. Then, by applying the Cauchy-Schwarz inequality to the integrals, we obtain that the modulus of $I_2$ satisfies

$$\begin{aligned} |I_2| &\le \max_{|z|=1}\left|\widetilde{Q}_{n+k-1,k}(z)^2\right| \sum_{j=0}^{\infty}\left[\frac{1}{2\pi}\int_{|\xi|=1} |g_{n+j}(\xi)|^2\,|d\xi|\right]^{1/2} \\ &= \max_{|z|=1}\left|\widetilde{Q}_{n+k-1,k}(z)^2\right| \sum_{j=0}^{\infty} \|E_{n+j}(f_2)\|_2. \end{aligned} \tag{4.12}$$

Using (4.2), (4.1), (4.3), and the first inequality in (3.9), we obtain, for $n \ge N_k$ and $|z| = 1$,

$$\begin{aligned} \left|\widetilde{Q}_{n+k-1,k}(z)\right|^2 &= \left|\sum_{j=0}^{k} a_{j,k,n}p_{j,n}(z)\right|^2 \le 2\left(\nu_{k,n}^2 + \left|\sum_{j=0}^{k-1} a_{j,k,n}p_{j,n}(z)\right|^2\right) \\ &\le 2\nu_{k,n}^2 + 8\nu_{k,n}^2\|E_n(f_2)\|_2^2\left(\sum_{j=0}^{k-1}\nu_{j,n}^2\right)^2 \le 4\nu_{k,n}^2. \end{aligned}$$

Hence, inequality (4.9) follows from (4.11) and (4.12). □

**Proof of Theorem 4.1.** Since $f_2^{(m)} \in L_1$, we know from the Riemann-Lebesgue Lemma [Ru87, (5.14)] that its Fourier coefficients satisfy

$$|a_j(f_2^{(m)})| + |b_j(f_2^{(m)})| = o(1) \quad \text{as} \quad j \to \infty.$$

Taking into account that $|a_j(f_2^{(m)})| + |b_j(f_2^{(m)})| = j^m(|a_j(f_2)| + |b_j(f_2)|)$ for $j \ge 1$, we may conclude that, as $n \to \infty$,

$$\|E_n(f_2)\|_2 = o(n^{1/2-m}), \qquad \sum_{j=0}^{\infty}\|E_{n+j}(f_2)\|_2 = o(n^{3/2-m}).$$

On the other hand, we know from (3.9) that $\nu_{k,n}^2 = \mathcal{O}(n^{2k+1+\beta})$ as $n \to \infty$ for fixed $\alpha, \beta, k$, and $1/2 - m + 2k + 1 + \beta \le \tau - 1 \le 0$, by assumption on $f_2$ and $\tau$. Hence, as $n \to \infty$,

$$\|E_n(f_2)\|_2\left(\sum_{j=0}^{k}\nu_{j,n}^2\right) = o(1), \quad \nu_{k,n}^2\sum_{j=0}^{\infty}\|E_{n+j}(f_2)\|_2 = o(n^{\tau}),$$

and the assumption of Lemma 4.2 is true for sufficiently large $N_k$ by the choice of $m$. We also observe that

$$\left[\frac{\delta}{n^{\tau}}, \pi\right] \subset \left[\frac{\pi}{3}\frac{\sqrt{k(\beta+k)}}{n-1}, \pi\right],$$

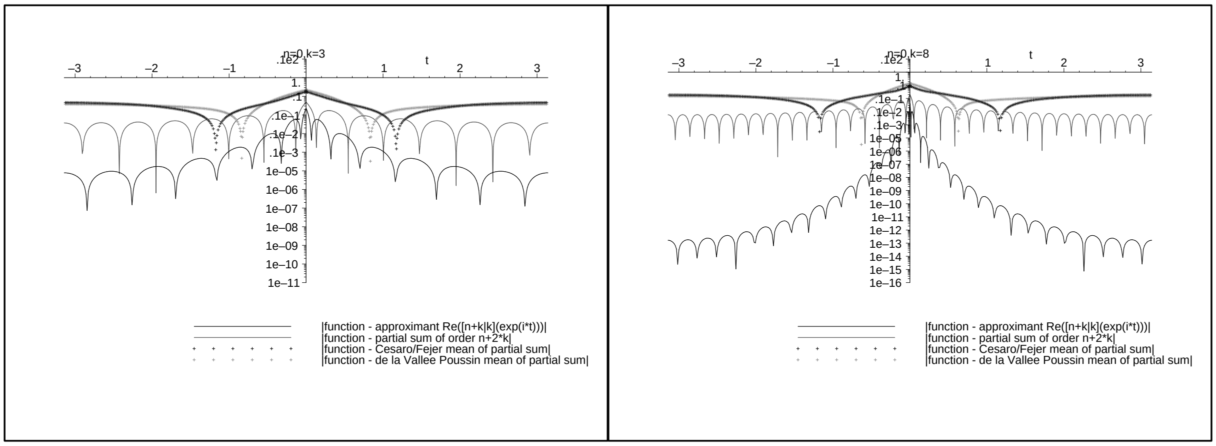


Figure 3: *The modulus of the error of approximation on a logarithmic scale of the Fourier series of* $f(t) = |\sin(\frac{t}{2})|$. *On the left we use the first 7 and on the right the first 17 coefficients of the Fourier series.*

for sufficiently large $n$ by assumption on $\delta, \tau$.

Using Lemmas 4.3 and 4.4 we get for the Padé error that

$$\max_{\delta/n^\tau \le |t| \le \pi} \left| e_{n,k}(e^{it}) \right| \le \max_{\delta/n^\tau \le |t| \le \pi} \frac{4}{|p_{n,k}(e^{it})|^2} \left[ \frac{2}{\mathrm{dist}\,(e^{i\delta/n^\tau}, [0,1])} + o(n^\tau)_{n\to\infty} \right],$$

and hence Theorem 4.1 follows from (3.10).

□

Notice that, according to the explicit form of Lemmas 4.3 and 4.4, it would be possible to give more explicit bounds for the Padé error in case where explicit expressions for $\|E_n(f_2)\|_2$ are available.

# 5 Numerical results

We have seen already in Figure 1 that for the saw tooth function $s$ of (1.2) and hence

$$G(s)(z) = -2i \log(1-z) = 2izG^{(0,0)}(z)$$

(compare with Table 1($d_1$)) we have an impressive acceleration of convergence via the $\epsilon$-algorithm even for low order. Indeed, as shown in Corollary 3.3, the error is dominated by the Padé error on the unit circle of the diagonal approximants $[3|3]_{G(s)}(z) = 2iz\,[2|3]_{G^{(0,0)}}(z)$ and $[8|8]_{G(s)}(z) = 2iz\,[7|8]_{G^{(0,0)}}(z)$, which is quite small: for the second approximant we obtain for $z = e^{it}$, $\delta = \pi/4 \le |t| \le \pi$ from Corollary 3.3 the upper bound $3.57\,10^{-8}/[\sin\delta(1-\cos\delta)^8] = 8.08\,10^{-7}$, which is approximately attained for $t = \delta$.

Let us compare in Figure 3 these findings with a $2\pi$-periodic function being $\mathcal{C}^0$ but having a derivative with a discontinuity at zero, namely

$$f(t) = |\sin(\tfrac{t}{2})| = \tfrac{2}{\pi} - \tfrac{4\cos(t)}{3\times5\times\pi} - \tfrac{4\cos(2t)}{5\times7\times\pi} - \tfrac{4\cos(3t)}{7\times9\times\pi} - \ldots \quad \Longrightarrow \quad G(f)(z) = \tfrac{2}{\pi} - \tfrac{4z}{3\pi}G^{(-\frac{1}{2},1)}(z)$$

(compare with Table 1($d_1$)). We again observe that the error for the partial sums is strongly oscillating, and remains quite large even for higher order Fourier sums, namely about 1/100 for order 7, and 1/1000 for order 17 (smaller as for the saw tooth function in Figure 1, since of cause the latter function is less regular). In this example we see that both linear acceleration procedures, namely the Cesaro means and the de la Vallée-Poussin mean, give very disappointing results (in what follows we will no longer display them). According to Table 1(c), the error

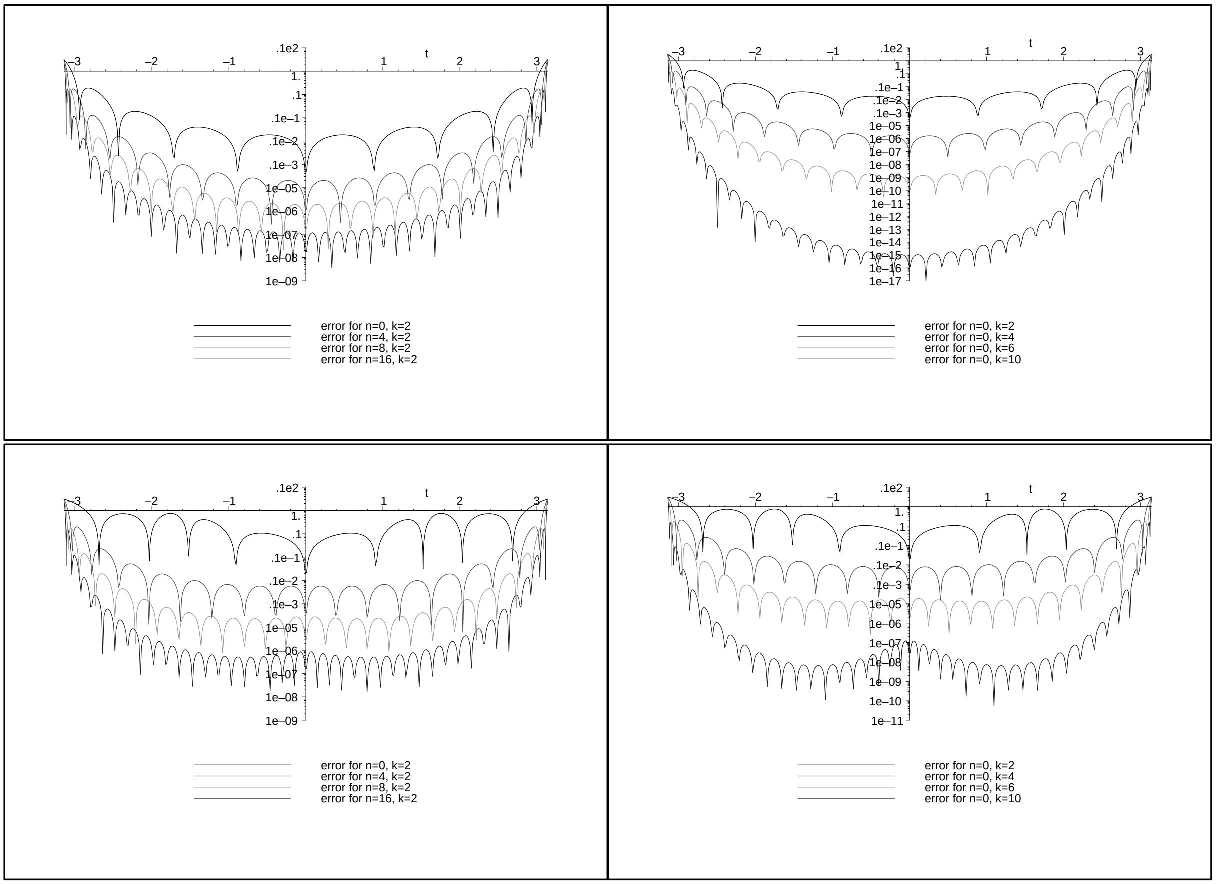


Figure 4: *The modulus of the error of approximation on a logarithmic scale of the Fourier series of $f_1(t) = s(t-\pi)$ (on the top) and of $f(t) = f_1(t) + (1-\cos(t))^3 s(t)$ (on the bottom). On the left we display the error for $\epsilon_4^{(2\ell)}$ and on the right for $\epsilon_{4+2\ell}^{(0)}$, $\ell = 0, 2, 4, 8$.*

obtained by the real part of $\epsilon_{2k}^{(n)}(t)$ (here for $k = 3$ and $k = 8$) is dominated by $\frac{4}{3\pi}$ times the error on the unit circle of $[k-1|k]_{G^{(-\frac{1}{2},1)}}$, the latter being estimated in Corollary 3.3. Again, even for arguments close to the singularity $t = 0$ we have a quite impressive convergence improvement. We should mention that all numerical experiments have been performed using Maple with sufficiently high precision such that rounding errors can be neglected.

In our last example in Figure 4 we consider the functions

$$f_1(t) = s(t-\pi) \notin \mathcal{C}^0, \quad f_2(t) = (1-\cos(t))^3 s(t) \in \mathcal{C}^5 \setminus \mathcal{C}^6$$

and hence $G(f_1)(z) = -2izG^{(0,0)}(-z)$. We compare the improvements obtained for columns of the Padé table (here $k = 2$ and $n = 2\ell$ for $\ell \in \{0, 2, 4, 8\}$) and diagonals (here $n = 0$ and $k = 2 + \ell$ for $\ell \in \{0, 2, 4, 8\}$). Notice that the number of terms of the Fourier series required for $\epsilon_4^{(2\ell)}$ and for $\epsilon_{4+2\ell}^{(0)}$ is the same (namely $5 + 2\ell$). We observe that in both cases there is improvement of convergence for increasing $\ell$, however, the rate is much more interesting for our diagonal sequence $\epsilon_{2+2\ell}^{(0)}$, in particular for $f(t) = f_1(t)$.

The error for $\epsilon_2^{(2\ell)}$ on the bottom of Figure 4 (that is, $f(t) = f_1(t) + f_2(t)$) and sufficiently large $\ell$ has been discussed (implicitly) in Theorem 4.1: since $G(f_1)(z) = -2izG^{(0,0)}(-z)$, we replace $z = e^{it}$ by $-z = e^{i(\pi-t)}$, and set $k = 2$, $\alpha = \beta = 0$. Also, $f_2 \in \mathcal{C}^5$, and $f_2^{(6)} \in L_1$ (with one jump), such that $m = 6$, showing that the asymptotic rate $\mathcal{O}(n^{-5(1-\tau)})$ of Theorem 4.1 is valid for all $0 \le \tau < 1$. In order to be more precise for finite $n$, we have to to compute explicitly the quantity $N_2$ (or even $N_k$ for diagonal sequences) in the hypothesis of Lemma 4.2. Observe that, by (3.8) and (3.11),

$$\sum_{j=0}^{2} \nu_{j,n}^2 = (n+1) + (n+3)(2n+3)^2 + (n+5)(n+2)^2(2n+5)^2 \leq 5(n+5)(n+2)^4,$$

and from Table $1(e)_1$ for $n \geq 3$

$$||f_2 - S_{n-1}(f_2)||^2_{L_2([-\pi,\pi])} \leq 4(\frac{2^{-3}(1)_6}{(2)_6})^2 \sum_{j=n-3}^{\infty} (\frac{(1)_j}{(8)_j})^2 \leq (\frac{1}{28})^2 \sum_{j=n-2}^{\infty} (\frac{(1)_7}{(j)_7})^2 \leq \frac{180^2}{13}(n-3)^{-13}.$$

Thus, with the very rough choice $N_2 \geq 83$, the hypothesis of Lemma 4.2 is true, and a combination of Lemma 4.3, Lemma 4.4, and (3.10) enables us to establish more explicit bounds for $n \geq N_2$.

Finally, we should comment on the peak of the error on the lower right plot of Figure 4 around $t = 0$: indeed, the influence of $f_2$ on $[n+k|k]_{G(f_1+f_2)}$ is negligible for fixed $k$ and $n \to \infty$, but this is no longer true for fixed $n$ and $k \to \infty$: here the zeros of the Padé denominator also detect the singularities of $G(f_2)$.

# 6 Conclusion

In the present paper we have established a link between the complex $\epsilon$-algorithm applied to partial Fourier sums, and the non-linear Padé-Chebyshev and Padé-Fourier approximants. We were able to show by deriving explicit error estimates for a class of hypergeometric functions that the complex $\epsilon$-algorithm allows one to accelerate convergence of partial Fourier sums. In particular, as observed numerically by Brezinski [Br02], this technique allows one to smooth the Gibbs phenomenon for functions which either themselves or their higher order derivatives have a jump. Finally we have shown that the rate of convergence for columns is preserved even after smooth perturbations of the underlying function.

There are several open questions in this field of research: we have seen in our numerical experiments that, for ray sequences of the form $k = [\lambda n]$ with $\lambda > 0$, we get a rate of convergence better than that of columns ($k$ fixed). Here one could derive a result similar to Theorem 4.1 by using the strong asymptotics of Jacobi polynomials with varying parameters as derived in [GS91, MO05]. However, then for the hypothesis (4.1) we would require very smooth $f_2$ with exponentially decaying $\|E_n(f_2)\|_2$, obtained for instance for rational $G(f_2)$. It would also be interesting to combine our findings with those of Rakhmanov [Ra77] who discusses the error of diagonal Padé approximants where $G(f_1)$ is a Stieltjes function and $G(f_2)$ is rational.

A nice test function $s_m$ not included in our class of hypergeometric functions would be the $m$-th primitive of the saw tooth function $s$ of (1.1), with $j$-th derivative being continuous for all $j \neq m$, and having one jump for $j = m$. Notice that, by (1.2),

$$G(s_m)(z) = 2i^{4m+1} \sum_{j=1}^{\infty} \frac{z^j}{j^{m+1}} = \frac{2i^{4m+1}}{m!} z \int_0^1 \frac{(\log(1/y))^m}{1 - yz} dy,$$

that is, we essentially get a Stieltjes function. Therefore, it would be interesting to extend Corollary 3.3 and Theorem 4.1 to general Stieltjes functions.

Also it would be nice to understand the convergence behavior for functions $f$ having several jumps, like $t \mapsto f_0(t) = s(t-t_0) - s(t+t_0)$, having two jumps at $\pm t_0$, and reducing to a multiple of $t \mapsto \text{sign}(\cos(t))$ for $t_0 = \pi/2$. One may derive an explicit formula for the diagonal Padé denominators of $G(f_0)$, showing that the poles stay outside the unit disk, but are no longer on the real axis but now on a circle orthogonal to the unit circle and intersecting the unit circle at $e^{\pm it_0}$.

In addition, for spectral methods in PDEs, the data are non necessarily available as partial sums of Fourier or Chebyshev series, but also of Legendre series. We suspect that by exploiting the link with Baker-Gammel approximants [BGM96, Section 7.2] we should get similar convergence results.

Finally, a different approach for smoothing the Gibbs phenomenon for functions $f$ with jumps at $t = 0$ is discussed by Driscoll and Fornberg [DF01] who construct approximants of the form

$$[p_1(t) + S(t)p_2(t)]/p_3(t)$$

with $p_1, p_2, p_3$ suitable trigonometric polynomials. Though numerical experiences show that this approach seems promising, no error estimates have been given so far. We expect that the technique of Hermite-Padé approximation of Nikishin systems should give more insight in the convergence behavior.

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