A lenticular version of a von Neumann inequality

Bernhard Beckermann* and Michel Crouzeix

Abstract

We generalize to lens-shaped domains the classical von Neumann inequality for the disk.

1 Introduction

We will say that L is a convex lens-shaped domain of the complex plane, with vertices σ and σ', if

- either there exist two disks
  $D_1 := \{z \in \mathbb{C}\,; |z-\alpha_1| < r_1\}$ and $D_2 := \{z \in \mathbb{C}\,; |z-\alpha_2| < r_2\}$
  such that $L = D_1 \cap D_2$, $\sigma \neq \sigma'$ and $\{\sigma, \sigma'\} = \partial D_1 \cap \partial D_2$,
- either there exist a disk and a half-plane
  $D_1 := \{z \in \mathbb{C}\,; |z-\alpha_1| < r_1\}$ and $\Pi_2 := \{z \in \mathbb{C}\,; \operatorname{Re} e^{i\theta}(z-\sigma) < 0\}$
  such that $L = D_1 \cap \Pi_2$, $\sigma \neq \sigma'$ and $\{\sigma, \sigma'\} = \partial D_1 \cap \partial \Pi_2$.

We will denote by $2\alpha \in]0, \pi]$ the angle of the lens L at the vertices. We will consider also as a lens the limit case where $L = D_1 = D_2$ is a disk. Then, any point of the boundary may be considered as a vertex and $\alpha = \frac{\pi}{2}$.

Now let us consider a bounded linear operator $A \in \mathcal{B}(H)$ on a complex Hilbert space H. We will say that the operator A is of the lenticular L-type if we have

- $\|A-\alpha_1 I\| \le r_1$ and $\|A-\alpha_2 I\| \le r_2$, if $L = D_1 \cap D_2$,
- $\|A-\alpha_1 I\| \le r_1$ and $\operatorname{Re} e^{i\theta}((A-\sigma)v, v) \le 0$, $\forall\, v \in H$, if $L = D_1 \cap \Pi_2$.

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In this paper, the norm used for a linear operator on a Hilbert space H (as well as for a matrix) is always the operator norm induced by the hilbertian structure.

The aim of this paper is to prove the following result.

Theorem 1. *Let L be a convex lens-shaped domain of the complex plane with angle 2α. There exists a best constant $C(\alpha) \in \mathbb{R}$ such that the inequality*

$$\|p(A)\| \leq C(\alpha) \sup_{z \in L} |p(z)|, \tag{1}$$

holds for all polynomials $p : \mathbb{C} \to \mathbb{C}$, for all linear operators $A \in \mathcal{B}(H)$ of L-type and for all Hilbert spaces H. Furthermore this constant, which is only depending on the angle α, is a continuous decreasing function of $\alpha \in (0, \frac{\pi}{2}]$ and we have the estimate

$$\frac{\pi}{2\alpha} \sin\alpha \leq C(\alpha) \leq \min(2 + 2/\sqrt{3}, \frac{\pi - \alpha}{\alpha}). \tag{2}$$

Note that for $\alpha = \frac{\pi}{2}$, which corresponds to the case where L is a disk, we have $C(\frac{\pi}{2}) = 1$ and we recover a famous von Neumann inequality [4]. Theorem 1 can be generalized in several directions. For instance, by Mergelyan's Theorem, the inequality (1) remains valid if instead of polynomials we take p holomorphic in L and continuous in $\overline{L}$. The theorem is also valid in a completely bounded form. More precisely, there exists a continuous decreasing function $C_{cb}(\alpha)$ (which satisfies the bounds (2)) such that the inequality

$$\|P(A)\| \leq C_{cb}(\alpha) \sup_{z \in L} \|P(z)\|,$$

holds for all polynomials with matrix values $P : \mathbb{C} \to \mathbb{C}^{n,n}$, for all $n \geq 1$, for all linear operators $A \in \mathcal{B}(H)$ of type L and for all Hilbert spaces H.

Except for the value $\alpha = \frac{\pi}{2}$, we do not know the exact values of $C(\alpha)$ and $C_{cb}(\alpha)$. Even more, we do not know if $C(\alpha) = C_{cb}(\alpha)$ or not... A small improvement

$$C_{cb}(\alpha) \leq \frac{\pi - \alpha}{\pi} \Big(2 - \frac{2}{\pi} \log\tan\big(\frac{\alpha\,\pi}{4(\pi - \alpha)} \big) \Big),$$

of the upper bound in (2) can be deduced from Theorem 4.2 in [1].

We should mention that a preliminary version of this theorem, in the particular case where L has a straight face, has been implicitly used in [2] to study the convergence of the GMRES method.

2 The proof

Our proof of Theorem 1 is heavily based on the result of the paper [3], that we recall now. Let S_α be a convex sector of the complex plane with angle 2α. An operator $B \in \mathcal{B}(H)$ is said S_α-accretive iff $(Bv, v) \in \overline{S_\alpha}$, for all $v \in H$ satisfying $\|v\| = 1$. The result proved in [3] is

there exists a best constant $C_\alpha \in \mathbb{R}$ such that the inequality

$$\|r(B)\| \leq C_\alpha \sup_{z \in S_\alpha} |r(z)|, \tag{3}$$

holds for all rational functions bounded in S_α and for all S_α-accretive operators B (it is easily seen that this definition is not depending on the particular choice of the sector). Furthermore this constant C_α is a continuous and decreasing function of α and it satisfies the estimates

$$\frac{\pi}{2\alpha} \sin\alpha \leq C_\alpha \leq \min(2 + 2/\sqrt{3}, \frac{\pi - \alpha}{\alpha}).$$

Therefore it is sufficient to prove that $C_\alpha = C(\alpha)$ for getting the theorem.

We turn now to the proof of this equality. Without loss of generality, we can assume that the vertices of L are $\sigma = 0$ and $\sigma' = 1$, and that $\operatorname{Im}\alpha_1 < 0$. We introduce the rational function $g(z) := \frac{z}{z-1}$. It is easily seen that g is an involution and that g realizes a bijection of the disk $D_j := \{z \in \mathbb{C}\,; |z - \alpha_j| < |\alpha_j|\}$ onto the half-plane $P_j := \{z \in \mathbb{C}\,; \operatorname{Re}\bar{\alpha}_j z < 0\}$. In the case where the lens has a straight face $L = D_1 \cap \Pi_2$ with $\Pi_2 := \{z \in \mathbb{C}\,; \operatorname{Re} i\, z < 0\}$, we remark also that g realizes a bijection of the half-plane Π_2 onto the half-plane $P_2 := \{z \in \mathbb{C}\,; \operatorname{Re} i\, z > 0\}$. Therefore g is a bijection of the lens L onto the sector $S_\alpha = P_1 \cap P_2$. Note that the sector and the lens have the same angle 2α and that $1 \notin S_\alpha$.

Let us consider now a linear operator A such that 1 does not belong to its spectrum $\sigma(A)$, and we set $B = g(A) = A(A-I)^{-1}$. It is easily seen that $(B-I)(A-I) = I$, thus $1 \notin \sigma(B)$, and $A = g(B)$.

Using that $\operatorname{Re}\alpha_j = \frac{1}{2}$, we remark by setting $v = (A-I)w$ that

$$\begin{aligned}
& |\alpha_j|^2\,\|w\|^2 - \|(A - \alpha_j I)w\|^2 \geq 0, \forall\, w \in H, \\
\Longleftrightarrow \quad & \|Aw\|^2 - 2\operatorname{Re}\bar{\alpha}_j(Aw, w) \leq 0, \forall\, w \in H, \\
\Longleftrightarrow \quad & 2\operatorname{Re}\bar{\alpha}_j(Aw, (A-I)w) \leq 0, \forall\, w \in H, \\
\Longleftrightarrow \quad & \operatorname{Re}\bar{\alpha}_j(Bv, v) \leq 0, \forall\, v \in H.
\end{aligned}$$

Therefore if the linear operator A is of L-type, then the operator B is S_α-accretive. Conversely if B is S_α-accretive then $1 \notin \sigma(B)$ (since $1 \notin S_\alpha$) and $A = g(B)$ is of L-type.

Let us consider now a polynomial p and set $r(z) = p(g(z))$, then we have $p(A) = r(B)$ and $\sup_{z\in S_\alpha} |r(z)| = \sup_{\zeta\in L} |p(\zeta)|$. We deduce from (3) that

$$\|p(A)\| \leq C_\alpha \sup_{\zeta\in L} |p(\zeta)|.$$

Note that, if $1 \in \sigma(A)$, then for $0 < \varepsilon < 1$, the operator $A_\varepsilon := (1-\varepsilon)A$ is of L-type and $1 \notin \sigma(A_\varepsilon)$, which shows that the previous inequality is still valid by using a limit argument. Therefore we have $C(\alpha) \leq C_\alpha$.

Conversely if we consider a rational function r bounded in S_α, $p(z) = r(g(z))$ is a rational function bounded in L, therefore we deduce from (1) (which is valid for such rational functions) that

$$\|r(B)\| \leq C(\alpha) \sup_{z\in S_\alpha} |r(z)|,$$

which implies $C(\alpha) \geq C_\alpha$, and thus finally $C(\alpha) = C_\alpha$.

The proofs would be the same for the completely bounded form of our estimates.

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e-mail: Bernhard.Beckermann@univ-lille1.fr,

Address: Laboratoire de Mathématiques Paul Painlevé, UMR CNRS 8524,
Bâtiment M2 - Cité Scientifique,
Université des Sciences et Technologies de Lille,
F-59655 VILLENEUVE D'ASCQ CEDEX, France

e-mail: michel.crouzeix@univ-rennes1.fr,

Address: IRMAR, UMR CNRS 6625, Université de Rennes 1,
Campus de Beaulieu, F-35042 RENNES CEDEX, France