Operators with numerical range in a conic domain

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Abstract

Given a convex domain $\Omega \subset \mathbb{C}$ with conic boundary, a linear operator A with numerical range contained in Ω, and a rational function bounded on Ω, we are interested in estimating the norm of $r(A)$ in terms of the supremum of r in Ω. In particular, we show that ellipses, hyperbolas and parabolas are K-spectral sets with $K = 2 + 2/\sqrt{3} \leq 3.16$.

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1 Introduction

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. The numerical range (or field of values) of a linear operator A acting on $\mathcal{H}$ is the set

$$W(A) := \{\langle Av, v\rangle \,;\, v \in \mathcal{H},\ \|v\| = 1\},$$

a subset of the complex plane $\mathbb{C}$, which by the Toeplitz-Hausdorff Theorem is known to be convex. In this paper we are concerned with the following problem: given a convex open set $\Omega \subset \mathbb{C}$, find upper bounds for the smallest constant $C(\Omega)$ depending only on Ω such that for any bounded linear operator A on $\mathcal{H}$ with $\overline{W(A)} \subset \Omega$ and for any rational function r there holds

$$\|r(A)\| \leq C(\Omega) \sup_{z \in \Omega} |r(z)|. \tag{1}$$

John von Neumann [8] has shown that $C(\Omega) = 1$ if Ω is a half-plane, or equivalently, along his terminology, a half-plane is a spectral set for any operator with numerical range contained in it. More generally [7], if (1) holds for every rational function r with $C(\Omega)$ replaced by K, then Ω is called a K-spectral set for A. Thus Ω is an $C(\Omega)$-spectral set for any operator with $W(A) \subset \overline{\Omega}$.

Starting with the paper [6] showing the finiteness of $C(\Omega)$ for bounded Ω, there were a number of publications [1, 2, 3, 4, 5] over the last years dealing with the problem of estimating $C(\Omega)$. The general bound $C(\Omega) \leq 11.1$ has been established in [5]. This bound is considered to be pessimistic: for instance, for a disk D it is known [1] that $C(D) = 2$, and Crouzeix conjectures in [3] that $C(\Omega) \leq 2$ for any open convex set Ω.

Estimates of type (1) have various applications. In numerical analysis for instance, they are useful for the convergence theory of Krylov subspace methods in numerical linear algebra, or for studying the time discretization of partial differential equations. Here one often requires sharp estimates for $C(\Omega)$ for particular sets Ω. Using techniques different from those in [5], it is shown in [4] that $C(\Omega) \leq 4.75$ for a parabolic domain Ω, and in [1, 2] that

$$C(\mathcal{S}_\alpha) \leq \min\Big(2+\frac{2}{\sqrt{3}},\ \frac{\pi-\alpha}{\pi}\big(2-\frac{2}{\pi}\log\tan\big(\frac{\alpha\,\pi}{4(\pi-\alpha)}\big)\big)\Big) \tag{2}$$

for a sector

$$\mathcal{S}_\alpha := \{z \in \mathbb{C}\,;\, z \neq 0,\ |\arg(z)| < \alpha\}$$

including the limiting case $\mathcal{S}_0$ of a strip.

In the present paper we are concerned with convex domains with conic boundaries. By definition, the constant $C(\Omega)$ is invariant under displacement or scaling of the set Ω, and thus our bounds will only depend on the eccentricity of the conic domain under consideration: we will show in Theorem 1 that for the interior $\mathcal{E}_\alpha$ of an ellipse with eccentricity $e = 1/\operatorname{ch}\alpha$, $\alpha > 0$ there holds

$$C(\mathcal{E}_\alpha) \leq 2 + \frac{2}{\sqrt{4-e^2}} \tag{3}$$

and hence $C(\mathcal{E}_\alpha) \leq 2 + 2/\sqrt{3}$. As a limiting case, we obtain in Corollary 4 the bound $C(\mathcal{P}) \leq 2 + 2/\sqrt{3} \leq 3.16$ for a parabola, which improves [4]. Secondly, we consider the case where the convex domain $\mathcal{H}_\alpha$ has a boundary given by the branch of a hyperbola with eccentricity $e = 1/\cos\alpha$, $\alpha \in (0, \pi/2)$, and thus with asymptotics forming an angle 2α. Using the same techniques of proof as in the case on an ellipse, we will show in Theorem 5 that

$$C(\mathcal{H}_\alpha) \leq 2\,\frac{\pi-\alpha}{\pi} + \mu(\alpha), \quad \mu(\alpha) := \frac{\sin 2\alpha}{\pi}\int_0^\infty \frac{dy}{y^2\cos\alpha - 2\,y\,\cos 2\alpha + \cos\alpha}. \tag{4}$$

As shown in Corollary 6 via a limiting argument, we obtain the same upper bound for the sector $\mathcal{S}_\alpha$, which is an improvement of (2) as long as $0 < \alpha < .22\,\pi$.

The paper is organized as follows: Section 2 deals with the case of ellipses/parabolas whereas in Section 3 we discuss the case of a hyperbola. Finally we compare in Section 4 our findings for a sector with those from [1, 2].

Applying the same arguments as in [5, Section 2], one may show that the bound (1) remains valid (with the same constant) for a closed linear and not necessarily bounded operator A satisfying $\sigma(A) \subset \overline{W(A)} \subset \overline{\Omega}$, and for functions r which are holomorphic in Ω, and continuous and bounded on the closure $\overline{\Omega}$. Also, in this paper, we do not consider the completely bounded version $C_{cb}(\Omega)$ of our constants (see for instance [1] for the definition), but the reader familiar with this notion will easily notice that all our estimates are still valid with $C_{cb}(\Omega)$ in place of $C(\Omega)$.

2 The case of an ellipse or a parabola

The aim of this section is to show the following result.

Theorem 1. *Let $\mathcal{E}_\alpha$ be the interior of an ellipse with eccentricity $e = 1/\operatorname{ch}\alpha$, $\alpha > 0$. Then $C(\mathcal{E}_\alpha) \leq 2 + 2/\sqrt{4-e^2}$.*

Proof. Since $C(\Omega)$ is invariant under displacement and scaling, we may suppose that the ellipse is defined by

$$\Omega = \mathcal{E}_\alpha = \{x+iy\,;\, \frac{x^2}{\operatorname{ch}^2\alpha} + \frac{y^2}{\operatorname{sh}^2\alpha} < 1\}.$$

Let r be a rational function bounded by 1 in Ω and let A be a linear operator with $\overline{W(A)} \subset \Omega$. Then we may represent $r(A)$ via the Cauchy formula

$$r(A) = \frac{1}{2\pi i} \int_{\partial\Omega} r(\sigma)\,(\sigma - A)^{-1}\,d\sigma.$$

Following [1, 2, 4, 5], we rewrite $r(A)$ using the splitting

$$r(A) = \int_{\partial\Omega} r(\sigma)\,\mu(\sigma, A)\,ds + \frac{1}{2\pi i} \int_{\partial\Omega} r(\sigma)\,(\overline{\sigma} - A^*)^{-1}\,d\overline{\sigma}, \tag{5}$$

where in the first integral $\sigma = \sigma(s)$ is an arclength parametrisation of $\partial\Omega$, and

$$\mu(\sigma, A) = \frac{1}{2\pi}\big(\nu(\sigma - A)^{-1} + \bar{\nu}(\bar{\sigma} - A^*)^{-1}\big), \quad \nu = \frac{1}{i}\frac{d\sigma}{ds}.$$

Based on the observation that $W(A) \subset \Omega$ implies that $\mu(\sigma, A)$ is self-adjoint and positive definite for any $\sigma \in \partial\Omega$, we have for the first term the estimate

$$\Big\| \int_{\partial\Omega} r(\sigma)\,\mu(\sigma, A)\,ds \Big\| \leq \Big\| \int_{\partial\Omega} \mu(\sigma, A)\,ds \Big\| = 2,$$

see [1] or [5]. Thus it suffices to show that

$$\Big\| \frac{1}{2\pi i} \int_{\partial\Omega} r(\sigma)\,(\overline{\sigma} - A^*)^{-1}\,d\overline{\sigma} \Big\| \leq \frac{2}{\sqrt{4-e^2}}.$$

For that we remark that the points $\sigma = x+iy$ on the boundary of the ellipse satisfy the equation

$$\frac{x^2}{\mathrm{ch}^2\alpha} + \frac{y^2}{\mathrm{sh}^2\alpha} = 1, \quad \text{or, equivalently,} \quad \overline{\sigma}^2 - 2\,\mathrm{ch}(2\alpha)\,\overline{\sigma}\,\sigma + \sigma^2 + \mathrm{sh}^2(2\alpha) = 0.$$

Let $\Delta = (-1, 1)$ be the focal axis. Denoting by $\sqrt{z^2 - 1}$ the continuous determination of the square root in $\mathbb{C} \setminus \Delta$ with value $z + O(z^{-1})$ as $z \to \infty$, we obtain the formula

$$\overline{\sigma} = g(\sigma) := \mathrm{ch}(2\alpha)\sigma - \mathrm{sh}(2\alpha)\sqrt{\sigma^2 - 1} \qquad \text{for} \;\; \sigma \in \partial\mathcal{E}_\alpha.$$

Note that the function g is analytic in $\mathbb{C} \setminus \overline{\Delta}$, and admits the boundary values

$$g(x \pm i0) = g_\pm(x) = \mathrm{ch}(2\alpha)x \mp i\;\mathrm{sh}(2\alpha)\sqrt{1 - x^2}$$

for σ approaching the cut Δ from above or from below. A simple calculation shows that, for $\beta, \theta \in \mathbb{R}$,

$$g(\mathrm{ch}\,\beta\,\cos\theta + i\;\mathrm{sh}\,\beta\,\sin\theta) = \mathrm{ch}(2\alpha - \beta)\,\cos\theta - i\,\mathrm{sh}(2\alpha - \beta)\,\sin\theta.$$

Since the point $\mathrm{ch}\,\beta\,\cos\theta + i\;\mathrm{sh}\,\beta\,\sin\theta$ belongs to $\mathcal{E}_\alpha \setminus \overline{\Delta}$ if and only if $0 < \beta < \alpha$, we conclude that g maps $\mathcal{E}_\alpha \setminus \overline{\Delta}$ into the exterior of $\mathcal{E}_\alpha$, and thus the function $r(\cdot)(g(\cdot) - A^*)^{-1}g'(\cdot)$ is holomorphic in $\mathcal{E}_\alpha \setminus \overline{\Delta}$. This justifies to deform the path of integration from $\partial\mathcal{E}_\alpha$ to Δ

$$\begin{aligned} \frac{1}{2\pi i} \int_{\partial\mathcal{E}_\alpha} r(\sigma)\,(\overline{\sigma} - A^*)^{-1}\,d\overline{\sigma} &= \frac{1}{2\pi i} \int_{\partial\mathcal{E}_\alpha} r(\sigma)\,(g(\sigma) - A^*)^{-1}\,g'(\sigma)\,d\sigma \\ &= -\int_{-1}^{1} r(x)\,\psi(x, A^*)^{-1}\,dx, \end{aligned}$$

where

$$\psi(x, A^*)^{-1} = \frac{1}{2\pi i}\,\big(g'_+(x)(g_+(x) - A^*)^{-1} - g'_-(x)(g_-(x) - A^*)^{-1}\big).$$

Notice for later use that $\psi(\cdot, A^*)^{-1}$ is continuous in $(-1,1)$, and that

$$\psi(0,A^*)^{-1} = \frac{\mathrm{sh}(2\alpha)\ \mathrm{ch}(2\alpha)}{\pi}(-i\ \mathrm{sh}(2\alpha) - A^*)^{-1}(i\ \mathrm{sh}(2\alpha) - A^*)^{-1}$$

is clearly invertible. Hence $\psi(x, A^*)$ is continuous in $x = 0$.

Simple calculations give, setting $a = \mathrm{ch}(2\alpha)$ and $b = \mathrm{sh}(2\alpha)$,

$$\begin{aligned}
\psi(x,z) &= -2\pi i \left[\frac{g_-'(x)}{g_-(x)-z} - \frac{g_+'(x)}{g_+(x)-z}\right]^{-1} \\
&= -2\pi i \left[\frac{a - i\frac{bx}{\sqrt{1-x^2}}}{ax + ib\sqrt{1-x^2} - z} - \frac{a + i\frac{bx}{\sqrt{1-x^2}}}{ax - ib\sqrt{1-x^2} - z}\right]^{-1} \\
&= -\pi \frac{(z-ax)^2 + (1-x^2)b^2}{(z-ax)b\frac{x}{\sqrt{1-x^2}} - ab\sqrt{1-x^2}} \\
&= -\pi \frac{\sqrt{1-x^2}}{bx}\,\frac{(z-ax)^2 + (1-x^2)b^2}{z - a/x} \\
&= -\pi \frac{\sqrt{1-x^2}}{bx}\left[(z-a/x) + 2a\frac{1-x^2}{x} + \frac{1-x^2}{x^2}\,\frac{a^2-x^2}{z-a/x}\right]. \qquad (6)
\end{aligned}$$

We observe that Theorem 1 follows provided that

$$\left\| \int_{-1}^{1} r(x)\, \psi(x,A^*)^{-1}\, dx \right\| \le \frac{2}{\sqrt{4-e^2}}. \qquad (7)$$

In what follows we will write $\mathrm{Re}\, M = \frac{1}{2i}(M - M^*)$. The two basic ingredients of our proof of (7) are stated in the following two lemmas, where for the second result the numerical range assumption $W(A) \subset \mathcal{E}_\alpha$ is used in a crucial way.

Lemma 2. *Suppose that* $\mathrm{Re}\,\psi(x,A^*)$ *is positive definite and* $|r(x)| \le 1$ *for all* $x \in (c,d)$, *then*

$$\left\| \int_{c}^{d} r(x)\, \psi(x,A^*)^{-1}\, dx \right\| \le \left\| \int_{c}^{d} \big(\mathrm{Re}\,\psi(x,A^*)\big)^{-1}\, dx \right\|. \qquad (8)$$

Lemma 3. *If* $W(A) \subset \mathcal{E}_\alpha$ *then for all* $m < -\mathrm{ch}\,\alpha$ *there holds*

$$\mathrm{Re}((A^* - m)^{-1}) \ge \frac{m^2 - \mathrm{ch}^2\alpha}{m^2 - 1}(\mathrm{Re}\, A - m)^{-1}. \qquad (9)$$

Before giving a proof of these two lemmas, let us first show how to deduce the bound (7). Writing $B = \mathrm{Re}\, A$, we get from (6)

$$\mathrm{Re}\,\psi(x,A^*) = -\pi \frac{\sqrt{1-x^2}}{bx}\Big((B - \tfrac{a}{x}) + 2a\frac{1-x^2}{x} + \frac{1-x^2}{x^2}\,(a^2 - x^2)\,\mathrm{Re}((A^* - \tfrac{a}{x})^{-1})\Big).$$

For $x \in (-1,0)$, we set $m = a/x$ and deduce from (9) that

$$\begin{aligned}
\mathrm{Re}\,\psi(x,A^*) &\ge -\pi \frac{\sqrt{1-x^2}}{bx}\Big((B - \tfrac{a}{x}) + 2a\frac{1-x^2}{x} + \frac{1-x^2}{x^2}\,(a^2 - x^2\,\mathrm{ch}^2\alpha)(B - \tfrac{a}{x})^{-1}\Big) \\
&= \pi \frac{\sqrt{1-x^2}}{b}\,(a - xB)^{-1}(B^2 - 2a\,x\,B + a^2 - (1-x^2)\,\mathrm{ch}^2\alpha).
\end{aligned}$$

$(M + M^*)$ for the self-adjoint part of M, and Im M

Notice that $\overline{W(A)} \subset \mathcal{E}_\alpha$ implies $\overline{W(-A)} \subset \mathcal{E}_\alpha$. Hence after substitution of x and A by $-x$ and $-A$, respectively, we find that the the previous inequality is also valid for $x \in (0,1)$, and, by continuity, it still holds for $x = 0$. We set now

$$\phi(x,\lambda) := \frac{\pi\sqrt{1-x^2}}{b(a-\lambda x)}(\lambda^2 - 2a\,\lambda\,x + a^2 - (1-x^2)\,\mathrm{ch}^2\alpha),$$

and note that (recall that $a = \mathrm{ch}(2\alpha) > \mathrm{ch}\,\alpha$)

$$\phi(x,\lambda) = \frac{\pi\sqrt{1-x^2}}{b(a-\lambda x)}\big((\lambda - ax)^2 + (1-x^2)(a^2 - \mathrm{ch}^2\alpha)\big) > 0, \quad \forall\, x \in \Delta,\ \forall\, \lambda \in (-\mathrm{ch}\,\alpha, \mathrm{ch}\,\alpha).$$

Thus the self-adjoint operator $\phi(x,B)$ is well defined and positive definite for $x \in \Delta$. We have proved that $\mathrm{Re}\,\psi(x,A^*) \geq \phi(x,B) > 0$, therefore we can apply (8) and obtain

$$\Big\|\int_{-1}^{1} r(x)\,\psi(x,A^*)^{-1}\,dx\Big\| \;\leq\; \Big\|\int_{-1}^{1} \mathrm{Re}\,\psi(x,A^*)^{-1}\,dx\Big\| \leq \Big\|\int_{-1}^{1} \phi(x,B)^{-1}\,dx\Big\|.$$

In order to evaluate the integral on the right-hand side, let us show that, for $\lambda \in (-\mathrm{ch}\,\alpha, \mathrm{ch}\,\alpha)$,

$$\int_{-1}^{1} \frac{dx}{\phi(x,\lambda)} = \frac{b}{\sqrt{a^2 - \mathrm{ch}^2\alpha}}, \tag{10}$$

and thus this integral is independent of λ. Since the spectrum of our self-adjoint operator B satisfies $\sigma(B) \subset (-\mathrm{ch}\,\alpha, \mathrm{ch}\,\alpha)$, the relation (10) implies that

$$\int_{-1}^{1} \phi(x,B)^{-1}\,dx = \frac{\mathrm{sh}(2\alpha)}{\sqrt{\mathrm{ch}^2(2\alpha) - \mathrm{ch}^2\alpha}} = \frac{2}{\sqrt{4-e^2}},$$

which shows that (7) holds. For proving (10), we write

$$\frac{1}{\phi(x,\lambda)} = \frac{b}{\pi\sqrt{1-x^2}}\,\frac{a - \lambda\,x}{(a-\lambda x)^2 - (1-x^2)(\mathrm{ch}^2\alpha - \lambda^2)},$$

and use the change of variables $x = \cos t$, $\lambda = \mathrm{ch}\,\alpha\,\cos u$ in order to obtain

$$\begin{aligned}\frac{1}{\phi(x,\lambda)} &= \frac{b}{\pi\,\sin t}\,\frac{a - \mathrm{ch}\,\alpha\,\cos u\,\cos t}{(a - \mathrm{ch}\,\alpha\,\cos u\,\cos t)^2 - \sin^2 t\,\mathrm{ch}^2\alpha\,\sin^2 u)} \\ &= \frac{b}{2\,\pi\,\sin t}\Big(\frac{1}{a - \mathrm{ch}\,\alpha\,\cos(u+t)} + \frac{1}{a - \mathrm{ch}\,\alpha\,\cos(u-t)}\Big).\end{aligned}$$

This yields

$$\begin{aligned}\int_{-1}^{1}\frac{dx}{\phi(x,\lambda)} &= \frac{b}{2\,\pi}\int_0^{\pi}\frac{dt}{a - \mathrm{ch}\,\alpha\,\cos(u+t)} + \frac{b}{2\,\pi}\int_0^{\pi}\frac{dt}{a - \mathrm{ch}\,\alpha\,\cos(u-t)} \\ &= \frac{b}{2\,\pi}\int_{-\pi}^{\pi}\frac{dt}{a - \mathrm{ch}\,\alpha\,\cos(u+t)} = \frac{b}{\sqrt{a^2 - \mathrm{ch}^2\alpha}},\end{aligned}$$

which completes the proof of (7) and thus of Theorem 1. □

We still have to give a proof for the two assertions of Lemma 2 and Lemma 3.

Proof of Lemma 2. By assumption, the operators $\operatorname{Re}\psi(x, A^*)$ and $M(x) := (\operatorname{Re}\psi(x, A^*))^{-1}$ are self-adjoint and positive definite. This allows us to write

$$\psi(x, A^*) = M(x)^{-1/2}(I+iE(x))M(x)^{-1/2},$$

with

$$E(x) := M(x)^{1/2} \operatorname{Im}\psi(x, A^*)M(x)^{1/2}.$$

Let us consider u and $v \in \mathcal{H}$. Since E is self-adjoint, we have $\|(I+iE(x))^{-1}\| \leq 1$, and thus

$$\begin{aligned} |\langle \psi(x, A^*)^{-1} u, v\rangle| &= |\langle (I+iE(x))^{-1}M(x)^{1/2}u, M(x)^{1/2}v\rangle| \\ &\leq \langle M(x)\, u, u\rangle^{1/2}\langle M(x)\, v, v\rangle^{1/2}. \end{aligned}$$

Using the assumption $|r(x)| \leq 1$ and the Cauchy-Schwarz inequality we get

$$\begin{aligned} \Big|\Big\langle \int_c^d r(x)\, \psi(x, A^*)^{-1}\, dx\, u, v\Big\rangle\Big| &\leq \Big(\int_c^d \langle M(x)\, u, u\rangle\, dx\Big)^{1/2}\Big(\int_c^d \langle M(x)\, v, v\rangle\, dx\Big)^{1/2} \\ &\leq \Big(\Big\langle \int_c^d M(x) dx\, u, u\Big\rangle\Big)^{1/2}\Big(\Big\langle \int_c^d M(x) dx\, v, v\Big\rangle\Big)^{1/2} \\ &\leq \Big\| \int_c^d M(x)\, dx\Big\|\, \|u\|\, \|v\|. \end{aligned}$$

This yields

$$\Big\| \int_c^d r(x)\, \psi(x, A^*)^{-1}\, dx\Big\| \leq \Big\| \int_c^d M(x)\, dx\Big\|,$$

and completes the proof. □

The numerical range assumption $W(A) \subset \mathcal{E}_\alpha$ is used in a crucial way in the proof of Lemma 3.

Proof of Lemma 3. The point $m < -\operatorname{ch}\alpha$ is exterior to the ellipse $\mathcal{E}_\alpha$. Thus there exists a smallest angle $\theta \in (0, \pi/2)$ such that the sector $m + \mathcal{S}_\theta = m + \{z \in \mathbb{C}\,; z \neq 0,\ |\arg z| < \theta\}$ contains the ellipse, $m + \mathcal{S}_\theta \supset \mathcal{E}_\alpha$. Since $B-m$ is positive definite, we can set

$$D = (B-m)^{-1/2}(\operatorname{Im} A)\,(B-m)^{-1/2}.$$

Notice that D is self-adjoint, and $A-m = (B-m)^{1/2}(I+iD)(B-m)^{1/2}$. The condition $W(A) \subset \mathcal{E}_\alpha \subset m + \mathcal{S}_\theta$ yields

$$\langle (I+iD)(B-m)^{1/2}u, (B-m)^{1/2}u\rangle \in \mathcal{S}_\theta, \qquad \forall\, u \in \mathcal{H}, u \neq 0.$$

Setting $v = (B-m)^{1/2}u$ we deduce that $\|Dv\| \leq \tan\theta\ \|v\|$, $\forall v \in \mathcal{H}$, and thus $\|D\| \leq \tan\theta$. Consequently,

$$\operatorname{Re}(I + iD)^{-1} \geq \inf_{\lambda\in\sigma(D)} \operatorname{Re}\frac{1}{1+i\lambda} \geq \min_{\lambda\in[-\tan\theta,\tan\theta]} \operatorname{Re}\frac{1}{1+i\lambda} = \cos^2\theta,$$

and thus

$$\operatorname{Re}\big((A-m)^{-1}\big) = \operatorname{Re}\big((B-m)^{-1/2}(I + iD)^{-1}(B-m)^{-1/2}\big) \geq \cos^2\theta\ (B-m)^{-1}.$$

The value of θ may be obtained by writing that the straight line $y = \tan\theta\,(x-m)$ is a tangent of the boundary $\partial\mathcal{E}_\alpha = \{(\operatorname{ch}\alpha\, \cos t, \operatorname{sh}\alpha\, \sin t)\,; t \in [0, 2\pi]\}$, and hence

$$\operatorname{sh}\alpha\, \sin t - \tan\theta\,(\operatorname{ch}\alpha\, \cos t - m) = 0, \qquad \operatorname{sh}\alpha\, \cos t + \tan\theta\ \operatorname{ch}\alpha\, \sin t = 0.$$

This implies

$$\operatorname{sh}\alpha + m\,\tan\theta\,\sin t = 0 \quad \text{and} \quad \operatorname{ch}\alpha = m\,\cos t,$$

and finally

$$\cos^2\theta = \frac{1}{1+\tan^2\theta} = \frac{m^2\sin^2 t}{m^2\sin^2 t + \operatorname{sh}^2\alpha} = \frac{m^2 - \operatorname{ch}^2\alpha}{m^2-1}.$$

□

It remains to discuss the case of a parabola.

Corollary 4. *Denote by $\mathcal{P}$ the unbounded domain with boundary given by a parabola. Then $C(\mathcal{P}) \le \liminf_{\alpha\to 0+} C(\mathcal{E}_\alpha) \le 2 + 2/\sqrt{3}$.*

Proof. Since $C(\Omega)$ is invariant under displacement and scaling, we may suppose that

$$\mathcal{P} = \{x+iy : x, y \in \mathbb{R}, 2x > y^2\}.$$

Notice that the ellipses

$$\Omega_\alpha = \{x+iy : x,y\in\mathbb{R}, \operatorname{th}^2(\alpha)x^2 + y^2 - 2x < 0\} = \frac{1}{\operatorname{th}^2(\alpha)} + \frac{\operatorname{ch}(\alpha)}{\operatorname{sh}^2\alpha}\mathcal{E}_\alpha$$

for $\alpha > 0$ are decreasing in α, and $\Omega_\alpha \subset \mathcal{P}$, $\bigcup_{\alpha>0}\Omega_\alpha = \mathcal{P}$. Let r be a rational function bounded by 1 in $\mathcal{P}$ and let A be a bounded linear operator satisfying $\overline{W(A)} \subset \mathcal{P}$. Since $\overline{W(A)}$ is a compact subset of $\mathcal{P}$, there exists an $\alpha' > 0$ with $W(A) \subset \Omega_\alpha$ for all $\alpha \in (0, \alpha']$, and

$$\|r(A)\| \le C(\Omega_\alpha) \sup_{z\in\Omega_\alpha} |r(z)| \le C(\Omega_\alpha) = C(\mathcal{E}_\alpha) \le 2 + \frac{2}{\sqrt{3}},$$

the last inequality following from Theorem 1. Hence the assertion follows for $\alpha \to 0+$. □

We notice that we could have given also a proof for Corollary 4 following the lines of the proof of Theorem 1, with $g(\sigma) = \sigma - 2 + 2\sqrt{1-2\sigma}$ and $\Delta = (1/2, +\infty)$.

3 The case of a hyperbola

We now turn to the case where the boundary of the convex domain is a branch of a hyperbola with asymptotics forming an angle 2α, or, equivalently, with an eccentricity $e = 1/\cos\alpha$, $\alpha \in (0, \pi/2)$. We have

Theorem 5. *Let $\mathcal{H}_\alpha$ be a convex domain with boundary given by a branch of a hyperbola with asymptotics forming an angle 2α. Then we have*

$$C(\mathcal{H}_\alpha) \le 2\,\frac{\pi-\alpha}{\pi} + \mu(\alpha), \quad \mu(\alpha) := \frac{\sin 2\alpha}{\pi}\int_0^\infty \frac{dy}{y^2\cos\alpha - 2\,y\,\cos 2\alpha + \cos\alpha}.$$

Proof. Without loss of generality, we may suppose that the domain $\mathcal{H}_\alpha$ is given by

$$\mathcal{H}_\alpha := \{x+iy\,;\, \frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} > 1,\ x > 0\}.$$

We follow the lines of the proof of Theorem 1 (i.e., formally we replace α by $i\alpha$ in this proof and change some signs). For $\sigma \in \partial\mathcal{H}_\alpha$ we have $\overline{\sigma}^2 - 2\cos(2\alpha)\,\overline{\sigma}\,\sigma + \sigma^2 = \sin^2(2\alpha)$. We set $\Delta =$

$(1,+\infty)$. Denoting by $\sqrt{1-z^2}$ the continuous determination of the square root in $\mathbb{C}\setminus(\Delta\cup -\Delta)$ with value 1 for $z=0$, we obtain the formula

$$\overline{\sigma} = g(\sigma) := \cos(2\alpha)\sigma + \sin(2\alpha)\sqrt{1-\sigma^2}, \quad \text{if } \sigma \in \partial\mathcal{H}_\alpha.$$

The function g is analytic in $\mathcal{H}_\alpha \setminus \overline{\Delta}$, and admits the boundary values

$$g(x \pm i0) =: g_\pm(x) = \cos(2\alpha)x \mp i\,\sin(2\alpha)\sqrt{x^2-1},$$

for σ approaching the cut Δ from above or from below. Let r be a rational function bounded by 1 in $\mathcal{H}_\alpha$ and let A be a bounded operator with $\overline{W(A)} \subset \mathcal{H}_\alpha$. We have to show that

$$\|r(A)\| \le 2\,\frac{\pi-\alpha}{\pi} + \mu(\alpha).$$

We first remark that it is sufficient to prove this estimate for rational functions satisfying furthermore the condition $r(\infty)=0$. Indeed, otherwise we can introduce, with $\varepsilon>0$, the function $r_\varepsilon(z) = r(z)/(1+\varepsilon z)$. We still have r_ε bounded by 1 in $\mathcal{H}_\alpha$ and the result then follows from $r(A) = \lim_{\varepsilon\to 0} r_\varepsilon(A)$.

Thus we assume from now that $r(\infty)=0$ and r is bounded by 1 in $\mathcal{H}_\alpha$. We represent $r(A)$ via the Cauchy formula

$$r(A) = \frac{1}{2\pi i}\int_{\partial\mathcal{H}_\alpha} r(\sigma)\,(\sigma - A)^{-1}\,d\sigma.$$

With the same notation as in the proof of Theorem 1 we have the splitting

$$r(A) = \int_{\partial\mathcal{H}_\alpha} r(\sigma)\,\mu(\sigma,A)\,ds + \frac{1}{2\pi i}\int_{\partial\mathcal{H}_\alpha} r(\sigma)\,(\overline{\sigma} - A^*)^{-1}\,d\overline{\sigma},$$

but for our unbounded domain $\mathcal{H}_\alpha$ containing the sector $\cos(\alpha)+\mathcal{S}_\alpha$ we have the refined estimate

$$\Big\|\int_{\partial\mathcal{H}_\alpha} r(\sigma)\,\mu(\sigma,A)\,ds\Big\| \le \Big\|\int_{\partial\mathcal{H}_\alpha} \mu(\sigma,A)\,ds\Big\| = 2\,\frac{\pi-\alpha}{\pi},$$

compare with [1, Section 3]. Thus it is sufficient to show that

$$\Big\|\frac{1}{2\pi i}\int_{\partial\mathcal{H}_\alpha} r(\sigma)\,(\overline{\sigma} - A^*)^{-1}\,d\overline{\sigma}\Big\| \le \mu(\alpha).$$

Deforming the path of integration from $\partial\mathcal{H}_\alpha$ to Δ and using that $r(\infty)=0$ we get

$$\frac{1}{2\pi i}\int_{\partial\mathcal{H}_\alpha} r(\sigma)\,(\overline{\sigma} - A^*)^{-1}\,d\overline{\sigma} = -\int_1^\infty r(x)\,\psi(x,A^*)^{-1}\,dx,$$

where

$$\psi(x,z) = -2\pi i\left[\frac{g_-'(x)}{g_-(x)-z} - \frac{g_+'(x)}{g_+(x)-z}\right]^{-1}.$$

Simple calculations give, setting $a=\cos(2\alpha)$ and $b=\sin(2\alpha)$,

$$\begin{aligned}
\psi(x,z) &= 2\pi i\left[\frac{a+i\frac{bx}{\sqrt{x^2-1}}}{ax+ib\sqrt{x^2-1}-z} - \frac{a-i\frac{bx}{\sqrt{x^2-1}}}{ax-ib\sqrt{x^2-1}-z}\right]^{-1}\\
&= \pi\,\frac{\sqrt{x^2-1}}{b}\,\frac{(z-ax)^2+(x^2-1)b^2}{zx-a}\\
&= \pi\,\frac{\sqrt{x^2-1}}{bx}\left[(z-a/x) - 2a\frac{x^2-1}{x} + \frac{x^2-1}{x^2}\,\frac{x^2-a^2}{z-a/x}\right].
\end{aligned}$$

This yields, with $B := \operatorname{Re} A$,

$$\operatorname{Re}\psi(x,A^*) = \pi\,\frac{\sqrt{x^2-1}}{bx}\,\Big((B-\tfrac{a}{x}) - 2a\frac{x^2-1}{x} + \frac{x^2-1}{x^2}\,(x^2-a^2)\operatorname{Re}((A^*-\tfrac{a}{x})^{-1})\Big).$$

We admit for a while that for all $-1 < m < \cos\alpha$ we have

$$\operatorname{Re}((A^*-m)^{-1}) \geq \frac{\cos^2\!\alpha - m^2}{1-m^2}(\operatorname{Re} A - m)^{-1}. \tag{11}$$

Using this inequality with $m = a/x$ we get, for $x > 1$,

$$\begin{aligned}\operatorname{Re}\psi(x,A^*) &\geq \pi\,\frac{\sqrt{x^2-1}}{bx}\,\Big((B-\tfrac{a}{x}) - 2a\frac{x^2-1}{x} + \frac{x^2-1}{x^2}\,(x^2\cos^2\!\alpha - a^2)(B-\tfrac{a}{x})^{-1}\Big)\\ &\geq \pi\,\frac{\sqrt{x^2-1}}{b}\,(xB-a)^{-1}(B^2 - 2a\,x\,B + a^2 + (x^2-1)\cos^2\!\alpha).\end{aligned}$$

We now set

$$\begin{aligned}\phi(x,\lambda) &:= \frac{\pi\sqrt{x^2-1}}{b(\lambda x - a)}(\lambda^2 - 2a\,\lambda\,x + a^2 + (x^2-1)\cos^2\!\alpha)\\ &= \frac{\pi\sqrt{x^2-1}}{b}\,\frac{(\lambda x - a)^2 - (x^2-1)(\lambda^2 - \cos^2\!\alpha)}{\lambda x - a}.\end{aligned}$$

In case $x > 1$ and $\lambda > \cos\alpha$ we use the change of variables $x = \operatorname{ch} t$, $\lambda = \cos\alpha\,\operatorname{ch} u$, and obtain

$$\begin{aligned}\frac{1}{\phi(x,\lambda)} &= \frac{b}{\pi\,\operatorname{sh} t}\,\frac{\cos\alpha\,\operatorname{ch} u\,\operatorname{ch} t - a}{(\cos\alpha\,\operatorname{ch} u\,\operatorname{ch} t - a)^2 - \operatorname{sh}^2 t\,\cos^2\!\alpha\,\operatorname{sh}^2 u}\\ &= \frac{b}{2\pi\,\operatorname{sh} t}\Big(\frac{1}{\cos\alpha\,\operatorname{ch}(u-t) - a} + \frac{1}{\cos\alpha\,\operatorname{ch}(u+t) - a}\Big).\end{aligned} \tag{12}$$

This yields for $\lambda > \cos\alpha$

$$\begin{aligned}\int_1^\infty \frac{dx}{\phi(x,\lambda)} &= \frac{b}{2\pi}\int_0^\infty \frac{dt}{\cos\alpha\,\operatorname{ch}(u-t) - a} + \frac{b}{2\pi}\int_0^\infty \frac{dt}{\cos\alpha\,\operatorname{ch}(u+t) - a}\\ &= \frac{b}{2\pi}\int_{-\infty}^\infty \frac{dt}{\cos\alpha\,\operatorname{ch}(u+t) - a} = \mu(\alpha),\end{aligned}$$

where in the last equality we have used the substitution $y = \exp(t+u)$. Thus, as in the proof of Theorem 1, the above integral does not depend on λ.

We now observe that the spectrum of the self-adjoint operator B satisfies $\sigma(B) \subset \mathcal{H}_\alpha \cap \mathbb{R} = (\cos\alpha, +\infty)$. Using (12) and the relations $\cos\alpha > 0$, $\cos\alpha > \cos(2\alpha) = a$, we conclude that the self-adjoint operator $\phi(x,B)$ is well defined and positive definite for $x > 1$. Therefore, by Lemma 2,

$$\Big\|\int_1^\infty r(x)\,\psi(x,A^*)^{-1}\,dx\Big\| \;\leq\; \Big\|\int_1^\infty \operatorname{Re}\psi(x,A^*)^{-1}\,dx\Big\| \leq \Big\|\int_1^\infty \phi(x,B)^{-1}\,dx\Big\| = \mu(\alpha),$$

as claimed in Theorem 5.

It remains to show (11). For $m < \cos\alpha$ there exists a smallest angle θ such that $\mathcal{H}_\alpha \subset m + \mathcal{S}_\theta$. Then, as in the proof of Lemma 3, the assumption $W(A) \subset \mathcal{H}_\alpha \subset m + \mathcal{S}_\theta$ infers

$$\operatorname{Re}((A^* - m)^{-1}) \geq \cos^2\!\theta\,(\operatorname{Re} A - m)^{-1}.$$

The same calculations (mutatis mutandis) as in the proof of this lemma give the formula

$$\cos^2\!\theta = \frac{\cos^2\!\alpha - m^2}{1-m^2}, \quad \text{if } 0 < m < \cos\alpha,$$

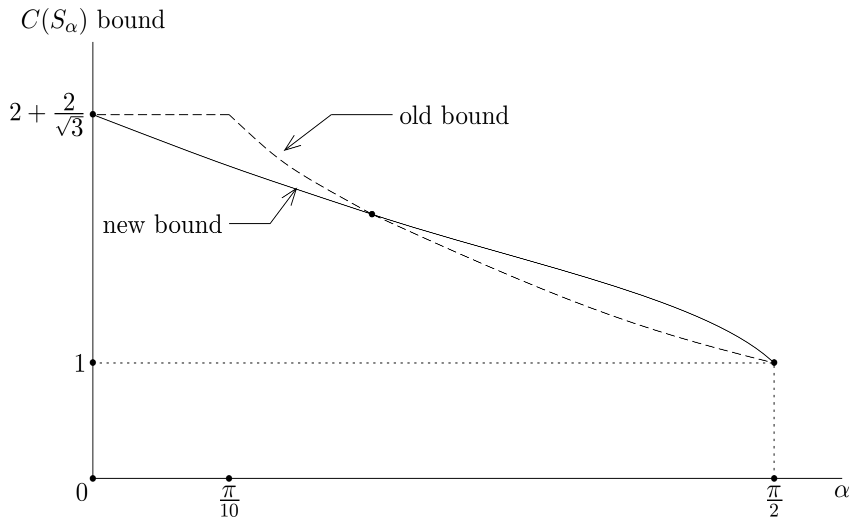

Figure 1: The new bound versus the old one (2) for the sector $\mathcal{S}_\alpha$.

which shows (11) in this case. If $-1 < m \leq 0$, it is easily seen that $\theta = \alpha$ and then the inequality (11) follows from $\cos^2\alpha \geq (\cos^2\alpha - m^2)/(1-m^2)$. □

4 The case of a sector

Corollary 6. *For each $\alpha \in (0, \pi/2)$ there holds $C(\mathcal{S}_\alpha) \leq C(\mathcal{H}_\alpha)$.*

Proof. We argue as in the proof of Corollary 4: the sets $\Omega_q := q\mathcal{H}_\alpha$ are decreasing in $q > 0$, and $\bigcup_{q>0} \Omega_q = \mathcal{S}_\alpha$. Let r be a rational function bounded by 1 in $\mathcal{S}_\alpha$ and let A be a bounded operator such that $\overline{W(A)} \subset \mathcal{S}_\alpha$. By compactness of $\overline{W(A)}$ there exists a $q > 0$ with $\overline{W(A)} \subset \Omega_q$, and thus

$$\|r(A)\| \leq C(\Omega_q) \sup_{z \in \Omega_q} |r(z)| \leq C(\Omega_q) = C(\mathcal{H}_\alpha).$$

□

In order to compare our findings of Corollary 6 (and Theorem 5) with the bounds (2) obtained in [1, 2], we have drawn in Figure 1 the curve $\alpha \mapsto 2 - \frac{2\alpha}{\pi} + \mu(\alpha)$ together with the bound

$$\alpha \mapsto \min\Big(2 + \frac{2}{\sqrt{3}}, \frac{\pi - \alpha}{\pi}\big(2 - \frac{2}{\pi}\log\tan\big(\frac{\alpha\,\pi}{4(\pi-\alpha)}\big)\big)\Big)$$

of (2). We observe that the new bound is sharper for $\alpha \leq .22\,\pi$.

Finally we notice that the quantity $\mu(\alpha)$ of (4) can be written in a more explicit form in terms of the eccentricity $e = 1/\cos\alpha$. We have

$$\mu(\alpha) = \frac{2}{\sqrt{4-e^2}}\big(1 - \frac{1}{\pi}\arccos(\tfrac{2}{e} - e)\big), \quad \text{if } 0 < \alpha \leq \tfrac{\pi}{3},$$

and

$$\mu(\alpha) = \frac{2}{\pi\sqrt{e^2-4}} \log\big(\frac{e^2-2+\sqrt{e^2-4}\sqrt{e^2-1}}{e}\big), \quad \text{if } \tfrac{\pi}{3} < \alpha < \tfrac{\pi}{2}.$$

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