The interval of equilibrium for the constrained energy problem in the presence of an external field

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Abstract

A system of integral equations for determination of the set of equilibrium for the constrained energy problem of the logarithmic potential with external field is obtained. Also the continuity of the family of the equilibrium sets is studied.

1 Introduction.

We shall investigate the interval of equilibrium for the constrained energy problem in the presence of an external field. This problem has a physical interpretation in terms of the density of an electric charge on the line conductor in some external field. In mathematical terms, we are looking for a measure on the interval which minimize the energy depending on the external field. The problem with constraint has been considered first by E.A.Rahmanov in [R], and further analyzed in [DS]. In physical terms, such a constraint corresponds to imposing an upper bound for the maximal density for the unknown charge. An important application of the constrained energy problems is the dispersion regularization of some hyperbolic PDEs, see [DM], [AV].

In this section we briefly recall the classical theorems dealing with weighted energy problems and constrained energy problems in logarithmic potential theory. Subsequently, we state our main findings.

Let $Q : \Sigma \to \mathbb{R}$ be a continuous function on some compact $\Sigma \subset \mathbb{R}$. We denote by $\mathcal{M}$ the collection of finite Borel measures, and by $\mathcal{M}^x_\Sigma \subset \mathcal{M}$ the set of measures μ with support S_μ in Σ and total mass x. For a fixed measure σ with support $S_\sigma = \Sigma$ let $\mathcal{M}^{x,\sigma}$ be the following subset of $\mathcal{M}^x_{S_\sigma}$

$$\mathcal{M}^{x,\sigma} = \{\mu : 0 \le \mu \le \sigma, \mu(\mathbb{R}) = x\}. \tag{1}$$

where $\mu \le \sigma$ means that $\sigma - \mu$ is a measure.

The *logarithmic potential* of a measure μ from $\mathcal{M}$ is defined by

$$U^\mu(z) = \int \log \frac{1}{|z-y|} d\mu(y)$$

and the *weighted energy integral* is defined by

$$I_Q(\mu) = \int\int \log \frac{1}{|z-y|} d\mu(y) d\mu(z) + 2\int Q d\mu \tag{2}$$

*This work was partly supported by INTAS network NeCCA 03-51-6637, the first author was also supported by the Program N1 of the Department of Mathematical Sciences of RAS and by the grants of Scientific Schools - 1551.2003.1, RFBR 05-01-00522.

and Q is called an *external field.*

If the weighted energy functionals are considered on an unbounded $\Sigma \subsetneqq \overline{\mathbb{R}}$, then the external field Q has to satisfy the additional growth condition

$$\lim_{|z|\to\infty, z\in\Sigma} Q(z) - \log(|z|) = +\infty.$$

We shall drop the dependency on Σ in our notations in the cases not providing misunderstanding.

Let us consider an *extremal energy problem*, i.e. the problem of finding a measure $\mu_Q^x \in \mathcal{M}_\Sigma^x$ such that

$$I_Q(\mu_Q^x) = W_Q^x = \inf_{\mu\in\mathcal{M}_\Sigma^x} I_Q(\mu). \tag{3}$$

If in addition the measure σ has finite logarithmic energy (i.e. the energy integral (2) with $Q = 0$ is finite) over all compact sets then σ is called a *constraint.* We can define a *constrained extremal energy problem* in the class (1), i.e., the problem of finding a measure $\mu_Q^{x,\sigma} \in \mathcal{M}^{x,\sigma}$ such that

$$I_Q(\mu_Q^{x,\sigma}) = W_Q^{x,\sigma} = \inf_{\mu\in\mathcal{M}^{x,\sigma}} I_Q(\mu) \tag{4}$$

provided that our constraint σ has a total mass $> x$. An introduction in the theory of constrained extremal problems is given in the paper by Dragnev and Saff [DS]. For the case without constraint we refer to the monograph by Saff and Totik [ST] and the references therein. We recall here some known facts.

If $W_Q^x \le \infty$ then there exists unique measure $\mu_Q^x \in \mathcal{M}^x$ such that $I_Q(\mu_Q^x) = W_Q^x$. Also if $W_Q^{x,\sigma} \le \infty$ then there exists unique measure $\mu_Q^{x,\sigma} \in \mathcal{M}^{x,\sigma}$ such that $I_Q(\mu_Q^{x,\sigma}) = W_Q^{x,\sigma}$. The extremal measures are called the *equilibrium measures* because they have the following properties (when U^σ and Q are continuous):

$$U^{\mu_Q^x} + Q \begin{cases} \ge F_Q^x & on \quad \Sigma \\ = F_Q^x & on \quad S_{\mu_Q^x} \end{cases} \tag{5}$$

for some constant F_Q^x, and correspondingly

$$U^{\mu_Q^{x,\sigma}} + Q \begin{cases} \le F_Q^{x,\sigma} & on \quad S_{\mu_Q^{x,\sigma}} \\ \ge F_Q^{x,\sigma} & on \quad S_{\sigma-\mu_Q^{x,\sigma}} \end{cases} \tag{6}$$

for some constant $F_Q^{x,\sigma}$.

Properties (5) and (6) are sufficient conditions for being equilibrium measures, namely if there is a constant such that the first or the second relation above holds for some measure with total mass x then the measure is solution of the extremal problems (3) or (4) and it is called respectively *equilibrium measure associated with Q with total mass x* or $\sigma-$*constrained equilibrium measure with total mass x associated with Q.*

In the paper we use continuous external fields and potentials of constraints, but they may be lower semi-continuous (see [BR, ST, DS]) and there exists similar equilibrium conditions.

There is a functional considered in [BR]

$$F_Q^x(K) := -x \log cap(K) + \int Q \, d\omega_K, \tag{7}$$

where ω_K denotes the equilibrium measure associated with the regular compact set K, i.e. the solution of the extremal problem (3) with $\Sigma = K$ and $Q = 0$. For the first time the functional (7) for $x = 1$ was introduced in [MS]. The support of the equilibrium measure in $\mathbb{R}$ minimizes the functional, more precisely

$$F_Q^x(K) \begin{cases} = F_Q^x & if \quad S_{\mu_Q^x} \subseteqq K \subseteqq S^{\mu_Q^x} \\ > F_Q^x & otherwise \end{cases} \tag{8}$$

were $S^{\mu_Q^x} = \{y : (U^{\mu_Q^x} + Q)(y) = F_Q^x\}$. One can derive this by integration of the above equilibrium conditions with respect to measure ω_K. However, a similar result for the constrained case is unknown.

In this paper we investigate the dependency on x of the set $S_{\mu_x} \cap S_{\sigma-\mu_x}$, where we fix the external field Q and the constraint σ, and write shorter $\mu_x = \mu_Q^{x,\sigma}$ for the extremal measure, i.e., the solution of (4). The set $S_{\mu_x} \cap S_{\sigma-\mu_x}$ is the set where the constraint is not hit. We will refer to is also as the *set of equilibrium*, since on this set we have equality in the equilibrium conditions (6). Here we restrict ourselves to the case where $S_{\mu_x} \cap S_{\sigma-\mu_x}$ is an interval. In Theorem 1 below we describe a system of nonlinear equations for obtaining the endpoints of the set of equilibrium. In the proof of Theorem 1, presented in Section 2, we will introduce a generalization of the linear functional (7) in the constrained case. Secondly, we discuss in Theorem 2 below the continuity the endpoints of the interval of equilibrium with respect to x for fixed σ and Q. The proof of this assertion, given in Section 3, uses several results from [K]. We finally show at the end of Section 3 that the endpoints of the interval of equilibrium are solutions of a system of partial differential equations, the so-called continuum limit of the Toda lattice [DM, AV].

Theorem 1. *Let $Q : [A, B] \to \mathbb{R}$, $(-\infty < A < B < +\infty)$ be a continuous and differentiable a. e. function on the interval $[A, B]$, and $Q' \in L^\infty([A, B])$. Let σ be a constraint on $[A, B]$ ($[A, B] = S_\sigma$), such that U^σ is continuous, and $\int_{\mathbb{R}} \frac{1}{\sqrt{|y-\gamma|}} d\sigma(y) < +\infty$ for all $\gamma \in (A, B)$.*

Let $\mu_x = \mu_Q^{\sigma,x}$ be the equilibrium measure for external field Q and constraint σ with total mass $\mu_x(\mathbb{R}) = x \leq \sigma(\mathbb{R})$, i.e. μ_x is the solution of (4). If the set of equilibrium is an interval $S_{\mu_x} \cap S_{\sigma-\mu_x} = [\alpha(x), \beta(x)] =: [a, b]$, $A \leq \alpha(x) < \beta(x) \leq B$, then

$$\begin{aligned} &\int_{[A,B]\setminus S_{\sigma-\mu_x}} \sqrt{\frac{\lambda-a}{\lambda-b}}\, d\sigma(\lambda) + \frac{1}{\pi}\int_a^b Q'(\lambda)\sqrt{\frac{\lambda-a}{b-\lambda}}\, d\lambda \begin{cases} = x & if \quad b < B \\ \leq x & if \quad b = B \end{cases} \\ -&\int_{[A,B]\setminus S_{\sigma-\mu_x}} \sqrt{\frac{\lambda-b}{\lambda-a}}\, d\sigma(\lambda) + \frac{1}{\pi}\int_a^b Q'(\lambda)\sqrt{\frac{b-\lambda}{\lambda-a}}\, d\lambda \begin{cases} = -x & if \quad a > A \\ \geq -x & if \quad a = A \end{cases} \end{aligned} \tag{9}$$

where $[A, B] \setminus S_{\sigma-\mu_x} = S_{\mu_x} \setminus S_{\sigma-\mu_x}$ is the part of S_{μ_x} where the constraint is hit.

Notice that the nonlinear system of equations of Theorem 1 reduces to the one given in [ST, Theorem IV.1.11] in the special case where the constraint is never active (that is $S_{\sigma-\mu_x} = [A, B]$). Such a system is obtained by expressing the fact that the gradient of the function $(\alpha, \beta) \mapsto F_Q^x([\alpha, \beta])$ should vanish. Here additional sufficient conditions (like convexity of Q) are known which insure that the system has a unique solution. For the more general case with constrains, the above system of non-linear equations has been mentioned without proof in [BK, Theorem 2.8] and [KL, Proof of Lemma 6.2] for the special case when $[A, B] \setminus S_{\sigma-\mu_x} = [A, a)$ (the so-called left ansatz). In both papers the authors refer to [DM, Chapter 4] for a proof, the latter reference being quite involved and requiring far more restrictive assumptions on Q and σ compared to Theorem 1. Notice also that, for the left ansatz, sufficient conditions for the existence of a unique solution of the above system are known, see [K, Proposition 4.1] and [KL, Lemma 3.1 and Lemma 3.3].

In many cases, we were able to solve quite successfully the system of non-linear equations resulting from Theorem 1 numerically by means of the Newton method (after a change of variables in order to eliminate the square root singularities). However, it may happen that this non-linear system does not have a unique solution, as for instance in the following example.

Example. Consider $\sigma = \omega_{[-2,2]} + \omega_{[-2,3]}$, $S_\sigma = [A, B] = [-2, 3]$, and the external field

$$Q(\lambda) = g_{[-2,2]}(\lambda) = \begin{cases} 0 & \textit{on} \quad [-2,2], \\ \operatorname{acosh}(\lambda/2) & \textit{on} \quad [2,3], \end{cases}$$

where $g_{[a,b]}(\lambda)$ is the Green function of the interval $[a, b]$ with pole at infinity, i.e.

$$g_{[a,b]}(\lambda) = \log \left| \Big(\frac{2\lambda - a - b}{b-a}\Big) + \sqrt{\Big(\frac{2\lambda - a - b}{b-a}\Big)^2 - 1} \right| = \log\Big(\frac{1}{\operatorname{cap}([a,b])}\Big) - U^{\omega_{[a,b]}}(\lambda). \quad (10)$$

Since $\operatorname{cap}([a, b]) = \frac{b-a}{4}$, we find that

$$U^{\omega_{[-2,2]}}(\lambda) + Q(\lambda) = 0 \qquad \lambda \in [-2, 3].$$

From (6) we conclude that the extremal measure with total mass 1 for the external field Q and the constraint σ is the Chebyshev measure of $[-2, 2]$, i.e.

$$\mu_1 = \mu_Q^{1,\sigma} = \omega_{[-2,2]}, \quad \text{with weight} \quad \frac{d\omega_{[-2,2]}}{d\lambda}(\lambda) = \frac{1}{\pi\sqrt{4-\lambda^2}},$$

and $S_{\sigma-\mu_1} = [A, B]$. Here the nonlinear system of Theorem 1 has an infinite number of solutions since for $a = -2$ and for any $b \in (2, 3]$

$$\begin{aligned} \frac{1}{\pi}\int_a^b Q'(\lambda)\sqrt{\frac{\lambda-a}{b-\lambda}}\,d\lambda &= \frac{1}{\pi}\int_2^b \sqrt{\frac{\lambda+2}{b-\lambda}}\frac{d\lambda}{\sqrt{\lambda^2-4}} = 1 = x, \\ \frac{1}{\pi}\int_a^b Q'(\lambda)\sqrt{\frac{b-\lambda}{\lambda-a}}\,d\lambda &= \frac{1}{\pi}\int_2^b \sqrt{\frac{b-\lambda}{\lambda+2}}\frac{d\lambda}{\sqrt{\lambda^2-4}} > 0 > -1 = -x. \end{aligned}$$

Notice however that $Q' \notin L^\infty([-2, 3])$ and also the assumptions of Theorem 1 on σ fail to be true. Consider the slightly modified input data $\sigma = \omega + \omega_{[-2,3]}$ with ω having the density $\sqrt{4-\lambda^2}/(2\pi)$ on $S_\omega = [-2, 2]$, and $Q = -U^\omega \in \mathcal{C}^1([-2, 3])$, compare with [ST, Theorem IV.5.1]. Here again $\mu_1 = \omega$, the assumptions of Theorem 1 hold for $x = 1$, and the nonlinear system has the solutions $a = -2$ and $b \in [2, 3]$. Since the underlying computations are quite involved, we omit the details.

Concerning the continuity of the endpoints of the set of equilibrium we have the following result.

Theorem 2. *Suppose that there is an interval $I \subset (0, \sigma(\mathbb{R}))$ such that $S_{\mu_Q^{x,\sigma}} \cap S_{\sigma-\mu_Q^{x,\sigma}}$ is a non-empty interval of the form $[\alpha(x), \beta(x)]$ for all $x \in I$.*

Then there exists an x_A such that $x \mapsto \alpha(x)$ is decreasing and lower semi-continuous in $I \cap (-\infty, x_A)$, and increasing and upper semi-continuous in $I \cap (x_A, +\infty)$.

Similarly, there exists an x_B such that $x \mapsto \beta(x)$ is decreasing and lower semi-continuous in $I \cap (x_B, +\infty)$, and increasing and upper semi-continuous in $I \cap (-\infty, x_B)$.

Finally, both functions are continuous at $x \in I$ if and only if there is no point in $[A, B] \setminus [\alpha(x), \beta(x)]$ where we have equality in (6).

2 Proof of Theorem 1

In this section we will fix x, consider an external field Q and a constraint σ as in Theorem 1, and write shorter $\mu_x = \mu_Q^{x,\sigma}$ for the corresponding constrained equilibrium measure.

By assumption of Theorem 1, both supports S_{μ_x} and $S_{\sigma-\mu_x}$ are closed subsets of the compact interval $[A, B]$, with their union being equal to $[A, B]$, and their intersection given by the compact interval $[a, b] := [\alpha(x), \beta(x)]$. It is not difficult to check that therefore both sets are themselves intervals with endpoints in $\{A, \alpha(x), \beta(x), b, B\}$: more precisely, exactly one of the following four cases is true

$$S_{\mu_x} = [\alpha(x), \beta(x)], \quad S_{\sigma-\mu_x} = [A, B], \quad \text{and } A \leq \alpha(x) < \beta(x) \leq B, \tag{11}$$
$$S_{\mu_x} = [\alpha(x), B], \quad S_{\sigma-\mu_x} = [A, \beta(x)], \quad \text{and } A \leq \alpha(x) < \beta(x) < B, \tag{12}$$
$$S_{\mu_x} = [A, \beta(x)], \quad S_{\sigma-\mu_x} = [\alpha(x), B], \quad \text{and } A < \alpha(x) < \beta(x) \leq B, \tag{13}$$
$$S_{\mu_x} = [A, B], \quad S_{\sigma-\mu_x} = [\alpha(x), \beta(x)], \quad \text{and } A < \alpha(x) < \beta(x) < B. \tag{14}$$

In the unconstrained case, provided that the extremal measure has the support $[a, b]$, we know from (8) that the function

$$(\alpha, \beta) \mapsto F_Q^x([\alpha, \beta]) = -x \log(\frac{\beta - \alpha}{4}) + \frac{1}{\pi} \int_\alpha^\beta \frac{Q(\lambda)}{\sqrt{(\lambda - \alpha)(\beta - \lambda)}} \, d\lambda$$

has a global minimum in $A \leq \alpha < \beta \leq B$ at the point (a, b). The same result can be shown to hold true in case (11), however, in the other cases, similar (but weaker) results can only be obtained after modifying the above function. For $A \leq \alpha < \beta \leq B$, let

$$F(\alpha, \beta) := -x \log(\frac{\beta - \alpha}{4}) + \frac{1}{\pi} \int_\alpha^\beta \frac{Q(\lambda)}{\sqrt{(\lambda - \alpha)(\beta - \lambda)}} \, d\lambda - \int_{I(\alpha,\beta)} g_{[\alpha,\beta]}(\lambda) \, d\sigma(\lambda), \tag{15}$$

where $I(\alpha, \beta)$ is defined to be empty in case (11), and to be equal to the sets $(\beta, B]$, $[A, \alpha)$ and $[A, \alpha) \cup (\beta, B]$, respectively, in the cases (12), (13), and (14). Notice that $I(a, b) = [A, B] \setminus S_{\sigma-\mu_x}$ by construction.

We have the following result on local minima and maxima of F.

Lemma 1. *The univariate function $\beta \mapsto F(a, \beta)$ has a minimum in $(a, B]$ at $\beta = b$ if b is an interior point of $S_{\sigma-\mu_x}$ or if $b = B$, and a maximum else. Similarly, the univariate function $\alpha \mapsto F(\alpha, b)$ has a minimum in $[A, b)$ at $\alpha = a$ if a is an interior point of $S_{\sigma-\mu_x}$ or if $a = A$, and a maximum else.*

Proof. We consider only the first function, the reasoning for the second is similar.

In both cases $b = B$ or $b \in \text{int}(S_{\sigma-\mu_x})$ we find that one of the cases (11) or (13) is true. In particular, $I(a, \beta) = S_{\mu_x} \setminus [a, b]$, and for all $\beta \in (a, B]$ we have that $[a, \beta] \subset S_{\sigma-\mu_x}$. Hence for all $\lambda \in [a, \beta]$ we get $F_Q^{x,\sigma} \leq U^{\mu_x}(\lambda) + Q(\lambda)$ from the equilibrium conditions for the extremal measure μ_x, and thus, by applying the Fubini Theorem and (10),

$$\begin{aligned}
F_Q^{x,\sigma} &\leq \int (U^{\mu_x} + Q) d\omega_{[a,\beta]} = \int Q d\omega_{[a,\beta]} + \int U^{\omega_{[a,\beta]}} \, d\mu_x \\
&= \int Q d\omega_{[a,\beta]} + \int \Big(\log(\frac{1}{\text{cap}([a, \beta])} - g_{[a,\beta]}(\lambda)\Big) \, d\mu_x(\lambda) \\
&= F(a, \beta) - \int_a^b g_{[a,\beta]}(\lambda) \, d\mu_x(\lambda) \leq F(a, \beta),
\end{aligned}$$

with equality if $\beta = b$, as claimed in the assertion of Lemma 1.

On the other hand, if neither $b = B$ nor $b \in \mathrm{int}(S_{\sigma-\mu_x})$, then necessarily one of the cases (12) or (14) is true, and in particular $b \in \mathrm{Int}(S_{\mu_x})$. In this case $I(a,\beta) = S_{\mu_x} \setminus [a,\beta]$, and the equilibrium conditions give the relation $F_Q^{x,\sigma} \geq U^{\mu_x}(\lambda) + Q(\lambda)$ for all $\lambda \in [a,\beta]$ for all $\beta \in (\alpha, B]$. Thus, again using Fubini and (10),

$$\begin{aligned} F_Q^{x,\sigma} &\geq \int (U^{\mu_x} + Q) d\omega_{[a,\beta]} = \int Q d\omega_{[a,\beta]} - x \log(\frac{\beta-\alpha}{4}) - \int g_{[a,\beta]}(\lambda) d\mu_x(\lambda) \\ &= F(a,\beta) + \int_{I(\alpha,\beta)} g_{[a,\beta]}(\lambda) d(\sigma - \mu_x)(\lambda) \geq F(a,\beta), \end{aligned}$$

with equality if $\beta = b$. $\square$

Lemma 1 tells us that, for $a, b \in (A,B) = \mathrm{int}(S_{\sigma-\mu_x}) \cup \mathrm{int}(S_{\mu_x})$,

$$\frac{\partial F}{\partial \alpha}(a,b) = \frac{\partial F}{\partial \beta}(a,b) = 0,$$

provided that F has partial derivatives. Notice that F is not always differentiable, for instance in the example after Theorem 1 the reader may verify that $F(-2,b) = \max(-\log((b+2)/4), 0)$, taking its minimum at $b \in [2,3]$, but being clearly not differentiable at $b = 2$. In our case, the additional smoothness assumptions of Theorem 1 on σ and Q do enable us to show in the next two technical lemmas below the differentiability.

Lemma 2. *Suppose that Q is continuous on $[A,B]$ and differentiable almost everywhere on $[A,B]$, with $Q' \in L^\infty([A,B])$. Furthermore, let I_1, I_2 be some compact sets, and $y : I_1 \times I_2 \mapsto [A,B]$, with $y \in C^1(I_1 \times I_2)$. Then for all $\beta \in I_1$*

$$\frac{\partial}{\partial \beta} \int_{I_2} Q(y(\beta,\theta))\, d\theta = \int_{I_2} Q'(y(\beta,\theta)) \frac{\partial y}{\partial \beta}(\beta,\theta)\, d\theta,$$

that is, the above integral is differentiable with respect to β, and the derivative is obtained by exchanging integration and differentiation.

Proof. We first notice that $\beta \mapsto Q(y(\beta,\theta))$ is Lipschitz continuous in $\beta \in I_1$ uniformly for $\theta \in I_2$. Indeed, since $Q' \in L^1([A,B])$, we have for $[\beta_1,\beta_2] \subset I_1$

$$Q(y(\beta_2,\theta)) - Q(y(\beta_1,\theta)) = \int_{\beta_1}^{\beta_2} Q'(y(\beta,\theta)) \frac{\partial y}{\partial \beta}(\beta,\theta)\, d\beta.$$

Consequently,

$$|Q(y(\beta_2,\theta)) - Q(y(\beta_1,\theta))| \leq \| Q' \|_{L^\infty([A,B])} \cdot \| \frac{\partial y}{\partial \beta} \|_{L^\infty(I_1 \times I_2)} \cdot |\beta_2 - \beta_1|.$$

Thus, for fixed $\beta \in [A,B)$, the functions $f_n(\theta) = [Q(y(\beta + \frac{1}{n}, \theta)) - Q(y(\beta,\theta))]/(1/n)$ are bounded uniformly in n, θ, and we get for the right derivative

$$\frac{\partial}{\partial_+ \beta} \int_{I_2} Q(y(\beta,\theta))\, d\theta = \lim_{n\to\infty} \int_{I_2} f_n(\theta)\, d\theta = \int_{I_2} \lim_{n\to\infty} f_n(\theta)\, d\theta = \int_{I_2} Q'(y(\beta,\theta)) \frac{\partial y}{\partial \beta}(\beta,\theta)\, d\theta,$$

the exchange of the integral and the limit being justified by Lebesgue's Dominated Convergence Theorem. A similar argument yields the same formula for the left derivative for $\beta \in (A,B]$. $\square$

The next Lemma deals with the differentiation of a Green potential of a non degenerate interval $[\alpha,\beta]$ with respect to one of the endpoints α or β.

Lemma 3. *Let ν be some measure with compact support $S(\nu)$, and $\alpha<\beta$. Provided that*

$$\int_{\mathbb{R}} \frac{1}{\sqrt{|y-\beta|}} d\nu(y) < \infty,$$

the Green potential $\int g_{[\alpha,\beta]}(y)\,d\nu(y)$ can be differentiated with respect to β, and

$$\begin{aligned}\frac{\partial}{\partial\beta}\int g_{[\alpha,\beta]}(y)\,d\nu(y) &= \int \frac{\partial}{\partial\beta} g_{[\alpha,\beta]}(y)\,d\nu(y)\\ &= \frac{-1}{\beta-\alpha}\int_{\mathbb{R}\setminus[\alpha,\beta]}\sqrt{\frac{y-\alpha}{y-\beta}}\,d\nu(y).\end{aligned}$$

Similarly, provided that

$$\int_{\mathbb{R}} \frac{1}{\sqrt{|y-\alpha|}} d\nu(y) < \infty,$$

the Green potential $\int g_{[\alpha,\beta]}(y)\,d\nu(y)$ can be differentiated with respect to α, and

$$\begin{aligned}\frac{\partial}{\partial\alpha}\int g_{[\alpha,\beta]}(y)\,d\nu(y) &= \int \frac{\partial}{\partial\alpha} g_{[\alpha,\beta]}(y)\,d\nu(y)\\ &= \frac{1}{\beta-\alpha}\int_{\mathbb{R}\setminus[\alpha,\beta]}\sqrt{\frac{y-\beta}{y-\alpha}}\,d\nu(y).\end{aligned}$$

Proof. We will only show the first assertion of the Lemma, the second part is similar. Notice first that, for $y\notin[\alpha,\beta]$,

$$g_{[\alpha,\beta]}(y) = \log\Big(\Big|\frac{2y-\alpha-\beta}{\beta-\alpha}\Big| + \sqrt{\Big(\frac{2y-\alpha-\beta}{\beta-\alpha}\Big)^2-1}\Big)$$

(here we take real square roots such that $\sqrt{1}=1$), and thus

$$\frac{\partial}{\partial\beta} g_{[\alpha,\beta]}(y) = -\frac{|y-\alpha|}{\beta-\alpha}\frac{1}{\sqrt{(y-\alpha)(y-\beta)}} = -\frac{1}{\beta-\alpha}\sqrt{\frac{y-\alpha}{y-\beta}}.$$

In the case $\beta\notin S(\nu)$, we notice that this derivative is smooth for $y\in S(\nu)\setminus[\alpha,\beta]$, and clearly we may exchange the order of differentiation and integration, see for instance the argument used in the preceding Lemma. By splitting if necessary the Green potential into two parts, we see that it only remains to consider the case $S(\nu)\subset[\alpha,\frac{3\beta-\alpha}{2}]$.

Consider the two sequences of functions

$$h_n^{\pm}(y) = \pm\frac{g_{[\alpha,\beta\pm 1/n]}(y)-g_{[\alpha,\beta]}(y)}{1/n},$$

converging point-wise to the derivative of $g_{[\alpha,\beta]}(y)$ with respect to β for almost all y. We will show below that, for all $y\in S(\nu)$ and $n\geq 1$,

$$|h_n^{\pm}(y)| \leq h(y) := \frac{5}{\sqrt{(\beta-\alpha)|y-\beta|}}. \tag{16}$$

Also, $\int h(y)\,d\nu(y) < \infty$ by assumption on ν. Thus the assertion of the Lemma and in particular the differentiability of the Green potential follows by observing that, by Lebesgue's Dominated Convergence Theorem,

$$\lim_{n\to\infty}\int h_n^{\pm}(y)\,d\nu(y) = \int \lim_{n\to\infty} h_n^{\pm}(y)\,d\nu(y).$$

In order to show (16), we start with the observation that, for $y \in S(\nu)\setminus[\alpha,\beta]$ (and thus $\frac{2y-\alpha-\beta}{\beta-\alpha} \in [1,2]$),

$$\begin{aligned} 0 \;\leq\; g_{[\alpha,\beta]}(y) &\leq \frac{2y-\alpha-\beta}{\beta-\alpha} - 1 + \sqrt{\Big(\frac{2y-\alpha-\beta}{\beta-\alpha}\Big)^2 - 1} \\ = \;& \sqrt{\frac{2y-\alpha-\beta}{\beta-\alpha}-1}\Big(\sqrt{\frac{2y-\alpha-\beta}{\beta-\alpha}-1} + \sqrt{\frac{2y-\alpha-\beta}{\beta-\alpha}+1}\Big) \leq 5\sqrt{\frac{y-\beta}{\beta-\alpha}}. \end{aligned}$$

As a consequence, $h_n^+(y) = 0$ for $y \in [\alpha,\beta]$, and for $y \in [\beta, \beta+1/n]$ we find that

$$|h_n^+(y)| = \frac{g_{[\alpha,\beta]}(y)}{1/n} \leq \frac{g_{[\alpha,\beta]}(y)}{y-\beta} \leq h(y).$$

Similarly, $h_n^-(y) = 0$ for $y \in [\alpha, \beta - 1/n]$, and for $y \in [\beta - 1/n, \beta]$ we find that

$$|h_n^-(y)| = \frac{g_{[\alpha,\beta-1/n]}(y)}{1/n} \leq \frac{g_{[\alpha,\beta-1/n]}(y)}{\sqrt{y-(\beta-1/n)}\sqrt{|y-\beta|}} \leq h(y).$$

In order to consider the remaining part of $S(\nu)$, we observe that, for $y \in S(\nu)\setminus[\alpha,\beta]$,

$$[\frac{\partial}{\partial\beta}]^2 g_{[\alpha,\beta]}(y) = -\frac{\partial}{\partial\beta}\frac{1}{\beta-\alpha}\sqrt{\frac{y-\alpha}{y-\beta}} = \frac{1}{(\beta-\alpha)^2}\sqrt{\frac{y-\alpha}{y-\beta}}\frac{y-3\beta/2+\alpha/2}{y-\beta} < 0.$$

Consequently, the function $\beta \mapsto g_{[\alpha,\beta]}(y)$ for fixed α, y is concave and decreasing in (α, y). Hence for $y \in S(\nu)\setminus[\alpha,\beta+1/n]$

$$|h_n^+(y)| = -\frac{g_{[\alpha,\beta+1/n]}(y) - g_{[\alpha,\beta]}(y)}{(\beta+1/n)-\beta} \leq -\frac{g_{[\alpha,y]}(y) - g_{[\alpha,\beta]}(y)}{y-\beta} \leq h(y).$$

Finally, for $y \in S(\nu)\setminus[\alpha,\beta]$ we find again by concavity

$$|h_n^-(y)| = -\frac{g_{[\alpha,\beta]}(y) - g_{[\alpha,\beta-1/n]}(y)}{\beta-(\beta-1/n)} \leq -\frac{g_{[\alpha,y]}(y) - g_{[\alpha,\beta]}(y)}{y-\beta} \leq h(y).$$

Thus inequality (16) holds. □

We are now prepared to conclude the proof of Theorem 1. Let $A \leq \alpha < \beta \leq B$. Notice first that

$$\int Q\,d\omega_{[\alpha,\beta]} = \frac{1}{\pi}\int_0^\pi Q(\frac{\beta+\alpha}{2} + \frac{\beta-\alpha}{2}\cos(\theta))\,d\theta,$$

and the assumptions of Theorem 1 on Q allow us to apply Lemma 2. Hence the following derivatives exist

$$\begin{aligned} \frac{\partial}{\partial\alpha}\int Q\,d\omega_{[\alpha,\beta]} &= \frac{1}{\pi}\int_\alpha^\beta \frac{\beta-\lambda}{\beta-\alpha}\frac{Q'(\lambda)}{\sqrt{(\lambda-\alpha)(\beta-\lambda)}}d\lambda, \\ \frac{\partial}{\partial\beta}\int Q\,d\omega_{[\alpha,\beta]} &= \frac{1}{\pi}\int_\alpha^\beta \frac{\lambda-\alpha}{\beta-\alpha}\frac{Q'(\lambda)}{\sqrt{(\lambda-\alpha)(\beta-\lambda)}}d\lambda. \end{aligned}$$

Since in addition in case $\alpha = A$ we have $I(\alpha,\beta) \subset (\beta, B]$ and similarly for $\beta = B$ there holds $I(\alpha,\beta) \subset [A,\alpha)$, we also find using the assumption of Theorem 1 on σ that the assumptions of Lemma 3 for $\nu = \sigma|_{I(\alpha,\beta)}$ are satisfied. Consequently, the function F defined in (15) has partial derivatives at the point $(\alpha,\beta) = (a,b)$, given by

$$\frac{\partial F}{\partial \alpha}(a,b) = \frac{x}{b-a} + \frac{1}{\pi}\int_a^b \frac{b-\lambda}{b-a}\frac{Q'(\lambda)}{\sqrt{(\lambda-a)(b-\lambda)}}d\lambda - \int_{I(a,b)} \sqrt{\frac{\lambda-b}{\lambda-a}}\frac{d\sigma(\lambda)}{b-a},$$
$$\frac{\partial F}{\partial \beta}(a,b) = \frac{-x}{b-a} + \frac{1}{\pi}\int_a^b \frac{\lambda-a}{b-a}\frac{Q'(\lambda)}{\sqrt{(\lambda-a)(b-\lambda)}}d\lambda + \int_{I(a,b)} \sqrt{\frac{\lambda-a}{\lambda-b}}\frac{d\sigma(\lambda)}{b-a}$$

(more precisely the first expression is a right derivative for $\alpha = A$, and the second one a left derivative for $\beta = B$).

Recall also that $I(a,b) = [A,B] \setminus S_{\sigma-\mu_x}$. Hence the assertion of Theorem 1 follows by observing that either $\alpha = A$ and thus by Lemma 1

$$\frac{\partial F}{\partial_+ \alpha}(a,b) \geq 0$$

or otherwise $\alpha \in (A,B) = \mathrm{Int}(S_{\mu_x}) \cup \mathrm{Int}(S_{\sigma-\mu_x})$ and thus

$$\frac{\partial F}{\partial \alpha}F(a,b) = 0.$$

Similarly, again by Lemma 1 we have that either $b = B$ and $\frac{\partial F}{\partial_- \beta}(a,b) \leq 0$, or otherwise $b \in (A,B)$ and $\frac{\partial F}{\partial \beta}(a,b) = 0$, implying Theorem 1.

3 Smoothness of the endpoints of the set of equilibrium

As before we will write shorter $\mu_x = \mu_Q^{x,\sigma}$. The proof of Theorem 2 relies essentially on the following result which has been shown first by Kuijlaars [K, Proposition 4.1(a)]. For the sake of completeness we give an alternate shorter proof.

Lemma 4. *The relation $y > x$ implies $\mu_y - \mu_x \geq 0$.*

Proof. Consider the new constraint and external field

$$\overline{\sigma} := \sigma - \mu_x, \quad \overline{Q}(\lambda) = Q(\lambda) + U^{\mu_x}(\lambda),$$

with extremal measure $\nu = \mu_{\overline{Q}}^{y-x,\overline{\sigma}}$. We want to show that $\nu^* := \mu_x + \nu$ is a candidate for the solution of the extremal problem related to μ_y. In this case, $\mu_x + \nu = \mu_y$ by uniqueness of the extremal measure, and hence the assertion of the lemma follows.

First one trivially observes that ν^* is a positive measure of total mass y, and $\nu^* \leq \sigma$ by definition of the constraint $\overline{\sigma}$. Hence $\nu^* \in \mathcal{M}^{y,\sigma}$. From the equilibrium conditions (6) for ν we get that

$$U^{\nu^*}(\lambda) + Q(\lambda) = U^{\nu}(\lambda) + U^{\mu_x}(\lambda) + Q(\lambda) \geq F := F_{\overline{Q}}^{y-x,\overline{\sigma}}, \qquad \lambda \in S_{\overline{\sigma}-\nu} = S_{\sigma-\nu^*},$$

and

$$U^{\nu^*}(\lambda) + Q(\lambda) \leq F, \qquad \lambda \in S_\nu \subset S_{\nu^*}.$$

It only remains to show that the last inequality is also true in the remaining part $S_{\nu^*} \setminus S_\nu = S_{\mu_x} \setminus S_\nu$ of S_{ν^*}. From (6) for μ_x we may conclude using the maximum principle for logarithmic potentials [ST, Corollary II.3.3] that, for all $\lambda \in S_{\mu_x} \setminus S_\nu$,

$$\begin{aligned} U^{\nu^*}(\lambda) + Q(\lambda) &= U^\nu(\lambda) + U^{\mu_x}(\lambda) + Q(\lambda) \leq F_Q^{x,\sigma} + U^\nu(\lambda) \\ &\leq F_Q^{x,\sigma} + \max\{U^\nu(\lambda') : \lambda' \in S_\nu\} \\ &= F_Q^{x,\sigma} + \max\{F - U^{\mu_x}(\lambda') - Q(\lambda) : \lambda' \in S_\nu\} \leq F, \end{aligned}$$

where in the final inequality we again have used (6) for μ_x and the fact that $\nu \leq \overline{\sigma} = \sigma - \mu_x$, and hence $S_\nu \subset S_{\sigma-\mu_x}$. Thus our candidate satisfies the equilibrium conditions (6) for μ_y, and hence $\nu = \mu_y$. □

From Lemma 4 we know in particular that $\mu_y - \mu_x$ is of total mass $|y - x|$, and hence $\mu_y \to \mu_x$ in weak* topology, as mentioned already in [K, Proposition 4.1(b)]. Using again Lemma 4 we may conclude that S_{μ_x} is increasing in x, and more precisely

$$S_{\mu_x} = \mathrm{Clos}\Big(\bigcup_{y<x} S_{\mu_y}\Big), \tag{17}$$

see [K, Eqn. (4.12)]. By adapting the language of continuity of families of sets parameterized by a real parameter (see for instance [Ku, Chapter 2]), the relation (17) tells us that the family $\{S_{\mu_x}\}_x$ is lower semi-continuous in x, with the upper limit given by the compact set

$$S^{\mu_x} := \bigcap_{y>x} S_{\mu_y}.$$

This last set may be larger than S_{μ_x}, we claim that

$$S^{\mu_x} = \bigcap_{y>x} S_{\mu_y} = \{\lambda \in S_\sigma : U^{\mu_x}(\lambda) + Q(\lambda) \leq F_Q^{x,\sigma}\}, \tag{18}$$

provided that $S_{\mu_x} \cap S_{\sigma-\mu_x}$ is non-empty. Notice that this last condition, which is true in the setting of Theorem 2, implies that the extremal constant $F_Q^{x,\sigma}$ is uniquely determined by (6). We should also mention that we have equality in the inequality of (18) in the special case $S_\sigma = S_{\sigma-\mu}$ where the constraint is not hit: here we recover an identity given in [BR] for the unconstrained case.

In order to show the claim (18), we recall from the proof of Lemma 4 that $\mu_y - \mu_x$ is the extremal measure with respect to the external field $\overline{Q}(\lambda) = Q(\lambda) + U^{\mu_x}(\lambda)$ and the constraint $\overline{\sigma} = \sigma - \mu_x$, with total mass $y - x$. In [K, Lemma 4.4.(c)] it is shown that

$$\bigcap_{y-x>0} S_{\mu_y-\mu_x} = \Big\{\lambda \in S_{\overline{\sigma}} : \overline{Q}(\lambda) = \min\{\overline{Q}(\lambda') : \lambda' \in S_{\overline{\sigma}}\}\Big\},$$

and the minimum equals $F_Q^{x,\sigma}$ by (6). Consequently,

$$\bigcap_{y>x} S_{\mu_y} = \bigcap_{y>x} S_{\mu_x} \cup S_{\mu_y-\mu_x} = S_{\mu_x} \cup \{\lambda \in S_{\sigma-\mu_x} : U^{\mu_x}(\lambda) + Q(\lambda) = F_Q^{x,Q}\},$$

and together with (6) we arrive at (18). By recalling the duality relation following for instance from (6) that $\sigma - \mu_x$ is extremal with respect to the constraint σ and the external field $-Q - U^\sigma$, we also obtain from (17) and (18) that $S_{\sigma-\mu_x}$ is a decreasing family of sets, with lower and upper limits

$$S_{\sigma-\mu_x} = \mathrm{Clos}\Big(\bigcup_{y>x} S_{\sigma-\mu_y}\Big), \quad S^{\sigma-\mu_x} := \bigcap_{y<x} S^{\sigma-\mu_y} = \{\lambda \in S_\sigma : U^{\mu_x}(\lambda) + Q(\lambda) \geq F_Q^{x,\sigma}\}. \tag{19}$$

We now are prepared to show Theorem 2: first observe that, with S_{μ_x} and $S_{\sigma-\mu_x}$, also the two sets S^{μ_x} and $S^{\sigma-\mu_x}$ are intervals for $x \in I$ by (18), (19). Define the quantities x_A, x_B as follows: if there is no $x \in I$ with $A \in S_{\mu_x}$ (and $B \in S_{\sigma-\mu_x}$, respectively), then put x_A to be equal to the right endpoint of I (and x_B equal to the left endpoint of I), and else

$$x_A = \inf\{x \in I : A \in S_{\mu_x}\}, \qquad x_B = \sup\{x \in I : B \in S_{\sigma-\mu_x}\}.$$

For $x \in I \cap (-\infty, x_A)$ we have $A \notin S_{\mu_x}$ by definition of x_A, and thus one of the cases (11) or (12) is true. In particular, S_{μ_x} is an interval, with left endpoint $\alpha(x)$. Since the sets S_{μ_x} are increasing, we therefore may conclude that $x \mapsto \alpha(x)$ decreases in $I \cap (-\infty, x_A)$, and

$$\alpha(x) = \inf_{y<x} \alpha(y) = \lim_{y \to x-0} \alpha(y)$$

by (17), showing the semi-continuity claimed in Theorem 2. Notice also that the upper limit $\alpha(x+0)$ is the left endpoint of the interval S^{μ_x} or $S^{\mu_x} \cap S^{\sigma-\mu_x}$ by (18).

For $x \in I \cap (x_A, \infty)$ we have $A \in S_{\mu_x}$ by definition of x_A, and thus either $\alpha(x) = A$ or one of the cases (13) or (14) is true. In particular, $\alpha(x)$ is the left endpoint of the interval $S_{\sigma-\mu_x}$, and the claimed monotony and upper semi-continuity follows from Lemma 4 and (19) as before. Here in addition the lower limit $\alpha(x-0)$ is the left endpoint of the interval $S^{\sigma-\mu_x}$ or $S^{\mu_x} \cap S^{\sigma-\mu_x}$ by (19).

Finally, for $x = x_A$, the left endpoint for the interval $S^{\mu_x} \cap S^{\sigma-\mu_x}$ is A by definition of x_A, and the left endpoint for the two intervals S_{μ_x} and $S_{\sigma-\mu_x}$ is given by the lower limit $\alpha(x_A - 0)$ and the upper limit $\alpha(x_A + 0)$, respectively.

Similar properties for the map $x \mapsto \beta(x)$ in I are established, we omit the details. The above considerations can be summarized as follows: for $x \in I$, there is equality in the equilibrium conditions (6) in the set

$$S^{\mu_x} \cap S^{\sigma-\mu_x} = [\min\{\alpha(x+0), \alpha(x-0)\}, \max\{\beta(x+0), \beta(x-0)\}],$$

which coincides with $[\alpha(x), \beta(x)]$ if and only if both functions α and β are continuous in x. Thus Theorem 2 is shown.

Let us relate the findings of Theorem 1 and Theorem 2. Revisiting the proof of Lemma 1 allows to conclude that

$$\frac{\partial F}{\partial \alpha}(a,b) = \frac{\partial F}{\partial \beta}(a,b) = 0$$

for $a \in [\min\{\alpha(x+0), \alpha(x-0)\}, \alpha(x)]$ and $b \in [\beta(x), \max\{\beta(x+0), \beta(x-0)\}]$. Thus, in case of discontinuities of α or β, the system of nonlinear equations in Theorem 1 will not have a unique solution.

We conclude this section by recalling the well-known relationship between intervals of equilibrium and the so-called continuum limit of the Toda lattice, a system of hyperbolic partial differential equations, see [DM, AV]. As in [AV], consider an external field depending in addition on the time variable t

$$Q(\lambda, t) = Q(\lambda, 0) - \frac{\lambda t}{2},$$

and a constraint σ independent of t, where $\lambda \mapsto Q(\lambda, 0)$ and σ satisfy the conditions of Theorem 1. We also suppose that there is neighborhood U of some (x_0, t_0) such that, for all $(x, t) \in U$, the equilibrium set $S_\mu \cap S_{\sigma-\mu}$ for the extremal measure $\mu = \mu^{x,\sigma}_{Q(\cdot,t)}$ is an interval $[\alpha(x, t), \beta(x, t)]$ with $A < \alpha(x, t) < \beta(x, t) < B$. Finally, we suppose that[1] the map $G : (x, t) \mapsto (\alpha, \beta)$ is of class $\mathcal{C}^1(U)$, with nonsingular Jacobian at the point (x_0, t_0). Our claim is that, in a neighborhood of (x_0, t_0), the two bivariate functions α, β are related as follows

$$\frac{\partial \alpha}{\partial t} = -\frac{\beta - \alpha}{4}\frac{\partial \alpha}{\partial x}, \quad \frac{\partial \beta}{\partial t} = \frac{\beta - \alpha}{4}\frac{\partial \beta}{\partial x}. \tag{20}$$

First notice that, by the theorem of local inversion, G has an inverse map $G^{-1} : (\alpha, \beta) \mapsto (x, t)$ of class $\mathcal{C}^1$ in some neighborhood of $(\alpha_0, \beta_0) = (\alpha(x_0, t_0), \beta(x_0, t_0))$, with nonsingular Jacobian. For showing (20) it is sufficient prove the identities

$$\frac{\partial x}{\partial \alpha} = -\frac{\beta - \alpha}{4}\frac{\partial t}{\partial \alpha}, \quad \frac{\partial x}{\partial \beta} = \frac{\beta - \alpha}{4}\frac{\partial t}{\partial \beta}. \tag{21}$$

Indeed, by multiplying the second identity of (21) by $\frac{\partial \alpha}{\partial x}$ and taking into account that $\frac{\partial \alpha}{\partial x}\frac{\partial x}{\partial \beta} + \frac{\partial \alpha}{\partial t}\frac{\partial t}{\partial \beta} = 0$, we arrive at

$$0 = \frac{\partial \alpha}{\partial x}\Big(-\frac{\partial x}{\partial \beta} + \frac{\beta - \alpha}{4}\frac{\partial t}{\partial \beta}\Big) = \frac{\partial t}{\partial \beta}\Big(\frac{\partial \alpha}{\partial t} + \frac{\beta - \alpha}{4}\frac{\partial \alpha}{\partial x}\Big).$$

Since, according to (21), the determininant of the Jacobian of G^{-1} is given by $-\frac{\beta-\alpha}{2}\frac{\partial t}{\partial \alpha}\frac{\partial t}{\partial \beta} \neq 0$, we obtain the first equation of (20), and the second is shown similarly. It remains to show (21). Taking into account the relations

$$\frac{1}{\pi}\int_a^b \frac{t}{2}\sqrt{\frac{\lambda - a}{b - \lambda}}\, d\lambda = \frac{b - a}{4}t = \frac{1}{\pi}\int_a^b \frac{t}{2}\sqrt{\frac{b - \lambda}{\lambda - a}}\, d\lambda,$$

the system of nonlinear equations of Theorem 1 for $\alpha = \alpha(x, t)$, $\beta = \beta(x, t)$ takes the form

$$\int_{[A,B]\setminus S_{\sigma-\mu}} \sqrt{\frac{\lambda - \alpha}{\lambda - \beta}}\, d\sigma(\lambda) + \frac{1}{\pi}\int_\alpha^\beta \frac{\partial Q}{\partial \lambda}(\lambda, 0)\sqrt{\frac{\lambda - \alpha}{\beta - \lambda}}\, d\lambda = x + \frac{\beta - \alpha}{4}t, \tag{22}$$

$$\int_{[A,B]\setminus S_{\sigma-\mu}} \sqrt{\frac{\lambda - \beta}{\lambda - \alpha}}\, d\sigma(\lambda) - \frac{1}{\pi}\int_\alpha^\beta \frac{\partial Q}{\partial \lambda}(\lambda, 0)\sqrt{\frac{\beta - \lambda}{\lambda - \alpha}}\, d\lambda = x - \frac{\beta - \alpha}{4}t. \tag{23}$$

Writing $\sqrt{\frac{\lambda-\alpha}{\lambda-\beta}} = \epsilon(\lambda)\frac{\lambda-\alpha}{\sqrt{(\lambda-\alpha)(\lambda-\beta)}}$ with $\epsilon(\lambda) = 1$ for $\lambda > \beta$ and $\epsilon(\lambda) = -1$ for $\lambda < \alpha$, and similarly for $\sqrt{\frac{\beta-\lambda}{\lambda-\alpha}}$, we get by subtracting (23) from (22) and by dividing the resulting equation by $2(\beta - \alpha)$ that

$$\frac{t}{4} = \frac{1}{2}\int_{[A,B]\setminus S_{\sigma-\mu}} \frac{\epsilon(\lambda)\, d\sigma(\lambda)}{\sqrt{(\lambda - \alpha)(\lambda - \beta)}} + \frac{1}{2\pi}\int_\alpha^\beta \frac{\partial Q}{\partial \lambda}(\lambda, 0)\frac{d\lambda}{\sqrt{(\lambda - \alpha)(\beta - \lambda)}}. \tag{24}$$

Comparing the right-hand sides of (22), (23) with (21), the equations (21) are obtained by observing that the left-hand side of (22) is differentiable with respect to α, the left-hand side

[1]Notice that this regularity assumption is not covered by our Theorem 2. However, according to (20), the determinant of the Jacobian equals $\frac{\beta-\alpha}{2}\frac{\partial \alpha}{\partial x}\frac{\partial \beta}{\partial x}$, and thus our assumption on the Jacobian is in accordance with the monotony of $x \mapsto \alpha(x, t)$ and $x \mapsto \beta(x, t)$ for fixed t.

of (23) is differentiable with respect to β, and that both partial derivatives are equal to the expression of $t/4$ given on the right-hand side of equation (24). Here the differentiability follows from the smoothness of G^{-1}, and the remaining straightforward computations are left to the reader.

Finally, observe that by adding equation (22) to (23), we also obtain an explicit expression for $x = x(\alpha, \beta)$, namely

$$x = \int_{[A,B]\setminus S_{\sigma-\mu}} \frac{\epsilon(\lambda)(\lambda - \frac{\alpha+\beta}{2})}{\sqrt{(\lambda-\alpha)(\lambda-\beta)}}\, d\sigma(\lambda) + \frac{1}{\pi}\int_\alpha^\beta \frac{\partial Q}{\partial \lambda}(\lambda, 0)\frac{\lambda - \frac{\alpha+\beta}{2}}{\sqrt{(\lambda-\alpha)(\beta-\lambda)}}\, d\lambda.$$

Thus, provided that $\lambda \mapsto Q(\lambda, 0)$ is of class $\mathcal{C}^2([A,B])$ and σ has a density function of class $\mathcal{C}^1([A,B])$, we could show also directly via Lebesgue's Dominated Convergence Theorem following the arguments of Section 2 that the map $(\alpha,\beta) \mapsto (x,t)$ is of class $\mathcal{C}^1$, and that (21) holds. Thus, at least at points where the the determinant $-\frac{\beta-\alpha}{2}\frac{\partial t}{\partial \alpha}\frac{\partial t}{\partial \beta}$ of the Jacobian is non-zero, we have the converse assertion complementing Theorem 2 that the endpoints $\alpha(x,t)$ and $\beta(x,t)$ of the equilibrium set are locally C^1, with non-vanishing partial derivatives.

Acknowledgments.
The authors gratefully acknowledge useful discussions with A.I. Aptekarev.

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