Gaussian, Lobatto and Radau positive quadrature rules with a prescribed abscissa

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Abstract

For a given $\theta \in (a, b)$, we investigate the question whether there exists a positive quadrature formula with maximal degree of precision which has the prescribed abscissa θ plus possibly a and/or b, the endpoints of the interval of integration. This study relies on recent results on the location of roots of quasi-orthogonal polynomials. The above positive quadrature formulae are useful in studying problems in one-sided polynomial L_1 approximation.

Keywords: Positive quadrature formulas, Lobatto-Radau quadrature formulas, orthogonal polynomials, Interlacing property.

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1 Introduction

Let σ be a positive measure on a compact interval, say, $[a, b]$, such that the support of $d\sigma$ contains an infinite set of points. In what follows we will be interested in n-point quadrature formulas having the form

$$I_\sigma(f) := \int_a^b f(x) d\sigma(x) = Q_n(f) + R_n(f), \quad Q_n(f) = \sum_{j=1}^{n} \lambda_{n,j} f(x_{n,j}), \tag{1}$$

with $x_{n,j} \in [a, b]$ distinct called the abscissas, and $\lambda_{n,j} \in \mathbb{R}$ the weights. This formula is said to be positive if $\lambda_{n,j} > 0$ for all j, and to have a degree of precision m if m is the maximal integer such that $R_n(f) = 0$ for all polynomials of degree $\leq m$. In what follows, all our formulas will have degree of precision $m \geq n - 1$, and thus the weights are computable from the abscissas by integrating Lagrange polynomials. Any polynomial w_n of degree n with roots $x_{n,1}, \ldots, x_{n,n}$ will be called a generating polynomial of Q_n.

It is well known that there exist unique n-point quadrature formulas of degree of precision $2n - 1$, the so-called Gauss rules. Also, by relaxing the degree of precision, one may prescribe one or both endpoints as abscissas: there exist unique so-called (left or right) n-point Radau rules with abscissa a or b and degree of precision $2n - 2$, and a unique n-point Lobatto rule with abscissas a and b and degree of precision $2n - 3$. The existence and uniqueness of such formulas can be seen from the fact that Q_n has a degree of precision $m \geq n - 1$ if and only if we have the orthogonality

$$\int_a^b x^j w_n(x) d\sigma(x) = 0, \qquad j = 0, 1, \ldots, m - n \tag{2}$$

for a generating polynomial w_n, which should have simple roots in $[a, b]$. Denoting by $\omega(x) \in \{1, b-x, x-a, (b-x)(x-a)\}$ the prescribed part of the generating polynomial, we see from (2) that for finding the remaining abscissas we have to find the roots of the orthogonal polynomial w_n/ω of degree $n - \deg \omega$ with respect to the positive measure $\omega \, d\sigma$ on $[a, b]$ for any of these four quadrature formulas. It is well known that these roots are simple and in (a, b), and that all the resulting quadrature formulas are positive.

Motivated by some problems related with the best one-sided polynomial approximation of Heaviside functions [3], the aim of this paper is to study the question whether for a given $\theta \in (a, b)$ there exists a positive quadrature formula of the above type having the additional prescribed abscissa θ, if we are willing to lower the corresponding degree of precision by 1. For Gauss rules this question has been fully treated already in [2, Theorem 2.9], we will give below an equivalent caracterization. In [2, Theorem 2.17] the authors discuss the situation of two precribed abscissa $\theta \in (a, b)$ and $c \in \{a, b\}$, but it seems for us that for the θ given by the authors the weight corresponding to the abscissa c might be begative.

As in [1, 2, 4], a central role in this study are played by quasi-orthogonal polynomials as introduced by Shohat [7]. Given a $\theta \in (a, b)$, we call an n-point quadrature rule a quasi Gauss rule (and quasi left Radau rule, quasi right Radau rule or quasi Lobatto rule, respectively) with abscissa θ if it has the prescribed abscissas in $\{\theta\}$ (and $\{\theta, a\}$, $\{\theta, b\}$, and $\{\theta, a, b\}$, respectively), all other abscissas in $(a, b) \setminus \{\theta\}$, and has degree of precision $2n-2$ (and $2n-3$, $2n-3$, and $2n-4$, respectively).

Denote by

$$a < x_{n,1}^G < ... < x_{n,n}^G < b, \quad a = x_{n,1}^{LR} < ... < x_{n,n}^{LR} < b,$$
$$a < x_{n,1}^{RR} < ... < x_{n,n}^{RR} = b, \quad a = x_{n,1}^{L} < ... < x_{n,n}^{L} = b,$$

the abscissas of the classical Gauss, the left Radau, the right Radau and the Lobatto n-point rules for the measure σ, respectively. It is shown implicitly in the proof of Theorem 1.1 below (and can be alternatively established directly using [8, Theorem 3.3.4] due to A. Markov) that

$$j = 1, ..., n: \qquad x_{n,j}^{LR} < x_{n,j}^G < x_{n,j}^{RR}, \tag{3}$$
$$j = 1, ..., n-1: \qquad x_{n-1,j}^{G} < x_{n,j+1}^{LR} < x_{n,j+1}^{L}, \tag{4}$$
$$j = 1, ..., n-1: \qquad x_{n,j}^{L} < x_{n,j}^{RR} < x_{n-1,j}^{G}, \tag{5}$$
$$j = 1, ..., n-2: \qquad x_{n-1,j}^{RR} < x_{n,j+1}^{L} < x_{n-1,j+1}^{LR}. \tag{6}$$

We have the following main result, which will be proved in the next section.

Theorem 1.1. *(a) There exists a positive n-point quasi Gauss rule with abscissa θ for $n \geq 1$ if and only if*

$$\theta \in \bigcup_{j=1}^{n} (x_{n,j}^{LR}, x_{n,j}^{RR}) \setminus \{x_{n,j}^G\}.$$

(b) There exists a positive n-point quasi left Radau rule with abscissa θ for $n \geq 2$ if and only if

$$\theta \in \bigcup_{j=1}^{n-1} (x_{n-1,j}^{G}, x_{n,j+1}^{L}) \setminus \{x_{n,j+1}^{LR}\}.$$

(c) There exists a positive n-point quasi right Radau rule with abscissa θ for $n \geq 2$ if and only if

$$\theta \in \bigcup_{j=1}^{n-1} (x_{n,j}^{L}, x_{n-1,j}^{G}) \setminus \{x_{n,j}^{RR}\}.$$

(d) There exists a positive n-point quasi Lobatto rule with abscissa θ for $n \geq 3$ if and only if

$$\theta \in \bigcup_{j=1}^{n-2} (x_{n-1,j}^{RR}, x_{n-1,j+1}^{LR}) \setminus \{x_{n,j+1}^{L}\}.$$

(e) Defining $\omega(x) = 1, x-a, b-x, (b-x)(x-a)$, respectively, and $\nu = n - \deg\omega$, a generating polynomial for any such quadrature formula is given by

$$w_n(x) = \omega(x)\Big(p_\nu(x, \omega\sigma) - \frac{p_\nu(\theta, \omega\sigma)}{p_{\nu-1}(\theta, \omega\sigma)} p_{\nu-1}(x, \omega\sigma)\Big) \tag{7}$$

with $p_j(x, \omega\sigma)$ the jth orthonormal polynomial with respect to the positive measure $\omega d\sigma$, hence the above quadrature formulas are unique.

Since the intervals occurring in Theorem 1.1(b) and in Theorem 1.1(c) are distinct and the union of their closures give the full interval $[a, b]$ (and similarly those in Theorem 1.1(a) for n replaced by $n-1$ and in Theorem 1.1(d)), we may draw two different conclusions for any prescribed abscissa $\theta \in [a, b]$.

Corollary 1.2. *If $n \geq 3$ and*

$$\theta \in \bigcup_{j=1}^{n} \{x_{n,j}^{LR}, x_{n,j}^{RR}, x_{n,j}^{L}\} \cup \bigcup_{j=1}^{n-1} \{x_{n-1,j}^{G}\}$$

then there exists either an $(n-1)$-point Gauss rule, an n-point Radau rule, or an n-point Lobatto rule with abscissa θ. Else, there exists a positive n-point quasi either left or right Radau rule with abscissa θ.

If $n \geq 3$ and

$$\theta \in \bigcup_{j=1}^{n} \{x_{n,j}^{L}\} \cup \bigcup_{j=1}^{n-1} \{x_{n-1,j}^{G}, x_{n-1,j}^{LR}, x_{n-1,j}^{RR}\}$$

then there exists either an $(n-1)$-point Gauss rule, an $(n-1)$-point Radau rule or an n-point Lobatto rule with abscissa θ. Else, there exists either a positive $(n-1)$-point quasi Gauss rule with abscissa θ or a positive n-point quasi Lobatto rule with abscissa θ.

Remark 1.3. In order to understand better the restrictions for the parameter θ in Theorem 1.1, let us have a closer look at the quasi left Radau rule of Theorem 1.1(b). As θ approaches the exceptional point $x_{n,j+1}^{LR}$, our quadrature formula becomes the classical n-point left Radau rule, having a degree of precision $2n-2$ and not $2n-3$ as required for a quasi Radau rule. For θ approaching the left endpoint $x_{n-1,j}^{G}$ we obtain the classical $(n-1)$-point Gauss rule (with degree of precision $2n-3$) since the weight corresponding to the abscissa a does vanish, see proof of Lemma 2.3 below. Finally, for θ approaching the right endpoint $x_{n,j+1}^{L}$, the right-hand abscissa of our quadrature formula approaches b (see Lemma 2.2 below), and we obtain the classical n-point Lobatto formula (and degree of precision $2n-3$). Similar phenomena do occur for the other three quadrature formulas of Theorem 1.1, we omit details.

2 Proofs

Let $\{p_n(x, \sigma)\}$ be the family of orthonormal polynomials on $[a, b]$ with respect to $d\sigma$ normalized to have positive leading coefficients. The roots of $p_n(x, \sigma)$ are known to be simple and in (a, b), we will enumerate them more explicitly as

$$x_{n,1}(\sigma) < \ldots < x_{n,n}(\sigma)$$

and use the convention $x_{n,0}(\sigma) = a, x_{n,n+1}(\sigma) = b$. If $\omega(x) \in \{1, b-x, x-a, (b-x)(x-a)\}$, we use the notation $p_n(x, \omega\sigma)$ for the polynomials corresponding to the measure $\omega d\sigma$. We also introduce the rational function

$$f_n(x, \sigma) = \frac{p_n(x, \sigma)}{p_{n-1}(x, \sigma)}.$$

As usual $\mathbb{P}_n$ denotes the family of all algebraic polynomials of degree not greater than n.

In the next statement which goes back essentially to Shohat [7] we will enumerate some classical necessary and sufficient conditions for a certain quadrature formula to exist. For the sake of completeness, each time a proof is provided.

Lemma 2.1. *Let $\omega(x) \in \{1, x-a, b-x, (b-x)(x-a)\}$, $\nu = n - \deg \omega$, and $\theta \in (a, b)$.*

(a) If $p_{\nu-1}(\theta, \omega\sigma) = 0$ then there exists no n-point quadrature formula of degree of precision $\geq n + \nu - 2$ having as prescribed abscissas the roots of $(x - \theta)\omega(x)$.

(b) If $p_\nu(\theta, \omega\sigma) = 0$ then there exists a unique n-point quadrature formula of degree of precision $\geq n + \nu - 2$ having as prescribed abscissas the roots of $(x-\theta)\omega(x)$, namely the Gauss/Radau/Lobatto rule which has degree of precision $= n + \nu - 1$.

(c) If $p_{\nu-1}(\theta, \omega\sigma)p_\nu(\theta, \omega\sigma) \neq 0$ then there exists at most one n-point quadrature formula of degree of precision $\geq n+\nu-2$ having as prescribed abscissas the roots of $(x - \theta)\omega(x)$. Such a quadrature formula has the generating polynomial w_n defined in (7).

(d) Conversely, if $p_{\nu-1}(\theta, \omega\sigma)p_\nu(\theta, \omega\sigma) \neq 0$ and provided that the roots of w_n/ω with w_n as in (7) are simple and in (a, b) and that the weights corresponding to the roots of ω are positive, the polynomial w_n of (7) generates a positive n-point quadrature formula of degree of precision $n + \nu - 2$ having as prescribed abscissas the roots of $(x - \theta)\omega(x)$.

Proof. For a proof of parts (a)–(c), let Q_n be an n-point quadrature formula as in (1), of degree of precision $\geq n + \nu - 2$, with generating polynomial W having roots including θ and the roots of ω. Then for $j = 0, 1, ..., \nu - 2$ we have

$$R_n(x^j W) = 0 = I_\sigma(x^j W) = \int_a^b x^j W(x)\, d\sigma(x).$$

Expanding W/ω in the basis of the $p_j(x, \omega\sigma)$ we deduce the equivalent property that

$$W(x) = \omega(x)(c_1 p_\nu(x, \omega\sigma) + c_2 p_{\nu-1}(x, \omega\sigma))$$

for some real constants c_1, c_2. Notice that $c_1 \neq 0$ since otherwise Q_n would be an $(n - 1)$-point rule. Since $W(\theta) = 0 \neq \omega(\theta)$, we conclude that $c_1 p_\nu(\theta, \omega\sigma) + c_2 p_{\nu-1}(\theta, \omega\sigma) = 0$. By the interlacing property of orthogonal polynomials [8, p. 46], the quantities $p_\nu(\theta, \omega\sigma)$ and $p_{\nu-1}(\theta, \omega\sigma)$ do not vanish simultaneously. Thus $p_{\nu-1}(\theta, \omega\sigma) \neq 0$, as claimed in part (a). Also, notice that in case

$p_\nu(\theta,\omega\sigma) = 0$ the roots of W become the roots of $\omega(x)p_\nu(x,\omega\sigma)$, that is, we obtain the classical Gauss/Radau/Lobatto rule described in the introduction. This shows claim (b).

In the remaining case $p_{\nu-1}(\theta,\omega\sigma)p_\nu(\theta,\omega\sigma) \neq 0$ we have shown that $w_n = W/c_1$ as in (7) is a generating polynomial for Q_n. Observing that

$$R_n(p_{\nu-1}(x,\omega\sigma)w) = I_\sigma(p_{\nu-1}(x,\omega\sigma)w) = c_2 \int_a^b p_{\nu-1}(x,\omega\sigma)^2\omega(x)d\sigma(x) \neq 0,$$

we conclude that there is a unique such Q_n, which has degree of precision precisely $n+\nu-2$, as claimed in part (c).

Conversely, under the assumptions of part (d) we may construct a quadrature formula (1) with abscissas $x_{n,j}$ being the roots of w_n (which have been supposed to be distinct elements of $[a,b]$), and weights

$$\lambda_{n,j} = \int_a^b \frac{w_n(x)}{w_n'(x_{n,j})(x-x_{n,j})}\, d\sigma(x) \tag{8}$$

obtained by integrating Lagrange polynomials [5, p. 80], and thus Q_n has degree of precision $\geq n-1$. Since any $f \in \mathbb{P}_{n+\nu-2}$ can be written as $f = f_1 w_n + f_2$ with $f_1 \in \mathbb{P}_{\nu-2}, f_2 \in \mathcal{P}_{n-1}$, we find that $R_n(f) = R_n(f_1 w_n) + R_n(f_2) = R_n(f_1 w_n)$, the latter vanishing by orthogonality. Hence Q_n has degree of precision $\geq n+\nu-2$, and thus $n+\nu-2$ by part (c).

It remains to discuss the positivity of the weights. If $\omega(x_{n,j}) = 0$, the property $\lambda_{n,j} > 0$ has been included in the assumptions. Else we observe that

$$P_j(x) := \frac{w_n(x)}{w_n'(x_{n,j})(x-x_{n,j})}\frac{P(x)}{P'(x_{n,j})(x-x_{n,j})}, \quad P(x) = \frac{w_n(x)}{\omega(x)}$$

is an element of $\mathbb{P}_{n+\nu-2}$ which is ≥ 0 on $[a,b]$ and $= 0$ only at a finite number of points, implying that $\lambda_{n,j} = Q_n(P_j) = I_\sigma(P_j) > 0$. □

One learns from Lemma 2.1(c),(d) that there is a gap between our necessary and sufficient conditions for a certain quadrature formula to exist, namely the localization of the roots of w_n/ω in (a,b), and the positivity of certain weights. Each of these conditions will be further analyzed in the next two lemmas.

We start by recalling results from Peherstorfer [6] and Brezinski et al. [1, Theorem 3] on the localization of roots of the quasi-orthogonal polynomial w_n of (7). The second part can be found in [2, Remark 2.11].

Lemma 2.2. *Let $\omega(x) \in \{1, x-a, b-x, (b-x)(x-a)\}$, $\nu = n - \deg\omega$, and $\theta \in (a,b)$ be such that $p_{\nu-1}(\theta,\omega\sigma)p_\nu(\theta,\omega\sigma) \neq 0$.*

The roots of w_n/ω with w_n as in (7) are simple and in (a,b) if and only if

$$f_\nu(a,\omega\sigma) < f_\nu(\theta,\omega\sigma) < f_\nu(b,\omega\sigma), \tag{9}$$

which again is equivalent to

$$\theta \in \bigcup_{j=1}^{\nu}\Big(x_{\nu-1,j-1}(\omega_a\sigma), x_{\nu-1,j}(\omega_b\sigma)\Big), \tag{10}$$

where $\omega_a(y) = (y-a)\omega(y)$ *and* $\omega_b(y) = (b-y)\omega(y)$.

Proof. From [8, p. 45] or [7, p. 463] it is known that the ν roots of the quasi-orthogonal polynomial w_n/ω are real and distinct, and that at least $\nu-1$ of them lie in (a,b). Thus it remains to localize the remaining root. The sufficiency of condition (9) has been shown in [1, Theorem 3(v)], and the necessity in [1, Theorem 3(iii) and (iv)].

In order to make the link with condition (10), we write the partial fraction decomposition

$$f_\nu(x,\omega\sigma) = \alpha x + \beta + \sum_{j=1}^{\nu-1} \frac{c_j}{x - x_{\nu-1,j}(\omega\sigma)},$$

where from the interlacing property of the roots of $p_\nu(x,\omega\sigma)$ and $p_{\nu-1}(x,\omega\sigma)$ [8, p. 46] it follows that $\alpha, c_1, ..., c_{\nu-1} > 0$, $f_\nu(b,\omega\sigma) > 0$, and $f_\nu(a,\omega\sigma) < 0$. In particular, $x \mapsto f_\nu(x,\omega\sigma)$ is strictly increasing in each subinterval

$$(x_{\nu-1,j-1}(\omega\sigma), x_{\nu-1,j}(\omega\sigma))$$

for $j = 1, ..., \nu$, where we recall the convention $x_{\nu-1,0}(\omega\sigma) = a$ and $x_{\nu-1,\nu}(\omega\sigma) = b$. Since $f_\nu(a) < 0 < f_\nu(b)$, we may conclude that $\theta \in (x_{\nu-1,j-1}(\omega\sigma), x_{\nu-1,j}(\omega\sigma))$ satisfies (9) iff $\theta \in (x_{j-1}, y_j)$, with

$$x_{\nu-1,j-1}(\omega\sigma) \le x_{j-1} < x_{\nu,j}(\omega\sigma) < y_j \le x_{\nu-1,j}(\omega\sigma) \tag{11}$$

and x_{j-1} a root of $P_y(x) = p_\nu(x,\omega\sigma) - f_\nu(y,\omega\sigma)p_{\nu-1}(x,\omega\sigma)$ for $y = a$, and y_j for $y = b$, respectively. By construction, $P_a(x)/(x-a) \in \mathbb{P}_{\nu-1}$ is orthogonal to $\mathbb{P}_{\nu-2}$ with respect to the measure $\omega_a\sigma$. Thus $P_a(x)$ is a non-trivial multiple of $(x-a)p_{\nu-1}(x,\omega_a\sigma)$, showing that $x_0 = a$, and $x_j = x_{\nu-1,j}(\omega_a\sigma)$ for $j = 1, ..., \nu-1$. By a similar argument, $y_\nu = b$, and $y_j = x_{\nu-1,j}(\omega_b\sigma)$ for $j = 1, ..., \nu-1$. □

We learn from the proof of Lemma 2.2 that condition (10) implies the hypothesis $p_{\nu-1}(\theta,\omega\sigma) \neq 0$, but it may happen that $p_\nu(\theta,\omega\sigma) = 0$.

We finally need to discuss the positivity of weights corresponding to prescribed abscissas being roots of ω for the n-point quadrature formula with generating polynomial w_n as in (7). The following result seems to be new.

Lemma 2.3. *Let* $\omega(x) \in \{1, x-a, b-x, (b-x)(x-a)\}$, $\nu = n - \deg\omega$, *and* $\theta \in J_j := \big(x_{\nu-1,j-1}(\omega_a\sigma), x_{\nu-1,j}(\omega_b\sigma)\big)$ *for some* $j \in \{1, ..., \nu\}$ *such that* $p_\nu(\theta,\omega\sigma) \neq 0$, *where as before* $\omega_a(y) = (y-a)\omega(y)$ *and* $\omega_b(y) = (b-y)\omega(y)$.

(a) If $\omega(a) = 0$ *then the weight corresponding to the prescribed abscissa* a *is* > 0 *if and only if* $\theta > x_{\nu,j}(\widetilde{\omega}_a\sigma)$, *with* $\widetilde{\omega}_a(y) = \omega(y)/(y-a)$.

(b) If $\omega(b) = 0$ *then the weight corresponding to the prescribed abscissa* b *is* > 0 *if and only if* $\theta < x_{\nu,j}(\widetilde{\omega}_b\sigma)$, *with* $\widetilde{\omega}_b(y) = \omega(y)/(b-y)$.

Proof. Since the second statement follows from the first after replacing x by $-x$, we only show part (a). We will write shorter $p_j(x,\widetilde{\omega}_a\sigma) = p_j(x)$ and require in what follows the so-called Christoffel-Darboux formula [8, p. 41-42]: there exist scalars $a_m > 0$ such that for all $m \geq 0$

$$a_m \frac{p_{m+1}(x)p_m(y) - p_{m+1}(y)p_m(x)}{x-y} = \sum_{j=0}^{m} p_j(x)p_j(y) =: K_m(x,y). \tag{12}$$

As before we observe that $K_m(x,a) \in \mathbb{P}_m$ is orthogonal to $\mathbb{P}_{m-1}$ with respect to the measure $(x-a)\widetilde{\omega}_a\sigma = \omega\sigma$, and hence

$$\frac{K_m(x,a)}{K_m(a,a)} = \frac{p_m(x,\omega\sigma)}{p_m(a,\omega\sigma)}. \tag{13}$$

Using (12) as well as the orthonormality of the p_j we deduce that

$$\int_a^b \frac{p_m(x,\omega\sigma)}{p_m(a,\omega\sigma)} \widetilde{\omega}_a(x)\, d\sigma(x) = \frac{1}{K_m(a,a)} > 0.$$

According to (8) we may write the weight corresponding to the abscissa a as follows

$$\begin{aligned}
\lambda_{n,1} &= \int_a^b \frac{p_\nu(x,\omega\sigma) - f_\nu(\theta,\omega\sigma)p_{\nu-1}(x,\omega\sigma)}{p_\nu(a,\omega\sigma) - f_\nu(\theta,\omega\sigma)p_{\nu-1}(a,\omega\sigma)} \frac{\widetilde{\omega}_a(x)}{\widetilde{\omega}_a(a)} d\sigma(x) \\
&= \int_a^b \frac{\frac{p_\nu(x,\omega\sigma)}{p_\nu(a,\omega\sigma)} - \frac{f_\nu(\theta,\omega\sigma)}{f_\nu(a,\omega\sigma)} \frac{p_{\nu-1}(x,\omega\sigma)}{p_{\nu-1}(a,\omega\sigma)}}{f_\nu(a,\omega\sigma) - f_\nu(\theta,\omega\sigma)} \frac{\widetilde{\omega}_a(x)}{\widetilde{\omega}_a(a)} d\sigma(x) \\
&= \frac{1}{\widetilde{\omega}_a(a)} \frac{1/K_{\nu-1}(a,a)}{f_\nu(\theta,\omega\sigma) - f_\nu(a,\omega\sigma)} \Big(\frac{f_\nu(\theta,\omega\sigma)}{f_\nu(a,\omega\sigma)} - \frac{K_{\nu-1}(a,a)}{K_\nu(a,a)} \Big).
\end{aligned}$$

Recall from Lemma 2.2 that $f_\nu(\theta,\omega\sigma) > f_\nu(a,\omega\sigma) < 0$, and trivially

$$\widetilde{\omega}_a(a) K_{\nu-1}(a,a) > 0.$$

With the rational function

$$r(x) = \frac{f_\nu(x,\omega\sigma)}{f_\nu(a,\omega\sigma)} - \frac{K_{\nu-1}(a,a)}{K_\nu(a,a)},$$

we therefore may conclude that $\lambda_{n,1} > 0$ if and only if $r(\theta) > 0$. In order to discuss the sign of $r(\theta)$, recall from the proof of Lemma 2.2 that r is strictly decreasing in the interval J_j, and notice that we have the value $1 - K_{\nu-1}(a,a)/K_\nu(a,a) > 0$ at the left endpoint of J_j, and a strictly negative value at the right endpoint of J_j. Thus $\lambda_{n,1} > 0$ if and only if $\theta > x_j$, with x_j the unique root in J_j of r. Applying (12) and (13) we obtain the simplification

$$r(x) = \frac{K_{\nu-1}(a,a)}{K_\nu(a,a)} \Big(\frac{K_\nu(x,a)}{K_{\nu-1}(x,a)} - 1 \Big) = \frac{K_{\nu-1}(a,a)}{K_\nu(a,a)} \frac{p_\nu(a)p_\nu(x)}{K_{\nu-1}(x,a)}$$

with roots $x_{\nu,1}(\widetilde{\omega}_a\sigma), ..., x_{\nu,\nu}(\widetilde{\omega}_a\sigma)$. Comparing with the ordered and disjoint intervals $J_1, ..., J_\nu$, we find that $x_{\nu,j}(\widetilde{\omega}_a\sigma) \in J_j$, and thus $x_j = x_{\nu,j}(\widetilde{\omega}_a\sigma)$, as claimed above. □

We are now prepared to show our main Theorem.

Proof of Theorem 1.1. Statement (a). We set $\omega(x) = 1$, $\nu = n$, and w_n as in (7). Provided that θ is as indicated in Theorem 1.1(a), we conclude from Lemma 2.2 with

$$x_{n-1,j-1}(\omega_a\sigma) = x^{LR}_{n,j} < x_{n,j}(\omega\sigma) = x^{G}_{n,j} < x_{n-1,j}(\omega_b\sigma) = x^{RR}_{n,j}$$

that $w_n = w_n/\omega$ has all its roots in (a, b). Thus the existence of a positive n-point quasi Gaussian quadrature with abscissa θ follows from Lemma 2.1(d).

Conversely, if $\theta = x_{n,j}(\omega\sigma) = x^G_{n,j}$ for some $j \in \{1, ..., n\}$ then $p_n(\theta, \omega\sigma) = 0$, and there does not exist a n-point quasi Gaussian quadrature by Lemma 2.1(b). For any other value of $\theta \in (a, b)$, the polynomial $w_n = w_n/\omega$ has one of its roots outside (a, b) by Lemma 2.2, and thus an n-point quasi Gaussian quadrature cannot exist by Lemma 2.1(c).

Statement (b). We set $\omega(x) = x - a$, $\nu = n - 1$, and w_n as in (7). Provided that θ is as indicated in Theorem 1.1(b), we conclude from Lemma 2.2 and Lemma 2.3(a) with

$$\begin{aligned}&x_{n-2,j-1}(\omega_a\sigma) < x_{n-1,j}(\widetilde{\omega}_a\sigma) = x^{G}_{n-1,j}\\&< x_{n-1,j}(\omega\sigma) = x^{LR}_{n,j+1} < x_{n-2,j}(\omega_b\sigma) = x^{L}_{n,j+1}\end{aligned}$$

that w_n/ω has all its roots in (a, b), and that the weight $\lambda_{n,1}$ corresponding to $x_{n,1} = a$ is > 0. Thus the existence of a positive n-point quasi left Radau quadrature with abscissa θ follows from Lemma 2.1(d).

The non-existence for other values of θ follows as above using again Lemma 2.1, Lemma 2.2, and Lemma 2.3(a).

Statement (c) follows from Theorem 1.1(b) after replacing x by $-x$, or, alternatively from Lemma 2.1, Lemma 2.2, and Lemma 2.3(b) with $\omega(x) = b-x$, $\nu = n - 1$, and

$$\begin{aligned}&x_{n-2,j-1}(\omega_a\sigma) = x^{L}_{n,j} < x_{n-1,j}(\omega\sigma) = x^{RR}_{n,j}\\&< x_{n-1,j}(\widetilde{\omega}_b\sigma) = x^{G}_{n-1,j} < x_{n-2,j}(\omega_b\sigma).\end{aligned}$$

Statement (d) follows from Lemma 2.1, Lemma 2.2, and Lemma 2.3(a),(b) with $\omega(x) = (x - a)(b - x)$, $\nu = n - 2$, and

$$\begin{aligned}&x_{n-3,j-1}(\omega_a\sigma) < x_{n-2,j}(\widetilde{\omega}_a\sigma) = x^{RR}_{n-1,j} < x_{n-2,j}(\omega\sigma) = x^{L}_{n,j+1}\\&< x_{n-2,j}(\widetilde{\omega}_b\sigma) = x^{LR}_{n-1,j+1} < x_{n-3,j}(\omega_b\sigma).\end{aligned}$$

Statement (e) follows from Lemma 2.1(c). □

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References

[1] C. Brezinski, K.A. Driver and M. Redivo-Zagliac, Quasi-orthogonality with applications to some families of classical orthogonal polynomials, Applied Numer. Math., 48 (2004), 157-168.

[2] A. Bultheel, R. Cruz-Barroso and M. Van Barel, On Gauss-type quadrature formulas with prescribed nodes anywhere on the real line, Calcolo 47 (2010), 21-48.

[3] J. Bustamante, J.M. Quesada, R. Martínez-Cruz, Best one-sided L_1 approximation to the Heaviside and sign functions, J. Approx. Theory 164-6 (2012), 791-802.

[4] K. Driver, K. Jordaan and N. Mbuyi, Interlacing of the zeros of Jacobi polynomials with different parameters, Numerical Algorithms, 49 (2008), 143-152.

[5] V. I. Krylov, *Approximate Calculation of Integrals*, Dover Publications, Inc., New York, 2005.

[6] F. Peherstorfer, Linear combinations of orthogonal polynomials generating positive quadrature formulas, Math. Comp. 55 (1990), 231-241.

[7] J.A. Shohat, On mechanical quadratures, in particular, with positive coefficients, Trans. Amer. Math. Soc. 42 (1937), 461-496.

[8] G. Szegö, Orthogonal Polynomials, American Mathematical Society, Providence, Rhode Island, 1939 (fourth edition, 1975).