Computing high precision Matrix Padé approximants

by

Bernhard Beckermann,[1] Daniel Bessis,[2] Luca Perotti,[3] Daniel Vrinceanu[4]

Dedicated to Claude Brezinski on the occasion of his 70th birthday.

Abstract

We describe a new method of computing matrix Padé approximants of series with integer data in a an efficient and fraction-free way, by controlling the growth of the size of intermediate coefficients. This algorithm is applied to compute high precision Padé approximants of matrix-valued generating functions of time series. As an illustration we show that we can successfully recover from noisy equidistant sampling data a joint damped signal of four antenna, even in the presence of background signals.

1 Introduction and application

Consider four independent antennas labeled by $\ell \in \{1, 2, 3, 4\}$, at different places receiving a damped signal

$$A^{(\ell)} e^{-\alpha t} \cos(2\pi\nu t + \varphi^{(\ell)})$$

with the same frequency $\nu > 0$ and damping factor $\alpha > 0$, but with different amplitudes $A^{(\ell)} \neq 0$ and phases $\varphi^{(\ell)} \in [-\pi, \pi]$. Each antenna has a characteristic spectrum that can be approximated by a finite number of discrete frequencies, contributing with a stationary background represented by

$$\sum_{j=1}^{n^{(\ell)}} A_j^{(\ell)} \cos(2\pi\nu_j^{(\ell)} t + \varphi_j^{(\ell)}).$$

Here the number of frequencies $n^{(\ell)} \geq 0$, the frequencies $\nu_j^{(\ell)} > 0$, amplitudes $A_j^{(\ell)} > 0$ and phases $\varphi_j^{(\ell)} \in [-\pi, \pi]$ might vary with the antenna ℓ.

Each antenna records $N + 1$ samples over the time interval $[0, T]$, in other words, we know the first $N + 1$ coefficients of the Taylor series at 0 of the following generating rational function associated with each time series

$$\begin{aligned} F^{(\ell)}(z) &= \sum_{k=0}^{\infty} z^k \Big(A^{(\ell)} e^{-\alpha \frac{kT}{N}} \cos(2\pi\nu \frac{kT}{N} + \varphi^{(\ell)}) + \sum_{j=1}^{n^{(\ell)}} A_j^{(\ell)} \cos(2\pi\nu_j^{(\ell)} \frac{kT}{N} + \varphi_j^{(\ell)}) \Big) \\ &= \frac{A^{(\ell)} e^{i\varphi^{(\ell)}}/2}{1 - z e^{(-\alpha + i2\pi\nu)T/N}} + \sum_{j=1}^{n^{(\ell)}} \frac{A_j^{(\ell)} e^{i\varphi_j^{(\ell)}}/2}{1 - z e^{i2\pi\nu_j^{(\ell)} T/N}} + cc, \end{aligned} \qquad (1)$$

[1]Laboratoire Paul Painlevé UMR CNRS 8524 (ANO-EDP), UFR Mathématiques – M3, UST Lille, F-59655 Villeneuve d'Ascq CEDEX, France, e-mail: bbecker@math.univ-lille1.fr

[2]Department of Physics, Texas Southern University, Houston, TX, USA, email : bessisd@comcast.net

[3]Department of Physics, Texas Southern University, Houston, TX, USA, e-mail: lperotti@hotmail.com

[4]Department of Physics, Texas Southern University, Houston, TX, USA, email: daniel.vrinceanu@gmail.com

where $+cc$ means that we have to add the complex conjugate of the whole expression, except z. By arranging these four generating functions in a 2×2 matrix we obtain a matrix Taylor expansion with coefficients $F_k \in \mathbb{R}^{2\times 2}$

$$\begin{aligned} F(z) &= \begin{bmatrix} F^{(1)}(z) & F^{(2)}(z) \\ F^{(3)}(z) & F^{(4)}(z) \end{bmatrix} = F_0 + F_1 z + ... + F_N z^N + O(z^{N+1})_{z\to 0} \\ &= \frac{\rho_0}{1 - z/z_0} + \sum_{\ell=1}^{4} \sum_{j=1}^{n^{(\ell)}} \frac{\rho_j^{(\ell)}}{1 - z/z_j^{(\ell)}} + cc. \end{aligned} \tag{2}$$

Notice that $F(z)$ is a matrix-valued rational function, with poles

$$z_0 = e^{(\alpha - i2\pi\nu)T/N}, \quad z_j^{(\ell)} = e^{-i2\pi\nu_j^{(\ell)} T/N}, \tag{3}$$

and their complex conjugates, and with

$$\rho_0 = \left[\begin{array}{cc} \frac{1}{2}A^{(1)}e^{i\varphi^{(1)}} & \frac{1}{2}A^{(2)}e^{i\varphi^{(2)}} \\ \frac{1}{2}A^{(3)}e^{i\varphi^{(3)}} & \frac{1}{2}A^{(4)}e^{i\varphi^{(4)}} \end{array} \right], \quad \rho_j^{(\ell)} = \frac{1}{2} A_j^{(\ell)} e^{i\varphi_j^{(\ell)}} \left[\begin{array}{cc} \delta_{\ell,1} & \delta_{\ell,2} \\ \delta_{\ell,3} & \delta_{\ell,4} \end{array} \right], \tag{4}$$

$\delta_{\ell,k}$ denoting the Kronecker symbol such that the matrices $\rho_j^{(\ell)}$ only have one entry different from 0. We will refer to the matrices in (4) as residual–type matrices since they are residuals of the rational function $\frac{1}{z}F(\frac{1}{z})$, a common change of variables in signal processing. Supposing that

$$\nu_j^{(\ell)} \text{ distincts}, \quad \nu < \frac{N}{2T}, \quad \nu_j^{(\ell)} < \frac{N}{2T}, \quad \text{and} \quad N \geq 3 + 4\max\{n^{(1)}, n^{(2)}, n^{(3)}, n^{(4)}\}, \tag{5}$$

we may reconstruct from the recorded sampling data $F_0, ..., F_N$ a unique partial fraction decomposition as in (2), and thus via the formulas (3) and (4) all frequencies, damping factors, amplitudes and (up to a sign) the phases. In particular, the desired frequency ν and damping factor α can be found from the (up to conjugation) unique pole of modulus $\exp(\alpha T/N) > 1$ (the other poles having modulus $= 1$), also in the case where the amplitude of the signal is smaller than the one of the characteristic oscillations of antennas. Moreover, even in case of a small damping factor $\alpha \approx 0$, we expect that the residual-type matrix ρ_0 is the only one with four entries different from zero, which enables us to identify the interesting pole and thus the desired frequency ν.

Since in general we do not know in advance the value of $n^{(\ell)}$, we will construct right matrix Padé approximants of type $[L|M]$, and in particular for the special case $L = M-1$, for different values of M. Such a matrix approximant for a power series $F(z)$ with coefficients $\in \mathbb{R}^{m\times m}$ (here $m = 2$) is defined to be the expression $P(z)Q(z)^{-1}$ with

- (i) $P(z)$ and $Q(z)$ are $m \times m$ matrix polynomials of degree $\leq L$, and $\leq M$, resp.,
- (ii) $Q(z)$ is invertible for almost all z, and
- (iii) the expansion of $F(z)Q(z) - P(z)$ at $z = 0$ starts with the power z^{L+M+1},

compare with [1, §8.2]. Then, for sufficiently large M, the matrix rational function $P(z)Q(z)^{-1}$ will coincide with $F(z)$, see Appendix B, and a partial fraction decomposition enables us to identify the desired frequency.

In the scalar case $m = 1$ it is well known that a Padé approximant always exists and that the fraction $P(z)Q(z)^{-1}$ is unique. However, for $m > 1$ it may happen that matrix

Padé approximants of a given type do not exist, essentially due to condition (ii), see [1, Section 8.2, Example 2]. In what follows we will exclude these cases by supposing that suitable underlying matrices are invertible, compare with [1, Theorem 8.2.2]. This can be justified by the fact hat the four antennas operate independently, or, as in Section 4, that there is random noise on the input. We should warn the reader that, due to lack of commutativity, we cannot just use the same algorithms as in the scalar case $m = 1$ to compute matrix Padé approximants for $m > 1$.

It is well known[1, §2.1] that the calculation of a scalar Padé approximant of type $[L|M]$ is fraught by numerical instability. A rule of thumb for this situation is that about M digits of precision from the initial data are lost, meaning that for large data sets one has to use numbers with at least M digits of precision. An efficient implementation of such a multi–precision algorithm is obtained by using only integer coefficients. For this we need of course to transform our floating point data into integer data, by multiplying $F(z)$ with a sufficiently large integer, which does not change the poles of the corresponding Padé approximant, and multiplies all residual matrices by that factor. Providing that the algorithm does not involve any division, then the results in turn remain within the ring of integers. The fraction–free computation of matrix Padé approximants has been discussed before in [5], using so–called order bases and Mahler systems, but this paper is quite involved since it covers more general approximation problems and in addition all different types of singular cases. It turns out that the normalization of the Padé numerators and denominators, essential for constructing fraction–free methods, is the same in [5] and in the present paper. One may consider the method presented here as a block version of that of [5] where one increases the order simultaneously for several rows. Such a block version allows us to generalize to the matrix setting several classical concepts like Jacobi matrices [13] and resolvent functions.

The reminder of the paper is organized as follows: in § 2 we establish our new fraction-free algorithm for computing matrix Padé approximants via matrix-valued three-term recurrence relations. Illustrating examples show that this new approach enables us to control coefficient growth. In § 3 we suggest a block Jacobi matrix approach in order to express these Padé approximants, and to compute their poles and residuals. § 4 shows how our new algorithm can help to detect the above desired time series data of four antennas from a finite number of recorded samplings, even in the presence of noise. Finally, Appendix A contains the proof of our main Theorem 2.3, whereas in Appendix B we address the question of how to choose the type of aatrix Padé approximants in order to reconstruct exactly our matrix-valued generating function.

2 Fraction–free computation of matrix Padé approximants

Let us consider $N+1$ real matrices F_k, $l = 0, ..., N$ of given dimension $m \times m$, representing the first $N + 1$ coefficients of the formal matrix Taylor series

$$F(z) = F_0 + F_1 z + ... + F_N z^N + \mathcal{O}(z^{N+1})_{z \to 0}. \tag{6}$$

In what follows we will suppose that each entry of each F_k is an integer, and we want to compute right matrix Padé approximants $R_k(z) = P_k(z)Q_k(z)^{-1}$ of $F(z)$ of a certain type to be specified later, and with integer coefficients.

With suitable $\alpha_k, \beta_k \in \mathbb{R}^{m\times m}$, consider sequences of matrix polynomials generated by the three–term recurrence relations for $k = 1, ..., N-1$

$$P_{k+1}(z) = P_k(z)\beta_{k+1} + zP_{k-1}(z)\alpha_{k+1}, \quad (7)$$
$$Q_{k+1}(z) = Q_k(z)\beta_{k+1} + zQ_{k-1}(z)\alpha_{k+1}, \quad (8)$$

together with the initializations

$$Q_0(z) = I, \quad P_0(z) = 0, \quad Q_1(z) = \beta_1, \quad P_1(z) = \alpha_1, \quad (9)$$

where here and in what follows I, and 0, denote the identity matrix, and the matrix containing only zeros, respectively, both of suitable size. In the scalar case $m = 1$, there is a close connection between three term recurrence relations and continued fractions: here the rational function $R_k(z) = P_k(z)Q_k(z)^{-1}$ may be written as the kth partial fraction (or convergent) of some infinite continued fraction, namely

$$R_k(z) = P_k(z)Q_k(z)^{-1} = \frac{\alpha_1 |}{| \beta_1} + \frac{\alpha_2 z |}{| \beta_2} + \frac{\alpha_3 z |}{| \beta_3} + ... + \frac{\alpha_k z |}{| \beta_k}. \quad (10)$$

In particular, we obtain the expansion in terms of a regular continued C–fraction by choosing all $\beta_k = 1$, see [1, §4.5] or [13]. Also, in the scalar case $m = 1$ there are known algorithms how to choose the α_k in such a way such that $R_{2k}(z)$, and $R_{2k+1}(z)$, respectively, are the Padé approximants of $F(z)$ of type $[k-1|k]$, and $[k|k]$, respectively, see, e.g., [1, Theorem 4.4.2]. In our case $m = 2$, due to the lack of commutativity, we prefer to stay with the recurrence relations (7), (8), (9) and do not use the formalism of continued fractions since it does not indicate how to divide by matrices, and in which order one has to multiply with the matrices α_k or β_k.

It is not difficult to verify by recurrence that the polynomials $P_k(z)$ and $Q_k(z)$ generated by (7), (8), (9) verify the degree constrains

$$\deg P_{2k}(z) \le k-1, \quad \deg P_{2k+1}(z) \le k, \quad \deg Q_{2k}(z) \le k, \quad \deg Q_{2k+1}(z) \le k. \quad (11)$$

Moreover, $Q_k(0) = Q_{k-1}(0)\beta_k = ... = \beta_1\beta_2...\beta_k$ is invertible as long as the β_k are invertible, which means that also the condition (ii) for right matrix Padé approximants is true. In order to identify $R_k(z)$ with right matrix Padé approximants of $F(z)$, we still require the accuracy–through–order condition

$$F(z)Q_k(z) - P_k(z) = E_k z^k + \mathcal{O}(z^{k+1})_{z\to 0} \quad (12)$$

for some matrix $E_k = \mathrm{coeff}(F(z)Q_k(z), z^k)$ of size $m \times m$. This relation is trivially true for $k = 0$ with

$$E_0 = \mathrm{coeff}(F(z)Q_0(z), z^0) = \mathrm{coeff}(F(z), z^0) = F_0, \quad (13)$$

whereas for $k = 1$ we require according to (9) that $F_0\beta_1 - \alpha_1 = 0$ or

$$E_0\beta_1 + E_{-1}\alpha_1 = 0, \quad E_{-1} := -I, \quad E_1 = \mathrm{coeff}(F(z)Q_1(z), z^1) = F_1\,\beta_1. \quad (14)$$

By recurrence, a similar relation is obtained in order to insure that $R_{k+1}(z)$ has the correct order, since by injecting (12) for k and $k-1$ into (7), (8) we obtain

$$\begin{aligned} & F(z)Q_{k+1}(z) - P_{k+1}(z) \\ = \; & \Big(F(z)Q_k(z) - P_k(z)\Big)\beta_{k+1} + z\Big(F(z)Q_{k-1}(z) - P_{k-1}(z)\Big)\alpha_{k+1} \\ = \; & z^k\Big(E_k\beta_{k+1} + E_{k-1}\alpha_{k+1}\Big) + \mathcal{O}(z^{k+1})_{z\to 0}. \end{aligned}$$

Thus $E_k\beta_{k+1} + E_{k-1}\alpha_{k+1} = 0$ implies that (12) holds for k replaced by $k+1$. We thus have shown the following result, compare with [1, p. 431].

Lemma 2.1 *Suppose that all matrices E_k in* (12) *are invertible, $E_{-1} := -I$, and define the matrix polynomials $P_k(z)$ and $Q_k(z)$ for $k = 0, 1, ..., N+1$ by* (7), (8), (9), *where we suppose that the $\beta_k \in \mathbb{R}^{m\times m}$ are invertible, and that*

$$E_k\beta_{k+1} + E_{k-1}\alpha_{k+1} = 0, \quad k = 0, ..., N. \tag{15}$$

Then $R_{2k}(z) = P_{2k}(z)Q_{2k}(z)^{-1}$ for $0 \le 2k \le N+1$ are right matrix Padé approximants of type $[k-1|k]$ of $F(z)$, and $R_{2k+1}(z) = P_{2k+1}(z)Q_{2k+1}(z)^{-1}$ for $0 \le 2k+1 \le N+1$ are right matrix Padé approximants of type $[k|k]$ of $F(z)$.

We may therefore identify our three–term recurrence relation as an identity between "neighbors" in the table of matrix Padé approximants. Such so–called Frobenius identities are well established [1, §3.5] in the scalar case $m = 1$, and have been generalized before to the matrix setting $m > 1$ by [8].

Equation (15) still gives a lot of liberties for computing matrix Padé approximants. This liberty corresponds to the question of how to normalize matrix Padé approximants, since the matrix fraction $R_k(z) = P_k(z)Q_k(z)^{-1}$ is invariant under multiplication on the right of both $P_k(z)$ and $Q_k(z)$ by some real or complex matrix of size $m\times m$. For instance, the choice

$$\beta_{k+1} = I, \quad \alpha_{k+1} = -E_{k-1}^{-1}E_k$$

leads to a normalization $Q_k(0) = I$ similar to the one used in regular C–fractions. The choice

$$\alpha_{k+1} = I, \quad \beta_{k+1} = -E_k^{-1}E_{k-1}$$

reminds of an equivalence transformation of S–fractions discussed in [13, §44,], it leads to the normalization of leading coefficients since then $\mathrm{coeff}(Q_{2k}, z^k) = I$ and $\mathrm{coeff}(P_{2k+1}, z^k) = I$. However, both choices will give in general matrix polynomials $P_k(z)$ and $Q_k(z)$ with rational and not integer coefficients.

Using the relation $\mathrm{adj}(A) = \det(A)A^{-1}$ for the adjoint of a matrix, we see that the choice

$$\alpha_{k+1} = \det(E_k)I, \quad \beta_{k+1} = -\mathrm{adj}(E_k)E_{k-1} \tag{16}$$

will enable us to obtain integer results from integer data, however, we will have to pay a very important price of exponential growth of coefficients which is expensive in multi–precision arithmetic: roughly speaking, the number of digits necessary to store our auxiliary quantities are multiplied by $m+1$ while stepping from index k to index $k+1$. This can be easily seen by writing down the first 3 approximants.

Example 2.2 With the choice (16) we get that

$$P_1(z) = \det(E_0)I = \det(F_0)I, \quad Q_1(z) = \mathrm{adj}(E_0) = \mathrm{adj}(F_0).$$

Since $\mathrm{adj}(\mathrm{adj}(M)) = \det(\mathrm{adj}(M))\mathrm{adj}(M)^{-1} = M\det(M)^{m-2}$, we find that $E_1 = F_1\mathrm{adj}(F_0)$, and hence

$$\begin{aligned}
\alpha_2 &= \det(E_1)I = \det(F_1)\det(F_0)^{m-1}I, \\
\beta_2 &= -\mathrm{adj}(E_1)E_0 = -\det(F_0)^{m-2}F_0\mathrm{adj}(F_1)F_0, \\
P_2(z) &= -P_1(z)\mathrm{adj}(E_1)E_0 + 0 = -\det(F_0)^{m-1}F_0\mathrm{adj}(F_1)F_0, \\
Q_2(z) &= -Q_1(z)\mathrm{adj}(E_1)E_0 + zQ_0(z)\det(E_1) \\
&= \det(F_0)^{m-1}\Big(-\mathrm{adj}(F_1)F_0 + zI\det(F_1)\Big)
\end{aligned}$$

implying that $F(z) - R_2(z) = \mathcal{O}(z^2)_{z\to 0}$ (after some computations). Notice that $P_2(z)$ and $Q_2(z)$ are indeed with integer coefficients, but there is a common factor $\det(F_0)^{m-1}$, so it seems that (16) is not an optimal choice for getting integer output, see Theorem 2.3 below.

In the next step we obtain

$$E_2 := \mathrm{coeff}(F(z)Q_2(z), z^2) = \det(F_0)^{m-1}\widetilde{E}_2, \quad \widetilde{E}_2 := -F_2\mathrm{adj}(F_1)F_0 + F_1\det(F_1),$$

where generically we do not expect $\widetilde{E}_2$ to have a non–trivial content, i.e., all entries are divisible by the same integer > 1. Then

$$\alpha_3 = \det(E_2)I = \det(F_0)^{m^2-m}\det(\widetilde{E}_2)I,$$
$$\beta_3 = -\mathrm{adj}(E_2)E_1 = -\det(F_0)^{(m-1)^2}\mathrm{adj}(\widetilde{E}_2)F_1\mathrm{adj}(F_0),$$

in particular we see that α_3 and β_3 contain at least a joint factor $\det(F_0)^{(m-1)^2}$, that is, the non–trivial content of $Q_2(z), P_2(z)$ is even magnified in later steps. Thus, even without explicitly writing down the complicated expressions for $P_3(z), Q_3(z)$, one sees that there should be a different and more efficient way of getting integer output. □

Exponential growth of coefficients (by cross multiplications) is also an important issue in several elementary tasks in Computer Algebra such as solving systems of equations [2] or GCD computations [7, 10]. The problem is that the cost for the multiplication of two integers heavily depends on the number of digits used to store these integers, see for instance [11]. One possibility to control the size of the intermediate quantities is to remove the content (i.e., a largest common integer factor) of $P_k(z)$, $Q_k(z)$, E_k, but finding the content may be quite expensive. Instead, one prefers in [2, 3, 4, 5, 7, 10] to predict common factors which in particular cases may not be optimal, but which can be removed with hardly any additional cost, leading to a linear growth of coefficients.

Theorem 2.3 *Suppose that the entries of the coefficients of $F(z)$ are integers, and define d_k recursively by $d_{-1} = d_0 = 1$, $d_1 = \det(F_0)$, and for $k = 1, 2, ..., N$ by factorizing*

$$d_{k+1}d_k^{m-1} = \det(E_k), \tag{17}$$

and the matrices $\alpha_{k+1}, \beta_{k+1}$ with rational entries for $k = 0, ..., N$ by

$$\alpha_{k+1} = \frac{1}{d_k^{m-1}d_{k-1}}\det(E_k)I, \quad \beta_{k+1} = -\frac{1}{d_k^{m-1}d_{k-1}}\mathrm{adj}(E_k)E_{k-1}. \tag{18}$$

Then d_k are integers, and moreover the matrix polynomials $P_k(z)$, $Q_k(z)$ and coefficients E_k have all integer entries, with the normalizations

$$Q_{2k}(z) = d_{2k}Iz^k + \textit{smaller degree}\ , \quad P_{2k+1}(z) = d_{2k+1}Iz^k + \textit{smaller degree.} \tag{19}$$

Finally, if the data are represented using d digits, then the quantities $P_k(z)$, $Q_k(z)$ and E_{k-1} may be represented using at most mkd digits.

We give a proof of this statement in Appendix A. Let us here have a look at the first approximants generated by our new recurrence relation.

Example 2.4 By definition (17), (18), the quantities $E_{-1} = -I, E_0 = F_0, \alpha_1, \beta_1$ are the same as in Example 2.2, and the same is true for $E_1, P_1(z), Q_1(z)$, that is,

$$P_1(z) = \det(E_0)I = \det(F_0)I, \quad Q_1(z) = \mathrm{adj}(E_0) = \mathrm{adj}(F_0), \quad E_1 = F_1\mathrm{adj}(F_0),$$

and

$$d_0 = 1, \quad d_1 = \det(F_0), \quad d_2 = \frac{\det(E_1)}{d_1^{m-1}} = \det(F_1),$$

in particular we find the normalization (19) for $P_1(z)$. Subsequently, we compute

$$\begin{aligned}
\alpha_2 &= \frac{\det(E_1)}{d_0 d_1^{m-1}} I = \det(F_1)I, \\
\beta_2 &= -\frac{\mathrm{adj}(E_1)E_0}{d_0 d_1^{m-1}} = -\frac{F_0\mathrm{adj}(F_1)F_0}{\det(F_0)}, \\
P_2(z) &= -P_1(z)\beta_2 + 0 = -F_0\mathrm{adj}(F_1)F_0, \\
Q_2(z) &= Q_1(z)\beta_2 + zQ_0(z)\alpha_2 = -\mathrm{adj}(F_1)F_0 + zd_2 I,
\end{aligned}$$

again with the normalization of $Q_2(z)$ predicted by (19). In particular, we succeeded to drop the common factor of Example 2.2.

In the next step we get

$$E_2 = \mathrm{coeff}(F(z)Q_2(z), z^2) = -F_2\mathrm{adj}(F_1)F_0 + F_1\det(F_1)$$

with E_2 having clearly integer entries. It is however far from evident that d_3 defined in (17) is an integer: to see this, notice that E_2 is a scalar times a Schur complement

$$E_2 = \det(F_1)\Big(F_1 - F_2F_1^{-1}F_0\Big)$$

from which it follows using classical properties of Schur complements that

$$\det(E_2) = \det(F_1)^m \frac{\det(\begin{bmatrix} F_1 & F_0 \\ F_2 & F_1 \end{bmatrix})}{\det(F_1)} = d_3 d_2^{m-1}, \quad d_3 = \det(\begin{bmatrix} F_1 & F_0 \\ F_2 & F_1 \end{bmatrix}).$$

In a similar way it is possible to show that the next matrix polynomials $P_3(z)$ and $Q_3(z)$ are indeed with integer coefficients. Since the involved matrices are quite complicated, we omit details. □

Example 2.5 Let us have a look at the special case $m = 1$ of Theorem 2.3, and hence $\det(M) = M$ and $\mathrm{adj}(M) = 1$ for any "matrix" $M \in \mathbb{C}^{1\times 1}$. From (17) we read that $d_{k+1} = \det(E_k) = E_k$. Thus combing (18) with the three–term recurrence relations of (8), (9), we obtain $Q_0(z) = 1$, $Q_1(z) = 1$, and for $k = 1, ..., N-1$

$$d_{k-1}Q_{k+1}(z) = -Q_k(z)d_k + zQ_{k-1}(z)d_{k+1}.$$

The same relation is obtained for the $P_k(z)$, with initialization $P_0(z) = 0$, $P_1(z) = F_0$. By eliminating all quantities with odd indices we obtain new three term recurrence relations between quantities with even indices. These are essentially the polynomials of the extended Euclidean algorithm corresponding to the fraction-free subresultant sequence, see [10, 7], at least in the special case where the degree in each step of the Euclidean algorithm decreases by 1. A similar fraction–free relation for scalar Padé approximants has been given in [9], and for scalar rational interpolants in [4]. □

We should insist on the fact that in general α_{k+1} and β_{k+1} have rational but not necessarily integer entries. For α_{k+1} we may even derive a simpler formula by injecting (17) in (18)

$$\alpha_{k+1} = \frac{d_{k+1} d_k^{m-1}}{d_{k-1} d_k^{m-1}} I = \frac{d_{k+1}}{d_{k-1}} I, \tag{20}$$

where in general d_{k-1} does not divide d_{k+1}. In Example 2.4 we have seen that we have the correct normalization (19) both for $P_1(z)$ and $Q_2(z)$. Since by construction and (11) the leading coefficients of $\alpha_{2k} z Q_{2k-2}(z)$ and of $Q_{2k}(z)$ as well as of $\alpha_{2k+1} z P_{2k-1}(z)$ and of $P_{2k+1}(z)$ coincident, it follows that (19) is true.

Remark 2.6 For large m, one efficient way of computing $X := -d_k^{m-1} d_{k-1} \beta_{k+1} = \mathrm{adj}(E_k) E_{k-1}$ together with $\det(E_k)$ is to apply fraction–free Gaussian elimination [2, 11] to the system of equations $E_k \widetilde{X} = E_{k-1}$ with m right–hand sides. One may show that the quantities after the first elimination step are divisible by d_k. By implementing this division, one obtains directly $X/d_k^{m-1} = -d_{k-1}\beta_{k+1}$ and determinant d_{k+1}. □

3 Block Jacobi matrices

In the scalar case $m = 1$ with the normalization $F(0) = 1$, we may write $R_{2k}(z) = (1, 0, ..., 0)(I - zJ_k)^{-1}(1, 0, ..., 0)^T$ for the Padé approximant of type $[k-1|k]$, with J_k a tridiagonal matrix of order k, the so–called Jacobi matrix. From such a formula we see that the poles of $R_{2k}(z)$ are obtained by computing eigenvalues, and the residuals by computing the first component of eigenvectors, both being classical tasks in numerical analysis where stable procedures are available.

One way of deriving such a matrix formula for Padé approximants is to contract the continued fraction representation (10) by recalling that the event part of a regular C–fraction with convergents $R_0, R_2, R_4, ...$ is a J–fraction (up to a change of variables $z \to 1/z$), and then use the classical matrix description of J–fractions, see for instance [13]. In terms of our three term recurrence relation, a contraction means that we eliminate from (7), (8) all polynomials with odd indices: recalling that with E_k, also α_k and β_k are invertible, we have that

$$Q_{2k+1}(z) = Q_{2k+2}(z)\beta_{2k+2}^{-1} - zQ_{2k}(z)\alpha_{2k+2}\beta_{2k+2}^{-1},$$

and thus

$$\begin{aligned} Q_{2k}(z)\beta_{2k+1} &= Q_{2k+1}(z) - zQ_{2k-1}(z)\alpha_{2k+1} \\ &= Q_{2k+2}(z)\beta_{2k+2}^{-1} - zQ_{2k}(z)\alpha_{2k+2}\beta_{2k+2}^{-1} - zQ_{2k}(z)\beta_{2k}^{-1}\alpha_{2k+1} + z^2 Q_{2k-2}(z)\alpha_{2k}\beta_{2k}^{-1}\alpha_{2k+1}. \end{aligned}$$

Injecting formulas (18), (20), and setting

$$\gamma_{k+1} := \beta_{k+1}^{-1} \frac{d_{k+1}}{d_k} = E_{k-1}^{-1} E_k \frac{d_{k-1}}{d_k} \tag{21}$$

we obtain

$$\frac{Q_{2k}(z)}{z^k d_{2k}} = z\Big(\frac{Q_{2k-2}(z)}{z^{k-1} d_{2k-2}} \gamma_{2k}\gamma_{2k+1} - \frac{Q_{2k}(z)}{z^k d_{2k}} (\gamma_{2k+2}\gamma_{2k+1} + \gamma_{2k}\gamma_{2k+1}) + \frac{Q_{2k+2}(z)}{z^{k+1} d_{2k+2}} \gamma_{2k+2}\gamma_{2k+1} \Big),$$

$Q_{-2}(z) = 0$, which may be rewritten as

$$\Big(\frac{Q_0(z)}{z^0 d_0}, ..., \frac{Q_{2k-2}(z)}{z^{k-1}d_{2k-2}}\Big)(I - zJ_k) = \Big(0, ..., 0, \frac{Q_{2k}(z)}{z^{k-1}d_{2k}}\gamma_{2k}\gamma_{2k-1}\Big) \qquad (22)$$

with the block tridiagonal Jacobi matrix

$$J_k = \begin{bmatrix} -\gamma_2\gamma_1 - \gamma_0\gamma_1 & \gamma_2\gamma_3 & 0 & \cdots & \cdots & 0 \\ \gamma_2\gamma_1 & -\gamma_4\gamma_3 - \gamma_2\gamma_3 & \gamma_4\gamma_5 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \gamma_{2k-2}\gamma_{2k-1} \\ 0 & \cdots & \cdots & 0 & \gamma_{2k-2}\gamma_{2k-3} & -\gamma_{2k}\gamma_{2k-1} - \gamma_{2k-2}\gamma_{2k-1} \end{bmatrix}.$$

We get the same recurrence for the numerator polynomials $P_{2k}(z)$ including the case $k = 0$ provided that we choose the initializations

$$P_0(z) = 0, \quad \frac{P_{-2}(z)}{z^{-1}d_{-2}}\gamma_0\gamma_1 = -\frac{P_2(z)}{z^1 d_2}\gamma_2\gamma_1 = -\frac{d_1}{d_2 z}\beta_2\gamma_2\gamma_1 = -\beta_1^{-1}\frac{d_1}{d_0 z} = -\frac{1}{z}F_0$$

(see Example 2.4), and hence the second relation

$$\Big(\frac{P_0(z)}{z^0 d_0}, ..., \frac{P_{2k-2}(z)}{z^{k-1}d_{2k-2}}\Big)(I - zJ_k) = \Big(-F_0, 0, ..., 0, \frac{P_{2k}(z)}{z^{k-1}d_{2k}}\gamma_{2k}\gamma_{2k-1}\Big). \qquad (23)$$

Multiplying (22) on the left by $R_{2k}(z)$ and subtracting (23), we obtain the left–hand side

$$\Big(\frac{R_{2k}(z)Q_0(z) - P_0(z)}{z^0 d_0}, *, ..., *\Big)(I - zJ_k) = R_{2k}(z)(I, *, ..., *)(I - zJ_k).$$

Hence we arrive at the following result.

Corollary 3.1 *With the block tridiagonal Jacobi matrix $J_k \in \mathbb{R}^{(mk)\times(mk)}$ as above, we have for the right matrix Padé approximant of type $[k-1|k]$ of $F(z)$*

$$R_{2k}(z) = \Big(F_0, 0, ..., 0\Big)(I - zJ_k)^{-1}\Big(I, 0, ..., 0\Big)^T$$

□

As a consequence, the poles of the Padé approximant $R_{2k}(z)$ of type $[k-1|k]$ are given by the reciprocal eigenvalues of J_k. Moreover, if $V^{-1}J_kV$ is diagonal, then one may construct from Corollary 3.1 the corresponding residuals: let the rows of $V_{1,j}$ span the left eigenspace of an eigenvalue $1/z_j$, and the columns of $V_{2,j}$ span the right eigenspace, with $V_{1,j}V_{2,j} = I$, then

$$R_{2k}(z) = \sum_j \frac{\rho_j}{1 - z/z_j}, \quad \rho_j = \Big(F_0, 0, ..., 0\Big)V_{2,j}V_{1,j}\Big(I, 0, ..., 0\Big)^T.$$

It is possible to construct $V_{1,j}$ using (22) at $z = z_j$, in particular we may deduce from (22) that, for diagonalizable J_k, the multiplicity of an eigenvalue is between 1 and m. An analog relation for $V_{2,j}$ is obtained by considering denominators of left matrix Padé approximants (leading to the same matrix rational function $R_{2k}(z)$, see [1, Theorem 8.2.1]). However, a numerically more interesting method consists in applying inverse subspace iterations with subspaces of dimension between 1 and m.

4 Application

This section presents results for a numerical example that demonstrates the efficacy of the proposed algorithm.

Our input data are a set of four $N = 127$ long time series, recorded with simple precision arithmetic (7 digits). The data are transformed into integers by rounding the numbers after multiplying with 10^7. All calculations are carried within the integer algebra, and, at the end, the residuals are recovered by dividing with the same factor. The poles of the Padé approximant are not affected by the overall multiplicative scale.

Each channel ℓ ($\ell = 1, 2, 3, 4$) is obtained by sampling with a rate of $N/T = 100$ Hz, over a period of $T = 1.27s$, a characteristic oscillator $\nu^{(\ell)}$ of unit amplitude superimposed on a signal, with frequency ν and decay constant α, that needs to be discovered. Our model also includes noise controlled by the parameter η and the random numbers $u_k^{(\ell)}$ sampled uniformly within $(-1, 1)$. Input time series are therefore obtained as

$$F_k^{(\ell)} = \sin(2\pi\nu^{(\ell)}kT/N) + A^{(\ell)}e^{-\alpha kT/N}\cos(2\pi\nu kT/N) + \eta u_k^{(\ell)}$$

with $k = 0, \ldots, N$ and $\ell = 1, 2, 3, 4$, and different signal amplitudes $A^{(\ell)}$ in each channel.

In all our experiments, the signal has frequency $\nu = 20$ Hz, decay constant $\alpha = 8$ Hz, and amplitudes $A^{(1)} = 0.1$, $A^{(2)} = 0.2$, $A^{(3)} = 0.3$ and $A^{(4)} = 0.4$. The vibration frequencies for each channel are $\nu^{(1)} = 24$ Hz, $\nu^{(2)} = 12$ Hz, $\nu^{(3)} = 32$ Hz and $\nu^{(4)} = 36$ Hz, such that assumption (5) holds.

In our first series of experiments reported in Fig. 1 and Table 1, we have a noise level of $\eta = 0.01$, which of course makes it impossible to recover exactly the desired 5 frequencies by some lower order matrix Padé approximant. We therefore decided to compute a [63|64] matrix Padé approximant $P(z)Q(z)^{-1}$ via the recurrence relations of Theorem 2.3, and determine its partial fraction decomposition as

$$P(z)Q(z)^{-1} = \sum_k \frac{\rho_k}{1 - z/z_k}, \quad \rho_k = -z_k \frac{P(z_k)\mathrm{adj}(Q(z_k))}{q'(z_k)} \in \mathbb{C}^{2\times 2}, \tag{24}$$

with the 128 poles z_k determined directly as roots of the polynomial $q(z) = \det(Q(z))$, that is, we did not use the alternate Jacobi matrix approach of Section 4. For time series longer than the one considered in this example the polynomial root finding procedure becomes unstable and an implementation of the Jacobi matrix method is necessary. In all our experiments, the roots of $q(z)$ have been simple. Only five poles, and their complex conjugate, in the partial fraction expansion (24) have significant residuals ρ_k, each of them representing one oscillator in the input data. The rest of the poles must be associated with noise. In a general time series, the number of oscillators is not known *a priori* and therefore we need to introduce a classification: in our case, we call a pole significant if it corresponding ρ_k matrix has at least one entry of modulus > 0.07. The exact relation between this threshold and the noise level η needs to be subject of a deeper statistical analysis, but this is outside the scope of our paper.

A typical distribution of the 128 poles in the complex plane can be found in Fig. 1, where poles z_k with non-significant ρ_k are represented by small circles, and for the others we display a circle with a quadrant filled according to its position in the matrix. As expected, there are exactly 10 poles with significant ρ_k. The other poles tend to cluster along the unit circle drawn with a dashed line. These poles seem to be related with the noise. It has been observed in [6] than a scalar ($m = 1$) random time series has all its

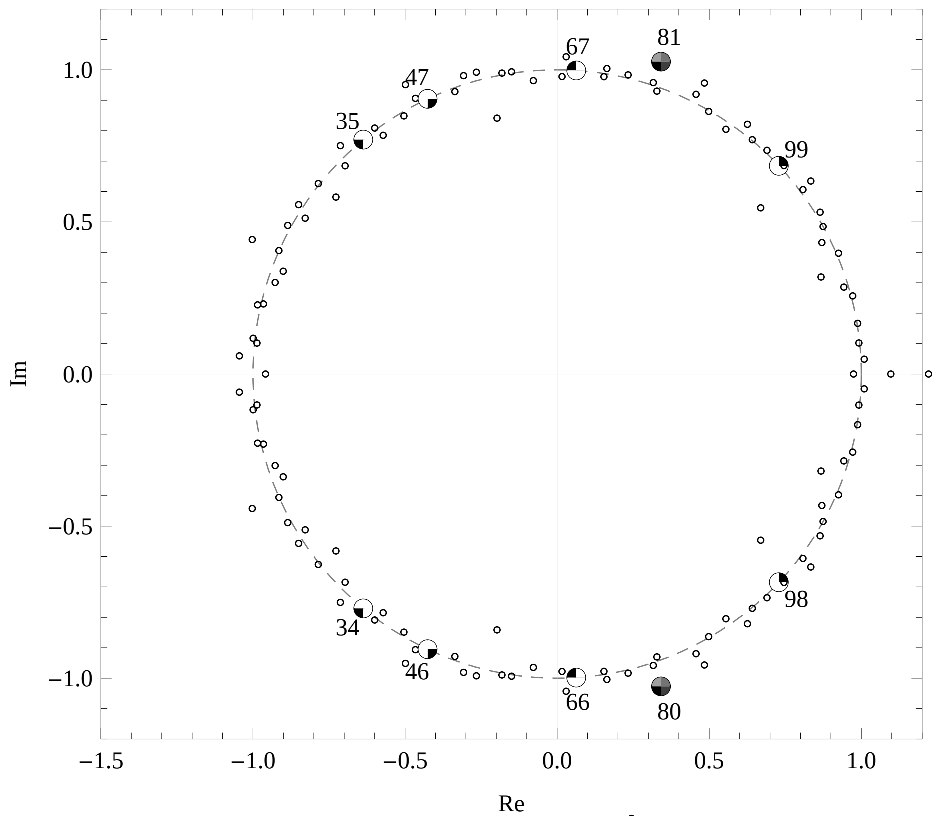

Figure 1: Poles of the right matrix Padé approximant [63|64] with noise level $\eta = 0.01$, which are represented as circles with quadrants filled according to their most significant residual matrix element, and labeled by their index. Poles with no significant element in their residuals are shown as small circles, and the unit circle is drawn with dashed lines.

poles distributed along the unit circle and the poles are paired with zeros of the Padé approximant, forming Froissart doublet. The distance between the pole and the zero in such a pair is proportional to the residual of the pole. Therefore poles with small residuals are always paired in a Froissart doublet. The probability distribution function of such poles is universal [6] because it does not depend on the specific distribution or characteristics of the noise.

More precise information about the quality of approximation of the time series data can be found in Table 1 where we only display poles z_k with significant ρ_k (and not their complex conjugates). According to (3) and (4), the magnitude and the phase of the pole z_k are related to the damping factor α and the frequency ν through $\alpha_k = N \log(|z_k|)/T$ and $\nu_k = N\,|\arg(z_k)|/(2\pi T)$, respectively. The matrix ρ_k associated with each pole z_k gives the amplitudes $A_k = |2\rho_k|$ and phases $\varphi_k = \arg(\rho_k)$, the magnitude and the argument functions being applied componentwise for each antenna.

The results tabulated in Table 1 show that the signal is correctly identified as the poles number 81, and its complex conjugate 80, which have four large elements in the residual-type matrix. The frequencies characteristic to each antenna are correctly identified by an amplitude close to unity in the corresponding position in the matrix and by negligible damping factor. The precision for frequency identification is in general compatible with the level of noise. As expected, the worse results are obtained for the common signal

Table 1: Poles with significant residuals and parameters of their corresponding oscillators

#	ν [Hz]	α [Hz]	A	φ
99	12.0017	-1.03×10^{-2}	$\begin{pmatrix} 3.96 \times 10^{-3} & 0.984 \\ 5.87 \times 10^{-6} & 1.46 \times 10^{-3} \end{pmatrix}$	$\begin{pmatrix} 2.22 & -1.58 \\ 0.541 & 3.02 \end{pmatrix}$
81	19.8911	7.91	$\begin{pmatrix} 0.139 & 0.202 \\ 0.27 & 0.404 \end{pmatrix}$	$\begin{pmatrix} 0.191 & 0.088 \\ 0.147 & 0.044 \end{pmatrix}$
67	24.0000	9.49×10^{-3}	$\begin{pmatrix} 1.01 & 8.51 \times 10^{-3} \\ 1.08 \times 10^{-3} & 9.19 \times 10^{-7} \end{pmatrix}$	$\begin{pmatrix} -1.57 & 0.48 \\ 1.76 & -2.46 \end{pmatrix}$
47	32.0020	-2.39×10^{-3}	$\begin{pmatrix} 2.04 \times 10^{-3} & 4.42 \times 10^{-6} \\ 0.99 & 2.16 \times 10^{-3} \end{pmatrix}$	$\begin{pmatrix} -1.57 & 0.48 \\ 1.76 & -2.46 \end{pmatrix}$
35	35.9987	1.07×10^{-2}	$\begin{pmatrix} 1.59 \times 10^{-6} & 5.84 \times 10^{-3} \\ 2.75 \times 10^{-3} & 1.01 \end{pmatrix}$	$\begin{pmatrix} -0.84 & -2.65 \\ 0.23 & -1.57 \end{pmatrix}$

since it has weak amplitudes and it is overshadowed by the other stronger oscillators in the system. However, any attempt to use traditional time-domain or frequency-domain filtering is likely to fail to detect reliably the common signal because its small amplitudes and because its frequency is within the range of characteristic frequencies.

We finally were interested in a more systematic study of the error in computing the joint frequency ν, the joint damping factor α, and the amplitudes $A^{(\ell)}$ by our approach as a function of the noise level η. Keeping all parameters the same, we repeated the numerical experiments 100 times, for various levels of noise $\eta \in [10^{-7}, 10^{-1}]$. Figure 2 shows that the error in determining the parameters for the oscillator common to all channels decreases linearly with the noise level. The error in finding the oscillator amplitude seems to be insensitive to the noise below a certain level, being significant even for very low noise levels. The cause of this behavior is the subject of further investigations.

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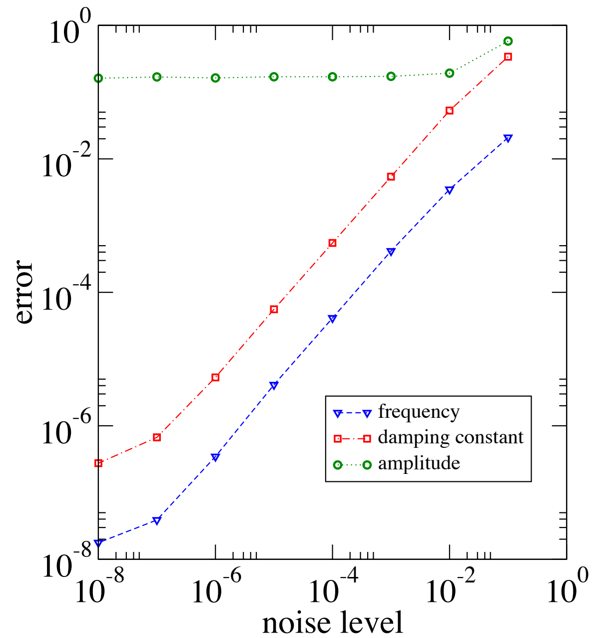

Figure 2: Error in computing the joint frequency ν, the joint damping factor α, and mean error for the the amplitudes $A^{(\ell)}$ for $\ell = 1, 2, 3, 4$, through a matrix Padé approximant of type [63|64] as a function of the noise level $\eta = 10^{-n}$, $n = 1, 2, ..., 7$. Each displayed data is obtained as a mean over 100 random realisations.

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Appendix

A Proof of Theorem 2.3

As in [4, 5], our proof of Theorem 2.3 is essentially based on Cramer's rule for the system of linear equations obtained by expanding $F(z)Q(z) - P(z)$ in powers of z.

More specifically, in order that $P(z)Q(z)^{-1}$ is a right matrix Padé approximant of type $[k-1|k]$ of $F(z)$ with a specific normalization, we need to find p_j, q_j complex matrices of size $m \times m$ obtained from

$$Q(z) = Iz^k + \sum_{j=0}^{k-1} q_j z^j, \quad P(z) = \sum_{j=0}^{k-1} p_j z^j,$$

such that

$$F(z)Q(z) - P(z) = z^{2k}E + \mathcal{O}(z^{2k+1})_{z\to 0},$$

or, equivalently

$$\begin{bmatrix} F_0 & I & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ F_1 & 0 & F_0 & I & 0 & 0 & \cdots & \cdots & 0 & 0 \\ F_2 & 0 & F_1 & 0 & F_0 & I & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & & 0 & 0 & 0 \\ F_{k-1} & 0 & F_{k-2} & 0 & F_{k-3} & \vdots & & F_0 & I & 0 \\ F_k & 0 & F_{k-1} & 0 & F_{k-2} & & & F_1 & 0 & F_0 \\ F_{k+1} & 0 & F_k & 0 & F_{k-1} & & & F_2 & 0 & F_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & & & \\ F_{2k} & 0 & F_{2k-1} & 0 & F_{2k-2} & \cdots & \cdots & F_{k+1} & 0 & F_k \end{bmatrix} \begin{bmatrix} X_{2k} \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ E \end{bmatrix}, \quad X_{2k} = \begin{bmatrix} q_0 \\ p_0 \\ q_1 \\ p_1 \\ \vdots \\ q_{k-1} \\ p_{k-1} \end{bmatrix}.$$

Denoting the matrix of coefficients by C_{2k+1} and the one obtained from C_{2k+1} by dropping the last m rows and columns by C_{2k}, and introducing the partitionning

$$C_{2k+1} = \begin{bmatrix} C_{2k} & A_{2k} \\ B_{2k} & F_k \end{bmatrix},$$

we see that we have to solve the system $C_{2k}X_{2k} = -A_{2k}$ with m right–hand sides, corresponding to the m columns of $Q(z)$ and $P(z)$. We suppose here and in what follows that all matrices C_k are invertible, and hence we find the unique solution

$$X_{2k} = -C_{2k}^{-1}A_{2k}, \quad E = F_k - B_{2k}C_{2k}^{-1}A_{2k} =: C_{2k+1}/C_{2k},$$

that is, we find a representation of E as a Schur complement. A generalization of Sylvester's determinental identity tells us that therefore $\det(E) = \det(C_{2k+1})/\det(C_{2k})$. Comparing with the normalization (19) of Theorem 2.3, we therefore find that $Q_{2k}(z) = d_{2k}Q(z)$, $P_{2k}(z) = d_{2k}P(z)$, implying that

$$E_{2k} = d_{2k}C_{2k+1}/C_{2k}, \quad \det(E_{2k}) = d_{2k}^m \frac{\det(C_{2k+1})}{\det(C_{2k})}. \tag{25}$$

Similarly, in order that $P(z)Q(z)^{-1}$ is a right matrix Padé approximant of type $[k|k]$ of $F(z)$ with a specific normalization, we need to find p_j, q_j complex matrices of size $m \times m$ obtained from

$$Q(z) = \sum_{j=0}^{k} q_j z^j, \quad P(z) = Iz^k + \sum_{j=0}^{k-1} p_j z^j,$$

such that

$$F(z)Q(z) - P(z) = z^{2k+1}E + \mathcal{O}(z^{2k+1})_{z\to 0},$$

or, equivalently,

$$C_{2k+2}\begin{bmatrix}X_{2k+1}\\ I\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ 0\\ E\end{bmatrix}, \quad X_{2k}=\begin{bmatrix}q_0\\ p_0\\ \vdots\\ q_{k-1}\\ p_{k-1}\\ q_k\end{bmatrix}.$$

Introducing the partitionning

$$C_{2k+2}=\begin{bmatrix}C_{2k+1} & A_{2k+1}\\ B_{2k+1} & 0\end{bmatrix},$$

we see that we have to solve the system $C_{2k+1}X_{2k+1}=-A_{2k+1}$ with m right–hand sides, corresponding to the m columns of $Q(z)$ and $P(z)$, with the unique solution

$$X_{2k+1}=-C_{2k+1}^{-1}A_{2k+1}, \quad E=0-B_{2k+1}C_{2k+1}^{-1}A_{2k+1}=:C_{2k+2}/C_{2k+1},$$

As before, we find that $Q_{2k+1}(z)=d_{2k+1}Q(z)$, $P_{2k+1}(z)=d_{2k+1}P(z)$, implying that

$$E_{2k+1}=d_{2k+1}C_{2k+2}/C_{2k+1}, \quad \det(E_{2k+1})=d_{2k+1}^{m}\frac{\det(C_{2k+2})}{\det(C_{2k+1})}. \tag{26}$$

Taking into account that $d_0=1$, the "determinant" of an empty matrix, we conclude from (25) and (26) that $d_k=\det(C_k)$ for all $k=0,...,N$, in particular we get the factorization (17) with integer factors.

Moreover, the coefficients of $P_k(z)$, $Q_k(z)$ different from those specified in (19) can be found in the vector

$$d_kX_k=-\det(C_k)C_k^{-1}A_k.$$

Since the entries of C_k and A_k are integers, we conclude using Cramer's rule that also these coefficients have a determinental representation with help of matrices of the same order as C_k, namely, mk. Hence they are integers, as claimed in Theorem 2.3.

Finally, the above determinental representation together with the Hadamard inequality [11, p. 299] allows to establish the claimed upper bound for the number of digits necessary for representing E_{k-1}, $P_k(z)$ and $Q_k(z)$. Roughly speaking, we may even divide divide this number of digits by 2 since half of the columns of C_k only contains 0 or 1.

B Reconstructing matrix-valued rational functions via matrix Padé

The assumptions of the following statement are true in our application, with $q(z)=(z-z_0)(z-\overline{z_0})$ and $m^{(\ell)}=2+2n^{(\ell)}$.

Lemma B.1 *Consider a matrix-valued rational function*

$$F(z)=\begin{bmatrix}\frac{p^{(1)}(z)}{q(z)q^{(1)}(z)} & \frac{p^{(2)}(z)}{q(z)q^{(2)}(z)}\\ \frac{p^{(3)}(z)}{q(z)q^{(3)}(z)} & \frac{p^{(4)}(z)}{q(z)q^{(4)}(z)}\end{bmatrix}$$

where we suppose that in each component the scalar polynomials $p^{(\ell)}(z)$ and $q(z)q^{(\ell)}(z)$ are coprime, $\deg p^{(\ell)}(z)+1 \leq m^{(\ell)} = \deg(q(z)q^{(\ell)}(z))$, and, in addition, any two of the polynomials $q^{(1)}(z), ..., q^{(4)}(z), q(z)$ have distinct roots, different from zero. Then, for

$$L+1 \geq M \geq \max\{m^{(1)}+m^{(2)}, m^{(1)}+m^{(3)}, m^{(2)}+m^{(4)}, m^{(3)}+m^{(4)}\} - \deg q(z), \quad (27)$$

any (right) matrix Padé approximant of $F(z)$ coincides with $F(z)$.

Proof. Consider the matrix polynomials of order 2

$$P_0(z) = \begin{bmatrix} p^{(1)}(z)q^{(2)}(z) & p^{(2)}(z)q^{(1)}(z) \\ p^{(3)}(z)q^{(4)}(z) & p^{(4)}(z)q^{(3)}(z) \end{bmatrix},$$
$$Q_0(z) = \begin{bmatrix} q(z)q^{(1)}(z)q^{(2)}(z) & 0 \\ 0 & q(z)q^{(3)}(z)q^{(4)}(z) \end{bmatrix},$$

and thus $F(z) = Q_0(z)^{-1}P_0(z)$. Provided that $L+1 \geq M \geq \max\{m^{(1)}+m^{(2)}, m^{(3)}+m^{(4)}\} - \deg q(z)$, the reader easily verifies that $Q_0(z)^{-1}P_0(z)$ is a left matrix Padé approximant of type $[L|M]$ of $F(z)$. Given any right matrix Padé approximant $P(z)Q(z)^{-1}$ of type $[L|M]$ of $F(z)$, we can follow the reasoning of [1, Theorem 8.2.1] and write

$$P_0(z)Q(z) - Q_0(z)P(z) = Q_0(z)(F(z)Q(z) - P(z))$$

with has order $L+M+1$, but is a matrix polynomial of degree at most $L+M$. Hence this expression vanishes, implying that $F(z) = P(z)Q(z)^{-1}$, as claimed above.

We claim in addition that in the case $M < \max\{m^{(1)}+m^{(3)}, m^{(2)}+m^{(4)}\} - \deg q(z)$ a right matrix Padé approximant $P(z)Q(z)^{-1}$ of type $[L|M]$ may not exist. To see this last claim, we have to enter more closely in the theory of matrix polynomials [12] : let

$$P_1(z) = \begin{bmatrix} p^{(1)}(z)q^{(3)}(z) & p^{(2)}(z)q^{(4)}(z) \\ p^{(3)}(z)q^{(1)}(z) & p^{(4)}(z)q^{(2)}(z) \end{bmatrix},$$
$$Q_1(z) = \begin{bmatrix} q(z)q^{(1)}(z)q^{(3)}(z) & 0 \\ 0 & q(z)q^{(2)}(z)q^{(4)}(z) \end{bmatrix},$$

and thus also $F(z) = P_1(z)Q_1(z)^{-1}$. It is possible to check that $P_1(z)$ and $Q_1(z)$ are right-coprime, and that both $Q(z)$ and the stacked matrix $S_1(z)$ defined below are column reduced. As a consequence [12, Section 6.5.4], the columns of $S_1(z)$ form a minimal basis of the right nullspace of $[-I, F(z)]$, in particular there is a matrix polynomial $V(z)$ such that

$$\begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} = S_1(z)V(z), \quad S_1(z) = \begin{bmatrix} P_1(z) \\ Q_1(z) \end{bmatrix}.$$

Notice that, with $Q(z)$, also $V(z)$ must be invertible for almost all z. The predictable degree property [12, Theorem 6.3-13.] implies that at least one column of $Q(z)$ must have degree $\geq \max\{m^{(1)}+m^{(3)}, m^{(2)}+m^{(4)}\} - \deg q(z)$, in contradiction with our choice of M and the degree constraints on $Q(z)$. □

In our application, (27) implies that we require recorded sampling data $F_0, ..., F_N$ with

$$N \geq L+M \geq 2M-1 \geq 3+4\max\{n^{(1)}+n^{(2)}, n^{(1)}+n^{(3)}, n^{(2)}+n^{(4)}, n^{(3)}+n^{(4)}\} \quad (28)$$

in order to reconstruct the generating function $F(z)$, a lower bound larger than the one found in (5).