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Results in Mathematics

Bernstein Polynomials and Operator Theory

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Abstract. Kelisky and Rivlin have proved that the iterates of the Bernstein operator (of fixed order) converge to L, the operator of linear interpolation at the endpoints of the interval $[0, 1]$. In this paper we provide a large class of (not necessarily positive) linear bounded operators T on $C[0, 1]$ for which the iterates T^n converge towards L in the operator norm. The proof uses methods from the spectral theory of linear operators.

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1. Introduction

1.1. Preamble

The Bernstein operator of order m associates to every continuous (real or complex-valued) function f on $[0, 1]$ the m-th Bernstein polynomial

$$B_m f(x) = \sum_{k=0}^{m} f\left(\frac{k}{m}\right) b_{m,k}(x) := \sum_{k=0}^{m} f\left(\frac{k}{m}\right) \binom{m}{k} x^k (1-x)^{m-k} \, .$$

These polynomials were introduced in 1912 in Bernstein's constructive proof of the Weierstrass approximation theorem. Since then they have been the object of multiple investigations, serving many times as a guide for several theorems in Approximation Theory. The Korovkin theorem [1] is a typical example. In this note, starting from several convergence results for the iterates of B_m, we prove the convergence towards L of the iterates of operators from a large class of continuous linear operators acting on $C[0, 1]$. Here L is the operator of linear interpolation at the endpoints of the interval $[0, 1]$. The proof uses spectral theory methods.

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1.2. Notation

Let X be a complex Banach space. We denote by $\mathcal{B}(X)$ the Banach space of all continuous (= bounded) linear operators on X. The Banach space we will work in will be $C[0,1]$, the Banach space of all complex-valued and continuous functions on the interval $[0,1]$, endowed with the supremum norm $\|\cdot\|_\infty$. For $T \in \mathcal{B}(C[0,1])$ we consider the operator norm

$$\|T\| = \sup\{\|Tf\|_\infty : \|f\|_\infty = 1\}.$$

We denote by $e_k \in C[0,1]$ the function $e_k(x) = x^k$ and by $\mathbb{P}_n = \operatorname{span}\{e_0, \ldots, e_n\}$ the space of all complex polynomials of degree less or equal than n.

We denote by I the identity operator, acting on possible different spaces. For $T \in \mathcal{B}(X)$ we denote by $\operatorname{Ker}(T)$ the kernel of T and by $\operatorname{Im}(T) = T(X)$ the range (image) of T. We recall that for $T \in \mathcal{B}(X)$ its *spectrum* $\sigma(T)$ is defined as the set of all complex numbers λ for which $T - \lambda I$ is not invertible in $\mathcal{B}(X)$. For $T \in \mathcal{B}(X)$ we will sometimes write $\sigma_X(T)$ instead of $\sigma(T)$, when we need to emphasize the space where T acts. The *spectral radius* $r(T)$ of T is the quantity

$$r(T) := \sup\big\{|\lambda| : \lambda \in \sigma(T)\big\} = \lim_{n\to\infty} \|T^n\|^{1/n}.$$

Here T^n is the n-th iterate of T, with $T^0 = I$ and $T^1 = T$. We refer for instance to [13] as a basic reference of spectral theory of linear operators.

1.3. Convergence of iterates of the Bernstein operator

The methods employed to study the convergence of iterates of some operators occuring in Approximation Theory include Matrix Theory methods, like stochastic matrices [11, 14, 16, 17], Korovkin-type theorems [1, 10], quantitative results about the approximation of functions by positive linear operators [6, 9], fixed point theorems [2, 8, 15], or methods from the theory of C_0-semigroups, like Trotter's approximation theorem [1, 10, 12].

As a motivation, we mention here the following results for the iterates of the Bernstein operator. Denote by $L \in \mathcal{B}(C[0,1])$ the operator of linear interpolation at the endpoints of the interval $[0,1]$:

$$Lf(x) = f(0) + \big(f(1) - f(0)\big)x = B_1 f(x). \tag{1.1}$$

It was proved by Kelisky and Rivlin [11] that we have, for each fixed $m \in \mathbb{N}$,

$$\lim_{n\to\infty} \|B_m^n f - Lf\|_\infty = 0 \quad \big(f \in C[0,1]\big).$$

In fact (as was firstly proved in [14]), we have convergence of the iterates of B_m towards L in the operator norm :

$$\lim_{n\to\infty} \|B_m^n - L\| = 0. \tag{1.2}$$

According to a result due to Karlin and Ziegler [10], if $(k(n))_{n\geq 1}$ is an increasing sequence of positive integers such that

$$\lim_{n\to\infty} \frac{k(n)}{n} = \infty, \tag{1.3}$$

then

$$\lim_{n\to\infty} \|B_n^{k(n)} - L\| = 0\,. \tag{1.4}$$

1.4. What this paper is about

The aim of the present note is to generalize the above-mentioned convergence results for the iterates of Bernstein polynomials to a large class of continuous linear operators T on $C[0,1]$. The conditions required about T are of approximation-theoretical flavour (degree of exactness, interpolation at endpoints, shape preserving properties), or of spectral nature (closed range condition, spectrum location). A spectral interpretation of condition (1.3) in the Karlin–Ziegler result is also given. The main novelty of the obtained results is that the linear operators are no longer supposed to be positive.

I have attempted to write this note so as to be accessible to both operator theorists ignorant of properties of Bernstein polynomials and approximation theorists unskilled in the nuances of spectral theory. This has sometimes influenced the exposition.

2. Main results

The following result is a generalization of the Kelisky–Rivlin theorem.

Theorem 2.1. *Let $T : C[0,1] \mapsto C[0,1]$ be a continuous linear operator such that*

1. $\mathrm{Ker}(T-I) = \mathrm{span}(e_0, e_1)$ (degree of exactness one)*;*
2. $T(f)(0) = f(0)$ *and* $T(f)(1) = f(1)$ *for every* $f \in C[0,1]$ (interpolation at endpoints)*;*
3. $\mathrm{Im}(T-I)$ *is closed* (closed range condition)*;*
4. $\sigma(T) \subset \mathbb{D} \cup \{1\}$ (spectrum location).

Then $\lim_{k\to\infty} \|T^k - L\| = 0$.

The corresponding version of the Karlin–Ziegler theorem is given by the following statement.

Theorem 2.2. *Let $T_n \in \mathcal{B}(C[0,1])$ be a sequence of continuous linear operators having degree of exactness one, interpolating at the endpoints* 0 *and* 1*, with all ranges* $\mathrm{Im}(T_n - I)$ *closed and verifying the spectral inclusion* $\sigma(T_n) \subset \mathbb{D} \cup \{1\}$*. Denote*

$$\gamma_n = \sup\left\{|\lambda| : \lambda \in \sigma(T_n) \setminus \{1\}\right\}.$$

Then $\gamma_n < 1$ and, if $k(n) \in \mathbb{N}$, with $k(1) < k(2) < k(3)\ldots$, and $k(n)(1-\gamma_n) \to \infty$ as $n \to \infty$, then $\lim_{n\to\infty} \|T_n^{k(n)} - L\| = 0$.

It is well known that for $T = B_n$, the n-th Bernstein polynomial, the first two conditions (degree of exactness one and interpolation at endpoints) are verified. The range of $T - I$ is closed since T is a compact operator (even a finite rank operator). We discuss the spectral condition $\sigma(T) \subset \mathbb{D} \cup \{1\}$ in the following lemma.

Lemma 2.3.
(a) *The spectrum of* $B_n : \mathbb{P}_n \mapsto \mathbb{P}_n$ *is given by*

$$\sigma_{\mathbb{P}_n}(B_n) = \left\{ \frac{n!}{(n-k)!} \frac{1}{n^k} : 0 \le k \le n \right\}.$$

(b) *The spectrum of* $B_n : C[0,1] \mapsto C[0,1]$ *is given by*

$$\sigma_{C[0,1]}(B_n) = \{0\} \cup \left\{ \frac{n!}{(n-k)!} \frac{1}{n^k} : 0 \le k \le n \right\}.$$

Thus $\sigma_{C[0,1]}(B_n) \subset \mathbb{D} \cup \{1\}$ *and*

$$\gamma_n = \sup \big\{ |\lambda| : \lambda \in \sigma(B_n) \setminus \{1\} \big\} = 1 - \frac{1}{n}.$$

Proof. The $(n+1)$ eigenvalues of B_n, acting on the finite dimensional space $\mathbb{P}_n$, were computed in [4]; see also [3,5]. For the spectrum computation in (b), note that B_n is a compact (even finite rank) operator acting on the infinite dimensional Banach space $C[0,1]$. Therefore $0 \in \sigma(B_n)$ and every non-zero point of the spectrum is an eigenvalue. If $\lambda \neq 0$ is an eigenvalue of B_n (viewed as an operator on $C[0,1]$), then $B_n g = \lambda g$ with $g \in C[0,1]$, $g \neq 0$. Since $B_n g$ is a polynomial of degree at most n and $\lambda \neq 0$, we have $g \in \mathbb{P}_n$. Thus λ is a non-zero eigenvalue of B_n, viewed as an operator on $\mathbb{P}_n$. We apply (a), and the formula for the "spectral gap" γ_n follows. In particular, the spectrum of $B_n \in \mathcal{B}(C[0,1])$ is included in $\mathbb{D} \cup \{1\}$. $\square$

Note that the condition $k(n)(1-\gamma_n) \to \infty$ of Theorem 2.2 reduces to the condition (1.3) of Karlin and Ziegler.

Remark 2.4.
(a) If T is positive, then the first condition of Theorem 2.1 (degree of exactness one) implies the second one (interpolation at endpoints). This follows for instance from [7, Theorem 3.1].
(b) If T is compact, or even a Fredholm operator, then the closed range condition is satisfied. Also, it is known (see for instance [18]) that if T is a compact operator with spectrum in $\mathbb{D} \cup \{1\}$, then the sequence of iterates (T^n) converges in the operator norm. Also, it can be proved that if T verifies $\sup_n \|T^n\| < \infty$, $\mathrm{Im}(T-I)$ is closed and $\sigma(T) \subset \mathbb{D} \cup \{1\}$, then (T^n) converges in the operator norm.

Corollary 2.5. *Let* $T_n : C[0,1] \mapsto C[0,1]$ *be a sequence of linear operators such that*

(i) $T_n f \in \mathbb{P}_n$ *for all* $f \in C[0,1]$ (polynomial operators)*;*
(ii) $\mathrm{Ker}(T_n - I) = \mathbb{P}_1$ (degree of exactness one)*;*
(iii) $[T_n(f)]^{(j)} \ge 0$ *if* $f \in C^j[0,1]$, $f^{(j)} \ge 0$, $j = 0, 1, \ldots, n$ (shape preserving).

Let $(k(n))_{n \ge 1}$ *be an increasing sequence of positive integers with* $\lim_{n\to\infty} \frac{k(n)}{n} = \infty$. *Then* $\lim_{n\to\infty} \|T_n^{k(n)} - L\| = 0$.

Proof. The condition (iii) implies in particular the positivity of T_n and thus (using the Remark (2.4a)), $T_n f(0) = f(0)$ and $T_n f(1) = f(1)$. Because of the condition (i), T_n is a finite rank operator and thus the range of $T_n - I$ is closed. The condition $\sigma(T_n) \subset \mathbb{D} \cup \{1\}$ follows from [3], as well as the inequality $\gamma_n \leq 1 - \frac{1}{n}$. Therefore

$$k(n)(1-\gamma_n) \geq \frac{k(n)}{n} \to +\infty$$

and we apply Theorem 2.2. □

3. The Proofs

There is a common part to the proofs of Theorems 2.1 and 2.2. Recall that $\mathbb{P}_1 = \operatorname{span}\{e_0, e_1\}$ is the space of all linear functions. Denote

$$C_{0,1} = \big\{g \in C[0,1] : g(0) = 0, g(1) = 0\big\}$$

the closed subspace of $C[0,1]$ of all functions vanishing at the endpoints of the interval.

Step 1. The decomposition of the space. Using the above notation, we have the following decomposition of the space as a topological direct sum

$$C[0,1] = \mathbb{P}_1 \oplus C_{0,1}\,. \tag{3.1}$$

Indeed, for every complex-valued continuous function $f \in C[0,1]$ we have

$$f = Lf + f - Lf\,,$$

with $Lf \in \mathbb{P}_1 = \operatorname{Ker}(T-I)$ and $f - Lf \in C_{0,1}$. Also, if $g \in \mathbb{P}_1 \cap C_{0,1}$, then g is a linear function vanishing at both endpoints. Therefore $\mathbb{P}_1 \cap C_{0,1} = \{0\}$.

Since T preserves linear functions we have $T(\mathbb{P}_1) \subset \mathbb{P}_1$. Because of the interpolation at endpoints, we also have $T(C_{0,1}) \subset C_{0,1}$. Hence, with respect to the decomposition (3.1), the operator T has the following matrix decomposition:

$$T = \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix} \in \mathcal{B}(\mathbb{P}_1 \oplus C_{0,1})\,. \tag{3.2}$$

Indeed, T acts as identity on the space of linear functions. Note also that in the above decomposition, $L = T\,|_{\mathbb{P}_1}$ is a projection ($L^2 = L$) with $\operatorname{Im}(L) = \mathbb{P}_1 = \operatorname{Ker}(T-I)$ and $\operatorname{Ker}(L) = C_{0,1}$.

Step 2. Properties of A from properties of T. The above matrix decomposition for T induces several properties of A, the restriction of T to the space $C_{0,1}$. It is not difficult to see that the spectra of the two operators are related by

$$\sigma(T) = \{1\} \cup \sigma(A)\,. \tag{3.3}$$

Using the spectrum location condition we obtain that $\sigma(A) \subset \mathbb{D} \cup \{1\}$.

We now show that $\operatorname{Im}(A-I)$ is closed. Indeed, suppose that g_k and g are functions from $C_{0,1}$ verifying $\|(A-I)g_k - g\|_\infty \to 0$, as k tends to infinity. As

$$T - I = \begin{bmatrix} 0 & 0 \\ 0 & A-I \end{bmatrix},$$

and the range of $T-I$ is closed, we get $g \in C_{0,1} \cap \mathrm{Im}(T-I) = \mathrm{Im}(A-I)$. Therefore $\mathrm{Im}(A-I)$ is closed.

The operator $A - I$ is also one-to-one, since $(A-I)g = 0$ with $g \in C_{0,1}$ implies $g \in \mathrm{Ker}(T-I) \cap C_{0,1} = \mathbb{P}_1 \cap C_{0,1} = \{0\}$.

Consider now $S : C[0,1] \mapsto \mathrm{Im}(A-I)$, $S = A - I$. Then $\mathrm{Im}(A-I) \subset C_{0,1} \subset C[0,1]$ is complete, as a closed set of a Banach space. Also S is onto, and thus an isomorphism since A is continuous. According to the Banach isomorphism theorem the inverse operator S^{-1} is continuous. We obtain

$$\|(A-I)g\|_\infty \geq \delta \|g\|_\infty \quad (g \in C_{0,1}) \tag{3.4}$$

with $\delta = \frac{1}{\|S^{-1}\|}$.

Step 3. The spectrum of A does not contain 1. Suppose to the contrary that $1 \in \sigma(A)$. Then $A - I$ is not invertible, but it is injective and with closed range. Thus $\mathrm{Im}(A-I)$ is not dense in $C_{0,1}$. We obtain the existence of a function $h \in C_{0,1}$ and of a constant $\varepsilon > 0$ such that

$$\|h - (A-I)g\|_\infty \geq \varepsilon \quad (g \in C_{0,1})\,. \tag{3.5}$$

On the other hand, the point 1 is in the boundary of $\sigma(A)$ and thus there exists a sequence $(\lambda_n)_{n\geq 1}$ in $\mathbb{C}$ with $|\lambda_n - 1| \to 0$ and $A - \lambda_n I$ invertible. Let

$$f_n = \frac{1}{\|(A-\lambda_n I)^{-1}h\|_\infty}(A-\lambda_n I)^{-1}h \quad (n \geq 1)\,.$$

Then $f_n \in C_{0,1}$, $\|f_n\|_\infty = 1$ and

$$\begin{aligned}
|\lambda_n - 1| &= \|(A-\lambda_n I)f_n - (A-I)f_n\|_\infty \\
&= \left\| \frac{1}{\|(A-\lambda_n I)^{-1}h\|_\infty} h - (A-I)f_n \right\|_\infty \\
&= \frac{1}{\|(A-\lambda_n I)^{-1}h\|_\infty} \left\| h - (A-I)\left(\|(A-\lambda_n I)^{-1}h\|_\infty f_n\right)\right\|_\infty \\
&\geq \frac{\varepsilon}{\|(A-\lambda_n I)^{-1}h\|_\infty} \quad \text{(using (3.5))}.
\end{aligned}$$

Hence

$$\lim_{n\to\infty} \|(A-\lambda_n I)^{-1}h\|_\infty = +\infty\,.$$

Using condition (3.4) we have

$$\begin{aligned}
\delta &\leq \|(A-I)f_n\|_\infty && (3.6)\\
&\leq \|(A-\lambda_n I)f_n - (A-I)f_n\|_\infty + \|(A-\lambda_n I)f_n\|_\infty && (3.7)\\
&\leq |\lambda_n - 1| + \frac{\|h\|}{\|(A-\lambda_n I)^{-1}h\|_\infty}\,. && (3.8)
\end{aligned}$$

As the last two terms tend to zero, we get a contradiction[1]. Therefore $1 \notin \sigma(A)$.

[1] This is a general argument in spectral theory, saying that the boundary of the spectrum is in the approximate point spectrum.

Proof of Theorem 2.1. Putting all things together, we obtain that the spectral radius of A is smaller than 1 and thus the iterates A^n of A converge to 0 in the operator norm as n tends to infinity. From the decomposition of T we get that T^n converges to L, the projection onto $\mathrm{Ker}(T-I)=\mathrm{span}(e_0,e_1)$. □

Proof of Theorem 2.2. Let (T_n) be a sequence of bounded linear operators satisfying the conditions of the theorem. With respect to the same decomposition $C[0,1]=\mathbb{P}_1\oplus C_{0,1}$ we have

$$T_n=\begin{bmatrix} I & 0 \\ 0 & A_n \end{bmatrix}\in\mathcal{B}(\mathbb{P}_1\oplus C_{0,1})\,.$$

Let $n\in\mathbb{N}$. As above, we have $\sigma(A_n)\subset\mathbb{D}$ and thus $r(A_n)=\gamma_n<1$. Since $r(A_n)=\lim_{m\to\infty}\|A_n^m\|^{1/m}$, we obtain that $\|A_n^m\|\leq C\gamma_n^m$, for every $m,n\in\mathbb{N}$, and with a suitable positive constant C. Therefore

$$\|T_n^{k(n)}-L\|=\|A_n^{k(n)}\|\leq C\gamma_n^{k(n)}=C\exp\left\{k(n)\log\left[1-(1-\gamma_n)\right]\right\}.$$

Using the condition $k(n)(1-\gamma_n)\to\infty$ we obtain the desired result. □

Remark 3.1.

(a) The above proof gives the known estimate

$$\|B_n^{k(n)}-L\|\leq C\left(1-\frac{1}{n}\right)^{k(n)}$$

for the Bernstein polynomials.

(b) The operator-theoretical method described here is also useful for related problems. For instance, a result similar to Theorem 2.1 can be given for operators preserving only the constant functions. In this case the limit of the iterates is of the form $f\mapsto\mu(f)e_0$, for a suitable continuous linear functional μ on $C[0,1]$. This is a generalization of [8, Theorem 2], for instance. We plan to return to this topic in the future.

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