Associated schemes and vertex algebras

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Abstract. These notes are written in preparation for the mini-course entitled "Associated schemes and vertex algebras" which will take place in the summer school "Current Topics in the Theory of Algebraic Groups" of the GDR TLAG.

Contents

Introduction

The goal of this series of lectures is to introduce the theory of vertex algebras, with emphasis on their geometrical aspects.

Roughly speaking, a vertex algebra is a vector space V, endowed with a distinguished vector, the vacuum vector, and the vertex operator map from V to the space of formal Laurent series with linear operators on V as coefficients. These data satisfy a number of axioms. Although the definition is purely algebraic, these axioms have deep geometric meaning. They reflect the fact that vertex algebras give an algebraic framework of the two-dimensional conformal field theory. The connections of this topic with other branches of mathematics and physics include algebraic geometry (moduli spaces), representation theory (modular representation theory, geometric Langlands correspondence), two dimensional conformal field theory, string theory (mirror symmetry) and four dimensional gauge theory (AGT conjecture).

To each vertex algebra V one can naturally attach a certain Poisson variety X_V called the associated variety of V. For an affine Poisson variety X, a vertex algebra V such that $X_V \cong X$ is called a *chiral quantization* of X.

A vertex algebra V is called lisse if $\dim X_V = 0$. Lisse vertex algebras are natural generalizations of finite-dimensional algebras and possess remarkable properties. For instance, the characters of simple V-modules form vector valued modular functions. More generally, vertex algebras whose associated variety has only finitely symplectic leaves, are also of great interest for several reasons that will be addressed in the lectures.

Important examples of vertex algebras are those coming from affine Kac-Moody algebra, which are called affine vertex algebras. They play a crucial role in the representation theory of affine Kac-Moody algebras, and of *W-algebras*. In the case that V is a simple affine vertex algebra, its associated variety is an invariant and conic subvariety of the corresponding simple Lie algebra. It plays an analog role to the associated variety of primitive ideals of the enveloping algebra of simple Lie algebras. However, associated varieties of affine vertex algebras are not necessarily contained in the nilpotent cone and it is difficult to describe them in general.

Associated varieties not only capture some of the important properties of vertex algebras but also have interesting relationship with the Higgs branches of four-dimensional $N = 2$ superconformal field theories (SCFTs). However, their general description is fairly open, except in a few cases.

It is only quite recently that the study of associated varieties of vertex algebras and their arc spaces, has been more intensively developed. In this mini-course I wish to highlight this aspect of the theory of vertex algebras which seems to be very promising. In particular, I will include open problems on associated varieties in the setting of affine vertex algebras (vertex algebras associated with Kac-Moody algebras) and W-algebras (they are certain vertex algebras attached with nilpotent elements of a simple Lie algebra) raised by my recent works with Tomoyuki Arakawa.

PART 1

Affine Kac-Moody algebras and their vacuum representations

References: [**Kac1**, **Moody-Pianzola**].

The goal of this lecture is to introduce the *universal vacuum representation* $V^k(\mathfrak{g})$ attached to an affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ and a complex number k. We will see next lecture that it has a natural vertex algebra structure. We also attach to *any* quotient $V^k(\mathfrak{g})$ a certain Poisson algebra and a corresponding Poisson variety.

1.1. Affine Kac-Moody algebras

1.1.1. Simple Lie algebras. Let $\mathfrak{g}$ be a complex simple Lie algebra, that is, the only ideals of $\mathfrak{g}$ are $\{0\}$ or $\mathfrak{g}$ and $\dim \mathfrak{g} \geqslant 3$. Hence $\mathfrak{g}$ is the Lie algebra of a certain linear algebraic group $\mathbf{G}$, $\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$.

Let $(\ |\ )_{Kil}$ be the *Killing form* of $\mathfrak{g}$,

$$(\ |\ )_{Kil} \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}, \quad (x, y) \mapsto \mathrm{tr}(\mathrm{ad}\, x\ \mathrm{ad}\, y).$$

It is a nondegenerate symmetric bilinear form of $\mathfrak{g}$ which is $\mathbf{G}$-invariant, that is,

$$(g.x|g.y)_{Kil}) = (x|y)_{Kil} \quad \text{for all } x, y \in \mathfrak{g},\ g \in \mathbf{G},$$

or else,

$$([x, y]|z)_{Kil} = (x|[y, z])_{Kil} \quad \text{for all } x, y, z \in \mathfrak{g}.$$

Since $\mathfrak{g}$ is semisimple, any other such bilinear form is a nonzero multiple of the Killing form.

Example 1.1. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}_n$, $n \geqslant 2$, which is the set of traceless complex n-size square matrices, with bracket $[A, B] = AB - BA$. The Lie algebra $\mathfrak{sl}_n$ is known to be simple and its Killing form is given by

$$(A, B) \mapsto 2n\, \mathrm{tr}(AB).$$

1.1.2. Dual Coxeter number. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha, \qquad \mathfrak{g}_\alpha := \{y \in \mathfrak{g} \mid [x, y] = \alpha(x)y \text{ for all } x \in \mathfrak{h}\},$$

be the corresponding root decomposition of $(\mathfrak{g}, \mathfrak{h})$, where Δ is the root system of $(\mathfrak{g}, \mathfrak{h})$. Let $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ be a basis of Δ, with r the *rank* of $\mathfrak{g}$, and let $\alpha_1^\vee, \ldots, \alpha_r^\vee$ be the coroots of $\alpha_1, \ldots, \alpha_r$ respectively. The element $\alpha_i^\vee$, $i = 1, \ldots, r$, viewed as an element of $(\mathfrak{h}^*)^* \cong \mathfrak{h}$, will be often denoted it by h_i. Let Δ_+ be the set of positive roots corresponding to Π, and let

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \tag{1}$$

be the corresponding triangular decomposition. Thus $\mathfrak{n}_+ = \bigoplus_{\alpha\in\Delta_+}\mathfrak{g}_\alpha$ and $\mathfrak{n}_- = \bigoplus_{\alpha\in\Delta_-}\mathfrak{g}_\alpha$ are both nilpotent Lie subalgebras of $\mathfrak{g}$.

Any positive root $\alpha \in \Delta_+$ can be written as

$$\alpha = \sum_{i=1}^{r} n_i\alpha_i, \qquad n_i \in \mathbb{Z}_{\geq 0}.$$

The *height* of α is $\mathrm{ht}(\alpha) = \sum_{i=1}^{r} n_i$. The *Coxeter number* of $\mathfrak{g}$ is

$$h := \mathrm{ht}(\theta) + 1,$$

where θ is the highest positive root of Δ, that is, the unique positive root $\theta \in \Delta_+$ such that $\theta + \alpha_i \notin \Delta \cup \{0\}$ for $i = 1, \dots, r$. Similarly, we define the *dual Coxeter number* $h^\vee$ of $\mathfrak{g}$ by:

$$h^\vee = \mathrm{ht}(\theta^\vee) + 1.$$

For example, if $\mathfrak{g} = \mathfrak{sl}_n$ then $h = h^\vee = n$.

We define the normalized bilinear form $(\ |\ )$ on $\mathfrak{g}$ by:

$$(\ |\ ) = \frac{1}{2h^\vee}(\ |\ )_{Kil}.$$

Thus with respect to the induced bilinear form on $\mathfrak{h}^*$, $(\theta|\theta) = 2$.

1.1.3. The loop algebra. Consider the *loop algebra of* $\mathfrak{g}$ which is the Lie algebra

$$\mathcal{L}\mathfrak{g} := \mathfrak{g}[t, t^{-1}] = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}],$$

with commutation relations

$$[xt^m, yt^n] = [x, y]t^{m+n}, \qquad x, y \in \mathfrak{g},\ m, n \in \mathbb{Z},$$

where xt^m stands for $x \otimes t^m$.

Remark 1.2. The Lie algebra $\mathcal{L}\mathfrak{g}$ is the Lie algebra of polynomial functions from the unit circle to $\mathfrak{g}$. This is the reason why it is called the loop algebra.

Definition 1.3. We define a bilinear map ν on $\mathcal{L}\mathfrak{g}$ by setting:

$$\nu(x \otimes f, y \otimes g) := (x|y)\mathrm{Res}_{t=0}(\frac{df}{dt}g),$$

for $x, y \in \mathfrak{g}$ and $f, g \in \mathbb{C}[t, t^{-1}]$, where the linear map $\mathrm{Res}_{t=0}\colon \mathbb{C}[t, t^{-1}] \to \mathbb{C}$ is defined by $\mathrm{Res}_{t=0}(t^m) = \delta_{m,-1}$ for $m \in \mathbb{Z}$.

The bilinear ν is a *2-cocycle on* $\mathcal{L}\mathfrak{g}$, that is, for any $a, b, c \in \mathcal{L}\mathfrak{g}$,

$$\nu(a, b) = -\nu(b, a), \tag{2}$$

$$\nu([a, b], c) + \nu([b, c], a) + \nu([c, a], b) = 0. \tag{3}$$

1.1.4. Affine Kac-Moody algebras.

Definition 1.4. We define the *affine Kac-Moody algebra* $\widehat{\mathfrak{g}}$ as the vector space $\widehat{\mathfrak{g}} := \mathcal{L}\mathfrak{g} \oplus \mathbb{C}K$, with the commutation relations $[K, \widehat{\mathfrak{g}}] = 0$ (so K is a *central element*), and

$$[x \otimes f, y \otimes g] = [x, y]_{\mathcal{L}\mathfrak{g}} + \nu(x \otimes f, y \otimes g)K, \quad x, y \in \mathfrak{g},\ f, g \in \mathbb{C}[t, t^{-1}], \tag{4}$$

where $[\ ,\ ]_{\mathcal{L}\mathfrak{g}}$ is the Lie bracket on $\mathcal{L}\mathfrak{g}$. In other words the commutation relations are given by:

$$[xt^m, yt^n] = [x,y]t^{m+n} + m\delta_{m+n,0}(x|y)K,$$
$$[K, \widehat{\mathfrak{g}}] = 0,$$

for $x, y \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$.

Exercise 1.5. Verify that the identifies (2) and (3) are true, and then that the above commutation relations indeed define a Lie bracket on $\widehat{\mathfrak{g}}$.

1.1.5. Triangular decomposition. Recall the triangular decomposition (1) of $\mathfrak{g}$, and consider the following subspaces of $\widehat{\mathfrak{g}}$:

$$\widehat{\mathfrak{n}}_+ := (\mathfrak{n}_- \oplus \mathfrak{h}) \otimes t\mathbb{C}[t] \oplus \mathfrak{n}_+ \otimes \mathbb{C}[t] = \mathfrak{n}_+ + t\mathfrak{g}[t],$$
$$\widehat{\mathfrak{n}}_- := (\mathfrak{n}_+ \oplus \mathfrak{h}) \otimes t^{-1}\mathbb{C}[t^{-1}] \oplus \mathfrak{n}_- \otimes \mathbb{C}[t^{-1}] = \mathfrak{n}_- + t^{-1}\mathfrak{g}[t^{-1}],$$
$$\widehat{\mathfrak{h}} := (\mathfrak{h} \otimes 1) \oplus \mathbb{C}K = \mathfrak{h} + \mathbb{C}K.$$

They are Lie subalgebras of $\widehat{\mathfrak{g}}$ and we have

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+. \tag{5}$$

1.2. The vacuum representations

Consider the Lie subalgebra $\mathfrak{g}[t] \oplus \mathbb{C}K$ of $\widehat{\mathfrak{g}}$. It is a parabolic subalgebra of $\mathfrak{g}$ since it contains the Borel subalgebra $\widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+$.

1.2.1. The universal vacuum representation. Fix $k \in \mathbb{C}$, and consider the one-dimensional representation $\mathbb{C}_k$ $\mathfrak{g}[t] \oplus \mathbb{C}K$ on which $\mathfrak{g}[t]$ acts by 0 and K acts as a multiplication by the scalar k.

We define the *universal vacuum representation of level k of $\widehat{\mathfrak{g}}$* as the representation induced from $\mathbb{C}_k$:

$$V^k(\widehat{\mathfrak{g}}) = \mathrm{Ind}_{\mathfrak{g}[t]\oplus\mathbb{C}K}^{\widehat{\mathfrak{g}}}\mathbb{C}_k = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}K)} \mathbb{C}_k. \tag{6}$$

It can be viewed as a generalized Verma module.

1.2.2. Level of the vacuum representation. The representation $V^k(\mathfrak{g})$ is a highest weight representation of $\widehat{\mathfrak{g}}$ with highest weight $k\Lambda_0$, with Λ_0 is the highest weight of the basic representation[1], and highest weight vector v_k, where v_k denotes the image of $1 \otimes 1$ in $V^k(\mathfrak{g})$. We will often denote by $|0\rangle$ the vector v_k (the notation will be justified next part when we will endow $V^k(\mathfrak{g})$ with a vertex algebra structure).

According to the well-known Schur Lemma, any central element of a Lie algebra acts as a scalar on a simple finite dimensional representation L. As the Schur Lemma extends to a representation with countable dimension, the result holds for highest weight $\widehat{\mathfrak{g}}$-modules.

Definition 1.6. A representation M is said to be *of level k* if K acts as $k\mathrm{Id}$ on M.

Then $V^k(\mathfrak{g})$ is by construction of level k.

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[1] that is, the dual of K in $\widehat{\mathfrak{h}}^*$ with respect to a basis of $\widehat{\mathfrak{h}}$ adapted to the decomposition $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K$.

1.2.3. PBW basis and grading. By the Poincaré-Birkhoff-Witt Theorem, the direct sum decomposition (as a vector space)

$$\widehat{\mathfrak{g}} = (\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]) \oplus (\mathfrak{g}[t] \oplus \mathbb{C}K)$$

gives us the isomorphism of vector spaces

$$U(\widehat{\mathfrak{g}}) \cong U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]) \otimes U(\mathfrak{g}[t] \oplus \mathbb{C}K),$$

whence

$$V^k(\widehat{\mathfrak{g}}) \cong U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]).$$

Let $\{x^1, \dots, x^d\}$, $d = \dim \mathfrak{g}$, be an ordered basis of $\mathfrak{g}$. For any $x \in \mathfrak{g}$ and $n \in \mathbb{Z}$, set

$$x_{(n)} := x \otimes t^n = xt^n \in \mathcal{L}(\mathfrak{g}).$$

Then $\{K, x^i_{(n)}, i = 1, \dots, d, n \in \mathbb{Z}\}$ forms a basis of $\widehat{\mathfrak{g}}$ and $\{K, x^i_{(n)}, i = 1, \dots, d, n \in \mathbb{Z}_{\geqslant 0}\}$ forms a basis of $\mathfrak{g}[t] \oplus \mathbb{C}K$. By the PBW Theorem, $V^k(\mathfrak{g})$ has a PBW basis of monomials of the form

$$x^{i_1}_{(n_1)} \dots x^{i_m}_{(n_m)}|0\rangle,$$

where $n_1 \leqslant n_2 \leqslant \dots \leqslant n_m < 0$, and if $n_j = n_{j+1}$, then $i_j \leqslant i_{j+1}$.

The space $V^k(\mathfrak{g})$ is naturally graded, $V^k(\mathfrak{g}) = \bigoplus_{\Delta \in \mathbb{Z}_{\geqslant 0}} V^k(\mathfrak{g})_\Delta$, where the grading is defined by

$$\deg x^{i_1}_{(n_1)} \dots x^{i_m}_{(n_m)}|0\rangle = -\sum_{i=1}^{m} n_i, \qquad \deg |0\rangle = 0.$$

We have $V^k(\mathfrak{g})_0 = \mathbb{C}|0\rangle$, and we identify $\mathfrak{g}$ with $V^k(\mathfrak{g})_1$ via the linear isomorphism defined by $x \mapsto xt^{-1}|0\rangle$.

Any graded quotient V of $V^k(\mathfrak{g})$ (i.e., a quotient by a proper submodule of $V^k(\mathfrak{g})$) is again a highest weight representation of $\widehat{\mathfrak{g}}$ with highest weight $k\Lambda_0$, and of level k. In particular, $V^k(\mathfrak{g})$ has a unique maximal proper graded submodule N_k and so

$$L_k(\mathfrak{g}) := V^k(\mathfrak{g})/N_k$$

is an irreducible highest weight representation of $\widehat{\mathfrak{g}}$ with highest weight $k\Lambda_0$, and of level k. Note that, as a $\widehat{\mathfrak{g}}$-representation, we have

$$L_k(\mathfrak{g}) \cong L(k\Lambda_0),$$

where for $\lambda \in \widehat{\mathfrak{h}}^*$, $L(\lambda)$ denotes the highest weight representation of $\widehat{\mathfrak{g}}$ of highest weight λ.

Note that $L_k(\mathfrak{g})$ is of level k too.

1.3. Associated varieties of vacuum representations

Recall that the *nilpotent cone of* $\mathfrak{g}$ is the (reduced) subscheme of $\mathfrak{g}^*$ associated with the augmentation ideal $\mathbb{C}[\mathfrak{g}^*]^{\mathbf{G}}_+$ of the ring of invariants $\mathbb{C}[\mathfrak{g}^*]^{\mathbf{G}}$.

We will see that $V^k(\mathfrak{g})$ plays the analogue of the enveloping algebra of $\mathfrak{g}$ for the representation theory. Because of this, it would be nice to have analogs of the associated varieties of primitive ideals in this context. Unfortunately, one cannot expect exactly the same theory. One of the main reason, is that the center of $U(\widehat{\mathfrak{g}})$ is trivial (unless for the *critical level* $k = -h^\vee$), and so we do not have analog of the *nilpotent cone* (for the critical level, the analog is played by the arc space of the nilpotent cone). However, what is very interesting with primitive ideals is that their associated variety in contained in the nilpotent cone (cf. §1.3.3).

To encounter this problem, we consider the associated variety of a certain Poisson algebra, this is ous next purpose.

1.3.1. Poisson algebras. Recall that $\mathbb{C}[\mathfrak{g}^*]$ has naturally a Poisson structure induced from the Kirillov-Kostant-Souriau Poisson structure on $\mathfrak{g}^*$. Namely, for $f, g \in \mathbb{C}[\mathfrak{g}^*]$, $x \in \mathfrak{g}^*$,

$$\{f, g\}(x) = \langle x, [d_x f, d_x g]\rangle,$$

where $d_x f, d_x g$ are the differentials of f, g at $x \in \mathfrak{g}^*$. They are elements of $(\mathfrak{g}^*)^* \cong \mathfrak{g}$. In particular, for $f, g \in \mathfrak{g} = (\mathfrak{g}^*)^* \subset \mathbb{C}[\mathfrak{g}^*]$, $\{f, g\} = [f, g]$.

Set

$$R_{V^k(\mathfrak{g})} = V^k(\mathfrak{g})/t^{-2}\mathfrak{g}[t^{-1}]V^k(\mathfrak{g}).$$

We an algebra isomorphism

$$\begin{aligned} \mathbb{C}[\mathfrak{g}^*] &\xrightarrow{\simeq} R_{V^k(\mathfrak{g})} = V^k(\mathfrak{g})/t^{-2}\mathfrak{g}[t^{-1}]V^k(\mathfrak{g}) \\ x_1 \dots x_m &\longmapsto (x_1 t^{-1}) \dots (x_m t^{-1})|0\rangle + t^{-2}\mathfrak{g}[t^{-1}]V^k(\mathfrak{g}) \quad (x_i \in \mathfrak{g}). \end{aligned} \tag{7}$$

Thus $R_{V^k(\mathfrak{g})}$ inherits a Poisson structure given by

$$\{\overline{a_{(-1)}|0\rangle}, \overline{b_{(-1)}|0\rangle}\} = \overline{a_{(0)}b_{(-1)}|0\rangle} = \overline{[a,b]_{(-1)}|0\rangle}, \qquad a, b \in \mathfrak{g}.$$

Identifying $V^k(\mathfrak{g})_1$ with $\mathfrak{g}$, it gives

$$\{\bar{a}, \bar{b}\} = \overline{a_{(0)}b}, \qquad a, b \in \mathfrak{g} = V^k(\mathfrak{g})_1 \cong \mathfrak{g}$$

since in $V^k(\mathfrak{g})$,

$$a_{(0)}b_{(-1)}|0\rangle = b_{(-1)}a_{(0)}|0\rangle + [a,b]_{(-1)}|0\rangle = 0 + [a,b]_{(-1)}|0\rangle.$$

More generally, if V is a graded quotient of $V^k(\mathfrak{g})$, then one can set

$$R_V = V/t^{-2}\mathfrak{g}[t^{-1}]V,$$

and we get a surjective morphism of Poisson algebras,

$$\begin{aligned} \mathbb{C}[\mathfrak{g}^*] &\longrightarrow R_V = V/t^{-2}\mathfrak{g}[t^{-1}]V \\ x_1 \dots x_r &\longmapsto \overline{x_{1,(-1)} \dots x_{r,(-r)}|0\rangle} + t^{-2}\mathfrak{g}[t^{-1}]V \quad (x_i \in \mathfrak{g}), \end{aligned} \tag{8}$$

the Poisson algebra structure on R_V being defined as before. This map is surjective but not an isomorphism in general.

1.3.2. Associated varieties. Continue to assume that V is a graded quotient of $V^k(\mathfrak{g})$.

Definition 1.7. We define the *associated variety* X_V *of* V to be the zero locus in $\mathfrak{g}^*$ of the kernel of the above map (8).

The Poisson variety X_V is then a $\mathbf{G}$-invariant, Poisson, and conic subvariety of $\mathfrak{g}^*$. We obviously have $X_{V^k(\mathfrak{g})} = \mathfrak{g}^*$ since the map (8) is an isomorphism for $V = V^k(\mathfrak{g})$.

For $V = L_k(\mathfrak{g})$, X_V can be viewed as an analog of the associated variety of primitive ideals. However, we will see that there are substantial differences.

1.3.3. Digression on primitive ideal of the enveloping algebra of $\mathfrak{g}$. Let I be a two-sided ideal of $U(\mathfrak{g})$. The PBW filtration on $U(\mathfrak{g})$ induces a filtration on I, so that $\operatorname{gr} I$ becomes a graded Poisson ideal in $\mathbb{C}[\mathfrak{g}^*]$. Thus, $U(\mathfrak{g})/I$ is a quantization of the Poisson $\mathbf{G}$-scheme $\widetilde{\mathscr{V}}(\operatorname{gr} I) = \operatorname{Spec} \mathbb{C}[\mathfrak{g}^*]/\operatorname{gr} I$.

The variety

$$\mathscr{V}(\operatorname{gr} I) = \operatorname{Specm} \mathbb{C}[\mathfrak{g}^*]/\operatorname{gr} I = (\widetilde{\mathscr{V}}(\operatorname{gr} I))_{\mathrm{red}} \subset \mathfrak{g}^*$$

is usually referred to as the *associated variety* of I.

A proper two-sided ideal I of $U(\mathfrak{g})$ is called *primitive* if it is the annihilator of a simple left $U(\mathfrak{g})$-module. There are two important results on primitive ideals of $U(\mathfrak{g})$. The first result is the Duflo Theorem [**Duflo77**], stating that any primitive ideal in $U(\mathfrak{g})$ is the annihilator $\operatorname{Ann}_{U(\mathfrak{g})} L_{\mathfrak{g}}(\lambda)$ of some simple highest weight module $L_{\mathfrak{g}}(\lambda)$, $\lambda \in \mathfrak{h}^*$, of $\mathfrak{g}$.

The second result is the *Irreducibility Theorem*. Identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ through $(\ |\ )$, we shall often view associated varieties of ideals of $U(\mathfrak{g})$ as subsets of $\mathfrak{g}$. The Irreducibility Theorem says that the associated variety $\mathscr{V}(\operatorname{gr} I)$ of a primitive ideal I in $U(\mathfrak{g})$ is irreducible, specifically, it is the closure $\overline{\mathbb{O}}$ of some nilpotent orbit $\mathbb{O}$ in $\mathfrak{g}$. The latter theorem was first partially proved (by a case-by-case argument) in [**Borho-Brylinski82**], and in a more conceptual way in [**Kashiwara-Tanisaki84**] and [**Joseph85**] (independently), using many earlier deep results due to Joseph, Gabber, Lusztig, Vogan and others.

It is possible that different primitive ideals share the same associated variety. In addition, not all nilpotent orbit closures appear as associated variety of some primitive ideal of $U(\mathfrak{g})$.

1.3.4. Integrable representations. First of all, since $L_k(\mathfrak{g}) \cong V^k(\mathfrak{g})$ for $k \notin \mathbb{Q}$ (cf. [**KK79**]), we see that $X_{V^k(\mathfrak{g})}$ is not always contained in the nilpotent cone $\mathcal{N}$.

One knows that $L_{\mathfrak{g}}(\lambda)$ is finite-dimensional if and only if $V(\operatorname{gr} \operatorname{Ann}_{U(\mathfrak{g})} L_{\mathfrak{g}}(\lambda)) = \{0\}$, where $L_{\mathfrak{g}}(\lambda)$ denotes the irreducible highest weight representation of $\mathfrak{g}$, with highest weight $\lambda \in \mathfrak{h}^*$.

Contrary to irreducible highest weight representations of $\mathfrak{g}$, the representation $L(\lambda)$, $\lambda \in \widehat{\mathfrak{h}}^*$, is finite dimensional if and only if $\lambda = 0$, that is, $L(\lambda)$ is the trivial representation.

The notion of finite dimensional representations has to be replaced by the notion of *integrable representations* in the *category* $\mathcal{O}$.

We define the category $\mathcal{O}$ for $\widehat{\mathfrak{g}}$ in the similar way that for $\mathfrak{g}$, except that we do not require that the object are finitely generated by $\mathfrak{g}$ (cf. [**Moody-Pianzola**]).

DEFINITION 1.8. A representation M of $\widehat{\mathfrak{g}}$ is said to be *integrable* if

(1) M is $\widehat{\mathfrak{h}}$-diagonalizable,
(2) for $\lambda \in \widehat{\mathfrak{h}}^*$, M_λ is finite dimensional,
(3) for $i = 0, \dots, r$, E_i and F_i act locally nilpotently on M, where E_i, H_i, F_i are Chevalley generators[2] of $\widehat{\mathfrak{g}}$.

REMARK 1.9. As a $\mathfrak{a}_i$-module, $i = 0, \dots, r$, an integrable representation M decomposes into a direct sum of finite dimensional irreducible $\widehat{\mathfrak{h}}$-invariant modules,

[2]Namely, $E_i = e_i \otimes 1$, $F_i = f_i \otimes 1$, $H_i = f_i \otimes 1$, for $i = 1, \dots, r$, with e_i, f_i, h_i Chevalley generators of $\mathfrak{g}$, and $E_0 = e_0 \otimes t$, $F_0 = f_0 \otimes t^{-1}$, with $f_0 \in \mathfrak{g}_\theta$, $e_0 \in \mathfrak{g}_{-\theta}$ such that $(f_0|e_0) = 1$.

where $\mathfrak{a}_i \cong \mathfrak{sl}_2$ is the Lie algebra generated by the Chevalley generators E_i, F_i, H_i. Hence the action of $\mathfrak{a}_i$ on M can be "integrated" to the action of the group $SL_2(\mathbb{C})$.

The character of the simple integrable representations in the category $\mathcal{O}$ satisfy remarkable combinatorial identities (related to Macdonald identities).

Recall that the irreducible representation $L_{\mathfrak{g}}(\lambda)$, $\lambda \in \mathfrak{h}^*$, is finite-dimensional if and only if its associated variety $V(\mathrm{Ann}_{U(\mathfrak{g})}(L_{\mathfrak{g}}(\lambda)))$ is zero.

THEOREM 1.10. *The associated variety of $L_k(\mathfrak{g})$ is $\{0\}$ if and only if $L_k(\mathfrak{g})$ is integrable as a $\widehat{\mathfrak{g}}$-module, which is true if and only if $k \in \mathbb{Z}_{\geqslant 0}$.*

The last part of the statement is well-known.

LEMMA 1.11. *Let (R, ∂) be a differential algebra over $\mathbb{Q}$, I a differential ideal of R, i.e., I is an ideal of R such that $\partial I \subset I$. Then $\partial\sqrt{I} \subset \sqrt{I}$.*

PROOF. Let $a \in \sqrt{I}$, so that $a^m \in I$ for some $m \in \mathbb{Z}_{\geqslant 0}$. Since I is ∂-invariant, we have $\partial^m a^m \in I$. But

$$\partial^m a^m \equiv m!(\partial a)^m \pmod{\sqrt{I}}.$$

Hence $(\partial a)^m \in \sqrt{I}$, and therefore, $\partial a \in \sqrt{I}$. □

Recall that a *singular vector of* $\mathfrak{g}$ of a $\mathfrak{g}$-module M is a vector $v \in M$ such that $\mathfrak{n}_+.v = 0$, that is, $e_i.v = 0$ for $i = 1, \dots, r$. A *singular vector of* $\widehat{\mathfrak{g}}$ of a $\widehat{\mathfrak{g}}$-representation M is a vector $v \in M$ such that $\widehat{\mathfrak{n}}_+.v = 0$, that is, $e_i.v = 0$ for $i = 1, \dots, r$, and $(f_\theta t).v = 0$. In particular, regarding $V^k(\mathfrak{g})$ as a $\widehat{\mathfrak{g}}$-representation, a vector $v \in V^k(\mathfrak{g})$ is singular if and only if $\widehat{\mathfrak{n}}_+.v = 0$.

PROOF OF THE "IF" PART OF THEOREM 1.10. Suppose that $L_k(\mathfrak{g})$ is integrable. This condition is equivalent to that $k \in \mathbb{Z}_{\geqslant 0}$ and the maximal submodule $N_k(\mathfrak{g})$ of $V^k(\mathfrak{g})$ is generated by the singular vector $(e_\theta t^{-1})^{k+1}|0\rangle$ ([**Kac1**]). The exact sequence $0 \to N_k(\mathfrak{g}) \to V^k(\mathfrak{g}) \to L_k(\mathfrak{g}) \to 0$ induces the exact sequence

$$0 \to I_k \to R_{V^k(\mathfrak{g})} \to R_{L_k(\mathfrak{g})} \to 0,$$

where I_k is the image of N_k in $R_{V^k(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*]$, and so, $R_{L_k(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*]/I_k$. The image of the singular vector in I_k is given by e_θ^{k+1}. Therefore, $e_\theta \in \sqrt{I_k}$. On the other hand, by Lemma 1.11, $\sqrt{I_k}$ is preserved by the adjoint action of $\mathfrak{g}$. Since $\mathfrak{g}$ is simple, $\mathfrak{g} \subset \sqrt{I_k}$. This proves that $X_{L_k(\mathfrak{g})} = \{0\}$ as required. □

The proof of the "only if" part follows from [**Dong-Li-Mason06**]. It can also be proved using W-algebras.

PART 2

Vertex algebras, definitions, first properties and examples

References: [**Frenkel-BenZvi**, **Kac2**]

Vertex algebras were introduced by Borcherds in 1986 [**Borcherds86**]. They give the mathematical formalism of two-dimensional conformal field theory (CFT).

2.1. Operator product expansion and definition of vertex algebras

Let V be a vector space over $\mathbb{C}$. We denote by $(\operatorname{End} V)[[z, z^{-1}]]$ the set of all formal Laurent series in the variable z with coefficients in the space $\operatorname{End} V$. We call elements $a(z)$ of $(\operatorname{End} V)[[z, z^{-1}]]$ a *series on* V. For a series $a(z)$ on V, we set

$$a_{(n)} = \operatorname{Res}_{z=0} a(z) z^n$$

so that the expansion of $a(z)$ is

$$a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}.$$

The coefficient $a_{(n)}$ is called a *Fourier mode* of $a(z)$. We write

$$a(z)b = \sum_{n \in \mathbb{Z}} a_{(n)} b z^{-n-1}$$

for $b \in V$.

DEFINITION 2.1. A series $a(z) \in (\operatorname{End} V)[[z, z^{-1}]]$ is called a *field* on V if for any $b \in V$, $a(z)b \in V((z))$, that is, for any $b \in V$, $a_{(n)}b = 0$ for large enough n.

We denote by $\mathscr{F}(V)$ the space of all fields on V.

2.1.1. Definition. A *vertex algebra* is a vector space V equipped with the following data:

- (*the vacuum vector*) a vector $|0\rangle \in V$,
- (*the vertex operators*) a linear map

$$V \to \mathscr{F}(V), \quad a \mapsto a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1},$$

- (*the translation operator*) a linear map $T \colon V \to V$.

These data are subject to the following axioms:

- (*the vacuum axiom*) $|0\rangle(z) = \mathrm{id}_V$. Furthermore, for all $a \in V$,

$$a(z)|0\rangle \in V[[z]]$$

and $\lim_{z \to 0} a(z)|0\rangle = a$. In other words, $a_{(n)}|0\rangle = 0$ for $n \geqslant 0$ and $a_{(-1)}|0\rangle = a$.

- (*the translation axiom*) for any $a \in V$,

$$[T, a(z)] = \partial_z a(z), \qquad (Ta)(z) = \partial_z a(z)$$

and $T|0\rangle = 0$.
- (*the locality axiom*) for all $a, b \in V$, $(z-w)^{N_{a,b}}[a(z), b(w)] = 0$ for some $N_{a,b} \in \mathbb{Z}_{\geqslant 0}$.

When two fields $a(z), b(z)$ on a vector space V verify the condition of the locality axiom, we say that there are *mutually local*.

2.1.2. Operator product expansion (OPE).

PROPOSITION 2.2. *Let $a(z), b(z)$ be two fields on a vector space V. The following assertions are equivalent.*

(i) $(z-w)^{N_{a,b}}[a(z), b(w)] = 0$ *for some* $N = N_{a,b} \in \mathbb{Z}_{\geqslant 0}$.

(ii)

$$[a(z), b(w)] = \sum_{n=0}^{N_{a,b}-1} (a_{(n)}b)(w) \frac{1}{n!} \partial_w^n \delta(z-w), \tag{9}$$

where $\delta(z-w) := \sum_{n \in \mathbb{Z}} w^n z^{-n-1} \in \mathbb{C}[[z, w, z^{-1}, w^{-1}]]$ *is the* formal delta-function.

(iii)

$$a(z)b(w) = \sum_{n=0}^{N_{a,b}-1} (a_{(n)}b)(w) \tau_{z,w}\left(\frac{1}{(z-w)^{j+1}}\right) + :a(z)b(w):,$$

and

$$b(w)a(z) = \sum_{n=0}^{N_{a,b}-1} (a_{(n)}b)(w) \tau_{w,z}\left(\frac{1}{(z-w)^{j+1}}\right) + :a(z)b(w):,$$

where $:a(z)b(w):$ *and the maps* $\tau_{z,w}$ *and* $\tau_{w,z}$ *are defined below.*

For $a(z), b(z) \in \mathscr{F}(V)$, set

$$:a(z)b(w): = a(z)_+ b(w) + b(w) a(z)_-,$$

where

$$a(z)_+ = \sum_{n<0} a_{(n)} z^{-n-1}, \qquad a(z)_- = \sum_{n \geqslant 0} a_{(n)} z^{-n-1}.$$

The *normally ordered product* on a vertex algebra V is defined as $:ab := a_{(-1)}b$. Thus

$$:ab:(z) = :a(z)b(z):.$$

The normally ordered product is neither commutative nor associative. By definition, $:a(z)b(z)c(z):$ stands for $:a(z):b(z)c(z)::$.

The two maps $\tau_{z,w}$ and $\tau_{w,z}$ are the morphisms of algebras defined by:

$$\tau_{z,w} \colon \mathbb{C}[z, w, z^{-1}, w^{-1}, \frac{1}{z-w}] \to \mathbb{C}((z))((w)), \quad \frac{1}{z-w} \mapsto \frac{1}{z} \sum_{n \geqslant 0} \left(\frac{w}{z}\right)^n = \delta(z-w)_-,$$

$$\tau_{w,z} \colon \mathbb{C}[z, w, z^{-1}, w^{-1}, \frac{1}{z-w}] \to \mathbb{C}((w))((z)), \quad \frac{1}{z-w} \mapsto -\frac{1}{z} \sum_{n > 0} \left(\frac{z}{w}\right)^n = -\delta(z-w)_+.$$

Thus the map $\tau_{z,w}$ is the expansion of $\dfrac{1}{z-w}$ in $|z| > |w|$ and $\tau_{w,z}$ is the expansion of $\dfrac{1}{z-w}$ in $|w| > |z|$.

PROOF OF THE IMPLICATION (II)⇐(I). We verify only the converse implication (see for instance [**Frenkel-BenZvi**, Chap. 3] for more details).

Let us write

$$\delta(z-w) = \underbrace{\frac{1}{z}\sum_{n\geqslant 0}\left(\frac{w}{z}\right)^n}_{\delta_-} + \underbrace{\frac{1}{z}\sum_{n>0}\left(\frac{z}{w}\right)^n}_{\delta_+},$$

so that when $|z| > |w|$, the series $\delta(z-w)_-$ converges to the meromorphic function $\dfrac{1}{z-w}$ and when $|z| < |w|$, the series $\delta(z-w)_+$ converges to the meromorphic function $-\dfrac{1}{z-w}$.

We have

$$\delta(z-w) = \tau_{z,w}(\frac{1}{z-w}) - \tau_{w,z}(\frac{1}{z-w}).$$

Both morphisms $\tau_{z,w}$ and $\tau_{w,z}$ commutes with ∂_w and ∂_z. Therefore,

$$\frac{1}{j!}\partial_w^j\delta(z-w) = \tau_{z,w}(\frac{1}{(z-w)^{j+1}}) - \tau_{w,z}(\frac{1}{(z-w)^{j+1}}),$$

whence

$$\begin{aligned}(z-w)^{n+1}\frac{1}{n!}\partial_w^n\delta(z-w) &= (z-w)^{n+1}\left(\tau_{z,w}(\frac{1}{(z-w)^{n+1}}) - \tau_{w,z}(\frac{1}{(z-w)^{n+1}})\right)\\ &= \tau_{z,w}(1) - \tau_{w,z}(1) = 0.\end{aligned}$$

The implication (ii)⇒(i) is then clear. □

Note that the translation axiom says that

$$[T, a_{(n)}] = -na_{(n-1)}, \qquad n \in \mathbb{Z}, \tag{10}$$

and together with the vacuum axiom we get that

$$Ta = a_{(-2)}|0\rangle.$$

EXERCISE 2.3. Verify the above assertion.

Also, the vacuum axiom implies that the map $V \to \mathrm{End}(V)$ defined by the formula $a \mapsto a_{(-1)}$ is injective. Therefore the map $a \mapsto a(z)$ is also injective.

2.1.3. Borcherds identities. A consequence of the definition are the following relations, called *Borcherds identities*:

$$[a_{(m)}, b_{(n)}] = \sum_{i\geqslant 0}\binom{m}{i}(a_{(i)}b)_{(m+n-i)}, \tag{11}$$

$$(a_{(m)}b)_{(n)} = \sum_{j\geqslant 0}(-1)^j\binom{m}{j}\left(a_{(m-j)}b_{(n+j)} - (-1)^m b_{(m+n-j)}a_{(j)}\right), \tag{12}$$

for $m, n \in \mathbb{Z}$. In the above formulas, the notation $\binom{m}{i}$ for $i \geqslant 0$ and $m \in \mathbb{Z}$ means

$$\binom{m}{i} = \frac{m(m-1) \times \cdots \times (m-i+1)}{i(i-1) \times \cdots \times 1}.$$

2.2. First examples of vertex algebras

2.2.1. Commutative vertex algebras. A vertex algebra V is called *commutative* if all vertex operators $a(z)$, $a \in V$, commute each other (i.e., we have $N_{a,b} = 0$ in the locality axiom). This condition is equivalent to that

$$[a_{(m)}, b_{(n)}] = 0, \qquad \forall a, b \in \mathbb{Z},\ m, n \in \mathbb{Z}$$

by (11).

Hence if V is a commutative vertex algebra, then $a(z) \in \operatorname{End} V[[z]]$ for all $a \in V$. Then a commutative vertex algebra has a structure of a unital commutative algebra with the product:

$$a \cdot b = :ab: = a_{(-1)}b,$$

where the unit is given by the vacuum vector $|0\rangle$. The translation operator T of V acts on V as a derivation with respect to this product:

$$T(a \cdot b) = (Ta) \cdot b + a \cdot (Tb).$$

Therefore a commutative vertex algebra has the structure of a *differential algebra*, that is, a unital commutative algebra equipped with a derivation.

Conversely, there is a unique vertex algebra structure on a differential algebra R with derivation ∂ such that:

$$a(z)b = \left(e^{z\partial}a\right) b = \sum_{n \geqslant 0} \frac{z^n}{n!}(\partial^n a)b,$$

for all $a, b \in R$. We take the unit as the vacuum vector. This correspondence gives the following result.

THEOREM 2.4 ([**Borcherds86**]). *The category of commutative vertex algebras is the same as that of differential algebras.*

One important example of commutative vertex algebras are obtained by considering the function sheaf over arc spaces of a scheme (see Section 3.1).

2.2.2. Universal affine vertex algebras. Let us consider a slightly more general construction of the vacuum representation considered in Section 1.2. We will recover this special case with $\mathfrak{a} = \mathfrak{g}$, and $\kappa = k(\ |\ )$.

Let $\mathfrak{a}$ be a Lie algebra endowed with a symmetric invariant bilinear form κ. Let

$$\widehat{\mathfrak{a}} = \mathfrak{a}[t, t^{-1}] \oplus \mathbb{C}\mathbf{1}$$

be the *Kac-Moody affinization* of $\mathfrak{a}$. It is a Lie algebra with commutation relations

$$[xt^m, yt^n] = [x, y]t^{m+n} + m\delta_{m+n,0}\kappa(x, y)\mathbf{1}, \quad x, y \in \mathfrak{a},\ m, n \in \mathbb{Z}, \qquad [\mathbf{1}, \widehat{\mathfrak{a}}] = 0.$$

Let

$$V^{\kappa}(\mathfrak{a}) = U(\widehat{\mathfrak{a}}) \otimes_{U(\mathfrak{a}[t] \oplus \mathbb{C}\mathbf{1})} \mathbb{C}, \tag{13}$$

where $\mathbb{C}$ is a one-dimensional representation of $\mathfrak{a}[t] \oplus \mathbb{C}\mathbf{1}$ on which $\mathfrak{a}[t]$ acts trivially and $\mathbf{1}$ acts as the identity. By the PBW Theorem, we have the following isomorphism of vector spaces:

$$V^\kappa(\mathfrak{a}) \cong U(\mathfrak{a} \otimes t^{-1}\mathbb{C}[t^{-1}]).$$

The space $V^\kappa(\mathfrak{a})$ is naturally graded: $V^\kappa(\mathfrak{a}) = \bigoplus_{\Delta \in \mathbb{Z}_{\geqslant 0}} V^\kappa(\mathfrak{a})_\Delta$, where the grading is defined by setting $\deg(xt^n) = -n$, $\deg|0\rangle = 0$. Here $|0\rangle$ is the image of $1 \otimes 1$ in $V^\kappa(\mathfrak{a})$. We have $V^\kappa(\mathfrak{a})_0 = \mathbb{C}|0\rangle$. We identify $\mathfrak{a}$ with $V^\kappa(\mathfrak{a})_1$ via the linear isomorphism defined by $x \mapsto xt^{-1}|0\rangle$.

There is a unique vertex algebra structure on $V^\kappa(\mathfrak{a})$, such that $|0\rangle$ is the vacuum vector and

$$x(z) := \sum_{n \in \mathbb{Z}} (xt^n) z^{-n-1}, \quad x \in \mathfrak{a}.$$

(So $x_{(n)} = xt^n$ for $x \in \mathfrak{a} = V^\kappa(\mathfrak{a})_1$, $n \in \mathbb{Z}$.) The vertex algebra $V^\kappa(\mathfrak{a})$ is called the *universal affine vertex algebra associated with* $(\mathfrak{a}, \kappa)$.

Let us describe the vertex algebra structure in more details. Set

$$x_{(n)} = xt^n, \qquad \forall\, x \in \mathfrak{a},\, n \in \mathbb{Z},$$

and let $|0\rangle$ be the image of $1 \otimes 1 \in U(\hat{\mathfrak{a}}) \otimes \mathbb{C}$ in $V^\kappa(\mathfrak{a})$. Let $(x^i\,; i = 1 \dots, \dim \mathfrak{a})$ be an ordered basis of $\mathfrak{a}$. By the PBW Theorem, $V^\kappa(\mathfrak{a})$ has a basis of the form

$$x^{i_1}_{(n_1)} \dots x^{i_m}_{(n_m)}|0\rangle,$$

where $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_m < 0$, and if $n_j = n_{j+1}$, then $i_j \leqslant i_{j+1}$.

Then $(V^\kappa(\mathfrak{a}), |0\rangle, T, Y)$ is a vertex algebra where the translation operator T is given by

$$T|0\rangle = 0, \qquad [T, x^i_{(n)}] = -n x^i_{(n-1)},$$

for $n \in \mathbb{Z}$, and the vertex operators are defined inductively by:

$$\begin{aligned}
&|0\rangle(z) = \mathrm{Id}_{V^\kappa(\mathfrak{a})}, \qquad x^i_{(-1)}|0\rangle(z) = x^i(z) = \sum_{n \in \mathbb{Z}} x^i_{(n)} z^{-n-1},\\
&(x^{i_1}_{(n_1)} \dots x^{i_m}_{(n_m)}|0\rangle)(z)\\
&\qquad = \frac{1}{(-n_1-1)! \dots (-n_m-1)!} :\partial_z^{-n_1-1} x^{i_1}(z) \dots \partial_z^{-n_m-1} x^{i_m}(z):
\end{aligned}$$

THEOREM 2.5. *$V^\kappa(\mathfrak{a})$ is a $\mathbb{Z}_{\geqslant 0}$-graded vertex algebra.*

PROOF. We check only the locality axiom.

It is enough to check the locality on generator fields by Dong's lemma, which says that if $a(z)$, $b(z)$, $c(z)$ are three mutually local fields on a vector space V, then the fields $:a(z)b(z):$ and $c(z)$ are also mutually local.

Let $i, j \in \{A, \dots, d\}$. Then

$$\begin{aligned}
[x^i(z), x^j(w)] &= \sum_{n,m \in \mathbb{Z}} [x^i_{(n)}, x^j_{(m)}] z^{-n-1} w^{-m-1}\\
&= \sum_{n,m \in \mathbb{Z}} [x^i, x^j]_{(n+m)} z^{-n-1} w^{-m-1} + \sum_{n \in \mathbb{Z}} n\kappa(x^i, x^j) z^{-n-1} w^{-n-1}\\
&= \sum_{l \in \mathbb{Z}} [x^i, x^j]_{(l)} \left(\sum_{n \in \mathbb{Z}} z^{-n-1} w^n \right) w^{-l-1} + \kappa(x^i, x^j) \sum_{n \in \mathbb{Z}} n z^{-n-1} w^{n-1}\\
&= [x^i, x^j](w)\delta(z-w) + \kappa(x^i, x^j)\partial_w \delta(z-w).
\end{aligned}$$

Then it follows that for any i, j,

$$(z-w)^2[x^i(z), x^j(w)] = 0,$$

so the locality axiom holds for these fields.

REMARK 2.6. (1) In fact, the equality

$$[x^i(z), x^j(w)] = [x^i, x^j](w)\delta(z-w) + \kappa(x^i, x^j)\partial_w \delta(z-w)$$

is equivalent to the commutation relations in the Lie algebra $\widehat{\mathfrak{a}}$.

(2) Using the commutation relations, one can check directly on this example the OPE formula with $N = 2$:

$$[x^i(z), x^j(w)] = \sum_{n=0}^{N-1} (x^i_{(n)} x^j)(w) \frac{1}{n!} \partial_w^n \delta(z-w).$$

□

When $\mathfrak{a} = \mathfrak{g}$ is the simple Lie algebra as in Part 1, so that $\widehat{\mathfrak{a}} = \widehat{\mathfrak{g}}$ is the affine Kac-Moody algebra, and

$$\kappa = k(\ |\ ) = \frac{k}{2h^\vee} \times (\ |\ )_{\mathrm{Kil}}, \qquad \text{for } k \in \mathbb{C},$$

with $\kappa_{\mathfrak{g}}$ the Killing form of $\mathfrak{g}$, then we write $V^k(\mathfrak{g})$ for the universal affine vertex algebra vertex algebra $V^\kappa(\mathfrak{a})$. By what foregoes, $V^k(\mathfrak{g})$ has a natural vertex algebra structure, and it is called the *universal affine vertex algebra associated with $\widehat{\mathfrak{g}}$ of level k*.

When $\mathfrak{a} \cong \mathbb{C}$ is one-dimensional with κ any non-degenerate bilinear form, then $V^\kappa(\mathfrak{a})$ is the *Heisenberg vertex algebra*.

2.2.3. The Virasoro vertex algebra. Let $Vir = \mathbb{C}((t))\partial_t \oplus \mathbb{C}C$ be the *Virasoro Lie algebra*, with the commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3 - n}{12}\delta_{n+m,0}C, \tag{14}$$

$$[C, Vir] = 0, \tag{15}$$

where $L_n := -t^{n+1}\partial_t$ for $n \in \mathbb{Z}$.

Let $c \in \mathbb{C}$ and define the induced representation

$$\mathrm{Vir}^c = \mathrm{Ind}_{\mathbb{C}[[t]]\partial_t \oplus \mathbb{C}C}^{Vir} \mathbb{C}_c = U(Vir) \otimes_{\mathbb{C}[[t]]\partial_t \oplus \mathbb{C}C} \mathbb{C}_c,$$

where C acts as multiplication by c and $\mathbb{C}[[t]]\partial_t$ acts by 0 on the 1-dimensional module $\mathbb{C}_c$.

By the PBW Theorem, Vir^c has a basis of the form

$$L_{j_1} \dots L_{j_m}|0\rangle, \qquad j_1 \leqslant \dots \leqslant j_l \leqslant -2,$$

where $|0\rangle$ is the image of $1 \otimes 1$ in Vir^c. Then $(\mathrm{Vir}^c, |0\rangle, T, Y)$ is a vertex algebra, called the *universal Virasoro vertex algebra with central charge* c, such that $T = L_{-1}$ and:

$$Y(|0\rangle, z) = \mathrm{Id}_{\mathrm{Virc}_c}, \qquad Y(L_{-2}|0\rangle, z) =: L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},$$

$$Y(L_{j_1} \dots L_{j_m}|0\rangle, z) = \frac{1}{(-j_1-2)! \dots (-j_m-2)!} : \partial_z^{-j_1-2} L(z) \dots \partial_z^{-j_m-2} L(z) :$$

Moreover, Vir^c is $\mathbb{Z}_{\geqslant 0}$-graded by $\deg|0\rangle = 0$ and $\deg L_n|0\rangle = -n$.

2.2.4. Conformal vertex algebras. A *Hamiltonian* of V is a semisimple operator H on V satisfying

$$[H, a_{(n)}] = -(n+1)a_{(n)} + (Ha)_{(n)}$$

for all $a \in V$, $n \in \mathbb{Z}$.

DEFINITION 2.7. A vertex algebra equipped with a Hamiltonian H is called *graded*. Let $V_\Delta = \{a \in V \mid Ha = \Delta a\}$ for $\Delta \in \mathbb{C}$, so that $V = \bigoplus_{\Delta\in\mathbb{C}} V_\Delta$. For $a \in V_\Delta$, Δ is called the *conformal weight* of a and it is denoted by Δ_a. We have

$$a_{(n)}b \in V_{\Delta_a+\Delta_b-n-1} \tag{16}$$

for homogeneous elements $a, b \in V$.

For example, the universal affine vertex algebra $V^k(\mathfrak{g})$ is $\mathbb{Z}_{\geqslant 0}$-graded (that is, $V^k(\mathfrak{g})_\Delta = 0$ for $\Delta \notin \mathbb{Z}_{\geqslant 0}$) and the Hamiltonian H is defined letting H acts on $V^k(\mathfrak{g})_\Delta$ as $\Delta \mathrm{Id}_{V^k(\mathfrak{g})_\Delta}$ for any $\Delta \in \mathbb{Z}_{\geqslant 0}$.

DEFINITION 2.8. A graded vertex algebra $V = \bigoplus_{\Delta\in\mathbb{C}} V_\Delta$ is called *conformal* of central charge $c \in \mathbb{C}$ if there is a *conformal vector* $\omega \in V_2$ such that the *Fourier coefficients* L_n of the corresponding vertex operators

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

satisfy the defining relations (14) of the Virasoro algebra with central charge c, and if in addition we have

$$\begin{aligned} &\omega_{(0)} = L_{-1} = T, \\ &\omega_{(1)} = L_0 = H \quad \text{ i.e.,} \quad L_0|_{V_\Delta} = \Delta \mathrm{Id}_{V_\Delta} \quad \forall\, \Delta \in \mathbb{Z}. \end{aligned}$$

A $\mathbb{Z}$-graded conformal vertex algebra is also called a *vertex operator algebra*.

EXAMPLE 2.9. The Virasoro vertex algebra Vir^c is clearly conformal with central charge c and conformal vector $\omega = L_{-2}|0\rangle$.

EXAMPLE 2.10. The universal affine vertex algebra $V^k(\mathfrak{g})$ has a natural conformal vector, provided that $k \neq -h^\vee$. Set

$$S = \frac{1}{2}\sum_{i=1}^{\dim\mathfrak{g}} x_{i,(-1)} x^i_{(-1)}|0\rangle,$$

where $(x_i;\, i = 1, \ldots, \dim\mathfrak{g})$ is the dual basis of $(x^i;\, i = 1, \ldots, \dim\mathfrak{g})$ with respect to the bilinear form $(\ |\ )$, and

$$x^i(z) = \sum_{n\in\mathbb{Z}} x^i_{(n)} z^{-n-1}, \qquad x_i(z) = \sum_{n\in\mathbb{Z}} x_{i,(n)} z^{-n-1}.$$

EXERCISE 2.11. For $k \neq -h^\vee$, show that $L = \dfrac{S}{k+h^\vee}$ is a conformal vector of $V^k(\mathfrak{g})$, called the *Segal-Sugawara vector*, with central charge

$$c(k) = \frac{k\dim\mathfrak{g}}{k+h^\vee}.$$

(cf. [**Frenkel-BenZvi**, §3.4.8]).

Then we have

$$[L_m, x_{(n)}] = (m-n)x_{(m+n)} \quad x \in \mathfrak{g},\ m.n \in \mathbb{Z}. \tag{17}$$

2.3. Modules over vertex algebras

2.3.1. Definition. A *module* over the vertex algebra V is a vector space M together with a linear map

$$V \to \mathscr{F}(M), \quad a \mapsto a^M(z) = \sum_{n\in\mathbb{Z}} a^M_{(n)} z^{-n-1},$$

which satisfies the following axioms:

$$|0\rangle(z) = \mathrm{Id}_M, \tag{18}$$

$$(Ta)^M(z) = \partial_z a^M(z), \tag{19}$$

$$\sum_{j\leqslant 0} \binom{m}{j} (a_{(n+j)}b)^M_{(m+k-j)} = \sum_{j\geqslant 0} (-1)^j \binom{n}{j} (a^M_{(m+n-j)} b^M_{(k+j)} - (-1)^n b^M_{(n+k-j)} a^M_{(m+j)}). \tag{20}$$

Notice that (20) is equivalent to (11) and (12) for $M = V$.

Suppose in addition V is graded (cf. Definition 2.7). A V-module M is called *graded* if there is a compatible semisimple action of H on M, that is, $M = \bigoplus_{d\in\mathbb{C}} M_d$, where $M_d = \{m \in M \mid Hm = dm\}$ and $[H, a^M_{(n)}] = -(n+1)a^M_{(n)} + (Ha)^M_{(n)}$ for all $a \in V$. We have

$$a^M_{(n)} M_d \subset M_{d+\Delta_a - n-1} \tag{21}$$

for homogeneous $a \in V$.

When there is no ambiguity, we will often denote by $a_{(n)}$ the element $a^M_{(n)}$ of $\mathrm{End}(M)$.

The axioms imply that V is a module over itself (called the *adjoint module*). We have naturally the notions of *submodules*, *quotient module* and *vertex ideals*. A module whose only submodules are 0 and itself is called *simple*.

2.3.2. Modules of the universal affine vertex algebra. In the case that V is the universal affine vertex algebra $V^k(\mathfrak{g})$ associated with $\widehat{\mathfrak{g}}$ at level $k \in \mathbb{C}$, V-modules play a crucial important role in the representation theory of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$.

A $\widehat{\mathfrak{g}}$-module M of level k is called *smooth* if $x(z)$ is a field on M for $x \in \mathfrak{g}$, that is, for any $m \in M$ there is $N > 0$ such that $(xt^n)m = 0$ for all $x \in \mathfrak{g}$ and $n \geqslant N$.

Any $V^k(\mathfrak{g})$-module M is naturally a smooth $\widehat{\mathfrak{g}}$-module of level k. Conversely, any smooth $\widehat{\mathfrak{g}}$-module of level k can be regarded as a $V^k(\mathfrak{g})$-module. It follows that a $V^k(\mathfrak{g})$-module is the same as a smooth $\widehat{\mathfrak{g}}$-module of level k.

Namely, we have the following.

PROPOSITION 2.12 (See [**Frenkel-BenZvi**, §5.1.18] for a proof). *There is an equivalence of category between the category of $V^k(\mathfrak{g})$-modules and the category of smooth $\widehat{\mathfrak{g}}$-modules of level k.*

PART 3

Poisson vertex algebras, arc spaces, and associated varieties

3.1. Jet schemes and arc spaces

In this section, we present some general facts on jet schemes and arc spaces. Our main references on the topic are [**Mustata01**, **Ein-Mustata09**, **Ishii11**].

3.1.1. Definitions. Denote by Sch the category of schemes of finite type over $\mathbb{C}$. Let X be an object of this category, and $m \in \mathbb{Z}_{\geqslant 0}$.

Definition 3.1. An *m-jet of X* is a morphism

$$\operatorname{Spec} \mathbb{C}[t]/(t^{m+1}) \longrightarrow X.$$

The set of all m-jets of X carries the structure of a scheme $J_m(X)$, called the *m-th jet scheme of X*. It is a scheme of finite type over $\mathbb{C}$ characterized by the following functorial property: for every scheme Z over $\mathbb{C}$, we have

$$\operatorname{Hom}_{Sch}(Z, J_m(X)) = \operatorname{Hom}_{Sch}(Z \times_{\operatorname{Spec}\mathbb{C}} \operatorname{Spec} \mathbb{C}[t]/(t^{m+1}), X).$$

The $\mathbb{C}$-points of $J_m(X)$ are thus the $\mathbb{C}[t]/(t^{m+1})$-points of X. From Definition 3.1, we have for example that $J_0(X) \simeq X$ and that $J_1(X) \simeq \mathrm{T}X$ where $\mathrm{T}X$ denotes the total tangent bundle of X.

The canonical projection $\mathbb{C}[t]/(t^{m+1}) \to \mathbb{C}[t]/(t^{n+1})$, $m \geqslant n$, induces a *truncation morphism* $\pi_{X,m,n} \colon J_m(X) \to J_n(X)$. The canonical injection $\mathbb{C} \hookrightarrow \mathbb{C}[t]/(t^{m+1})$ induces a morphism $\iota_{X,m} \colon X \to J_m(X)$, and we have $\pi_{X,m} \circ \iota_{X,m,0} = \mathrm{id}_X$. Hence $\iota_{X,m}$ is injective and $\pi_{X,m,0}$ is surjective.

Define the (formal) disc as

$$D := \operatorname{Spec} \mathbb{C}[[t]].$$

The projections $\pi_{X,m,n}$ yield a projective system $\{J_m(X), \pi_{X,m,n}\}_{m \geqslant n}$ of schemes.

Definition 3.2. Denote by $J_\infty(X)$ its projective limit in the category of schemes,

$$J_\infty(X) = \varprojlim J_m(X).$$

It is called the *arc space*, or *the infinite jet scheme* of X.

Thus elements of $J_\infty(X)$ are the morphisms

$$\gamma \colon D \to \mathbb{C}[[t]],$$

and for every scheme Z over $\mathbb{C}$,

$$\operatorname{Hom}_{Sch}(Z, J_m(X)) = \operatorname{Hom}_{Sch}(Z \widehat{\times}_{\operatorname{Spec}\mathbb{C}} D, X),$$

where $Z \widehat{\times} D$ is the completion of $Z \times D$ with respect to the subscheme $Z \times \{0\}$. In other words, the contravariant functor

$$Sch \to Set, \quad Z \mapsto \mathrm{Hom}_{Sch}(Z \widehat{\times} D, X)$$

is represented by the scheme $J_\infty(X)$. The reason why we need the completion $Z \widehat{\times} D$ in the definition is that, for A an algebra, $A \otimes \mathbb{C}[[t]] \subsetneqq A[[t]] = A \widehat{\otimes} \mathbb{C}[[t]]$ in general.

We denote by $\pi_{X,\infty}$ the canonical projection:

$$\pi_{X,\infty} \colon J_\infty(X) \to X.$$

3.1.2. The affine case. In the case where X is affine, we have the following explicit description of $J_\infty(X)$. (We describe similarly $J_m(X)$.)

Let $N \in \mathbb{Z}_{>0}$ and $X \subset \mathbb{C}^N$ be an affine subscheme defined by an ideal $I = \langle f_1, \dots, f_r \rangle$ of $\mathbb{C}[x^1, \dots, x^N]$. Thus

$$X = \mathrm{Spec}\ \mathbb{C}[x^1, \dots, x^N]/I.$$

For $f \in \mathbb{C}[x^1, \dots, x^N]$, we extend f as a map from $\mathbb{C}[[t]]^N$ to $\mathbb{C}[[t]]$ via base extension. Then giving a morphism $\gamma \colon D \to X$ is equivalent to giving a morphism $\gamma^* \colon \mathbb{C}[x^1, \dots, x^N]/I \to \mathbb{C}[[t]]$, or to giving

$$\gamma^*(x^i) = \sum_{j \geqslant 0} \gamma^i_{(-j-1)} t^j, \qquad i = 1, \dots, N,$$

such that for any $k = 1, \dots, r$,

$$f_k(\gamma^*(x^1), \dots, \gamma^*(x^N)) = 0 \ \text{ in } \ \mathbb{C}[[t]].$$

For any $f \in \mathbb{C}[x^1, \dots, x^N]$, there exist functions $f^{(j)}$, $j \geqslant 0$, which only depend on f, in the variables $\gamma = (\gamma^1_{(-j-1)}, \dots, \gamma^N_{(-j-1)})_{j \geqslant 0}$ such that

$$f\big(\gamma^*(x^1), \dots, \gamma^*(x^N)\big) = \sum_{j \geqslant 0} \frac{f^{(j)}}{j!}(\gamma)\, t^j. \tag{22}$$

Regarding the coordinates x^i as functions over $\mathbb{C}^N$, we set:

$$x^i_{(-j-1)} := (x^i)^{(j)}, \quad \text{ that is, } \quad x^i_{(-j-1)}(\gamma) = j! \gamma^i_{(-j-1)},$$

for $i = 1, \dots, N$.

The jet scheme $J_\infty(X)$ is then the closed subscheme in $\mathrm{Spec}\, \mathbb{C}[x^i_{(-j-1)}\, ; i = 1, \dots, N,\, j \geqslant 0]$ defined by the ideal generated by the polynomials $f_k^{(j)}$, for $k = 1, \dots, r$ and $j \geqslant 0$, that is,

$$J_\infty(X) \cong \mathrm{Spec}\, \mathbb{C}[x^i_{(-j-1)}\, ; i = 1, \dots, N,\, j \geqslant 0]/\langle f_k^{(j)}\, ; k = 1, \dots, r,\, j \geqslant 0 \rangle.$$

In particular, if X is an N-dimensional vector space, then

$$J_\infty(X) \cong \mathrm{Spec}\, \mathbb{C}[x^i_{(-j-1)}\, ; i = 1, \dots, N,\, j \geqslant 0],$$

and for $m \in \mathbb{Z}_{\geqslant 0}$, the projection $J_\infty(X) \to J_m(X)$ corresponds to the projection onto the first $(m+1)N$ coordinates.

One can also define the functions $f_k^{(j)}$ using a derivation.

LEMMA 3.3. *Define the derivation T of $\mathbb{C}[x^i_{(-j-1)}\, ; i = 1, \dots, N,\, j \geqslant 0]$ by*

$$T x^i_{(-j)} = j x^i_{(-j-1)}, \qquad j \geqslant 0.$$

Then $f_k^{(j)} = T^j f_k$ for $k = 1, \dots, r$ and $j \geqslant 0$. Here we identify x^i with $x^i_{(-1)}$.

With the above lemma, we conclude that for the affine scheme $X = \operatorname{Spec} R$, with $R = \mathbb{C}[x^1, x^2, \cdots, x^N]/\langle f_1, f_2, \cdots, f_r\rangle$, its arc space $J_\infty X$ is the affine scheme $\operatorname{Spec}(J_\infty(R))$, where

$$J_\infty(R) := \frac{\mathbb{C}[x^i_{(-j)}\ ;\ i = 1, 2, \cdots, N,\ j \geqslant 0]}{\langle T^j f_i\ ;\ i = 1, \ldots, r,\ j \geqslant 0\rangle}$$

and T is as defined in the lemma.

The derivation T acts on the above quotient ring $J_\infty(R)$. Hence for an affine scheme $X = \operatorname{Spec} R$, the coordinate ring $J_\infty(R) = \mathbb{C}[J_\infty(X)]$ of its arc space $J_\infty(X)$ is a differential algebra, hence is a commutative vertex algebra by Theorem 2.4.

Remark 3.4. The differential algebra $(J_\infty(R), T)$ is universal in the following sense. We have a $\mathbb{C}$-algebra homomorphism $j\colon R \to J_\infty(R)$ such that if (A, ∂) is another differential algebra, and if $f\colon R \to A$ is a $\mathbb{C}$-algebra homomorphism, then there is a unique differential algebra homomorphism $h\colon J_\infty(R) \to A$ making the following diagram commutative.

$$\begin{tikzcd} R \arrow[rr, "j"] \arrow[dr, "f"'] & & (J_\infty(R), T) \arrow[dl, dashed, "h"] \\ & (A, \partial) & \end{tikzcd}$$

(The map h is a *differential algebra homomorphism* means that it is a $\mathbb{C}$-algebra homomorphism such that $\partial(h(u)) = h(T(u))$ for all $u \in J_\infty(R)$.)

Lemma 3.5 ([**Ein-Mustata09**]). *Let $m \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$. Then for every open subset U of X, $J_m(U) = \pi^{-1}_{X,m}(U)$.*

Then for a general scheme Y of finite type with an affine open covering $\{U_i\}_{i\in I}$, its arc space $J_\infty(Y)$ is obtained by glueing $J_\infty(U_i)$ (see [**Ein-Mustata09, Ishii11**]). In particular, the structure sheaf $\mathcal{O}_{J_\infty(Y)}$ is a sheaf of commutative vertex algebras.

The natural projection $\pi_{X,\infty}\colon J_\infty(X) \to X$ corresponds to the embedding $R \hookrightarrow J_\infty(R)$, $x^i \to x^i_{(-1)}$ in the case where $X = \operatorname{Spec} R$ is affine. In terms of arcs, $\pi_{X,\infty}(\alpha) = \alpha(0)$ for $\alpha \in \operatorname{Hom}_{Sch}(D, X)$, where 0 is the unique closed point of the formal disc D.

3.1.3. Basic properties. The map from a scheme to its jet schemes and arc space is functorial. If $f\colon X \to Y$ is a morphism of schemes, then we naturally obtain a morphism $J_m f\colon J_m(X) \to J_m(Y)$ making the following diagram commutative,

$$\begin{tikzcd} J_m(X) \arrow[r, "J_m f"] \arrow[d, "\pi_{X,m,0}"'] & J_m(Y) \arrow[d, "\pi_{Y,m,0}"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

In terms of arcs, it means that $J_m f(\alpha) = f \circ \alpha$ for $\alpha \in J_m(X)$. This also holds for $m = \infty$.

We have also the following for $m \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ and for every schemes X, Y,

$$J_m(X \times Y) \cong J_m(X) \times J_m(Y). \tag{23}$$

Indeed, for any scheme Z in Sch,

$$\begin{aligned}
\mathrm{Hom}(Z, J_m(X\times Y)) &= \mathrm{Hom}(Z\times_{\mathrm{Spec}_{\mathbb{C}}}\mathbb{C}[t]/(t^{m+1}), X\times Y)\\
&\cong \mathrm{Hom}(Z\times_{\mathrm{Spec}_{\mathbb{C}}}\mathbb{C}[t]/(t^{m+1}), X)\times \mathrm{Hom}(Z\times_{\mathrm{Spec}_{\mathbb{C}}}\mathbb{C}[t]/(t^{m+1}), Y)\\
&= \mathrm{Hom}(Z, J_m(X))\times \mathrm{Hom}(Z, J_m(Y))\\
&\cong \mathrm{Hom}(Z, J_m(X)\times J_m(Y)).
\end{aligned}$$

For $m=\infty$, just replace $\mathbb{C}[t]/(t^{m+1})$ with $\mathbb{C}[[t]]$ and take the completion in the product $Z\widehat{\times}\,\mathrm{Spec}\,\mathbb{C}[[t]] = Z\widehat{\times} D$.

If $\mathbf{A}$ is a group scheme over $\mathbb{C}$, then $J_m(\mathbf{A})$ is also a group scheme over $\mathbb{C}$. Moreover, by (23), if $\mathbf{A}$ acts on X, then $J_m(\mathbf{A})$ acts on $J_m(X)$.

Example 3.6. Consider the algebra

$$\mathfrak{g}_\infty := \mathfrak{g}[[t]] = \mathfrak{g}\otimes_{\mathbb{C}}\mathbb{C}[[t]] \cong J_\infty(\mathfrak{g}).$$

It is naturally a Lie algebra, with Lie bracket:

$$[xt^m, yt^n)] = [x,y]t^{m+n}, \qquad x,y\in\mathfrak{g},\ m,n\in\mathbb{Z}_{\geqslant 0}.$$

The arc space $J_\infty(\mathbf{G})$ of the algebraic group $\mathbf{G}$ is naturally a proalgebraic group. Regarding $J_\infty(\mathbf{G})$ as the set of $\mathbb{C}[[t]]$-points of $\mathbf{G}$, we have $J_\infty(\mathbf{G})=\mathbf{G}[[t]]$. As Lie algebras, we have

$$\mathfrak{g}_\infty \cong \mathrm{Lie}(J_\infty(\mathbf{G})).$$

The adjoint action of $\mathbf{G}$ on $\mathfrak{g}$ induces an action of $J_\infty(\mathbf{G})$ on $\mathfrak{g}_\infty$, and the coadjoint action of $\mathbf{G}$ on $\mathfrak{g}^*$ induces an action of $J_\infty(\mathbf{G})$ on $J_\infty(\mathfrak{g}^*)$, and so on $\mathbb{C}[J_\infty(\mathfrak{g}^*)]$.

We refer to [**Mustata01**, Appendix] for the following result.

Lemma 3.7. *For $f\in\mathbb{C}[\mathfrak{g}]^{\mathbf{G}}$, the polynomials $T^j f = f^{(j)}$, $j\geqslant 0$, are elements of $\mathbb{C}[\mathfrak{g}_\infty]^{J_\infty(\mathbf{G})}$. In particular, the arc space $J_\infty(\mathcal{N})$ of the nilpotent cone is the subscheme of $\mathfrak{g}_\infty$ defined by the equations T^jP_i, $i=1\ldots,r$ and $j\geqslant 0$, if $P_1,\ldots,P_r$ are homogeneous generators of $\mathbb{C}[\mathfrak{g}]^{\mathbf{G}}$, that is,*

$$J_\infty(\mathcal{N}) = \mathrm{Spec}\,\mathbb{C}[\mathfrak{g}_\infty]/(T^jP_i\ ;\ i=1\ldots,r,\ j\geqslant 0).$$

3.1.4. Geometrical results. So far, we have stated basic properties common for both jet schemes $J_m(X)$ and the arc space $J_\infty(X)$. For the geometry, arc spaces behave rather differently. The main reason is that $\mathbb{C}[[t]]$ is a domain, contrary to $\mathbb{C}[t]/(t^{m+1})$. Thereby the geometry of are spaces is somehow simpler.

However, although $J_m(X)$ is of finite type if X is, this is not anymore true for $J_\infty(X)$, and its coordinate ring $\mathbb{C}[J_\infty(X)]$ is not noetherian in general.

Lemma 3.8. *Denote by X_{red} the reduced scheme of X. The natural morphism $X_{\mathrm{red}}\to X$ induces an isomorphism $J_\infty X_{\mathrm{red}}\xrightarrow{\simeq}(J_\infty X)_{\mathrm{red}}$ of topological spaces.*

Proof. We may assume that $X=\mathrm{Spec}\,R$. An arc α of X corresponds to a ring homomorphism $\alpha^*\colon R\to\mathbb{C}[[t]]$. Since $\mathbb{C}[[t]]$ is an integral domain, it decomposes as $\alpha^*\colon R\to R/\sqrt{0}\to\mathbb{C}[[t]]$. Thus, α is an arc of X_{red}. □

Similarly, if $X=X_1\cup\ldots\cup X_r$, where all X_i are closed in X, then

$$J_\infty(X) = J_\infty(X_1)\cup\ldots\cup J_\infty(X_r).$$

(Note that Lemma 3.8 is false for the schemes $J_m(X)$.)

If X is a point, then $J_\infty(X)$ is also a point, since $\mathrm{Hom}(D,X)=\mathrm{Hom}(\mathbb{C},\mathbb{C}[[t]])$ consists of only one element. Thus, Lemma 3.8 implies the following.

COROLLARY 3.9. *If X is zero-dimensional then $J_\infty(X)$ is also zero-dimensional.*

THEOREM 3.10 ([**Kolchin73**]). *The scheme $J_\infty(X)$ is irreducible if X is irreducible.*

Theorem 3.10 is false for the jet schemes $J_m(X)$: see for instance [**Moreau-Yu16**] for counter-examples in the setting of nilpotent orbit closures. We refer to *loc. cit.*, and reference therein, for more about existing relations between the geometry of the jet schemes $J_m(X)$, $m \in \mathbb{Z}_{\geqslant 0}$, and the singularities of X.

3.2. Poisson vertex algebras

Let V be a commutative vertex algebra (cf. §2.2.1), or equivalently, a differential algebra. Recall that this means: $a_{(n)} = 0$ in $\mathrm{End}(V)$ for all $n \geqslant 0$.

3.2.1. Definition. A commutative vertex algebra V is called a *Poisson vertex algebra* if it is also equipped with a linear operation,

$$V \to \mathrm{Hom}(V, z^{-1}V[z^{-1}]), \quad a \mapsto a_-(z),$$

such that

$$(Ta)_{(n)} = -na_{(n-1)}, \tag{24}$$

$$a_{(n)}b = \sum_{j\geqslant 0} (-1)^{n+j+1} \frac{1}{j!} T^j (b_{(n+j)}a), \tag{25}$$

$$[a_{(m)}, b_{(n)}] = \sum_{j\geqslant 0} \binom{m}{j} (a_{(j)}b)_{(m+n-j)}, \tag{26}$$

$$a_{(n)}(b \cdot c) = (a_{(n)}b) \cdot c + b \cdot (a_{(n)}c) \tag{27}$$

for $a, b, c \in V$ and $n, m \geqslant 0$. Here, by abuse of notations, we have set

$$a_-(z) = \sum_{n\geqslant 0} a_{(n)} z^{-n-1}$$

so that the $a_{(n)}$, $n \geqslant 0$, are "new" operators, the "old" ones given by the field $a(z)$ being zero for $n \geqslant 0$ since V is commutative.

The equation (27) says that $a_{(n)}$, $n \geqslant 0$, is a derivation of the ring V. (Do not confuse $a_{(n)} \in \mathrm{Der}(V)$, $n \geqslant 0$, with the multiplication $a_{(n)}$ as a vertex algebra, which should be zero for a commutative vertex algebra.)

3.2.2. Poisson vertex structure on arc spaces.

THEOREM 3.11 ([**Arakawa12**, Proposition 2.3.1]). *Let X be an affine Poisson scheme, that is, $X = \operatorname{Spec} R$ for some Poisson algebra R. Then there is a unique Poisson vertex algebra structure on $J_\infty(R) = \mathbb{C}[J_\infty(X)]$ such that*

$$a_{(n)}b = \begin{cases} \{a, b\} & \text{if } n = 0 \\ 0 & \text{if } n > 0, \end{cases}$$

for $a, b \in R$.

PROOF. The uniqueness is clear by (10) since $J_\infty(R)$ is generated by R as a differential algebra. We leave it to the reader to check the well-definedness. Since $J_\infty(R)$ is generated by R, the formula $a_{(n)}b = \delta_{n,0}\{a, b\}$ for $a, b \in R$ is sufficient to define the fields on $J_\infty(R)$ by formulas (24), whence the existence. $\square$

REMARK 3.12. More generally, let X be a Poisson scheme which is not necessarily affine. Then the structure sheaf $\mathcal{O}_{J_\infty(X)}$ carries a unique Poisson vertex algebra structure such that

$$f_{(n)}g = \delta_{n,0}\{f,g\}$$

for $f, g \in \mathcal{O}_X \subset \mathcal{O}_{J_\infty(X)}$, see [**Arakawa-Kuwabara-Malikov**, Lemma 2.1.3.1].

EXAMPLE 3.13. Recall that the affine space $\mathfrak{g}^*$ is a Poisson variety by the Kirillov-Kostant-Souriau Poisson structure. If $\{x^1, \dots, x^N\}$ is a basis of $\mathfrak{g}$, then

$$\mathbb{C}[\mathfrak{g}^*] = \mathbb{C}[x^1, \dots, x^N].$$

Thus

$$J_\infty(\mathfrak{g}^*) = \operatorname{Spec} \mathbb{C}[x^i_{(-n)}\ ;\ i = 1, \dots, N,\ n \geqslant 1]. \tag{28}$$

So we may identify $\mathbb{C}[J_\infty(\mathfrak{g}^*)]$ with the symmetric algebra $S(\mathfrak{g}[t^{-1}]t^{-1})$ via

$$x_{(-n)} \longmapsto xt^{-n}, \qquad x \in \mathfrak{g},\ n \geqslant 1.$$

For $x \in \mathfrak{g}$, identify x with $x_{(-1)}|0\rangle = (xt^{-1})|0\rangle$, where we denote by $|0\rangle$ the unit element in $S(\mathfrak{g}[t^{-1}]t^{-1})$. Then (26) gives that

$$[x_{(m)}, y_{(n)}] = (x_{(0)}y)_{m+n} = \{x, y\}_{(m+n)} = [x, y]_{(m+n)}, \tag{29}$$

for $x, y \in \mathfrak{g} \cong (\mathfrak{g}^*)^* \subset \mathbb{C}[\mathfrak{g}^*] \subset \mathbb{C}[J_\infty(\mathfrak{g}^*)]$ and $m, n \in \mathbb{Z}_{\geq 0}$. So the Lie algebra $J_\infty(\mathfrak{g}) = \mathfrak{g}[[t]]$ acts on $\mathbb{C}[J_\infty(\mathfrak{g}^*)]$. This action coincides with that obtained by differentiating the action of $J_\infty(\mathbf{G}) = \mathbf{G}[[t]]$ on $J_\infty(\mathfrak{g}^*)$ induced by the coadjoint action of $\mathbf{G}$ (see Example 3.6). In other words, the Poisson vertex algebra structure of $\mathbb{C}[J_\infty(\mathfrak{g}^*)]$ comes from the $J_\infty(\mathbf{G})$-action on $J_\infty(\mathfrak{g}^*)$.

3.2.3. Canonical filtration and Poisson vertex structure. Our second basic example of Poisson vertex algebras comes from the graded vertex algebra associated with the canonical filtration, that is, the *Li filtration*.

Haisheng Li [**Li05**] has shown that *every* vertex algebra is canonically filtered: For a vertex algebra V, let F^pV be the subspace of V spanned by the elements

$$a^1_{(-n_1-1)}a^2_{(-n_2-1)} \cdots a^r_{(-n_r-1)}|0\rangle$$

with $a^1, a^2, \cdots, a^r \in V$, $n_i \geqslant 0$, $n_1 + n_2 + \cdots + n_r \geqslant p$. Then

$$V = F^0V \supset F^1V \supset \dots.$$

It is clear that $TF^pV \subset F^{p+1}V$.

Set

$$(F^pV)_{(n)}F^qV := \operatorname{span}_{\mathbb{C}}\{a_{(n)}b\ ;\ a \in F^pV,\ b \in F^qV\}.$$

Note that $F^1V = \operatorname{span}_{\mathbb{C}}\{a_{(-2)}b \mid a, b \in V\} = C_2(V)$.

LEMMA 3.14. *We have*

$$F^pV = \sum_{j \geqslant 0} (F^0V)_{(-j-1)}F^{p-j}V.$$

PROPOSITION 3.15. (1) $(F^pV)_{(n)}(F^qV) \subset F^{p+q-n-1}V$. *Moreover, if* $n \geqslant 0$, *we have* $(F^pV)_{(n)}(F^qV) \subset F^{p+q-n}V$. *Here we have set* $F^pV = V$ *for* $p < 0$.

(2) *The filtration* $F^\bullet V$ *is* separated, *that is,* $\bigcap_{p \geqslant 0} F^pV = \{0\}$, *if* V *is a positive energy representation, i.e., positively graded over itself.*

EXERCISE 3.16. The verifications are straightforward and are left as an exercise.

DEFINITION 3.17. A vertex algebra V is called *finitely strongly generated* if there exist finitely many elements $a^1, \dots, a^r$ in V such that V is spanned by the elements of the form

$$a^{i_1}_{(-n_1)} \dots a^{i_s}_{(-n_s)}|0\rangle$$

with $s \geqslant 0$, $n_i \geqslant 1$.

For example, the universal affine vertex algebra and the Virasoro vertex algebra are strongly finitely generated.

In this note we always assume that a vertex algebra V is finitely strongly generated. We will assume that V is conformal and positively graded, $V = \bigoplus_{n\in\mathbb{Z}_{\geqslant 0}} V_n$, so that the filtration $F^\bullet V$ is separated.

Set

$$\mathrm{gr}^F V = \bigoplus_{p\geqslant 0} F^p V/F^{p+1}V.$$

We denote by $\sigma_p \colon F^p V \mapsto F^p V/F^{p+1}V$, for $p \geqslant 0$, the canonical quotient map. When the filtration F is obvious, we often denote simply by $\mathrm{gr}\, V$ the space $\mathrm{gr}^F V$.

PROPOSITION 3.18 ([Li05]). *The space $\mathrm{gr}^F V$ is a Poisson vertex algebra by*

$$\begin{aligned}\sigma_p(a)\cdot\sigma_q(b) &:= \sigma_{p+q}(a_{(-1)}b),\\ T\sigma_p(a) &:= \sigma_{p+1}(Ta),\\ \sigma_p(a)_{(n)}\sigma_q(b) &:= \sigma_{p+q-n}(a_{(n)}b),\end{aligned}$$

for $a \in F^pV \setminus F^{p-n}V$, $b \in F^qV$, $n > 0$.

Set

$$R_V := F^0V/F^1V = V/C_2(V) \subset \mathrm{gr}\, V.$$

DEFINITION 3.19. The algebra R_V is called the *Zhu's C_2-algebra* of V. The algebra structure is given by:

$$\bar a\cdot\bar b := \overline{a_{(-1)}b}, \tag{30}$$

where $\bar a = \sigma_0(a)$.

PROPOSITION 3.20 ([Zhu96, Li05]). *The restriction of the vertex Poisson structure on $\mathrm{gr}^F V$ gives to the Zhu's C_2-algebra R_V a Poisson algebra structure, that is, R_V is a Poisson algebra by*

$$\bar a\cdot\bar b := \overline{a_{(-1)}b}, \qquad \{\bar a,\bar b\} = \overline{a_{(0)}b},$$

where $\bar a = \sigma_0(a)$.

PROOF. It is straightforward from Proposition 3.18. □

3.3. Associated variety of a vertex algebra

We now in a position to define the main object of study of the lecture.

3.3.1. Associated variety and singular support.

DEFINITION 3.21. Define the *associated scheme* $\tilde{X}_V$ and the *associated variety* X_V of a vertex algebra V as

$$\tilde{X}_V := \operatorname{Spec} R_V, \qquad X_V := \operatorname{Specm} R_V = (\tilde{X}_V)_{\mathrm{red}}.$$

It was shown in [**Li05**, Lemma 4.2] that $\mathrm{gr}^F V$ is generated by the subring R_V as a differential algebra. Thus, we have a surjection $J_\infty(R_V) \to \mathrm{gr}^F V$ of differential algebras by Remark 3.4 since R_V generates $J_\infty(R_V)$ as a differential algebra either. This is in fact a homomorphism of Poisson vertex algebras.

THEOREM 3.22 ([**Li05**, Lemma 4.2], [**Arakawa12**, Proposition 2.5.1]). *The identity map $R_V \to R_V$ induces a surjective Poisson vertex algebra homomorphism*

$$J_\infty(R_V) = \mathbb{C}[J_\infty(\tilde{X}_V)] \twoheadrightarrow \mathrm{gr}^F V.$$

DEFINITION 3.23. Define the *singular support* of a vertex algebra V as

$$SS(V) := \operatorname{Spec}(\mathrm{gr}^F V) \subset J_\infty(\tilde{X}_V).$$

THEOREM 3.24. *We have* $\dim SS(V) = 0$ *if and only if* $\dim X_V = 0$.

PROOF. The "only if" part is obvious since $\pi_{\tilde{X}_V,\infty}(SS(V)) = \tilde{X}_V$. The "if" part follows from Corollary 3.9. □

3.3.2. The lisse condition. A vertex algebra V is called *lisse* (or C_2*-cofinite*) if $R_V = V/C_2(V)$ is finite dimensional.

Thus by Theorem 3.24 we get:

LEMMA 3.25. *The vertex algebra V is lisse if and only if* $\dim X_V = 0$, *that is, if and only if* $\dim SS(V) = 0$.

REMARK 3.26. Suppose that V is $\mathbb{Z}_{\geqslant 0}$-graded by some Hamiltonian H, i.e., $V = \bigoplus_{i \geqslant 0} V_i$ with $V_i = \{x \in V \mid Hx = ix\}$, and that $V_0 = \mathbb{C}|0\rangle$. Then $\mathrm{gr}^F V$ and R_V are equipped with the induced grading:

$$\mathrm{gr}^F V = \bigoplus_{i \geqslant 0} (\mathrm{gr}^F V)_i, \quad (\mathrm{gr}^F V)_0 = \mathbb{C},$$
$$R_V = \bigoplus_{i \geqslant 0} (R_V)_i, \quad (R_V)_0 = \mathbb{C}.$$

So the following conditions are equivalent:

(1) V is lisse,
(2) $X_V = \{\text{point}\}$,
(3) the image of any vector $a \in V_i$ for $i \geqslant 1$ in $\mathrm{gr}^F V$ is nilpotent,
(4) the image of any vector $a \in V_i$ for $i \geqslant 1$ in R_V is nilpotent.

Thus, lisse vertex algebras can be regarded as a generalization of finite-dimensional algebras.

3.3.3. Comparison with weight-depending filtration. Let V be a vertex algebra that is $\mathbb{Z}$-graded by some Hamiltonian H (see §2.2.4):

$$V = \bigoplus_{\Delta \in \mathbb{Z}} V_\Delta \quad \text{where} \quad V_\Delta := \{v \in V \mid Hv = \Delta v\}.$$

Then there is another natural filtration of V defined as follows [**Li04**].

Let G_pV be the subspace of V spanned by the vectors

$$a^1_{(-n_1-1)}a^2_{(-n_2-1)}\cdots a^r_{(-n_r-1)}|0\rangle$$

with $a^i \in V$ homogeneous, $\Delta_{a^1}+\cdots+\Delta_{a^r}\leqslant p$. Then $G_\bullet V$ defines an increasing filtration of V:

$$0=G_{-1}V\subset G_0V\subset \dots G_1V\subset \dots,\quad V=\bigcup_p G_pV.$$

Moreover we have

$$\begin{aligned}&TG_pV\subset G_pV,\\ &(G_p)_{(n)}G_qV\subset G_{p+q}V\quad \text{for } n\in\mathbb{Z},\\ &(G_p)_{(n)}G_qV\subset G_{p+q-1}V\quad \text{for } n\in\mathbb{Z}_{\geqslant 0},\end{aligned}$$

It follows that $\mathrm{gr}_G V=\bigoplus G_pV/G_{p-1}V$ is naturally a Poisson vertex algebras.

It is not too difficult to see the following.

Lemma 3.27 ([**Arakawa12**, Proposition 2.6.1]). *We have*

$$F^pV_\Delta = G_{\Delta-p}V_\Delta,$$

where $F^pV_\Delta=V_\Delta\cap F^pV$, $G_pV_\Delta=V_\Delta\cap G_pV$. *Therefore*

$$\mathrm{gr}^F V\cong \mathrm{gr}_G V$$

as Poisson vertex algebras.

Proposition 3.28 ([**Arakawa12**, Corollary 2.6.2]). *A vertex algebra V is finitely strongly generated if and only if R_V is finitely generated as a ring.*

If the images of some vectors $a^1,\dots,a^r\in V$ in R_V generate R_V, we say that V is *strongly generated by* $a^1,\dots,a^r$.

Proof. Suppose that $a^1,\dots,a^r$ are strong generators of V. By Lemma 3.27, $C_2(V)=F^1V$ is spanned by the vectors $a^{i_1}_{(-n_1-1)}\cdots a^{i_s}_{(-n_s-1)}|0\rangle$ with $s\geqslant 1$ and $n_1+\cdots+n_s\geqslant 1$. Thus $\{\bar a^1,\dots,\bar a^r\}$ generates R_V, where $\bar a^i$ is the image of a^i in R_V.

Conversely, suppose that $\{\bar a^1,\dots,\bar a^r\}$ generates R_V. Then by Theorem 3.22, $\{\bar a^1,\dots,\bar a^r\}$ generates $\mathrm{gr}^F V$ as a differential algebra. Since $\mathrm{gr}^F V\cong V$ as $\mathbb{C}$-vector spaces by the assumption that $F^\bullet V$ is separated, it follows that $\{a^1,\dots,a^r\}$ strongly generates V. $\square$

Remark 3.29. In fact a stronger fact is known: V is spanned by the above vectors with $r\geqslant 0$, $n_1>n_2>n_3>\ldots\geqslant 1$, see [**Gaberdiel-Neitzke03**], [**Li05**, Theorem 4.7].

3.3.4. Universal affine vertex algebras. Consider the universal affine vertex algebra $V^\kappa(\mathfrak{a})$ defined by (13) as in §2.2.2.

Recall that we have $F^1V^\kappa(\mathfrak{a})=\mathfrak{a}[t^{-1}]t^{-2}V^\kappa(\mathfrak{a})$, and a Poisson algebra isomorphism

$$\begin{aligned}\mathbb{C}[\mathfrak{a}^*]&\xrightarrow{\simeq} R_{V^\kappa(\mathfrak{a})}=V^\kappa(\mathfrak{a})/t^{-2}\mathfrak{a}[t^{-1}]V^\kappa(\mathfrak{a})\\ x_1\dots x_r&\longmapsto x_1t^{-1}\dots x_rt^{-1}|0\rangle+t^{-2}\mathfrak{a}[t^{-1}]V^\kappa(\mathfrak{a})\quad (x_i\in\mathfrak{a}).\end{aligned}\tag{31}$$

Thus

$$X_{V^{\kappa}(\mathfrak{a})} = \mathfrak{a}^*.$$

We have the isomorphism

$$\mathbb{C}[J_\infty(\mathfrak{a}^*)] \cong \operatorname{gr} V^{\kappa}(\mathfrak{a}). \tag{32}$$

Indeed, the graded dimensions of both sides coincide. Moreover,

$$G_p V^{\kappa}(\mathfrak{a}) = U_p(\mathfrak{a}[t^{-1}]t^{-1})|0\rangle,$$

where $\{U_p(\mathfrak{a}[t^{-1}]t^{-1})\}_p$ is the PBW filtration of $U(\mathfrak{a}[t^{-1}]t^{-1})$, and we have the isomorphisms

$$\operatorname{gr} U(\mathfrak{a}[t^{-1}]t^{-1}) \cong S(\mathfrak{a}[t^{-1}]t^{-1}) \cong \mathbb{C}[J_\infty(\mathfrak{a}^*)].$$

As a consequence of (32), we get

$$SS(V^{\kappa}(\mathfrak{a})) = J_\infty(\mathfrak{a}^*).$$

EXAMPLE 3.30. Let Vir^c be the universal Virasoro vertex algebra with central charge c as in §2.2.3. Any Vir-module with central c (i.e., the central element C of Vir acts as a multiplication by c) on which $L(z)$ is a field can be considered as a Vir^c-module.

EXERCISE 3.31. Show that

$$R_{\mathrm{Vir}^c} \cong \mathbb{C}[x],$$

with the trivial Poisson structure, where x is the image of $L_{-2}|0\rangle$.

3.4. Lisse and quasi-lisse vertex algebras

3.4.1. Lisse vertex algebras. A vertex algebra V is called *lisse* if $\dim X_V = 0$, or equivalently, if R_V is finite-dimensional.

EXAMPLE 3.32. The simple affine vertex algebra $L_k(\mathfrak{g})$ is lisse if and only if $L_k(\mathfrak{g})$ is integrable as a $\widehat{\mathfrak{g}}$-module, or equivalently, $k \in \mathbb{Z}_{\geqslant 0}$.

Therefore, the lisse condition generalizes the integrability to an arbitrary vertex algebra.

EXAMPLE 3.33. Let N_c be the unique maximal submodule of the Virasoro vertex algebra Vir^c, and $\mathrm{Vir}_c = \mathrm{Vir}^c/N_c$ the unique quotient.
By [**Arakawa12**, Proposition 3.4.1], the following are equivalent:

(i) Vir_c is lisse,
(ii) $c = 1 - \dfrac{6(p-q)^2}{pq}$ for some $p, q \in \mathbb{Z}_{\geqslant 2}$ such that $(p, q) = 1$. (These are precisely the central charge of the *minimal series representations* of the Virasoro algebra Vir.)

It is known that lisse vertex algebras have various nice properties. As an example, we state the following remarkable result.

THEOREM 3.34 ([**Abe-Buhl-Dong04**, **Zhu96**, **Miyamoto04**]). *Let V be lisse.*

(1) *Any simple V-module is a positive energy representation. Therefore the number of isomorphic classes of simple V-modules is finite.*

(2) *Let $M_1, \dots, M_s$ be representatives of these classes, and let for $i = 1, \dots, s$,*

$$\chi_{M_i}(\tau) = \mathrm{Tr}_{M_i}(q^{L_0 - \frac{c}{24}}) = \sum_{n \geqslant 0} \dim(M_i)_n q^{n - \frac{c}{24}}, \quad q = e^{2i\pi\tau},$$

be the normalized character of M_i. *Then $\chi_{M_i}(\tau)$ converges in the domain $\{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$, and the vector space generated by $SL_2(\mathbb{Z}).\chi_{M_i}(\tau)$ is finite-dimensional.*

Further, if it is also *rational*, it is known [**Huang08**] that under some mild assumptions, the category of V-modules forms a modular tensor category, which for instance yields an invariant of 3-manifolds, see [**Bakalov-Kirillov01**].

DEFINITION 3.35. A conformal vertex algebra V is called *rational* if every $\mathbb{Z}_{\geqslant 0}$-graded V-modules is completely reducible (i.e., isomorphic to a direct sum of simple V-modules).

It is known ([**Dong-Li-Mason98**]) that this condition implies that V has finitely many simple $\mathbb{Z}_{\geqslant 0}$-graded modules and that the graded components of each of these $\mathbb{Z}_{\geqslant 0}$-graded modules are finite dimensional.

In fact lisse vertex algebras also verify this property (see Theorem 3.34). It is actually conjectured by [**Zhu96**] that rational vertex algebras must be lisse (this conjecture is still open).

However, there are significant vertex algebras that do not satisfy the lisse condition. For instance, an *admissible* affine vertex algebra $L_k(\mathfrak{g})$ (see below) has a complete reducibility property ([**Arakawa15b**]) and the modular invariance property ([**Kac-Wakimoto89**]) in the category $\mathcal{O}$ still holds, although it is not lisse unless it is integrable.

So it is natural to try to relax the lisse condition.

3.4.2. Symplectic stratification and quasi-lisse vertex algebras. An *affine Poisson scheme* (resp. *affine Poisson variety)* is an affine scheme $X = \mathrm{Spec}\, A$ (resp. $X = \mathrm{Specm}\, A$) such that A is a Poisson algebra. Let X be a *Poisson scheme*, that is, a scheme such that the structure sheaf $\mathcal{O}_X$ is a sheaf of Poisson algebras.

If X is smooth, then one may view X as a complex-analytic manifold equipped with a holomorphic Poisson structure. For each point $x \in X$ one defines the *symplectic leaf $\mathcal{S}_x$ through* x to be the set of points that could be reached from x by going along Hamiltonian flows[1].

If X is not necessarily smooth, let $\mathrm{Sing}(X)$ be the singular locus of X, and for any $k \geqslant 1$ define inductively $\mathrm{Sing}^k(X) := \mathrm{Sing}(\mathrm{Sing}^{k-1}(X))$. We get a finite partition $X = \bigsqcup_k X^k$, where the strata $X^k := \mathrm{Sing}^{k-1}(X) \setminus \mathrm{Sing}^k(X)$ are smooth analytic varieties (by definition we put $X^0 = X \setminus \mathrm{Sing}(X)$). It is known (cf. e.g., [**Brown-Gordon03**]) that each X^k inherits a Poisson structure. So for any point $x \in X^k$ there is a well defined symplectic leaf $\mathcal{S}_x \subset X^k$. In this way one defines symplectic leaves on an arbitrary Poisson variety. In general, each symplectic leaf is a connected smooth analytic (but not necessarily algebraic) subset in X. However, if the algebraic variety X consists of finitely many symplectic leaves only, then it

[1] A *Hamiltonian flow* in X from x to x' is a curve γ defined on an open neighborhood of $[0,1]$ in $\mathbb{C}$, with $\gamma(0) = x$ and $\gamma(1) = x'$, which is an integral curve of a Hamiltonian vector field ξ_f, for some $f \in \mathcal{O}(X)$, defined on an open neighborhood of $\gamma([0,1])$. See for example [**Laurent-Pichereau-Vanhaecke**, Chapter 1] for more details.

was shown in [**Brown-Gordon03**] that each leaf is a smooth irreducible locally-closed algebraic subvariety in X, and the partition into symplectic leaves gives an algebraic stratification of X.

Example 3.36. The space $\mathfrak{g}^*$ is a (smooth) Poisson variety and the symplectic leaves of $\mathfrak{g}^*$ are the coadjoint orbits of $\mathfrak{g}^*$. The nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$ is an example of Poisson variety with finitely many symplectic leaves. The latter are precisely the nilpotent orbits of $\mathfrak{g}^* \cong \mathfrak{g}$.

Definition 3.37. A vertex algebra is called *quasi-lisse* if X_V has only finitely many symplectic leaves.

Clearly, lisse vertex algebras are quasi-lisse.

Example 3.38. Since symplectic leaves in $X_{L_k(\mathfrak{g})}$ are the coadjoint $\mathbf{G}$-orbits contained in $X_{L_k(\mathfrak{g})}$, it follows that $L_k(\mathfrak{g})$ is quasi-lisse if and only if $X_{L_k(\mathfrak{g})} \subset \mathcal{N}$.

Important examples of quasi-lisse simple affine vertex algebras are the *admisible representations* $L_k(\mathfrak{g})$, as we explain next paragraph.

3.4.3. Admissible representations. Let $\hat{\Delta}^{re}$ be the set of real roots of $\widehat{\mathfrak{g}}$, and $\hat{\Delta}^{re}_+$ the set of real positive roots with respect to the triangular decomposition (5) (cf. §1.1.5).

Definition 3.39 ([**Kac-Wakimoto89, Kac-Wakimoto08**]). A weight $\lambda \in \widehat{\mathfrak{h}}^*$ is called *admissible* if

(1) λ is *regular dominant*, that is,
$$\langle \lambda + \hat{\rho}, \alpha^\vee \rangle \notin -\mathbb{Z}_{\geqslant 0} \quad \text{for all} \quad \alpha \in \hat{\Delta}^{re}_+,$$

(2) $\mathbb{Q}\hat{\Delta}_\lambda = \mathbb{Q}\hat{\Delta}^{re}$, where $\hat{\Delta}_\lambda := \{\alpha \in \hat{\Delta}^{re} \mid \langle \lambda + \hat{\rho}, \alpha^\vee \rangle \in \mathbb{Z}\}$ with $\hat{\rho} = h^\vee \Lambda_0 + \rho$.

The irreducible highest weight representation $L(\lambda)$ of $\widehat{\mathfrak{g}}$ with highest weight $\lambda \in \widehat{\mathfrak{h}}^*$ is called *admissible* if λ is admissible. Note that an irreducible integrable representation of $\widehat{\mathfrak{g}}$ is admissible.

The simple affine vertex algebra $L_k(\mathfrak{g})$ is called *admissible* if it is admissible as a $\widehat{\mathfrak{g}}$-module. This happens if and only if k satisfies one of the following conditions:

(1) $k = -h^\vee + \dfrac{p}{q}$ where $p, q \in \mathbb{Z}_{>0}$, $(p,q) = 1$, and $p \geqslant h^\vee$,

(2) $k = -h^\vee + \dfrac{p}{r^\vee q}$ where $p, q \in \mathbb{Z}_{>0}$, $(p,q) = 1$, $(p, r^\vee) = 1$ and $p \geqslant h$.

Here $r^\vee$ is the *lacety* of $\mathfrak{g}$ (i.e., $r^\vee = 1$ for the types A, D, E, $r^\vee = 2$ for the types B, C, F and $r^\vee = 3$ for the type G_2), h and $h^\vee$ are the Coxeter and dual Coxeter numbers.

Definition 3.40. If k satisfies one of the conditions of Proposition **??**, we say that k is an *admissible level*.

The following fact was conjectured by Feigin and Frenkel and proved for the case that $\mathfrak{g} = \mathfrak{sl}_2$ by Feigin and Malikov [**Feigin-Malikov97**].

Theorem 3.41 ([**Arakawa15a**]). *If k is admissible then $SS(L_k(\mathfrak{g})) \subset J_\infty(\mathcal{N})$ or, equivalently, the associated variety $X_{L_k(\mathfrak{g})}$ is contained in $\mathcal{N}$.*

In fact, the following stronger result holds.

THEOREM 3.42 ([**Arakawa15a**]). *Assume that k is admissible. Then*

$$X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_q},$$

where $\mathbb{O}_q$ is a nilpotent orbit which only depends on q, with q as above.

REMARK 3.43. For $\mathfrak{g} = \mathfrak{sl}_n$, Theorem 3.42 gives the following. Let k be admissible, and let $q \in \mathbb{Z}_{>0}$ be the denominator of k, that is, $k + h^\vee = p/q$, with $p \in \mathbb{Z}_{>0}$ and $(p, q) = 1$. Then

$$X_{L_k(\mathfrak{g})} = \{x \in \mathfrak{g} \mid (\operatorname{ad} x)^{2q} = 0\} = \overline{\mathbb{O}_q},$$

where $\mathbb{O}_q$ is the nilpotent orbit corresponding to the partition

$$\begin{cases} (n) & \text{if } q \geqslant n, \\ (q, q, \ldots, q, s) \quad (0 \leqslant s \leqslant q-1) & \text{if } q < n. \end{cases}$$

Remind that $h^\vee = n$ for $\mathfrak{g} = \mathfrak{sl}_n$.

3.4.4. Exceptional Deligne series. There was actually a "strong Feigin-Frenkel conjecture" stating that k is admissible if and only if $X_{L_k(\mathfrak{g})} \subset \mathcal{N}$ (provided that k is not critical, that is, $k \neq -h^\vee$ in which case it is known that $X_{L_k(\mathfrak{g})} = \mathcal{N}$). Such a statement would be interesting because it would give a geometrical description of the admissible representations $L_k(\mathfrak{g})$.

This stronger conjecture is actually wrong, as shown the following.

THEOREM 3.44 ([**Arakawa-Moreau15**]). *Assume that $\mathfrak{g}$ belongs to the Deligne exceptional series* [**Deligne96**],

$$A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8,$$

and that $k = -\dfrac{h^\vee}{6} - 1 + n$, $n \in \mathbb{Z}_{\geqslant 0}$ such that $k \notin \mathbb{Z}_{\geqslant 0}$. Then

$$X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_{min}}.$$

Note that the level $k = -h^\vee/6 - 1$ is not admissible for the types D_4, E_6, E_7, E_8. Theorem 3.44 provides the first known examples of associated varieties contained in the nilpotent cone corresponding to non-admissible levels.

The proof of this result is closely related to the Joseph primitive ideal, and its description by Gan and Savin [**Gan-Savin04**], associated with the minimal nilpotent orbit.

Note that the condition $X_{L_k(\mathfrak{g})} \subset \mathcal{N}$ implies that $L_k(\mathfrak{g})$ has only finitely many simple objects in the category $\mathcal{O}$, and one can describe them thanks to Joseph's classification of irreducible highest weights representation $L_{\mathfrak{g}}(\lambda)$ whose associated variety is $\overline{\mathbb{O}_{min}}$.

3.4.5. Other examples. There are other examples of simple quasi-lisse affine vertex algebra $L_k(\mathfrak{g})$, with non-admissible level k, in type D_r, $r \geqslant 5$, and in type B_r, $r \geqslant 3$; see [**Arakawa-Moreau15, Arakawa-Moreau16, Arakawa-Moreau17**].

Except for $\mathfrak{g} = \mathfrak{sl}_2$, the classification problem of quasi-lisse vertex affine algebras is wide open.

3.4.6. Sheets as associated variety. In all the above examples, $X_{L_k(\mathfrak{g})}$ is a closure of a some nilpotent orbit $\mathbb{O} \subset \mathcal{N}$, or $X_{L_k(\mathfrak{g})} = \mathfrak{g}^*$. The later happens if k is generic, that is, $k \notin \mathbb{Q}$ in which case $L_k(\mathfrak{g}) = V^k(\mathfrak{g})$. Therefore it is natural to ask whether there are cases when $X_{L_k(\mathfrak{g})} \not\subset \mathcal{N}$ and $X_{L_k(\mathfrak{g})}$ is a proper subvariety of $\mathfrak{g}^*$.

Given $m \in \mathbb{N}$, let $\mathfrak{g}^{(m)}$ be the set of elements $x \in \mathfrak{g}$ such that $\dim \mathfrak{g}^x = m$, with $\mathfrak{g}^x$ the centralizer of x in $\mathfrak{g}$. A subset $\mathbb{S} \subset \mathfrak{g}$ is called a *sheet* of $\mathfrak{g}$ if it is an irreducible component of one of the locally closed sets $\mathfrak{g}^{(m)}$. It is G-invariant and conic and it is smooth if $\mathfrak{g}$ is classical. The description of sheets is closely related to the Jordan classes. A sheet is a finite disjoint union of *Jordan classes*, [**Borho-Kraft79**] (or [**Tauvel-Yu**, 39.3.4]). As a consequence, the sheet closures are the closures of certain Jordan classes and they are parameterized by the G-conjugacy classes of pairs $(\mathfrak{l}, \mathbb{O}_{\mathfrak{l}})$ where $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$ and $\mathbb{O}_{\mathfrak{l}}$ is a *rigid* nilpotent orbit of $\mathfrak{l}$, i.e., which cannot be properly induced in the sense of Lusztig-Spaltenstein [**Borho-Kraft79, Borho81**] (see also [**Tauvel-Yu**, §39]). The pair $(\mathfrak{l}, \mathbb{O}_{\mathfrak{l}})$ is called the *datum* of the corresponding sheet. When $\mathbb{O}_{\mathfrak{l}}$ is zero, the sheet is called *Dixmier*, meaning that it contains a semisimple element. We denote by $\mathbb{S}_{\mathfrak{l}}$ the sheet with datum $(\mathfrak{l}, \{0\})$.

It is known that sheets appear in the representation theory of finite-dimensional Lie algebras, see, e.g., [**Borho-Brylinski82, Borho-Brylinski85, Borho-Brylinski89**], and more recently of finite W-algebras, [**Premet-Topley14, Premet14**].

Next result is that sheets also appear as associated varieties of some affine vertex algebras.

THEOREM 3.45 ([**Arakawa-Moreau16**]). (1) *For $n \geqslant 4$,*

$$\tilde{X}_{V_{-1}(\mathfrak{sl}_n)} \cong \overline{\mathbb{S}_{min}}$$

as schemes, where $\mathbb{S}_{min}$ is the unique sheet containing $\mathbb{O}_{min}$. Moreover, as schemes,

$$SS(V_{-1}(\mathfrak{sl}_n) = J_\infty(\overline{\mathbb{S}_{min}}).$$

(2) *For $m \geqslant 2$,*

$$\tilde{X}_{V_{-m}(\mathfrak{sl}_{2m})} \cong \overline{\mathbb{S}_0}$$

as schemes, where $\mathbb{S}_0$ is the unique sheet containing the nilpotent orbit $\mathbb{O}_{(2^m)}$. Moreover, as schemes,

$$SS(V_{-m}(\mathfrak{sl}_{2m}) = J_\infty(\overline{\mathbb{S}_0}).$$

(3) *Let r be an odd integer. Then*

$$X_{V_{2-r}(\mathfrak{so}_{2r})} = \overline{\mathbb{S}_{I_r}},$$

where $\mathbb{S}_{I_r}$ is the unique sheets containing the nilpotent orbits $\mathbb{O}_{(2^{r-1},1^2)}$.

By the Irreducibility Theorem, associated varieties of primitive ideals are irreducible and contained in the nilpotent cone. Theorem 3.45 shows that this is not anymore the case for affine vertex algebras.

3.4.7. Conjectures, open problems. In view of the above results, and other results, particularly, on associated varieties of simple affine W-algebras (cf. Part 4), we formulate a conjecture.

CONJECTURE 1 ([**Arakawa-Moreau16**, Conjecture 1]). *Let $V = \oplus_{d \geqslant 0} V_d$ be a simple, finitely strongly generated, positively graded conformal vertex operator algebra such that $V_0 = \mathbb{C}$.*

(1) *X_V is equidimensional.*
(2) *Assume that X_V has finitely many symplectic leaves, that is, V is quasi-lisse. Then X_V is irreducible. In particular, $X_{V_k(\mathfrak{g})}$ is irreducible if $X_{V_k(\mathfrak{g})} \subset \mathcal{N}$.*

Part (1) of the conjecture is an analog of the equidimentionality theorems of Gabber and Kashiwara [**Ka76**]. Part (2) of the conjecture is a natural analog of the above mentioned irreducibility result for the associated variety of primitive ideals of $U(\mathfrak{g})$. Note that this irreducibility theorem has been generalized to a large class of Noetherian algebras by Ginzburg [**Gi03**].

Theorem 3.46 ([**Gi03**]). *Let A be a filtered unital $\mathbb{C}$-algebra. Assume furthermore that $\operatorname{gr} A \cong \mathbb{C}[X]$ is the coordinate ring of a reduced irreducible affine algebraic variety X, and assume that the Poisson variety $\operatorname{Spec}(\operatorname{gr} A)$ has only finitely many symplectic leaves. Then for any primitive ideal $I \subset A$, the variety $\mathscr{V}(I)$ is the closure of a single symplectic leaf. In particular, it is irreducible.*

However, our algebra is not Noetherian.

On the other hand, note that scheme-theoretic intersections of Slodowy slices with associated varieties of simple affine vertex algebras appear as associated varieties of W-algebras [**Arakawa15a**]. These intersections are either empty of complete intersections [**Ginzburg09**]. This gives an evidence to Part (1) of the conjecture. However, these intersections are not always irreducible, and Part (2) of the conjecture is discussed in [**Arakawa-Moreau17**] (see also Part 4).

To conclude this section, note that there are other known examples of associated varieties with finitely many symplectic leaves: apart from the above examples, it is the case when V is the (generalized) Drinfeld-Sokolov reduction (see Part 4) of the above affine vertex algebras provided that it is nonzero ([**Arakawa15a**]). This is also expected to happen for the vertex algebras obtained from four dimensional $N = 2$ superconformal field theories ([**BLL$^+$**]), where the associated variety is expected to coincide with the spectrum of the chiral ring of the Higgs branch of the four dimensional theory. Of course, it also happens when the associated variety of V is a point, that is, when V is lisse (see Section 4.4 for more details).

PART 4

Associated varieties of affine W-algebras

The study of affine W-algebras began with the work of Zamolodchikov in 1985. Mathematically, affine W-algebras are defined by the method of quantized Drinfeld-Sokolov reduction that was discovered by Feigin and Frenkel in the 1990s. The general definition of affine W-algebras were given by Kac, Roan and Wakimoto in 2003. Affine W-algebras are related with integrable systems, the two-dimensional conformal field theory and the geometric Langlands program. The most recent developments in representation theory of affine W-algebras were done by Kac-Wakimoto and Arakawa.

The affine W-algebras are certain vertex algebras associated with nilpotent elements of simple Lie algebras. They can be regarded as affinizations of finite W-algebras, and can also be considered as generalizations of affine Kac-Moody algebras and Virasoro algebras. They quantize the arc space of the Slodowy slices associated with nilpotent elements.

Since they are not finitely generated by Lie algebras, the formalism of vertex algebras is necessary to study them. In this context, associated varieties of W-algebras, and their quotients, are important tools to understand some properties, such as the lisse condition and even the rationality condition.

The definition of W-algebras is quite technical and need a number a backgrounds. We do not give the precise definition in this lecture, and refer for instance to [**Arakawa-lectures**] or [**Arakawa-Moreau-lectures**], and references therein, for more details.

4.1. Slodowy slices

Let f be a nilpotent element of $\mathfrak{g}$ that we embed into an $\mathfrak{sl}_2$-triple (e,h,f) of $\mathfrak{g}$. Let $\phi\colon \mathfrak{g}\to\mathfrak{g}^*$ be the isomorphism induced from the non-degenerate bilinear form $(\ |\ )$, and set

$$\chi := \phi(f) = (f|\cdot\,) \in \mathfrak{g}^*.$$

Then define the *Slodowy slice associated with* (e,h,f) to be,

$$\mathscr{S}_f := \phi(f+\mathfrak{g}^e) = \chi + \phi(\mathfrak{g}^e) \subset \mathfrak{g}^*.$$

Denote by $\mathfrak{g}_i$ the i-eigenspace of $\operatorname{ad}(h)$ for $i\in\mathbb{Z}$,

$$\mathfrak{g}_i = \{x\in\mathfrak{g} \mid [h,x]=ix\},\quad i\in\mathbb{Z}.$$

The restriction of the antisymmetric bilinear form,

$$\omega_\chi\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{C},\qquad (x,y)\mapsto (f|[x,y]),$$

to $\mathfrak{g}_{\frac12}\times\mathfrak{g}_{\frac12}$ is nondegenerate. This results from the paring between $\mathfrak{g}_{\frac12}$ and $\mathfrak{g}_{-\frac12}$, and from the injectivity of the map $\operatorname{ad} f\colon \mathfrak{g}_{\frac12}\to\mathfrak{g}_{-\frac12}$. It is called the *Kirillov form* *associated with* f. Let $\mathfrak{L}$ be a Lagrangian subspace of $\mathfrak{g}_{\frac{1}{2}}$, that is, $\mathfrak{L}$ is maximal isotropic which means $\omega_\chi(\mathfrak{L},\mathfrak{L}) = 0$ and $\dim\mathfrak{L} = \frac{1}{2}\dim\mathfrak{g}_{\frac{1}{2}}$. Set

$$\mathfrak{m} = \mathfrak{m}_{\chi,\mathfrak{L}} := \mathfrak{L} \oplus \bigoplus_{j>\frac{1}{2}} \mathfrak{g}_j.$$

Then $\mathfrak{m}$ is an ad-nilpotent[1], $\operatorname{ad} h$-graded subalgebra, of $\mathfrak{g}$. Moreover, the algebra $\mathfrak{m}$ verifies the following properties:

(χ1) $\chi([\mathfrak{m},\mathfrak{m}]) = (f|[\mathfrak{m},\mathfrak{m}]) = 0$,
(χ2) $\mathfrak{m} \cap \mathfrak{g}^f = \{0\}$;
(χ3) $\dim\mathfrak{m} = \frac{1}{2}\dim\mathbf{G}.f$.

Let $\mathbf{M}$ be the unipotent subgroup of $\mathbf{G}$ corresponding to $\mathfrak{m}$.

4.1.1. Contracting $\mathbb{C}^*$-action. Let us introduce a $\mathbb{C}^*$-action on $\mathfrak{g}$ which stabilizes $\mathscr{S}_f \cong f + \mathfrak{g}^e$. The embedding $\operatorname{span}_{\mathbb{C}}\{e,h,f\} \cong \mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$ exponentiates to a homomorphism $SL_2 \to \mathbf{G}$. By restriction to the 1-dimensional torus consisting of diagonal matrices, we obtain a one-parameter subgroup $\rho\colon \mathbb{C}^* \to \mathbf{G}$. Thus $\rho(t)x = t^{2j}x$ for any $x \in \mathfrak{g}_j$. For $t \in \mathbb{C}^*$ and $x \in \mathfrak{g}$, set

$$\tilde\rho(t)x := t^2\rho(t)(x). \tag{33}$$

So, for any $x \in \mathfrak{g}_j$, $\tilde\rho(t)x = t^{2+2j}x$. In particular, $\tilde\rho(t)f = f$ and the $\mathbb{C}^*$-action of $\tilde\rho$ stabilizes $\mathscr{S}_f$. Moreover, it is contracting to f on $\mathscr{S}_f$, that is,

$$\lim_{t\to 0}\tilde\rho(t)(f+x) = f$$

for any $x \in \mathfrak{g}^e$, because $\mathfrak{g}^e \subset \mathfrak{m}^\perp \subseteq \mathfrak{g}_{>-1}$. The same lines of arguments show that the action $\tilde\rho$ stabilizes $f + \mathfrak{m}^\perp$ and it is contracting to f on $f + \mathfrak{m}^\perp$, too.

The affine space $\mathscr{S}_f$ is a "slice" according to the following result.

THEOREM 4.1. *The affine space $\mathscr{S}_f$ is transversal to the coadjoint orbits of $\mathfrak{g}^*$. More precisely, for any $\xi \in \mathscr{S}_f$, $T_\xi(\mathbf{G}.\xi) + T_\xi(\mathscr{S}_f) = \mathfrak{g}^*$. An analogue statement holds for the affine variety $\chi + \mathfrak{m}^\perp$.*

SKETCH OF PROOF. We have to prove that $[\mathfrak{g},x] + \mathfrak{g}^e = \mathfrak{g}$ for any $x \in f + \mathfrak{g}^e$ since $T_x(\mathbf{G}.x) = [\mathfrak{g},x]$ and $T_x(f+\mathfrak{g}^e) = \mathfrak{g}^e$.

It suffices to verify that the map

$$\eta\colon \mathbf{G} \times (f + \mathfrak{g}^e) \to \mathfrak{g}$$

is a submersion at any point (g,x) of $\mathbf{G} \times (f+\mathfrak{g}^e)$, that is, the differential $d\eta_{(g,x)}$ of η at (g,x) is surjective for any point (g,x) of $\mathbf{G}\times(f+\mathfrak{g}^e)$. The differential of η is the linear map $\mathfrak{g}\times\mathfrak{g}^e \to \mathfrak{g}$, $(v,w) \mapsto g([v,x]) + g(w)$.

Thus $d\eta_{(\mathrm{Id},f)}(v,w) = [v,f]+w$. Hence $d\eta_{(\mathrm{Id},f)}$ is surjective since $[\mathfrak{g},f]+\mathfrak{g}^e = \mathfrak{g}$. Thus $d\eta_{(\mathrm{Id},x)}$ is surjective for any x in an open neighborhood Ω of f in $f+\mathfrak{g}^e$. Since the morphism η is $\mathbf{G}$-equivariant for the action by left multiplication, we deduce that $d\eta_{(g,x)}$ is surjective for any $g \in \mathbf{G}$ and any $x \in \Omega$.

In particular, for any $x \in \Omega$, we get

$$\mathfrak{g} = [\mathfrak{g},x] + \mathfrak{g}^e$$

Next, we use the contracting $\mathbb{C}^*$-action ρ on $f + \mathfrak{g}^e$ to show that η is actually a submersion at any point of $\mathbf{G}\times(f+\mathfrak{g}^e)$. $\square$

[1] i.e., $\mathfrak{m}$ only consists of nilpotent elements of $\mathfrak{g}$.

4.1.2. An isomorphism. Consider the adjoint map

$$\mathbf{M} \times (f + \mathfrak{m}^\perp) \to \mathfrak{g}, \quad (g, x) \mapsto g.x$$

It image is contained in $f+\mathfrak{m}^\perp$. Indeed, for any $x \in \mathfrak{n}$ and any $y \in \mathfrak{m}^\perp$, $\exp(\operatorname{ad} x)(f+y) \in f + \mathfrak{m}^\perp$ since $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{m}$ and $\chi([\mathfrak{m}, \mathfrak{m}]) = 0$, and this is enough to conclude because, $\mathfrak{m}$ being ad-nilpotent, $\mathbf{M}$ is generated by the elements $\exp(\operatorname{ad} x)$ for x running through $\mathfrak{m}$. As a result, by restriction, we get a map

$$\alpha \colon \mathbf{M} \times \mathscr{S}_f \to f + \mathfrak{m}^\perp.$$

THEOREM 4.2 ([**Gan-Ginzburg02**, §2.3]). *The map α is an isomorphism of affine varieties.*

PROOF. We have a contracting $\mathbb{C}^*$-action on $\mathbf{M} \times \mathscr{S}_f$ defined by:

$$\forall\, t \in \mathbb{C}^*,\ g \in \mathbf{M},\ x \in \mathscr{S}_f, \qquad t.(g, x) := (\rho(t^{-1}) g \rho(t), \tilde{\rho}(t) x).$$

The morphism α is $\mathbb{C}^*$-equivariant with respect to this contracting $\mathbb{C}^*$-action, and the $\mathbb{C}^*$-action $\tilde{\rho}$ on $f + \mathfrak{m}^\perp$.

Then we conclude thanks to the following result, formulated in [**Gan-Ginzburg02**, Proof of Lemma 2.1]:

"*a $\mathbb{C}^*$-equivariant morphism $\alpha \colon X_1 \to X_2$ of smooth affine $\mathbb{C}^*$-varieties with contracting $\mathbb{C}^*$-actions which induces an isomorphism between the tangent spaces of the $\mathbb{C}^*$-fixed points is an isomorphism.*" □

As a consequence of this result, we get the isomorphism:

$$\mathbb{C}[\mathscr{S}_f] \cong \mathbb{C}[f + \mathfrak{m}^\perp]^{\mathbf{M}}.$$

4.1.3. Hamiltonian reduction. We refer to [**Vaisman**] or [**Laurent-Pichereau-Vanhaecke**, Proposition 5.39 and Definition 5.9] for the following result.

THEOREM 4.3 (Marsden-Weinstein). *Let X be a Poisson variety. Assume that $\mathbf{A}$ is connected and that the action of $\mathbf{A}$ in X is Hamiltonian. Let $\gamma \in \mathfrak{a}^*$. Assume that γ is a regular value*[2] *of μ, that $\mu^{-1}(\gamma)$ is $\mathbf{A}$-stable and that $\mu^{-1}(\gamma)/\mathbf{A}$ is a variety. Let $\iota \colon \mu^{-1}(\gamma) \hookrightarrow X$ and $\pi \colon \mu^{-1}(\gamma) \twoheadrightarrow \mu^{-1}(\gamma)/\mathbf{A}$ be the natural maps: ι is the inclusion and π is the quotient map. Then the triple*

$$(X, \mu^{-1}(\gamma), \mu^{-1}(\gamma)/\mathbf{A})$$

is Poisson-reducible, *i.e., there exists a Poisson structure $\{\,,\}'$ on $\mu^{-1}(\gamma)/\mathbf{A}$ such that for all open subset $U \subset X$ and for all $f, g \in \mathcal{O}_X(\pi(U \cap \mu^{-1}(\gamma))$, on has*

$$\{f, g\}' \circ \pi(u) = \{\tilde{f}, \tilde{g}\} \circ \iota(u)$$

at any point $u \in U \cap \mu^{-1}(\gamma)$, where $\tilde{f}, \tilde{g} \in \mathcal{O}_X(U)$ are arbitrary extensions of $f \circ \pi|_{U \cap \mu^{-1}(\gamma)}, g \circ \pi|_{U \cap \mu^{-1}(\gamma)}$ to U.

We intend to apply the theorem to the connected Lie group $\mathbf{M}$ acting on the Poisson variety $\mathfrak{g}^*$ by the coadjoint action. The action is Hamiltonian and the moment map

$$\mu \colon \mathfrak{g}^* \to \mathfrak{m}^* \tag{34}$$

[2] If $f \colon X \to Y$ is a smooth map between varieties, we say that a point y is a *regular value* of f if for all $x \in f^{-1}(y)$, the map $d_x f \colon T_x(X) \to T_y(Y)$ is surjective. If so, then $f^{-1}(y)$ is a subvariety of X and the codimension of this variety in X is equal to the dimension of Y.

is the restriction of functions from $\mathfrak{g}$ to $\mathfrak{m}$. Recall that $\chi = (f|\cdot)$. Since $\chi|_{\mathfrak{m}}$ is a character on $\mathfrak{m}$, it is fixed by the coadjoint action of $\mathbf{M}$. As a consequence, the set

$$\mu^{-1}(\chi|_{\mathfrak{m}}) = \{\xi \in \mathfrak{g}^* \mid \mu(\xi) = \chi|_{\mathfrak{m}}\}$$

is $\mathbf{M}$-stable. Moreover, we have the following lemma:

LEMMA 4.4. *$\chi|_{\mathfrak{m}}$ is a regular value for the restriction of μ to each symplectic leaf of $\mathfrak{g}^*$.*

PROOF. Note that $\mu^{-1}(\chi|_{\mathfrak{m}}) = \chi + \mathfrak{m}^{\perp}$. Then we have to prove that for any $\xi \in \chi + \mathfrak{m}^{\perp}$, the map

$$d_{\xi}\mu \colon T_{\xi}(\mathbf{G}.\xi) \to T_{\chi|_{\mathfrak{m}}}(\mathfrak{m}^*)$$

is surjective. But $T_{\xi}(\mathbf{G}.\xi) \simeq [\mathfrak{g}, \xi]$ while $T_{\chi|_{\mathfrak{m}}}(\mathfrak{m}^*) = \mathfrak{m}^*$. Since $\chi + \mathfrak{m}^{\perp}$ is transversal to the coadjoint orbits in $\mathfrak{g}^*$ (cf. Theorem 4.1), we have

$$\mathfrak{g} = [\mathfrak{g}, \xi] + \mathfrak{m}^{\perp}.$$

Let $\gamma \in \mathfrak{m}^*$ and write $\gamma = x + x'$, with $x \in [\mathfrak{g}, \xi]$ and $x' \in \mathfrak{m}^{\perp}$, according to the above decomposition of $\mathfrak{g}$. Then $\mu(x) = \gamma$. $\square$

Since the map

$$\mathbf{M} \times \mathscr{S}_f \longrightarrow \chi + \mathfrak{m}^{\perp}$$

is an isomorphism of affine varieties (cf. Theorem 4.2),

$$\mathscr{S}_f \cong (\chi + \mathfrak{m}^{\perp})/\mathbf{M}.$$

Therefore, the conditions of Theorem 4.3 are fulfilled and we get a symplectic structure on $\mathscr{S}_f$.

In fact, thanks to Lemma 4.4, we have shown that the symplectic form on each leaf on $\mathscr{S}_f$ is obtained by symplectic reduction from the symplectic form of the corresponding leaf of $\mathfrak{g}^*$.

The Poisson structure on $\mathscr{S}_f$ is described as follows. Let $\pi \colon \chi + \mathfrak{m}^{\perp} \twoheadrightarrow (\chi + \mathfrak{m}^{\perp})/\mathbf{M} \simeq \mathscr{S}_f$ be the natural projection map, and $\iota \colon \chi + \mathfrak{m}^{\perp} \hookrightarrow \mathfrak{g}^*$ be the natural inclusion. Then for any $f, g \in \mathbb{C}[\mathscr{S}_f]$,

$$\{f, g\}_{\mathscr{S}_f} \circ \pi = \{\tilde{f}, \tilde{g}\} \circ \iota$$

where $\tilde{f}, \tilde{g}$ are arbitrary extensions of $f \circ \pi, g \circ \pi$ to $\mathfrak{g}^*$.

4.1.4. BRST reduction. One can also described the Hamiltonian reduction of §4.1.3 in a more factorial way, in terms of the *BRST cohomology* (where BRST refers to the physicists Becchi, Rouet, Stora and Tyutin). We do not detail here this construction, and refer to [**Arakawa-Moreau-lectures**] for more details.

Let us just say that there is a certain cohomology $H^i_{BRST,\chi}(\mathfrak{m}, \mathbb{C}[\mathfrak{g}^*])$ group depending on χ, the algebra $\mathfrak{m}$ and the Poisson algebra $\mathbb{C}[\mathfrak{g}^*])$, together with a certain complex, such that $H^i_{BRST,\chi}(\mathfrak{m}, \mathbb{C}[\mathfrak{g}^*]) = 0$ for $i \neq 0$, and

$$H^0_{BRST,\chi}(\mathfrak{m}, \mathbb{C}[\mathfrak{g}^*]) \cong \mathbb{C}[\mathscr{S}_f]$$

as Poisson algebras.

4.2. Affine W-algebras

For a nilpotent element f of $\mathfrak{g}$, let $\mathcal{W}^k(\mathfrak{g}, f)$ be the *W-algebra associated with $(\mathfrak{g}, f)$ at level k*:

$$\mathcal{W}^k(\mathfrak{g}, f) = H^0_{BRST,f}(\widehat{\mathfrak{m}}, V^k(\mathfrak{g})),$$

where $H^{\bullet}_{BRST,f}(\widehat{\mathfrak{m}}, ?)$ is the BRST functor of the generalized quantized Drinfeld-Sokolov reduction associated with $(\mathfrak{g}, f)$ with coefficients in a $\widehat{\mathfrak{g}}$-module M ([**Feigin-Frenkel90**, **Kac-Roan-Wakimoto03**]). Here $\widehat{\mathfrak{m}} = \mathfrak{m}[t, t^{-1}]$.

The W-algebras generalize both affine vertex algebras and Virasoro vertex algebras. Indeed, for $f = 0$, we get $\mathcal{W}^k(\mathfrak{g}, 0) \cong V^k(f)$, and for $\mathfrak{g} = \mathfrak{sl}_2$, and $f = f_{princ}$ (that is, f nonzero), then $\mathcal{W}^k(\mathfrak{sl}_2, f_{princ}) \cong \mathrm{Vir}^{c(k)}$, provided that $k \neq -2$, where $c(k) = \dfrac{k \dim \mathfrak{g}}{k + h^\vee} - 6k + h^\vee - 4$.

4.2.1. The BRST functor.

Let us denote simply by $H^0_f(?)$ the BRST functor $H^0_{BRST,f}(\widehat{\mathfrak{m}}, ?)$.

We have [**DeSole-Kac06**, **Arakawa15a**] a natural isomorphism $R_{\mathcal{W}^k(\mathfrak{g},f)} \cong \mathbb{C}[\mathscr{S}_f]$ of Poisson algebras, so that

$$\tilde{X}_{\mathcal{W}^k(\mathfrak{g},f)} = \mathscr{S}_f.$$

Moreover,

$$\mathrm{gr}\, \mathcal{W}^k(\mathfrak{g}, f) \cong \mathbb{C}[J_\infty \mathscr{S}_f],$$

so that $\mathcal{W}^k(\mathfrak{g}, f)$ is a quantization of $\mathbb{C}[J_\infty \mathscr{S}_f]$.

Let $\mathcal{W}_k(\mathfrak{g}, f)$ be the unique simple quotient of $\mathcal{W}^k(\mathfrak{g}, f)$. Then $X_{\mathcal{W}_k(\mathfrak{g},f)}$ is a $\mathbb{C}^*$-invariant Poisson subvariety of $\mathscr{S}_f$. Since it is $\mathbb{C}^*$-invariant, $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse if and only if $X_{\mathcal{W}_k(\mathfrak{g},f)} = \{f\}$.

THEOREM 4.5 ([**Arakawa15a**]). *For any quotient V of $V^k(\mathfrak{g})$ we have $\tilde{X}_{H^0_f(V)}$ is isomorphic to the scheme theoretic intersection $\tilde{X}_V \times_{\mathfrak{g}^*} \mathscr{S}_f$. So $X_{H^0_f(V)} = X_V \cap \mathscr{S}_f$.*

By Theorem 4.5, when $X_{L_k(\mathfrak{g})} \subset \mathcal{N}$, then $X_{H^0_f(L_k(\mathfrak{g}))}$ is contained in $\mathscr{S}_f \cap \mathcal{N}$ and so has finitely many symplectic leaves.

In fact, form the above theorem, we get that:

(1) $H^0_f(V) \neq 0$ if and only if $\overline{G.f} \subset X_V$,
(2) $H^0_f(V)$ is lisse if $X_V = \overline{G.f}$,
(3) $H^0_f(V)$ is quasi-lisse if $\overline{G.f} \subset X_V \subset \mathcal{N}$.

The vertex algebra $H^0_f(L_k(\mathfrak{g}))$ is a quotient vertex algebra of $\mathcal{W}^k(\mathfrak{g}, f)$ if it is nonzero. Conjecturally [**Kac-Roan-Wakimoto03**, **Kac-Wakimoto08**], we have

$$\mathcal{W}_k(\mathfrak{g}, f) \cong H^0_f(L_k(\mathfrak{g})) \text{ provided that } H^0_f(L_k(\mathfrak{g})) \neq 0.$$

(This conjecture has been verified in many cases [**Arakawa05**, **Arakawa11**].)

4.2.2. Lisse and quasi-lisse W-algebras. Theorem 4.5 implies that if $L_k(\mathfrak{g})$ is quasi-lisse and $f \in X_{L_k(\mathfrak{g})}$, then the W-algebra $H^0_f(L_k(\mathfrak{g}))$ is quasi-lisse as well, and so is its simple quotient $\mathcal{W}_k(\mathfrak{g}, f)$. In this way we obtain a huge number of quasi-lisse W-algebras.

We discuss next sections the problem of the irreducibility of the vertex algebras $V = H^0_f(L_k(\mathfrak{g}))$ with $X_{L_k(\mathfrak{g})} \subset \mathcal{N}$.

Moreover, if $X_{L_k(\mathfrak{g})} = \overline{G.f}$, then $X_{L_k(\mathfrak{g})} = \{f\}$ by the transversality of $\mathscr{S}_f$ to $\mathbf{G}$-orbits, so that $\mathcal{W}_k(\mathfrak{g}, f)$ in fact lisse. Thus, Conjecture 1 in particular says that a quasi-lisse affine vertex algebra produces exactly one lisse simple W-algebra.

For example, if k is an admissible level, then one knows that $X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}}$ for some nilpotent orbit (cf. Theorem 3.41). Picking $f \in \overline{\mathbb{O}}$, we obtain that $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse.

Note by Theorem 3.44, there are other lisse W-algebras, not coming from admissible level. Namely, let $(\mathfrak{g}, k)$ as in Theorem 3.44, and let $f \in \overline{\mathbb{O}_{min}}$. Then $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse. The statement is actually true for any $k = -\dfrac{h^\vee}{6} - 1 + n$, $n \in \mathbb{Z}_{\geqslant 0}$, for $\mathfrak{g} = D_4, E_6, E_7, E_8$, that is, $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse for such $(\mathfrak{g}, k)$, [**Arakawa-Moreau15**].

4.3. Branching and nilpotent Slodowy slices

We collect in this paragraph some results about branchings and nilpotent Slodowy slices. We refer to [**EGA61**, Chap. III, §4.3] for the definition of *unibranchness*, and to [**Kraft-Procesi82**] or [**Fu-et-al15**] for further details on branchings and nilpotent Slodowy slices.

Consider two varieties X, Y and two points $x \in X$, $y \in Y$. The singularity of X at x is called *smoothly equivalent* to the singularity of Y at y if there is a variety Z, a point $z \in Z$ and two maps

$$\begin{array}{ccc} Z & \xrightarrow{\varphi} & X \\ \Big\downarrow{\scriptstyle \psi} & & \\ Y & & \end{array}$$

such that $\varphi(z) = x$, $\psi(z) = y$, and φ and ψ are smooth in z ([**Hesselink76**]). This clearly defines an equivalence relation between pointed varieties (X, x). We denote the equivalence class of (X, x) by $\mathrm{Sing}(X, x)$.

Various geometric properties of X at x only depends on the equivalence class $\mathrm{Sing}(X, x)$, for example: *smoothness, normality, seminormality (cf.* [**Kraft-Procesi82**, §16.1]*), unibranchness, Cohen-Macaulay, rational singularities.*

Assume that the algebraic group $\mathbf{G}$ acts regularly on the variety X. Then $\mathrm{Sing}(X, x) = \mathrm{Sing}(X, x')$ if x and x' belongs to the same $\mathbf{G}$-orbit $\mathbb{O}$. In this case, we denote the equivalence class also by $\mathrm{Sing}(X, \mathbb{O})$.

A *cross section* (or *transverse slice*) at the point $x \in X$ is defined to be a locally closed subvariety $S \subset X$ such that $x \in S$ and the map

$$G \times S \longrightarrow X, \quad (g, s) \longmapsto g.s,$$

is smooth at the point $(1, x)$. We have $\mathrm{Sing}(S, x) = \mathrm{Sing}(X, x)$.

In the case where X is the closure of some nilpotent G-orbit of $\mathfrak{g}$, there is a natural choice of a cross section. Let $\mathbb{O}, \mathbb{O}'$ be two nonzero nilpotent orbits of $\mathfrak{g}$

and pick $f \in \mathbb{O}'$ that we embed f into an $\mathfrak{sl}_2$-triple (e, h, f) of $\mathfrak{g}$. The Slodowy slice $\mathscr{S}_f \cong f + \mathfrak{g}^e$ is a transverse slice of $\mathfrak{g}$ at f. It means that for any $\xi \in \mathscr{S}_f$,

$$T_\xi(\mathbf{G}.\xi) + T_\xi(\mathscr{S}_f) = \mathfrak{g}^*.$$

The variety

$$\mathscr{S}_{\mathbb{O},f} := \overline{\mathbb{O}} \cap \mathscr{S}_f$$

is then a transverse slice of $\overline{\mathbb{O}}$ at f, which we call, following the terminology of [**Fu-et-al15**], a *nilpotent Slodowy slice*.

Note that $\mathscr{S}_{\mathbb{O},f} = \{f\}$ if and only if $\mathbb{O} = G.f$. Moreover, since the $\mathbb{C}^*$-action of $\tilde{\rho}$ on $\mathscr{S}_f$ is contracting to f and stabilizes $\mathscr{S}_{f,\mathbb{O}}$, $\mathscr{S}_{\mathbb{O},f} = \varnothing$ if and only if $G.f \not\subseteq \overline{\mathbb{O}}$. Hence we can assume that $\mathbb{O}' \subseteq \overline{\mathbb{O}}$, that is, $\mathbb{O}' \leqslant \mathbb{O}$ for the Chevalley order on nilpotent orbits. The variety $\mathscr{S}_{\mathbb{O},f}$ is equidimensional, and

$$\dim \mathscr{S}_{\mathbb{O},f} = \operatorname{codim}(\mathbb{O}', \overline{\mathbb{O}}).$$

Since any two $\mathfrak{sl}_2$-triples containing f are conjugate by an element of the isotropy group of f in $\mathbf{G}$, the isomorphism type of $\mathscr{S}_{\mathbb{O},f}$ is independent of the choice of such $\mathfrak{sl}_2$-triples. Moreover, the isomorphism type of $\mathscr{S}_{\mathbb{O},f}$ is independent of the choice of $f \in \mathbb{O}'$. By focussing on $\mathscr{S}_{\mathbb{O},f}$, we reduce the study of $\operatorname{Sing}(\overline{\mathbb{O}}, \mathbb{O}')$ to the study of the singularity of $\mathscr{S}_{\mathbb{O},f}$ at f.

The variety $\mathscr{S}_{\mathbb{O},f}$ is not always irreducible. We are now interested in sufficient conditions for that $\mathscr{S}_{\mathbb{O},f}$ is irreducible.

Let X be an irreducible algebraic variety, and $x \in X$. We say that X is *unibranch at* x if the normalization $\pi\colon (\tilde{X}, x) \to (X, x)$ of (X, x) is locally a homeomorphism at x [**Fu-et-al15**, §2.4]. Otherwise, we say that X *has branches at* x and the number of branches of X at x is the number of connected components of $\pi^{-1}(x)$ [**Beynon-Spaltenstein84**, §5,(E)].

As it is explained in [**Fu-et-al15**, Section 2.4], the number of irreducible components of $\mathscr{S}_{\mathbb{O},f}$ is equal to the number of branches of $\overline{\mathbb{O}}$ at f.

If an irreducible algebraic variety X is normal, then it is obviously unibranch at any point $x \in X$. Hence we obtain the following result.

LEMMA 4.6. *Let $\mathbb{O}, \mathbb{O}'$ be nilpotent orbits of $\mathfrak{g}$, with $\mathbb{O}' \leqslant \mathbb{O}$ and $f \in \mathbb{O}'$. If $\overline{\mathbb{O}}$ is normal, then $\mathscr{S}_{\mathbb{O},f}$ is irreducible.*

The converse is not true. For instance, there is no branching in type G_2 but one knowns that the nilpotent orbit $\tilde{A}_1$ of G_2 of dimension 8 is not normal [**Levasseur-Smith88**].

The number of branches of $\overline{\mathbb{O}}$ at f, and so the number of irreducible components of $\mathscr{S}_{\mathbb{O},f}$, can be determined from the tables of Green functions in [**Shoji80, Beynon-Spaltenstein84**], as discussed in [**Beynon-Spaltenstein84**, Section 5,(E)-(F)]; see Meinolf Geck's lectures for more details about this.

We refer to Table 2 of [**Arakawa-Moreau17**] for the complete list of the nilpotent orbits $\mathbb{O}$ which have branchings in types F_4, E_6, E_7 and E_8 (there is no branching in type G_2), and Table 3 of [**Arakawa-Moreau17**] for the (conjectural) list a non-normal nilpotent orbit closures in the exceptional types. These results are extracted from [**Levasseur-Smith88, Kraft89, Broer98a, Broer98b, Sommers03**]. The list is known to be exhaustive for the types G_2, F_4 and E_6. It is only conjecturally exhaustive for the types E_7 and E_8.

The normality question of nilpotent orbit closures in the classical types is now completely answered ([**Kraft-Procesi79**, **Kraft-Procesi79**, **Sommers05**]). Note that, by [**Kraft-Procesi79**], if $\mathfrak{g} = \mathfrak{sl}_n$, then all nilpotent orbit closures are normal. In all the other types, there is at least one non-normal nilpotent closure.

Let $\mathbb{O}$ be a nilpotent orbit of $\mathfrak{g}$. Recall that the singular locus of $\overline{\mathbb{O}}$ is $\overline{\mathbb{O}} \setminus \mathbb{O}$. This was shown by Namikawa [**Namikawa04**] using results of Kaledin and Panyushev [**Kaledin06**, **Panyushev91**]; see [**Henderson15**, Section 2] for a recent review. This result also follows from Kraft and Procesi's work in the classical types [**Kraft-Procesi81**, **Kraft-Procesi82**], and from the main theorem of [**Fu-et-al15**] in the exceptional types.

THEOREM 4.7 ([**Kraft-Procesi82**, Theorem 1]). *Let $\mathbb{O}$ be a nilpotent orbit in $\mathfrak{o}_n$ or $\mathfrak{sp}_n$.*

(1) *$\overline{\mathbb{O}}$ is normal if and only if it is unibranch.*
(2) *$\overline{\mathbb{O}}$ is normal if and only if it is normal in codimension 2.*

In particular, $\overline{\mathbb{O}}$ is normal if it does not contain a nilpotent orbit $\mathbb{O}' \leqslant \mathbb{O}$ of codimension 2. Theorem 4.7 does not hold if $\mathfrak{g} = \mathfrak{so}_{2n}$ and if $\mathbb{O} = \mathbb{O}_{1,\boldsymbol{\lambda}}$, with $\boldsymbol{\lambda}$ very even. To determine the equivalence class $\mathrm{Sing}(\overline{\mathbb{O}_{\varepsilon,\boldsymbol{\lambda}}}, \mathbb{O}_{\varepsilon,\boldsymbol{\eta}})$, for $\varepsilon \in \{-1, 1\}$ and $\boldsymbol{\eta} < \boldsymbol{\lambda}$, there is a combinatorial method developed in [**Kraft-Procesi82**]. We refer to [**Arakawa-Moreau17**, Section 4] for more details about this.

Kraft and Procesi method together with Theorem 4.7 allow to deal with almost all nilpotent orbits, with exceptions for the very even nilpotent orbits in type $\mathfrak{so}_n$. For these orbits, the normality question was partially answered in [**Kraft-Procesi82**, Theorem 17.3], the remaining cases were dealt with in [**Sommers05**].

4.4. Conclusion, open problems

Using the results of the previous sections we can check the following: *for* all *known cases where the associated variety of the simple affine vertex algebra $L_k(\mathfrak{g})$ is the closure of some nilpotent orbit $\mathbb{O}$ of $\mathfrak{g}$, then for any $f \in \overline{\mathbb{O}}$, the variety $\mathscr{S}_{\mathbb{O},f}$ is irreducible.*

These known cases are summarized in the following table (cf. [**Arakawa-Moreau17**]).

	type of $\mathfrak{g}$	k	$X_{L_k(\mathfrak{g})}$
(1)	any	$-h^\vee$	$\mathcal{N}$
(2)	any	admissible	$\overline{\mathbb{O}}_q$
(3)	G_2	-1	$\overline{\mathbb{O}_{min}}$
(4)	D_4, E_6, E_7, E_8	$k \in \mathbb{Z}$, $-\frac{h^\vee}{6} - 1 \leq k \leq -1$	$\overline{\mathbb{O}_{min}}$
(5)	D_r with $r \geq 5$	$-2, -1$	$\overline{\mathbb{O}_{min}}$
(6)	D_r with r an even integer	$2 - r$	$\overline{\mathbb{O}_{(2^{r-2},1^4)}}$
(7)	B_r	-2	$\overline{\mathbb{O}_{(3,1^{2r-2})}}$

TABLE 1. Known pairs $(\mathfrak{g}, k)$ for which $X_{L_k(\mathfrak{g})} \subset \mathcal{N}$

In case (1) of Table 1, $V_{-h^\vee}(\mathfrak{g})$ does not satisfy the assumption of Conjecture 1,(2), since it is not conformal, but the irreducibility of $X_{V_{-h^\vee}(\mathfrak{g})}$ holds.

In case (2) of Table 1, $\mathbb{O}_q$ is a nilpotent orbit of $\mathfrak{g}$ described by Tables 2–10 of [**Arakawa15a**] which only depends on the denominator q of the admissible level $k \in \mathbb{Q}$.

As a consequence, we obtain that for all *known cases* of simple quasi-lisse vertex algebras satisfying the hypothesis of Conjecture 1, the associated variety is irreducible.

To conclude, we dress the list of all these *known cases*. Below is the list of all known, or expected, simple quasi-lisse vertex algebras V satisfying the hypothesis of Conjecture 1.

This happens when:

(1) V is *lisse* (or C_2-cofinite). Since V is assumed to be positively graded, the associated variety of V is then a point and so the conjecture is obviously true,
(2) V is a simple affine vertex algebra as in Table 1,
(3) V is a simple W-algebra $\mathcal{W}_k(\mathfrak{g}, f)$ with $L_k(\mathfrak{g})$ is as in Table 1,
(4) this is also expected to happen for the vertex algebras obtained from four dimensional $N = 2$ superconformal field theories ([**BLL$^+$**]), where the associated variety is expected to coincide with the spectrum of the chiral ring of the Higgs branch of the four dimensional theory.

    More precisely, the physicists Beem, Rastelli et al [**BLL$^+$**] showed that there is a remarkable map

$$\Phi\colon \{\text{4d N=2 SCFTs}\} \to \{\text{vertex algebras}\},$$

    which enjoys "nice properties". To such a 4d N=2 SCFT, let say T, there is important invariant, called the *Higgs branch*, which we denote by $Higgs(T)$. The Higgs branch $Higgs(T)$ is an affine hyperkähler variety, and hence, in particular a symplectic variety, possibly singular.

    Beem and Rastelli conjecture that for a 4d N=2 SCFT T, we have

$$Higgs(T) = X_{\Phi(T)}.$$

    The main examples of vertex algebras considered in [**BLL$^+$**] are the affine vertex algebras $L_k(\mathfrak{g})$ of types D_4, E_6, E_7, E_8 at level $k = -\frac{h^\vee}{4} - 1$. Namely, for T a 4d N=2 SCFT such that $\Phi(T) = L_k(\mathfrak{g})$, with $(\mathfrak{g}, k)$ as above, it is known that the Higgs branch of T is the closure of the minimal nilpotent orbit $\mathbb{O}_{min}$, which gives an evidence to their conjecture.

    We refer to the recent survey [**Arakawa-Higgs**] for more details about this conjecture.

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