Processes on random graphs: routing and attack vulnerability

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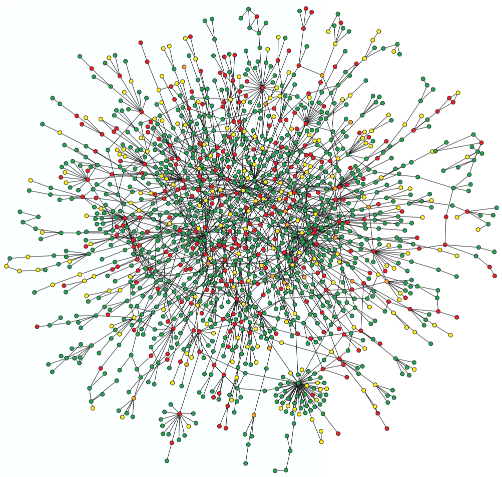
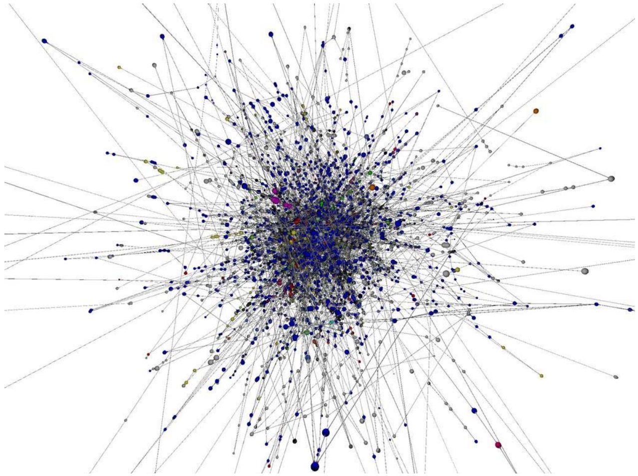
Part 1:

Real-world networks and models for them

Complex networks



Yeast protein interaction network



Internet topology in 2001

Network functions

Internet: e-mail

WWW: Information gathering

Friendship networks: gossiping, spread of information and disease

Power grids: reliability

Network functions

Internet: e-mail
Routing on networks, congestion, network failure

WWW: Information gathering
Crawling networks, motion on networks

Friendship networks: gossiping, spread of information and disease
Spread of diseases, motion on networks, consensus reaching

Power grids: reliability
Robustness to (random and deliberate) attacks

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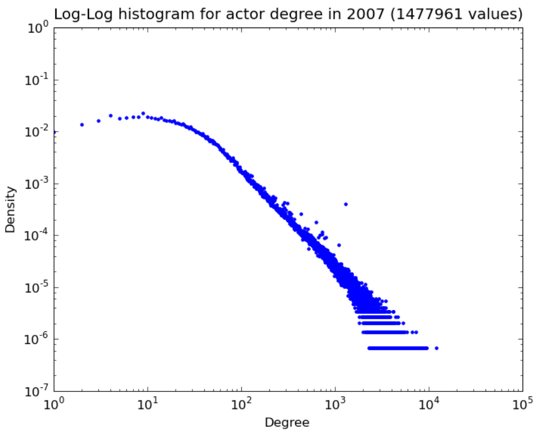
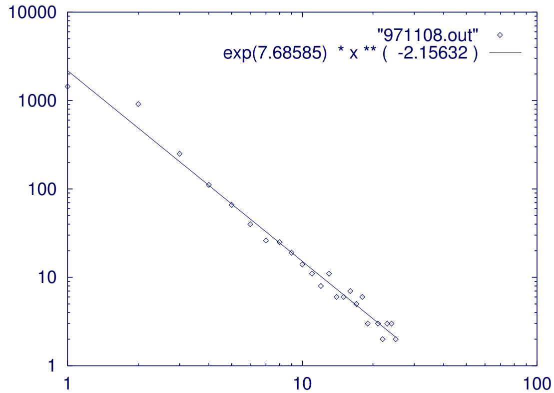
Processes on networks!

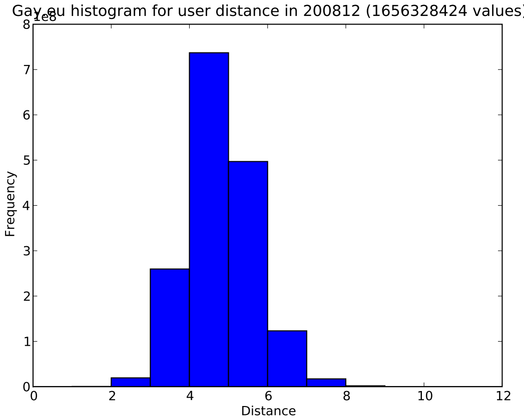
Scale-free paradigm

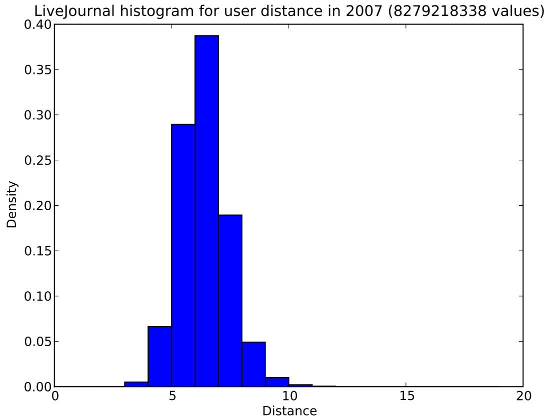




Loglog plot of degree sequences in Internet Movie Data Base (2007) and in the AS graph (FFF97)

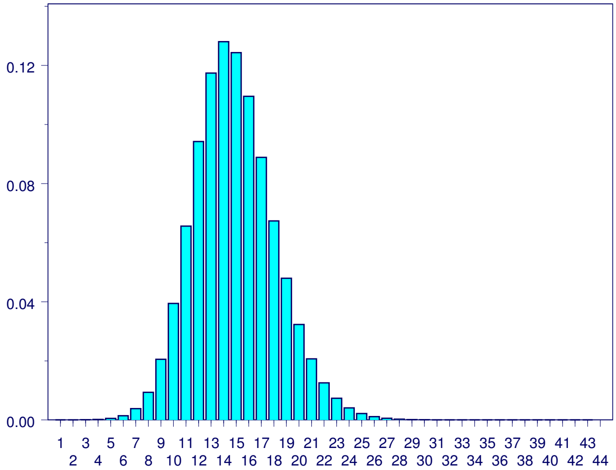
Small-world paradigm





Distances in social networks `gay.eu` on December 2008 and `livejournal` in 2007.

Distances in IP graph



Poisson distribution??

Modeling real networks

• Inhomogeneous Random Graphs:
Static random graph, independent edges with inhomogeneous edge occupation probabilities, yielding scale-free graphs.
(BJR07, CL02, CL03, BDM-L05, CL06, NR06, EHH06,...)

• Configuration Model:
Static random graph with prescribed degree sequence.
(MR95, MR98, RN04, HHV05, EHHZ06, HHZ07, JL07, FR07,...)

• Preferential Attachment Model:
Dynamic random graph, attachment proportional to degree plus constant.
(BA99, BRST01, BR03, BR04, M05, B07, HH07,...)

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Universality??

Part 2:

Routing on random graphs: First passage percolation on configuration model

Configuration model

Invented by Bollobás (1980), EJC: 285 cit. to study
number of graphs with given degree sequence.
Inspired by Bender+Canfield (1978), JCT(A): 300 cit.
Giant component studied by Molloy, Reed (1995), RSA: 664 cit.
Popularized by Newman, Strogatz, Watts (2001), Psys. Rev. E: 1190 cit.

Let n be number of vertices. Consider sequence of degrees $d_1, d_2, \ldots, d_n$.

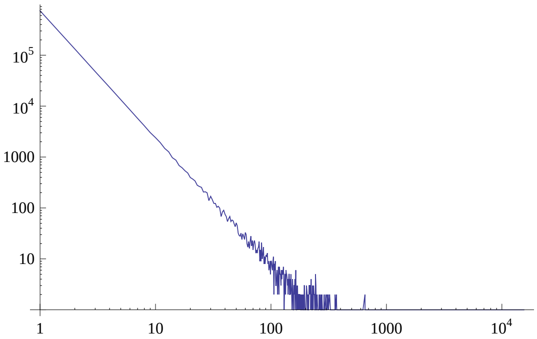
Often will take $d_i = D_i$, where $(D_i)_{i \in [n]}$ is sequence of independent and identically distributed (i.i.d.) random variables with a certain distribution.

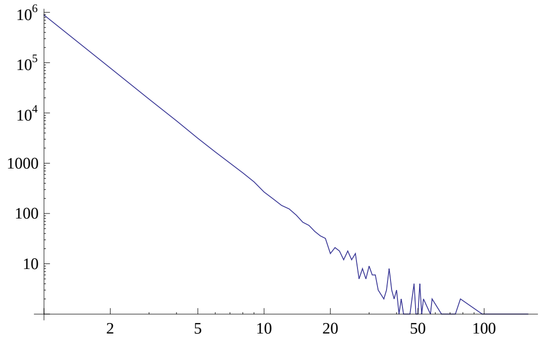
Special attention for power-law degrees, i.e., when

$$\mathbb{P}(D_1 \geq k) = c_\tau k^{-\tau+1}(1 + o(1)),$$

where c_τ is constant and $\tau > 1$.

Power-law degree sequence CM





Loglog plot of degree sequence CM with i.i.d. degrees $n = 1,000,000$ and $\tau = 2.5$ and $\tau = 3.5$, respectively.

Configuration model: graph construction

How to construct graph with above degree sequence?

- Assign to vertex j degree d_j.

$$\ell_n = \sum_{i \in [n]} d_i$$

is total degree. Assume ℓ_n is even.
Incident to vertex i have d_i ‘stubs’ or half edges.

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• Connect stubs to create edges as follows:
Number stubs from 1 to ℓ_n in any order.
First connect first stub at random with one of *other* $\ell_n - 1$ stubs.
Continue with second stub (when not connected to first) and so on, until all stubs are connected...

Properties configuration model

CM can have cycles and multiple edges, but these are relatively scarce compared to the number of edges.

Let D_n denote the degree of a uniformly chosen vertex.
We shall always assume that D_n converges in distribution to a limiting random variable D.

When $\mathbb{E}[D_n^2] \to \mathbb{E}[D^2] < \infty$, then the numbers of self-loops and multiple edges converges in distribution to two independent Poisson variables with parameters $\nu/2$ and $\nu^2/4$, respectively, where

$$\nu = \frac{\mathbb{E}[D(D-1)]}{\mathbb{E}[D]}.$$

Configuration model (CM) is locally tree-like.

Properties configuration model (Cont.)

Parameter ν arises as mean of size-biased distribution of D minus one.

This distribution is asymptotic distribution of forward degree of neighbor of uniformly chosen vertex.

$\nu > 1$ is equivalent to branching process approximation of connected components being supercritical, and giant component existing.

Shortest-weight problems

In many applications, edge weights represent cost structure of the graph, such as actual economic costs or congestion costs across edges.

Actual time delay experienced by vertices in the network is given by hop-count H_n which is the number of edges on shortest-weight path.

How does weight structure influence hopcount and weight SWP?

Assume that

edge weights are i.i.d. random variables:
Aldous' stochastic mean-field model of distance.

Problem with exponential edge weights has received tremendous attention on complete graph, here extend to general (random) graphs.

Results

Theorem 1. (BvdHH10). Let H_n be number of edges between two uniformly chosen vertices on CM with i.i.d. exponential edge weights.

Assume $D \geq 2$ a.s. and $\nu = \frac{\mathbb{E}[D(D-1)]}{\mathbb{E}[D]} > 1.$
For $\tau > 3$ or $\tau \in (2, 3)$,

$$\frac{H_n - \alpha \log n}{\sqrt{\alpha \log n}} \xrightarrow{d} Z,$$

where Z is standard normal, and

$$\alpha = \frac{\nu}{\nu - 1} > 1 \qquad \text{for} \quad \tau > 3,$$

$$\alpha = \frac{2(\tau - 2)}{\tau - 1} \in (0, 1) \qquad \text{for} \quad \tau \in (2, 3).$$

Results

Theorem 2. (BvdHH10). Let W_n be weight of shortest path between two uniformly chosen vertices on CM with i.i.d. exponential edge weights. Assume $D \geq 2$ a.s. and $\nu = \frac{\mathbb{E}[D(D-1)]}{\mathbb{E}[D]} > 1$.

Then, for some limiting random variable W, and for $\tau > 3$ or $\tau \in (2,3)$,

$$W_n - \gamma \log n \xrightarrow{d} W,$$

where

$$\begin{aligned} \gamma &= \frac{1}{\nu - 1} > 0 && \text{for} \quad \tau > 3, \\ \gamma &= 0 && \text{for} \quad \tau \in (2,3). \end{aligned}$$

Graph distances in configuration model

$\tilde{H}_n$ is graph distance between uniform pair of connected vertices in graph.

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$$\frac{\tilde{H}_n}{\log_\nu n} \xrightarrow{\mathbb{P}} 1,$$

and fluctuations are bounded.

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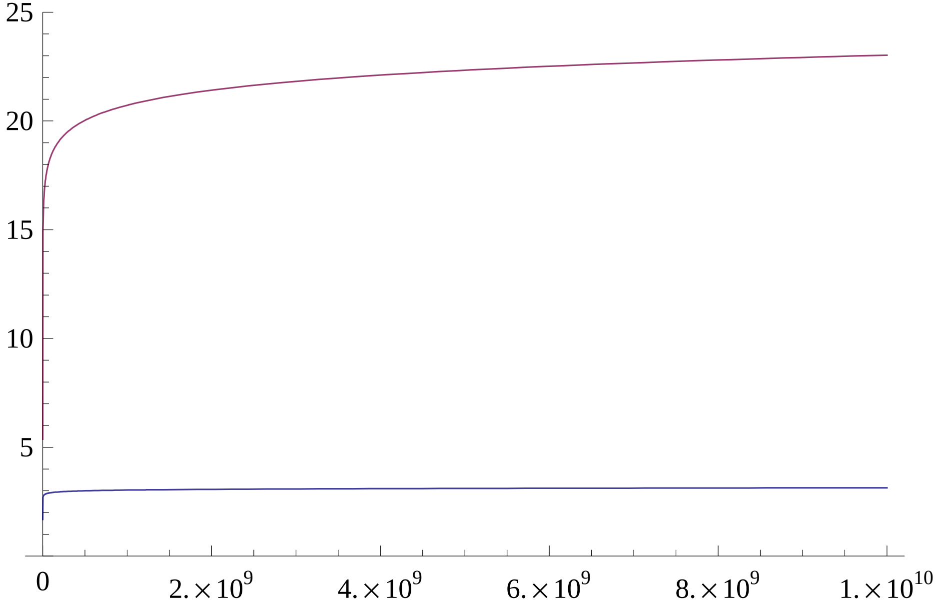
Theorem 4. (vdHHZ07, Norros+Reittu 04). When $\tau \in (2,3)$,

$$\frac{\tilde{H}_n}{\log\log n} \xrightarrow{\mathbb{P}} \frac{2}{|\log(\tau-2)|},$$

and fluctuations are bounded.

$x \mapsto \log\log x$ grows extremely slowly



Plots of $x \mapsto \log x$ and $x \mapsto \log\log x$.

Discussion Theorems 3-4

Proof relies on coupling of neighborhood of vertices to branching process.

Extensions:

Fluctuations around leading order are uniformly bounded, and 'limiting distribution' computed in terms of martingale limit of branching process. Interestingly, fluctuations are tight sequence of random variables that does not converge.

Diameter of graph is maximal distance between any pair of connected vertices.
Diameter CM is $\Theta(\log n)$ when $\mathbb{P}(D_i \geq 3) < 1$ (FR07, HHZ07),
while of order $\log \log n$ when $\tau \in (2, 3)$ and $\mathbb{P}(D_i \geq 3) = 1$ (HHZ07).

More information Erdős-Rényi + power-law degree random graphs:

www.win.tue.nl/∼rhofstad/NotesRGCN.pdf

Discussion Theorems 1-2

Random weights have marked effect on shortest-weight problem.

Proof Theorems 1-2: Comparison neighborhood uniform vertex to branching process, and use wealth of results on FPP on trees.

Surprisingly universal behavior for FPP on configuration model.
Universality is leading paradigm in statistical physics.
Only few examples where universality can be rigorously proved.
Extension to FPP on super-critical Erdős-Rényi random graph.

Key question:
To what extent is universality true for processes on random graphs models?

Cool application by Ding, Kim, Lubetzky, and Peres identifying distance between two random vertices in two-core of slightly supercritical ERRG.

Digression 1: Preferential attachment models

Albert-Barabási (1999):
Emergence of scaling in random networks (Science)
8737 citations on April 4, 2011.
Bollobas, Riordan, Spencer, Tusnády (2001):
The degree sequence of a scale-free random graph process (RSA)
371 citations in April 4, 2011.

In preferential attachment models, network is growing in time, in such a way that new vertices are more likely to be connected to vertices that already have high degree.

Rich-get-richer model.

Digression 1: Preferential attachment models

At time n, a single vertex is added to the graph with m edges emanating from it. Probability that an edge connects to the $i^{\rm th}$ vertex is proportional to

$$D_i(n-1)+\delta,$$

where $D_i(n)$ is degree vertex i at time n, $\delta > -m$ is parameter model.

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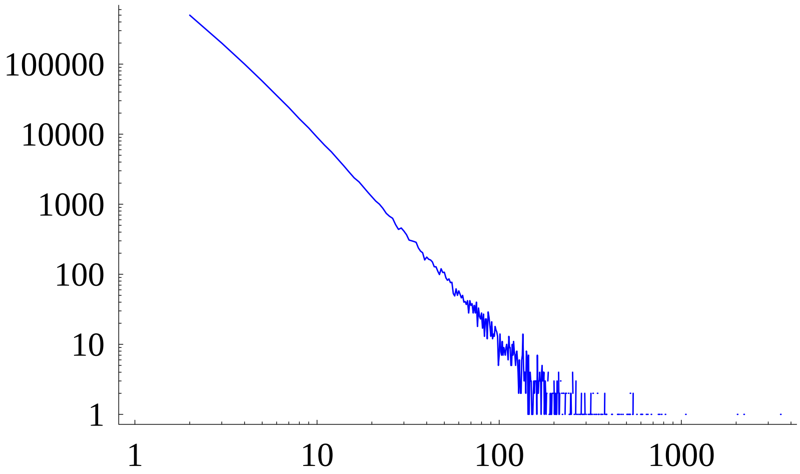
where $D_i(n)$ is degree vertex i at time n, $\delta > -m$ is parameter model.

Different edges can attach with different updating rules:
(a) intermediate updating degrees with self-loops (BA99, BR04, BRST01)
(b) intermediate updating degrees without self-loops;
(c) without intermediate updating degrees, i.e., independently.

(Graphs in cases (b-c) have advantage of being connected.)

Scale-free nature PA

Yields power-law degree sequence with power-law exponent $\tau = 3 + \delta/m \in (2, \infty)$.
(Bollobás, Riordan, Spencer, Tusnády (01) $\delta = 0$, Deijfen, vdE, vdH, Hoo (09),...)



$$(m = 2, \delta = 0, \tau = 3 + \frac{\delta}{m} = 3, n = 1,000,000)$$

Albert-László Barabási

"...the scale-free topology is evidence of organizing principles acting at each stage of the network formation. (...) No matter how large and complex a network becomes, as long as preferential attachment and growth are present it will maintain its hub-dominated scale-free topology."

Distances PA models

Non-rigorous physics literature predicts that scaling distances in preferential attachment models similar to the one in configuration model with equal power-law exponent degrees.

Distances PA models

Diam_n is diameter in PA model of size n.

Theorem 5 (Dommers-vdH-Hoo 10). For all $m \geq 2$ and $\tau \in (3, \infty)$,

$$\mathrm{Diam}_n, H_n = \Theta(\log n).$$

Theorem 6 (Dommers-vdH-Hoo 10, DerMonMor 11). For all $m \geq 2$ and $\tau \in (2, 3)$,

$$\frac{H_n}{\log \log n} \xrightarrow{\mathbb{P}} \frac{4}{|\log(\tau - 2)|},$$

and

$$\mathrm{Diam}_n = \Theta(\log \log n).$$

Distances PA models

Theorem 7 (Bol-Rio 04, Dommers-vdH-Hoo 10). For all $m \geq 2$ and $\tau = 3$,

$$\mathrm{Diam}_n, H_n \geq \frac{\log n}{\log \log n},$$

while, for model (a), matching upper bound exists (Bol-Rio 04).

Similar results can be proved for configuration model when $\tau = 3$.

Digression 2: FPP on complete graph

Consider complete graph $K_n = ([n], E_n)$ with edge weights E_e^s, where $(E_e)_{e \in E_n}$ are i.i.d. exponentials.

Theorem 8. (BvdH10). Let W_n and H_n be weight and number of edges of shortest path between two uniformly chosen vertices in K_n. Then, with

$$\lambda = \lambda(s) = \Gamma(1 + 1/s)^s,$$

there exists a limiting random variable W, such that

$$W_n - \frac{1}{\lambda} \log n \xrightarrow{d} W,$$

while

$$\frac{H_n - s \log n}{\sqrt{s^2 \log n}} \xrightarrow{d} Z,$$

where Z is standard normal.

Weights matter: $s < 0$

Not always CLT, even when weights have density:
Consider complete graph $K_n = ([n], \mathcal{E}_n)$ with edge weights E_e^s, where $(E_e)_{e \in \mathcal{E}_n}$ are i.i.d. exponentials and $s < 0$.

Theorem 9. (BvdHH10b). H_n converges in distribution. Limit is constant $k = k(s)$ for most $s \ldots$

What are universality classes FPP on complete graph?

Topology matters

Theorem 10. **(BvdHH in progress).** For configuration model with degree exponent $\tau > 3$, there exist $\alpha, \beta > 0$ such that

$$\frac{H_n - \alpha \log n}{\sqrt{\beta \log n}} \xrightarrow{d} Z.$$

Hopcount not always of order $\log n$:
Weights $(1 + E_e)_{e \in \mathcal{E}_n}$ and $\tau \in (2, 3)$, $H_n = \Theta(\log \log n)$.

What are universality classes FPP on random graph, and are they related to ones for FPP on complete graph?

Literature distances

[1] S. Bhamidi, R. van der Hofstad, and G. Hooghiemstra. First passage percolation on random graphs with finite mean degrees. *AoAP* **20**(5): 1907–1965, (2010).

[2] B. Bollobás and O. Riordan. The diameter of a scale-free random graph. *Combinatorica*, **24**(1):5–34, (2004).

[3] R. van der Hofstad, G. Hooghiemstra, and P. Van Mieghem. Distances in random graphs with finite variance degrees. *RSA*, **27**(1):76–123, (2005).

[4] R. van der Hofstad, G. Hooghiemstra, and D. Znamenski. Distances in random graphs with finite mean and infinite variance degrees. *EJP*, **12**(25):703–766, (2007).

[5] H. Reittu and I. Norros. On the power law random graph model of massive data networks. *Performance Evaluation*, **55**(1-2):3–23, (2004).

Part 3:

Attack vulnerability on random graphs: Critical inhomogeneous percolation.

Erdős-Rényi random graph

Vertex set $[n] := \{1, 2, \ldots, n\}$.

Erdős-Rényi random graph is random subgraph of complete graph on $[n]$ where each of $\binom{n}{2}$ edges is occupied with probab. p.

Simplest imaginable model of a random graph.

• Attracted tremendous attention since introduction 1959, mainly in combinatorics community.

Probabilistic method (Erdős et al).

Egalitarian: Every vertex has equal probability of being connected to. Misses hub-like structure of real networks.

Rank-1 inhomogeneous random graphs

Attach edge with probability p_{ij} between vertices i and j, where

$$p_{ij} = 1 - \mathrm{e}^{-w_i w_j / \ell_n},$$

and

$$\ell_n = \sum_{i \in [n]} w_i,$$

and different edges are independent.
Interpretation: w_i is close to expected degree vertex i.

When $w_i = -\log(1 - \lambda/n)$, we retrieve Erdős-Rényi random graph with $p = \lambda/n$.

Choice of weights

Take $\boldsymbol{w} = (w_1, \ldots, w_n)$ as

$$w_i = [1 - F]^{-1}(i/n),$$

where $F(x)$ is distribution function.
Interpretation: proportion of vertices i with $w_i \leq x$ is close to $F(x)$.

Simple example:

$$F(x) = \begin{cases} 0 & \text{for } x < a, \\ 1 - (a/x)^{\tau-1} & \text{for } x \geq a, \end{cases}$$

in which case

$$[1 - F]^{-1}(u) = a(1/u)^{-1/(\tau-1)}, \qquad \text{so that} \qquad w_j = a(n/j)^{1/(\tau-1)}.$$

Degree structure graph

Denote proportion of vertices with degree k by

$$P_k^{(n)} = \frac{1}{n} \sum_{i \in [n]} \mathbb{1}_{\{D_i = k\}},$$

where D_i denotes degree of vertex i.

Model is sparse, i.e., there exists probability distribution $(p_k)_{k=0}^{\infty}$ s.t.

$$P_k^{(n)} \xrightarrow{\mathbb{P}} p_k \qquad \text{where} \qquad p_k = \mathbb{E}\Big[\mathrm{e}^{-W} \frac{W^k}{k!}\Big],$$

for $w_i = [1 - F]^{-1}(i/n)$, with $W \sim F$.

In particular, $\sum_{l \geq k} p_k \sim ck^{-(\tau-1)}$ iff $\mathbb{P}(W \geq k) \sim ck^{-(\tau-1)}$.

Critical behavior Erdős-Rényi random graph

Double jump (Erdős and Rényi (60))
For $p = (1+\varepsilon)/n$, largest component is
(a) $\Theta_{\mathbb{P}}(\log n)$ for $\varepsilon < 0$;
(b) $\Theta_{\mathbb{P}}(n)$ for $\varepsilon > 0$;
(c) $\Theta_{\mathbb{P}}(n^{2/3})$ for $\varepsilon = 0$.

Scaling window: (Bollobás (84) and Łuczak (90))
For $p = (1/n)(1 + \lambda n^{-1/3})$, largest component is $\Theta_{\mathbb{P}}(n^{2/3})$.

Extension: Aldous (97): Weak convergence of ordered clusters.

Key question:
Degree ERRG with $p = c/n$ is Poisson with parameter c, not realistic!
How does critical behavior change when we let go of homogeneity vertices?

Critical value IRG

Bollobás-Janson-Riordan (07), Chung-Lu (02): Let $W \sim F$, then

- largest component $\sim \rho n$ with $\rho \in (0,1)$ for $\nu = \mathbb{E}[W^2]/\mathbb{E}[W] > 1$;
- largest component $o(n)$ for $\nu = \mathbb{E}[W^2]/\mathbb{E}[W] \leq 1$.

Identifies critical value IRG as

$$\nu = \mathbb{E}[W^2]/\mathbb{E}[W] = 1,$$

where ν is asymptotic expected number of forward neighbors, and W is asymptotic weight of uniform vertex.

In simple example $F(x) = 1 - (a/x)^{\tau-1}$ for $x \geq a$

$$\mathbb{E}[W] = \frac{a(\tau-1)}{\tau-2}, \qquad \mathbb{E}[W^2] = \frac{a^2(\tau-1)}{\tau-3},$$

so that critical case arises when $a = (\tau-3)/(\tau-2)$.

Robustness of networks

Above has important implications for robustness network under various attacks:

Random attack: Remove vertices uniformly at random with probability p. Obtain rank-1 IRG where now probability of edge ij between kept vertices equals

$$1 - \mathrm{e}^{-w_i w_j/\ell_n},$$

and otherwise equals 0.
Giant component exists whenever

$$(1-p)\nu > 1.$$

In particular, when $\nu = \infty$, always giant component:

Robust to random failure.

Robustness of networks

Above has important implications for robustness network under various attacks:

Deliberate attack: Remove proportion p of vertices with highest weight. Obtain rank-1 IRG where probability of edge ij for $i, j > np$ equals

$$1 - e^{-w_i w_j/\ell_n},$$

while otherwise probability equals 0.
Thus, giant component exists whenever

$$\frac{\sum_{i>np} w_i^2}{\ell_n} > 1.$$

In particular, even when $\nu = \infty$, for p large, no giant component:

Fragile to deliberate attacks.

Critical behavior

Let

$$1 - F(x) \sim cx^{-(\tau-1)} \qquad \text{for } x \text{ sufficiently large.}$$

Further, let $|\mathcal{C}_{\max}|$ denote largest connected component.

Theorem 11. (vdH 09) Assume that $\nu = 1$.

(a) Let $\tau > 4$. Then, there exists $b > 0$ such that for all $\omega \geq 1$

$$\mathbb{P}\Big(\frac{1}{\omega} n^{2/3} \leq |\mathcal{C}_{\max}| \leq \omega n^{2/3}\Big) \geq 1 - \frac{b}{\omega} \qquad \text{as } n \to \infty.$$

(b) Let $\tau \in (3, 4)$. Then, there exists $b > 0$ such that for all $\omega \geq 1$

$$\mathbb{P}\Big(\frac{1}{\omega} n^{(\tau-2)/(\tau-1)} \leq |\mathcal{C}_{\max}| \leq \omega n^{(\tau-2)/(\tau-1)}\Big) \geq 1 - \frac{b}{\omega} \qquad \text{as } n \to \infty.$$

Scaling limit for $\tau > 4$

Let $\mu = \mathbb{E}[W], \sigma^2 = \mathbb{E}[W^3]/\mathbb{E}[W]$. Consider

$$B_s^\lambda = \sigma B_s + s\lambda - s^2\sigma^2/(2\mu),$$

where B is standard Brownian motion. Let

$$R_s^\lambda = B_s^\lambda - \min_{0\leq u\leq s} B_s^\lambda.$$

Aldous (1997): Excursions of R^λ can be ranked in increasing order as $\gamma_1(\lambda) > \gamma_2(\lambda) > \dots$.

Let $|\mathcal{C}_{(1)}(\lambda)| \geq |\mathcal{C}_{(2)}(\lambda)| \geq |\mathcal{C}_{(3)}(\lambda)| \dots$ denote sizes of components with weights $\tilde{w}_i = (1 + \lambda n^{-1/3})w_i$ arranged in increasing order.

Theorem 12. (BvdHvL 10, Turova 09) Assume that $\nu = 1$, and $\mathbb{E}[W^3] < \infty$. Then

$$\big(n^{-2/3}|\mathcal{C}_{(i)}(\lambda)|\big)_{i\geq 1} \xrightarrow{d} \big(\gamma_i(\lambda)\big)_{i\geq 1}.$$

Scaling limit for $\tau \in (3,4)$

Let $|\mathcal{C}_{(1)}(\lambda)| \geq |\mathcal{C}_{(2)}(\lambda)| \geq |\mathcal{C}_{(3)}(\lambda)| \ldots$ denote sizes of components with weights $\tilde{w}_i = (1+\lambda n^{-(\tau-3)/(\tau-1)})w_i$ arranged in increasing order.

Theorem 13. (BvdHvL 09b) Assume that $\nu = 1$, and $\tau \in (3,4)$. Then,

$$\big(n^{-(\tau-2)/(\tau-1)}|\mathcal{C}_{(i)}(\lambda)|\big)_{i\geq 1} \xrightarrow{d} \big(H_i(\lambda)\big)_{i\geq 1}.$$

Moreover, for every i, j fixed

$$\mathbb{P}(i \longleftrightarrow j) \to q_{ij}(\lambda) \in (0,1).$$

Limits $H_i(\lambda)$ correspond to ordered hitting times of 0 of a certain fascinating 'thinned' Lévy process.

Multiplicative coalescents

Multiplicative coalescent is continuous-time Markov process $\lambda \mapsto \mathbf{X}(\lambda)$, where

$$\mathbf{X}(\lambda) \in \{\mathbf{x} = (x_i)_{i \geq 1} : x_i \geq x_{i+1}\},$$

where x_i corresponds to mass of i^{th} largest particle, and where particles with masses x_i and x_j merge to particle of mass $x_i + x_j$ at rate

$$x_i x_j.$$

Process describes evolution of masses where particles coalesce at rate equal to product of their masses.

Theorem 14. (Aldous97, BvdHvL 09b) As function of $\lambda \in \mathbb{R}$, processes $\lambda \mapsto \big(\gamma_i(\lambda)\big)_{i \geq 1}$ and $\lambda \mapsto \big(H_i(\lambda)\big)_{i \geq 1}$ are

multiplicative coalescents.

Distinction between $\tau > 4$ and $\tau \in (3, 4)$ arises through

entrance boundary at $\lambda = -\infty$.

Digression 3: Critical behavior CM

Theorem 15. (Joseph 11) Theorems 12 and 13 also hold for

configuration model

with i.i.d. degrees, under suitable conditions as for IRG, i.e.,

$$\nu = \mathbb{E}[D(D-1)]/\mathbb{E}[D] = 1.$$

Remarkably, the scaling limit is notably different.

Proof: weak convergence stochastic processes

Proof relies on three main ingredients:

(1) subsequent exploration of clusters;

(2) removal of possible further neighbors due to their exploration:

depletion of points effect;

(3) in critical window, these effects play at same scale, and

cluster exploration process weakly converges;

Cluster sizes correspond to excursion lengths limiting process having an increasing negative drift.

Proof: weak convergence stochastic processes

$\tau > 4$: exploration process has finite variance steps, so that Brownian motion appears in limit, and $\mathbb{P}(1 \in \mathcal{C}_{\max}) \to 0$:

‘power to the masses!’

$\tau \in (3, 4)$: exploration process is dominated by vertices with high weights, and $\mathbb{P}(1 \in \mathcal{C}_{\max}) \to q_1(\lambda) \in (0, 1)$:

‘power to the wealthy!’

Cluster exploration for $\tau > 4$

For all ordered pairs of vertices (i, j), let (i, j) be independent exponential random variables with rate $(1 + \lambda n^{-1/3}) w_j / \ell_n$.

Choose vertex $v(1)$ with probability proportional to $\boldsymbol{w}$, so that

$$\mathbb{P}(v(1) = i) = w_i / \ell_n.$$

Children of $v(1)$ are those vertices j for which

$$V(v(1), j) \leq w_{v(1)}.$$

Label children of $v(1)$ as $v(2), v(3), \dots v(c(1) + 1)$ in increasing order of their $V(v(1), \cdot)$ values.

Move to $v(2)$, explore all of its children, and label them as before. Children of $v(2)$ are those vertices j for which $V(v(2), j) \leq w_{v(2)}$, and continue!

Once we finish exploring one component, move onto next component by choosing starting vertex in size-biased manner amongst remaining vertices.

Size-biased reordering

Size-biased order $v^*(1), v^*(2), \ldots, v^*(n)$ is random reordering of vertex set $[n]$ where

- $v^*(1) = i$ with prob. w_i/ℓ_n;
- given $v^*(1), \ldots, v^*(i-1)$, $v^*(i) = j \in [n] \setminus \{v^*(1)\}$ with prob. proportional to w_j.

Key ingredient proof:

$$(v(i))_{i\in[n]} \qquad \text{is size-biased reordering.}$$

Number of new neighbors $c(i)$ of $v(i)$ is close to

$$c(i) = \mathrm{Poi}\Big(w_{v(i)} \sum_{j\in[n]\setminus\{v(1),\ldots,v(i)\}} w_j/\ell_n\Big).$$

Connected components

Recall number of new neighbors of $v(i)$ is close to

$$c(i) = \mathrm{Poi}\Big(w_{v(i)} \sum_{j\in[n]\setminus\{v(1),\ldots,v(i)\}} w_j/\ell_n\Big).$$

Denote cluster exploration process Z_n by $Z_n(0) = 0$ and

$$Z_n(i) = Z_n(i-1) + c(i) - 1.$$

Denote first hitting time of $-j$ by

$$\eta(j) = \min\{i : Z_n(i) = -j\}.$$

Then, all connected component sizes are given by successive excursions from past minima

$$\mathcal{C}^*(j) = \eta(j) - \eta(j-1).$$

Scaling limit of cluster exploration

Process $t \mapsto n^{-1/3} Z_n(sn^{2/3})$ is close to Brownian motion with changing drift given by

$$\mathbb{E}[n^{-1/3} Z_n(sn^{2/3})] \sim s\lambda - s^2\sigma^2/(2\mu),$$

while

$$n^{-1/3} Z_n(sn^{2/3}) - (s\lambda - s^2\sigma^2/(2\mu)) \xrightarrow{d} B_s.$$

Suggests that rescaled cluster sizes converge to successive excursions from past minima of process

$$B_s^\lambda = B_s + s\lambda - s^2\sigma^2/(2\mu).$$

Weak convergence of exploration process follows from

functional martingale central limit theorem.

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