Weak convergence of rescaled discrete objects in combinatorics

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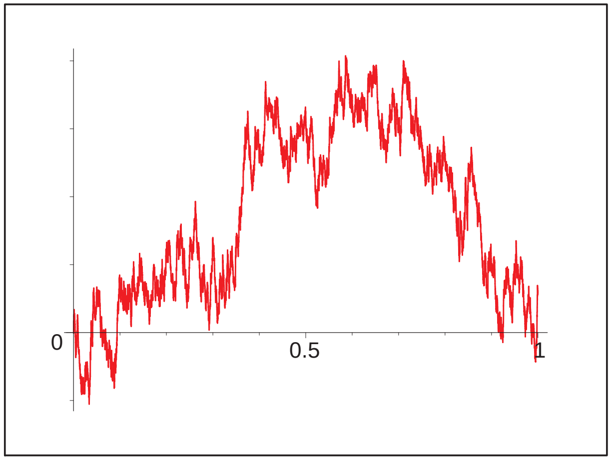
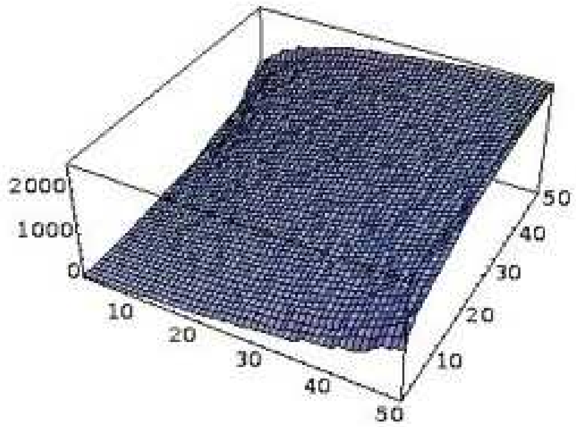
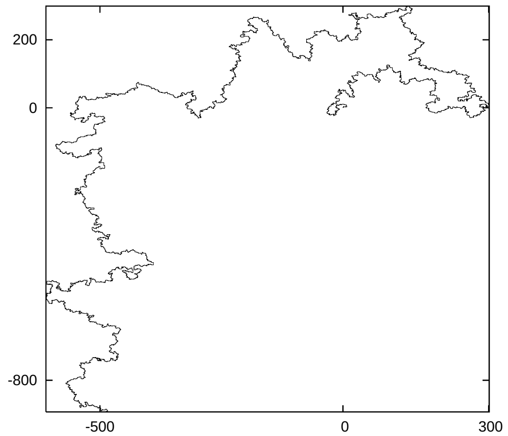
LILLE, 2011 !

O. What are we talking about? - Pictures







The talk deals with these situations when simulating random combinatorial objects with size $10^3, 10^6, 10^9$ in a window of fixed size, one sees essentially the same picture

Questions :

- What sense can we give to this:

– a sequence of (normalized) combinatorial structures converges?

– a sequence of random normalized combinatorial structures converges"?

- If we are able to prove such a result...:

– What can be deduced?

– What cannot be deduced?

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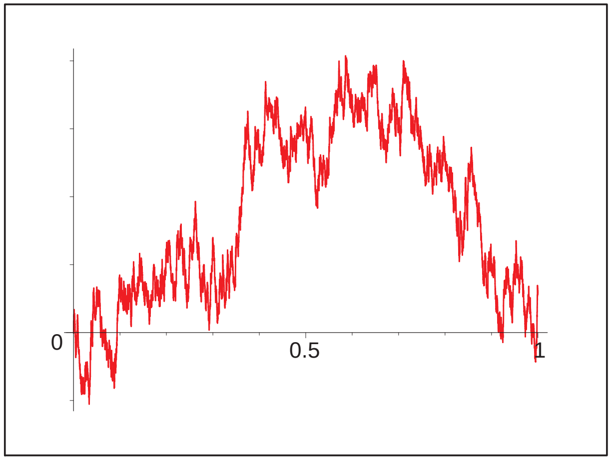

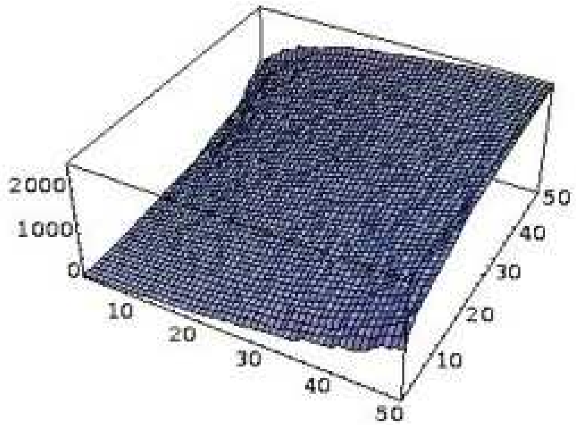

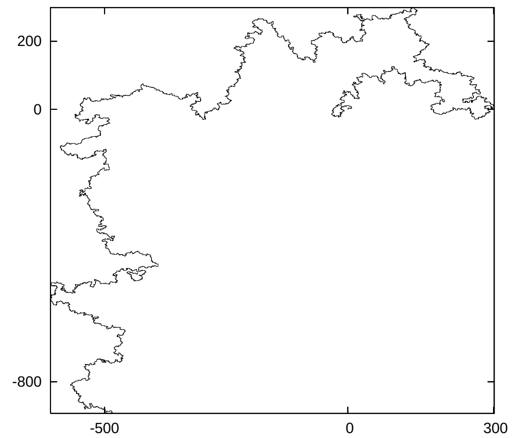

- What sense can we give to this:

– a sequence of normalized combinatorial structures converges?

answer: this is a question of topology...

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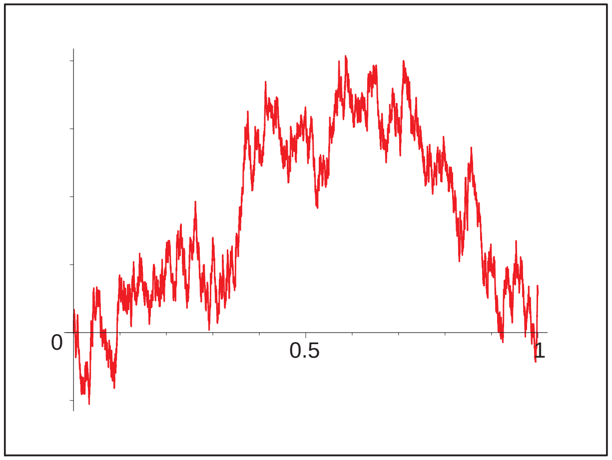
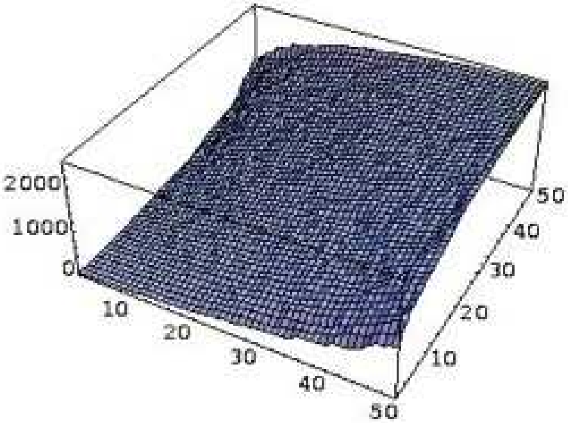
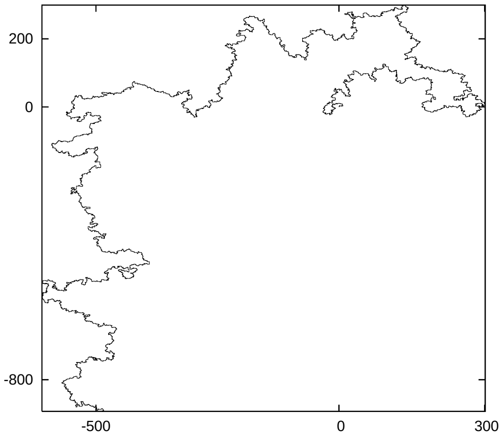






- What sense can we give to this:

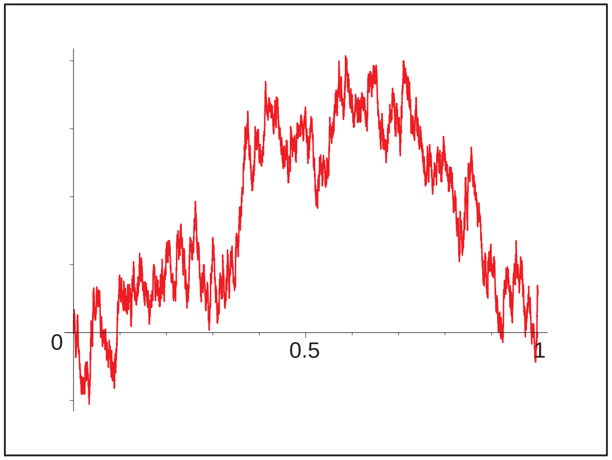
– a sequence of normalized combinatorial structures converges?

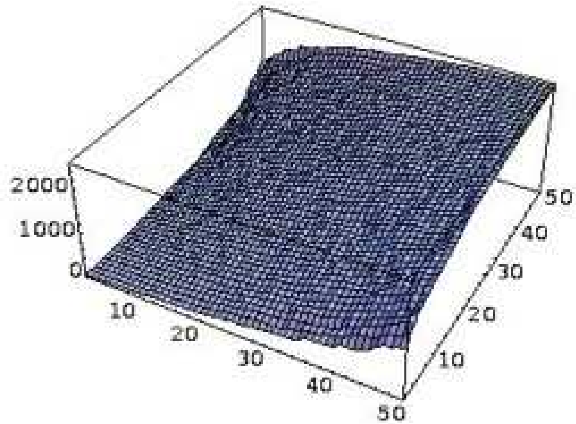
answer: this is a question of topology...

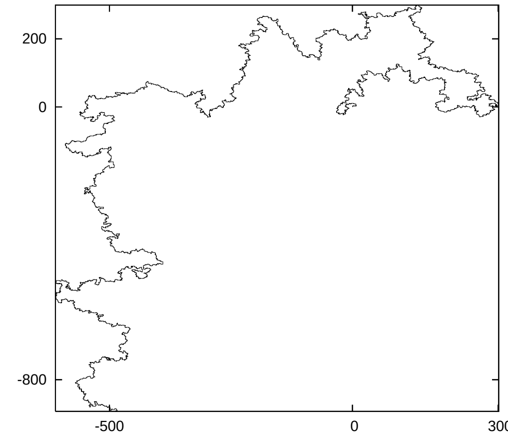
– a sequence of random normalized combinatorial structures converges"?

answer: this is a question of weak convergence associated with the topology.

O. What are we talking about? - Pictures







■ What sense can we give to this:

– a sequence of normalized combinatorial structure converges?

answer: this is a question of topology...

– a sequence of random normalized combinatorial structure converges"?

answer: this is a question of weak convergence associated with the topology.

■ If we are able to prove such a result...:

What can be deduced?

answer: infinitely many things... but it depends on the topology

What cannot be deduced?

answer: infinitely many things: but it depends on the topology

O. What are we talking about? - Pictures

First - we recall what means convergence in distribution

- in $\mathbb{R}$

- in a Polish space

– Then we treat examples... and see the byproducts

Random variables on $\mathbb{R}$

- A distribution μ on $\mathbb{R}$ is a (positive) measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with total mass 1.

- a real random variable X is a function $X : (\Omega, \mathcal{A}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, measurable.

- distribution of X: the measure μ,

$$\mu(A) = \mathbb{P}(X^{-1}(A)) = \mathbb{P}(X \in A).$$

Characterization of the distributions on $\mathbb{R}$

– the way they integrate some classes of functions (by duality)

$$f \mapsto \mathbb{E}(f(X)) = \int f(x) d\mu(x),$$

e.g. Continuous bounded functions

Other characterizations : Characteristic function $=$ Fourier transform, distribution function $x \mapsto F(x) = \mathbb{P}(X \leq x)$, Moments (sometimes)

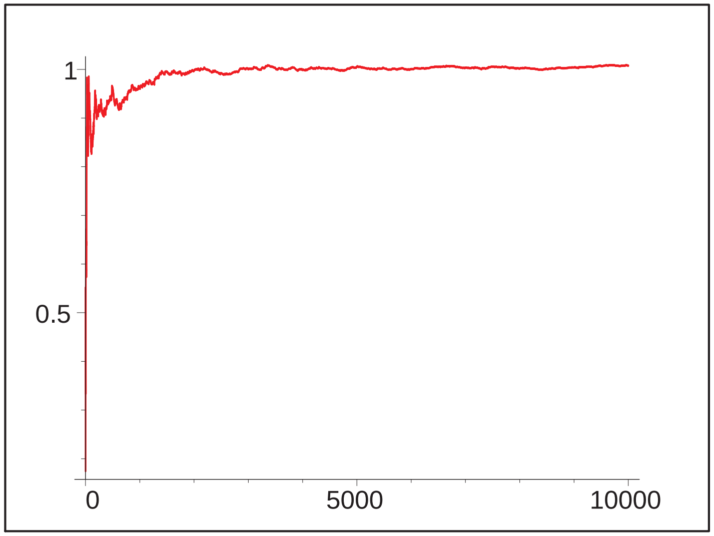
Convergence in probability

$$X_n \xrightarrow[n]{(proba.)} X \text{ if } \forall \varepsilon > 0, \quad \mathbb{P}(|X_n - X| \geq \varepsilon) \underset{n}{\to} 0.$$

$X, X_1, X_2, \ldots$ are to be defined on the same probability space Ω:

Archetype $=$ strong law of large number: if Y_i i.i.d. mean m,

$$X_n := \frac{\sum_{i=1}^{n} Y_i}{n} \xrightarrow[n]{(as.)} m$$



Convergence in distribution (duality)

$$X_n \xrightarrow[n]{(d)} X \quad \text{if} \quad \mathbb{E}(f(X_n)) \underset{n}{\rightarrow} \mathbb{E}(f(X))$$

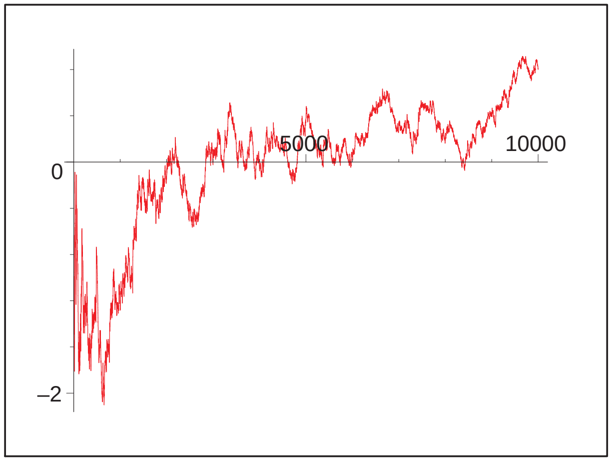
for any $f : \mathbb{R} \mapsto \mathbb{R}$ bounded, continuous

The variables need not to be defined on the same Ω

characterizations: Convergence of cumulative functions, Fourier tranforms, etc.

Archetype = the central limit theorem: if Y_i i.i.d. mean m, variance $\sigma^2 \in (0, +\infty)$

$$X_n := \frac{\sum_{i=1}^{n}(Y_i - m)}{\sigma\sqrt{n}} \xrightarrow[n]{(d)} \mathcal{N}(0,1)$$



The sequence (X_n) does not converge!
(Exercise)

Where define (weak) convergence of combinatorial structures?

we need a nice topological space :

– that contains the rescaled discrete objects and the continuous limits

– on which probability measures and weak convergences must be not too difficult!!

Nice topological spaces on which everything works like on $\mathbb{R}$ are Polish spaces.

Where define (weak) convergence of combinatorial structures?

$\boxed{\text{Nice topological spaces on which everything works like on } \mathbb{R} \text{ are Polish spaces.}}$

Polish space (S, ρ) : metric + separable + complete

$\rightarrow$ open balls, topology, Borelians, Borelian measures, integration theory, can be defined as on $\mathbb{R}$

Examples : – $\mathbb{R}^d$ with the usual distance,
– $(C[0,1], \|.\|_\infty)$, $d(f,g) = \|f-g\|_\infty$

Distribution μ on $(S, \mathcal{B}(S))$: measure with total mass 1.
S valued Random variable : $X : (\Omega, A, \mathbb{P}) \rightarrow (S, \mathcal{B}(S))$ measurable.
Distribution of X, $\mu(B) = \mathbb{P}(X \in B)$.

Characterization of measures (by duality)
– The way they integrate continuous bounded functions. $\mathbb{E}(f(X)) = \int f(x)d\mu(x)$.

f continuous in x_0 means: $\forall \varepsilon > 0, \exists \eta > 0,\ \rho(x, x_0) \leq \eta \Rightarrow |f(x) - f(x_0)| \leq \varepsilon$.

Random variables on a Polish space

Polish space (S, ρ) : metric + separable + complete

Convergence in probability

$$\forall \varepsilon > 0, \quad \mathbb{P}(\rho(X_n, X) \geq \varepsilon) \underset{n}{\to} 0.$$

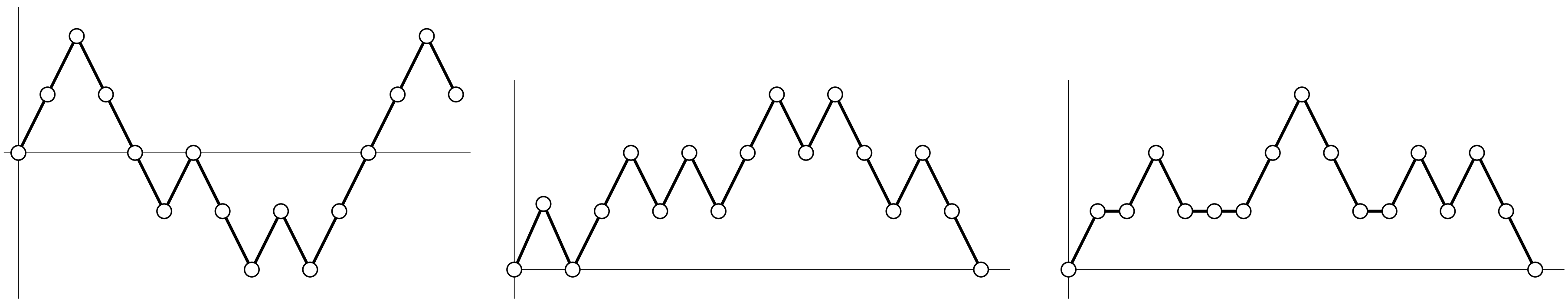
Convergence in distribution (duality)

$$\mathbb{E}(f(X_n)) \to \mathbb{E}(f(X)), \qquad \text{for any continuous bounded function } f : S \to \mathbb{R}$$

Byproduct : if $X_n \xrightarrow[n]{(d)} X$ then $h(X_n) \xrightarrow[n]{(d)} h(X)$ for any $h : S \to S'$ continuous

II. Convergence of rescaled paths

Paths are fundamental objects in combinatorics.
Paths with step ± 1, or other increments, Dyck paths, bridges,etc.



A question is :

$\boxed{\text{do they converge in distribution (after rescaling)?}}$

Here

$\boxed{\text{distribution} = \text{distribution on } C[0,1] \text{ (up to encoding + normalisation).}}$

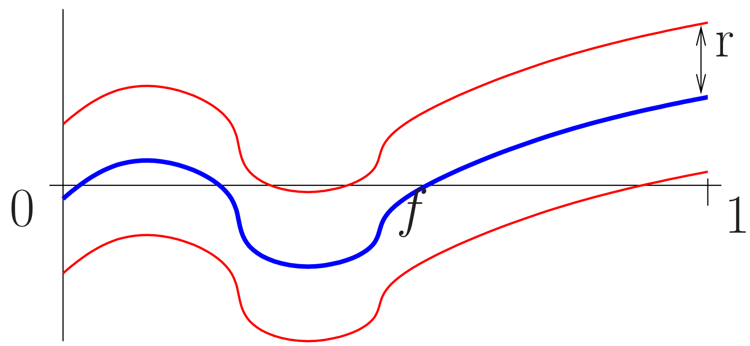
Here, we choose $C[0,1]$ as Polish space to work in...

II. Convergence of rescaled paths

How are characterized the distributions on $C[0,1]$?

$\rightarrow$ a distribution μ on $C[0,1]$ gives weight to the Borelians of $C[0,1]$.

Ball:$= B(f,r) = \{g \mid \|f-g\|_\infty < r\}$.



Let $X = (X(t), t \in [0,1])$ a process, with distribution μ.

> **Proposition 1** *The distribution of X is characterized by the finite dimensional distribution FDD:*
>
> $$i.e.\ \textit{the distribution of } (X(t_1), \ldots, X(t_k)), \quad k \geq 1, t_1 < \cdots < t_k.$$

II. Convergence of rescaled paths

How are characterized the convergence in distributions on $C[0,1]$?:

Convergence of FDD.

+ A tightness argument is needed

II. Convergence of rescaled paths

Convergence to Brownian processes

A) $X_1, \ldots, X_n$= i.i.d.random variables. $\mathbb{E}(X_1) = 0$, $\mathrm{Var}(X_i) = \sigma^2 \in (0, +\infty)$.

$$S_k = X_1 + \cdots + X_k$$

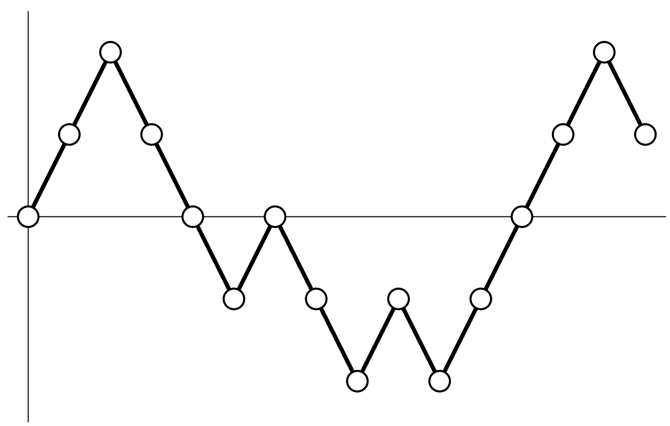
then

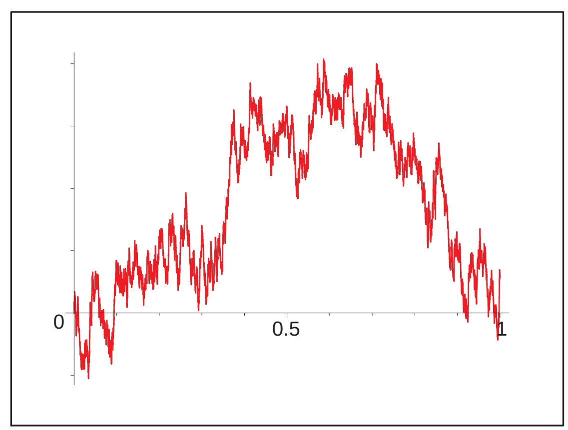
$$(Donsker's \;\; Theorem) \qquad \left(\frac{S_{nt}}{\sigma\sqrt{n}}\right)_{t\in[0,1]} \xrightarrow[n]{(d)} (B_t)_{t\in[0,1]}$$

where $(B_t)_{t\in[0,1]}$ is the **Brownian motion.**

The Brownian motion has for FDD: for $0 < t_1 < \cdots < t_k$, $B_{t_1} - B_0, \ldots, B_{t_k} - B_{t_{k-1}}$ are independent, $B_{t_j} - B_{t_{j-1}} \sim \mathcal{N}(0, t_j - t_{j-1})$.

$\left(\frac{S_{nt}}{\sqrt{n}}\right)_{t\in[0,1]}$ **does not converge in probability!**





II. Convergence of rescaled paths

Convergence to Brownian processes

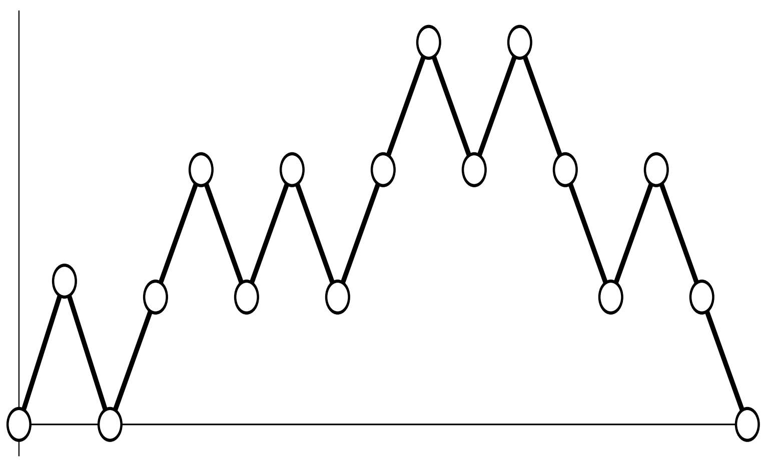
B) $X_1, \ldots, X_n$= i.i.d.random variables. $\mathbb{E}(X_1) = 0$, $\mathrm{Var}(X_i) = \sigma^2 \in (0, +\infty)$, $+$ X_i's lattice support.

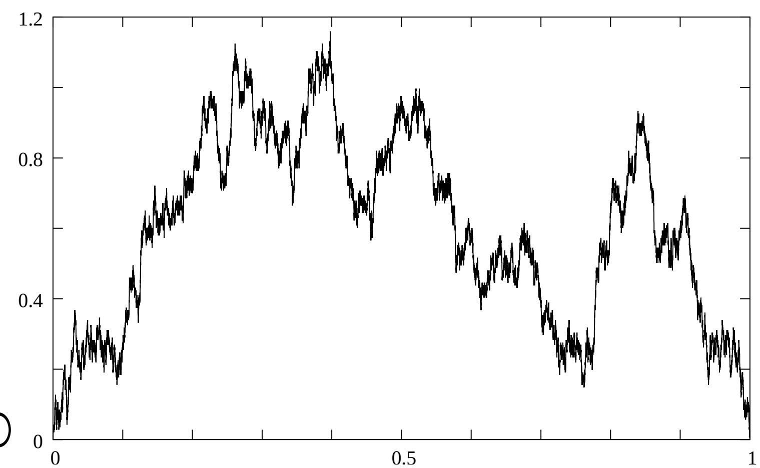
$$S_k = X_1 + \cdots + X_k$$

then under the condition $S_i \geq 0, i \leq n$, $S_n = 0$,

$$(Kaigh's\ \ Theorem) \qquad \left(\frac{S_{nt}}{\sigma\sqrt{n}}\right)_{t\in[0,1]} \xrightarrow[n]{(d)} (\mathbf{e}_t)_{t\in[0,1]}$$

where $(\mathbf{e}_t)_{t\in[0,1]}$ is the **Brownian excursion** .





Similar results for numerous models of random paths appearing in combinatorics

II. Convergence of rescaled paths

Byproducts of $X_n \xrightarrow[n]{(d)} X$ in $C[0,1]$.

1) $\mathbb{E}(f(X_n)) \to \mathbb{E}(f(X))$ for any f bounded continuous.

An infinity of byproducts (as much as bounded continuous functions)

$g \mapsto \|g\|_\infty \wedge 1$

$$\mathbb{E}\left(\int_0^1 \sin(X_n(t))dt\right) \underset{n}{\to} \mathbb{E}\left(\int_0^1 \sin(X(t))dt\right)$$

II. Convergence of rescaled paths

Byproducts of $X_n \xrightarrow[n]{(d)} X$ in $C[0,1]$

2) $f(X_n) \xrightarrow[n]{(d)} f(X)$ for any $f : C[0,1] \to S'$ continuous.

An infinity of byproducts (as much as continuous functions onto some Polish spaces).

Example :

$$g \mapsto \left(\max(g), \int_{1/2}^{2/3} g^{13}(t)dt, g(\pi/14), g^2 \right)$$

is continuous from $C[0,1]$ into $\mathbb{R}^3 \times C[0,1]$... Then

$$\left(\max X_n, \int_{1/2}^{2/3} X_n^{13}(t)dt, X_n(\pi/14), X_n^2 \right) \xrightarrow[n]{(d)} \left(\max X, \int_{1/2}^{2/3} X^{13}(t)dt, X(\pi/14), X^2 \right)$$

Examples of non-continuous functions :

$g \mapsto \min \operatorname{argmax}(g)$ (the first place where the max is reached),
$g \mapsto 1/g(1/3)$

II. Convergence of rescaled paths

"Contraction of information" at the limit :

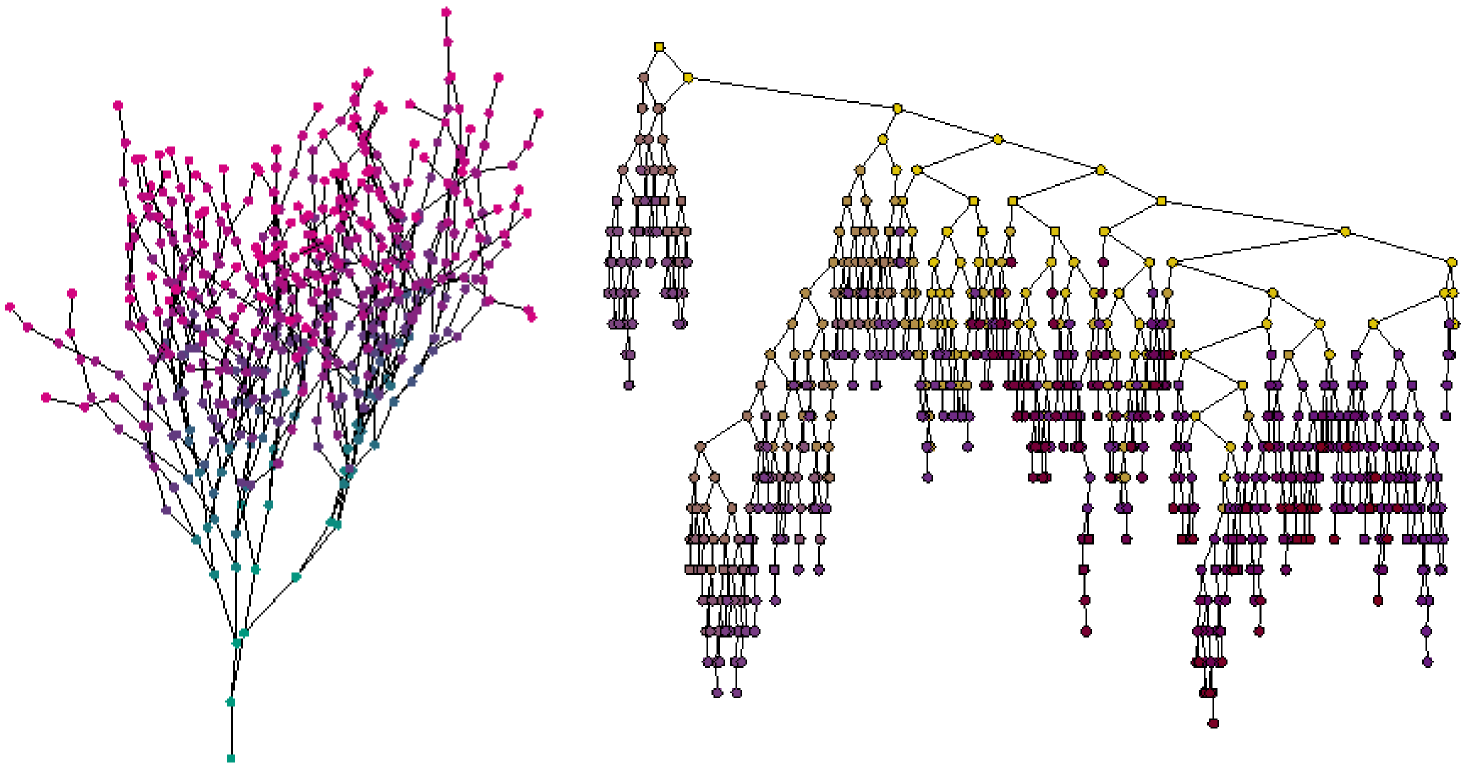
If X_n is a rescaled random discrete object, knowing $X_n \xrightarrow[n]{(d)} X$ in $C[0,1]$ says **nothing** about any phenomenon which is **not a the "same scale".**

Example: Almost surely the Brownian motion reaches is maximum once, traverses the origin an infinite number of times, is nowhere differentiable...

This is not the case in the discrete case

III. Convergence of trees... Convergence to continuum random trees

Question : do trees have a limit shape? How can we describe it?

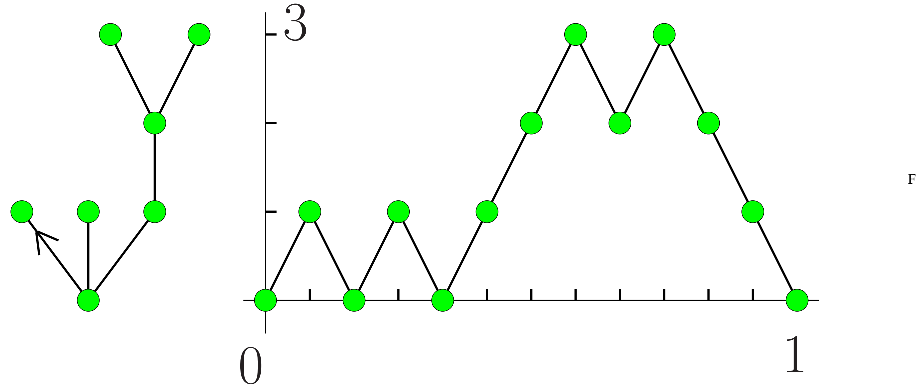


(Luc's Devroye trees)

To prove that rescaled trees converge we search a Polish space containing discrete trees and their limits (continuous trees).

Example of model of random trees : uniform rooted planar tree with n nodes

Trees as element of a Polish space : embedding in $C[0,1]$.



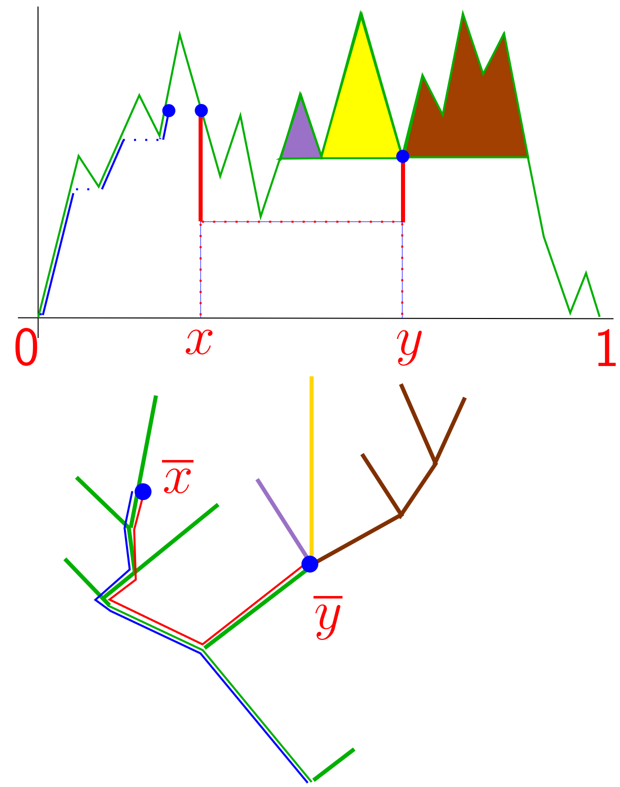
The contour process $(C(k), k = 0, \dots, 2(n-1))$.

The normalized contour process $\left(\dfrac{C(2(n-1)t)}{\sqrt{n}}\right)_{t\in[0,1]}$.

Notion of real tree

Let $C^+[0,1] = \{f \in C[0,1], f \geq 0, f(0) = f(1) = 0\}$.

With any function $f \in C^+[0,1]$, we associate a tree $A(f)$:



$$A(f) \ := [0,1]/ \underset{f}{\sim} \text{ where}$$

$$x \underset{f}{\sim} y \Longleftrightarrow f(x) = f(y) = \check{f}(x,y) \ := \min_{u \in [x \wedge y, x \vee y]} f(u)$$

$\star$ $A(f)$ equipped with the distance

$$d_f(\overline{x}, \overline{y}) = f(x) + f(y) - 2\check{f}(x,y)$$

is a compact metric space, loop free, connected: it is a tree!

The space $\mathcal{A}$ is equipped with the distance:

$$d(A(f), A(g)) = \|f - g\|_\infty.$$

It is then a Polish space

Convergence of rescaled tree in the space of real trees

Theorem [Aldous: Convergence of the rescaled contour process].

$$\left(\frac{C(2(n-1)t)}{\sqrt{n}}\right)_{t\in[0,1]} \xrightarrow[n]{(d)} \frac{2}{\sigma}\,(\mathsf{e}_t)_{t\in[0,1]}$$

RW: M & Mokkadem, Duquesne.

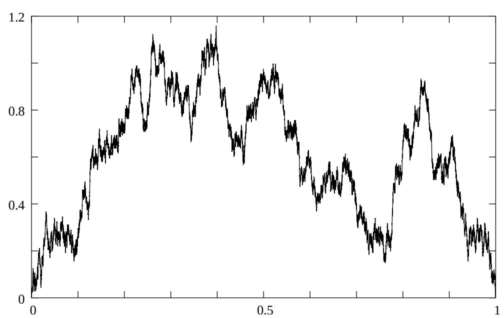
Result valid for critical GW tree conditioned by the size, including

Binary tree with n nodes, ...

Theorem [Aldous: convergence of rescaled tree to the Continuum random tree]

$$A\left(\frac{C(2(n-1).)}{\sqrt{n}}\right) \xrightarrow[n]{(d)} A(\frac{2}{\sigma}\mathsf{e}),$$

in the space of real trees.



This is a convergence (in distribution) of the whole macroscopic structure

Convergence of rescaled tree in the space of real trees

Byproducts : Explanation of most phenomenons at scale $\sqrt{n}$.

- Convergence of the height:

$$H_n/\sqrt{n} \xrightarrow[n]{(d)} \frac{2}{\sigma}\max \mathsf{e}$$

(Found before by Flajolet & Odlyzko (1982) + CV moments)

- convergence of the matrix of the distances $d(U_i, U_j)/\sqrt{n}$ of 12000 random nodes,

But : It does not explain the phenomenons at a different scale: the continuum random tree is a tree having only binary branching points, degree(root)=1...

Another topology à la mode : the Gromov-Hausdorff topology

The GH topology = topology on **compact metric spaces**.

> The GH distance is a distance on the set K of classes of isometry of compact metric spaces K.

$\to$ See the talk of Nicolas Broutin.

(K, d_{GH}) is a Polish space

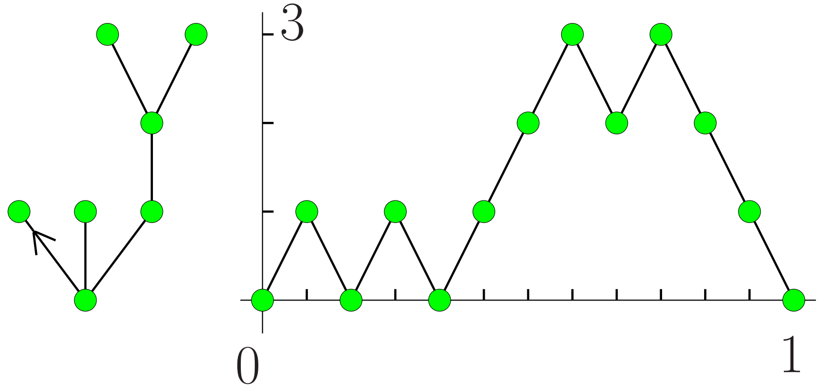
Intuition : take k points $U_i, i = 1, \ldots, k$ and show that the matrix of distance

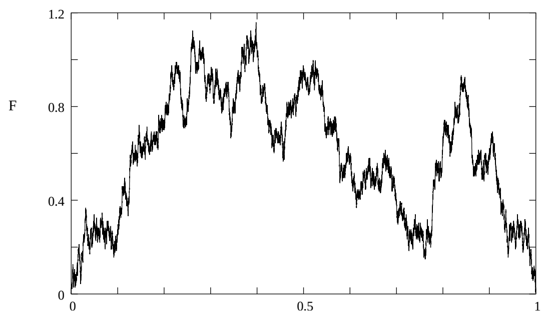
$$\Big(d_{S_n}(U_i, U_j)\Big)_{1 \le i,j \le k} \xrightarrow[n]{(d)} \Big(d_{S_\infty}(U_i, U_j)\Big)_{1 \le i,j \le k}.$$

Another topology à la mode : the Gromov-Hausdorff topology

The GH-topology is a quite weak topology...

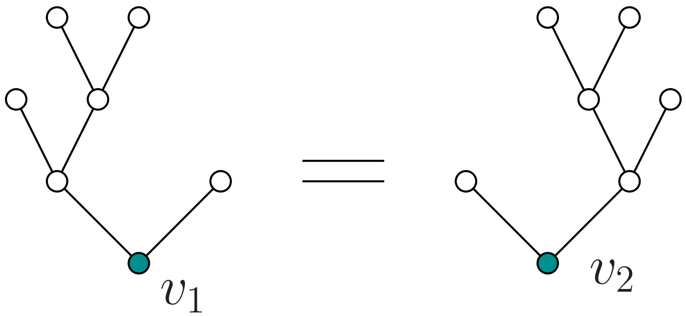
Theorem Normalized Galton-Watson trees converge to the CRT for the Gromov-Hausdorff topology.





Another topology à la mode : the Gromov-Hausdorff topology

Convergence of rooted non-planar binary trees for the GH topology



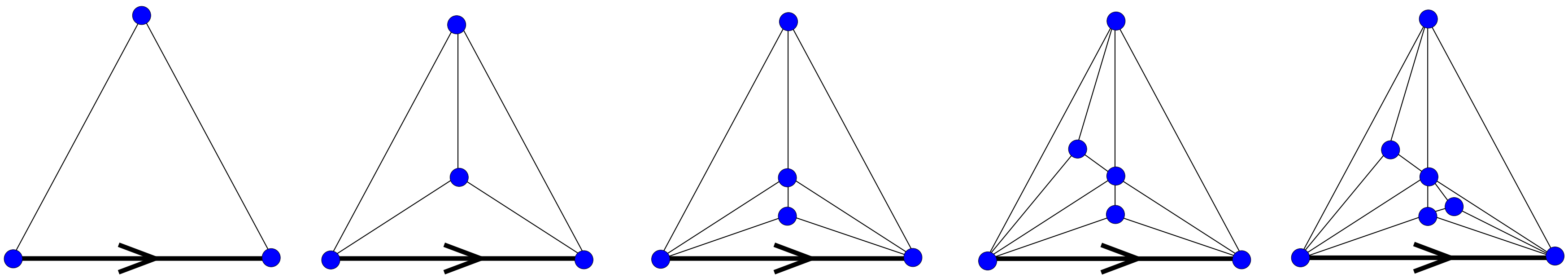
A non-planar-binary tree is a leaf or a multiset of two non-planar-binary trees

Theorem (M & Miermont). Under the uniform distr. on $\mathcal{U}_n$, the metric space $\left(\mathcal{T}_n, \frac{1}{\mathbf{c}\sqrt{n}} d_{\mathcal{T}_n}\right)$ converge in distribution to $(\mathcal{T}_{2\mathrm{e}}, d_{2\mathrm{e}})$ the CRT for the GH topology.

Related work: Otter, Drmota, Gittenberger, Broutin & Flajolet

Convergence to the CRT for objects that are not trees:

Model of uniform stacked triangulations



M_n= uniform stack-triangulation with $2n$ faces seen as a metric space;
D_{M_n}= graph-distance in M_n

Theorem (Albenque & M)

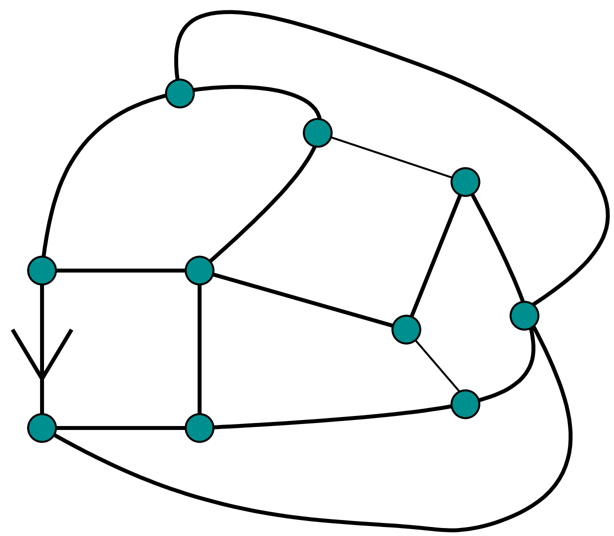
$$\left(m_n, \frac{D_{m_n}}{\sqrt{6n}/11}\right) \xrightarrow[n]{(d)} (\mathcal{T}_{2\mathrm{e}}, d_{2\mathrm{e}}),$$

for the Gromov-Hausdorff topology on compact metric spaces.

Related works: Bodini, Darasse, Soria

Another topology à la mode : the Gromov-Hausdorff topology

Convergence of quadrangulations with n faces?



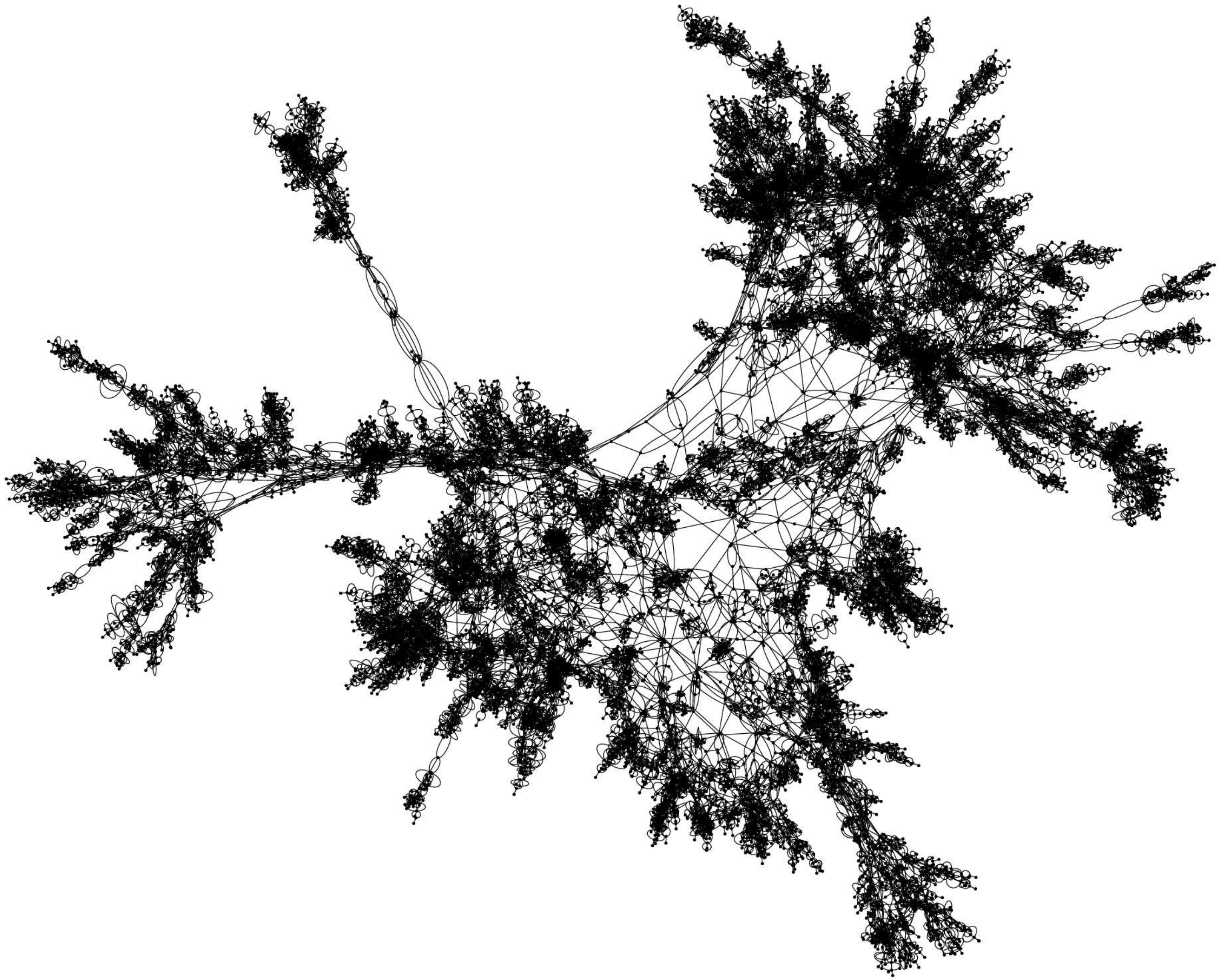
Seen as metric spaces, do they converges in distribution ?

What is known : subsequences converge in distribution to some random metric on the sphere (Le Gall, Miermont) for GH.

Related works : Chassaing-Schaeffer, M-Mokkadem, Bouttier - Di Francesco - Guitter, Miermont, Le Gall...

Another topology à la mode : the Gromov-Hausdorff topology

Convergence of quadrangulations with n faces?



Another topology à la mode : the Gromov-Hausdorff topology

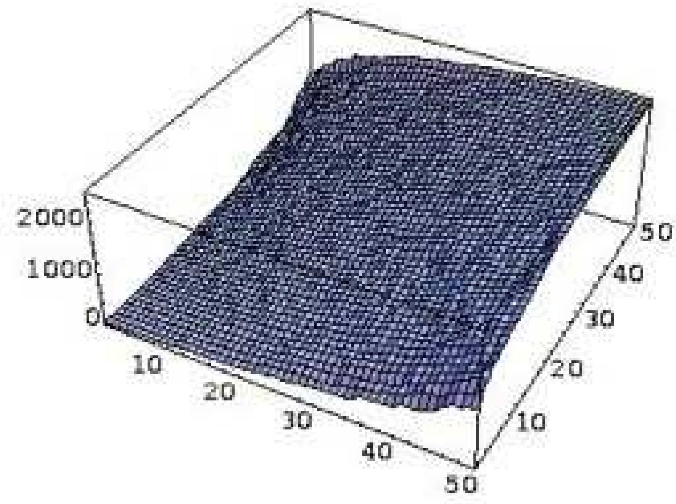
Connected component in “critical $G(n,p)$”

See the talk of Nicolas Broutin

IV. Other examples!

convergence of rescaled combinatorial structures to deterministic limit

- Limit shape of a uniform square Young-tableau: Pittel-Romik



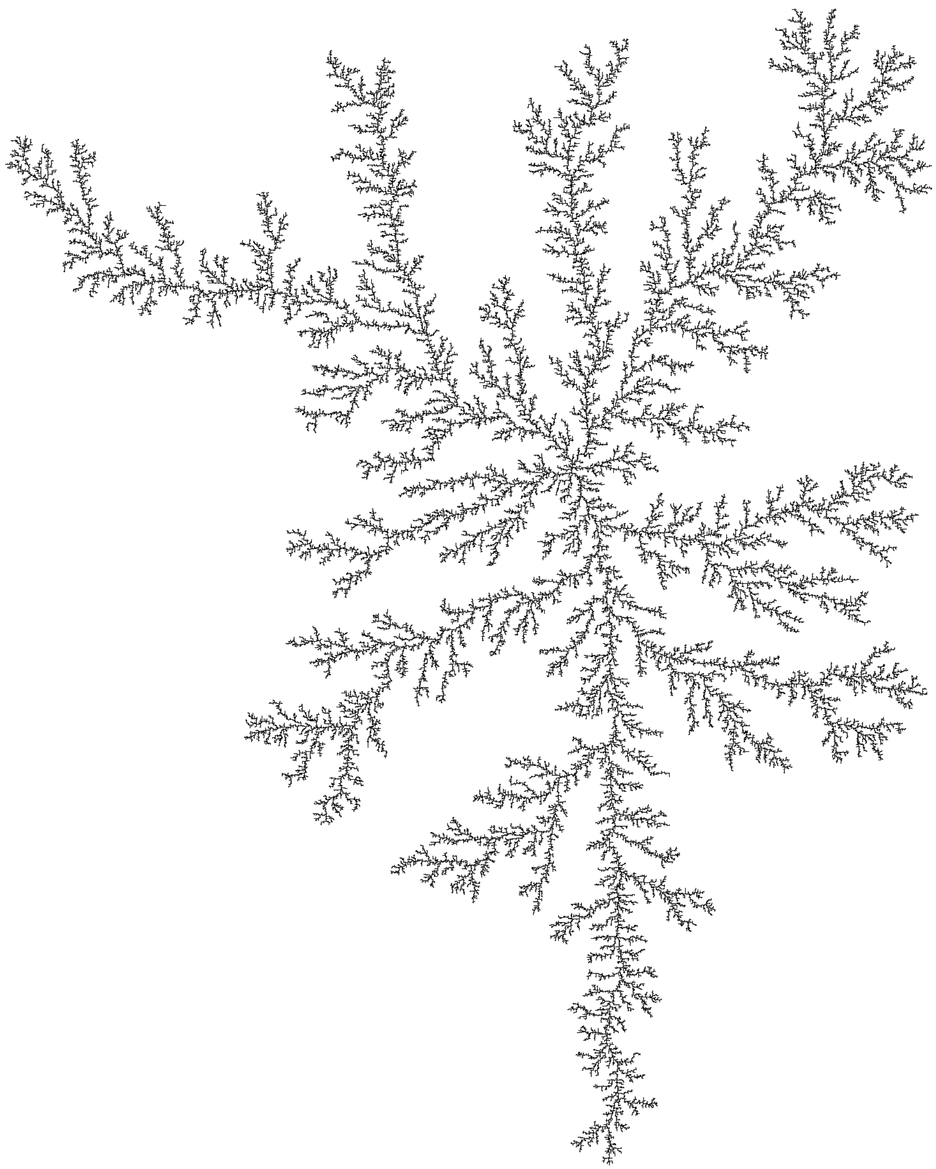
source: Dan Romik's page

Convergence for the topology of uniform convergence (functions defined on $[0,1]^2$).

- limit for Ferrer diagram (Pittel)

- Limit shape for plane partitions in a box (Cohn, Larsen, Propp)

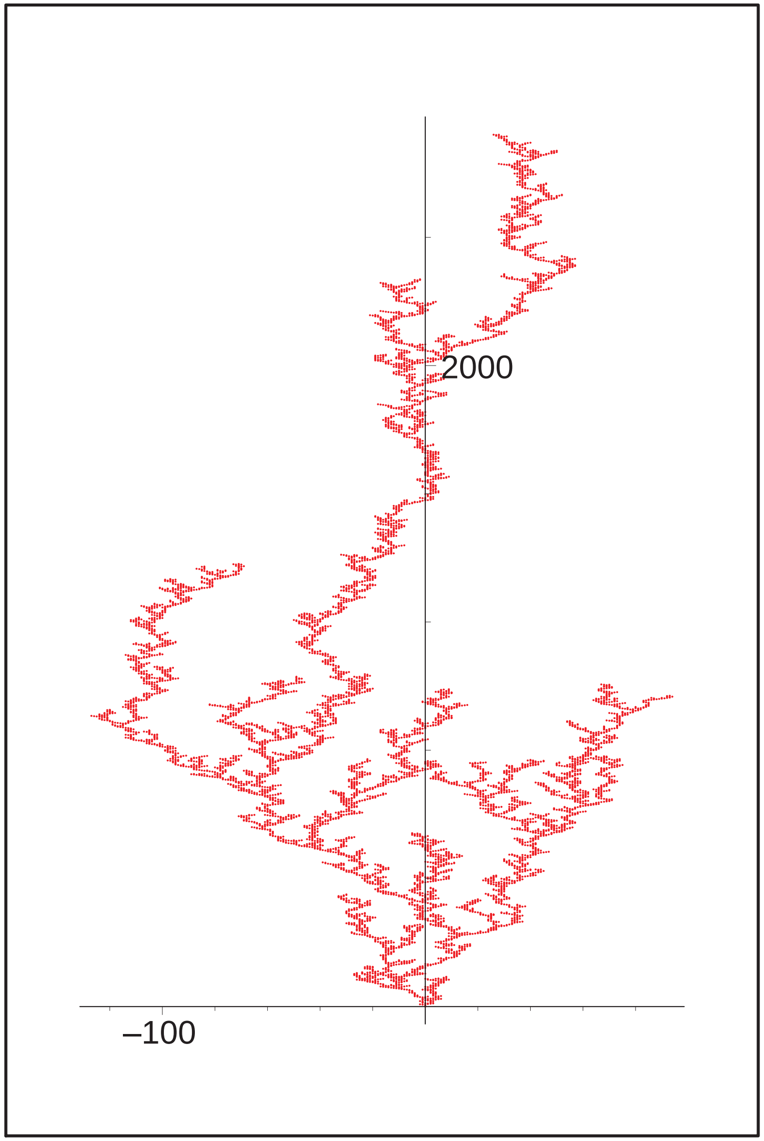
Unknown limits



DLA: diffusion limited aggregation
source:Vincent Beffara's page

Other model: internal DLA; the limit is the circle (CV in proba), Bramson, Griffeath, Lawler

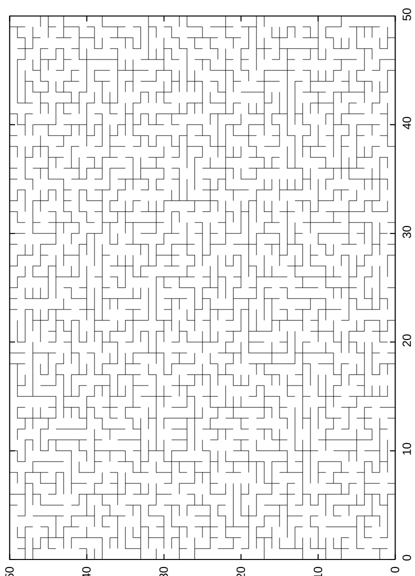
Unknown limits

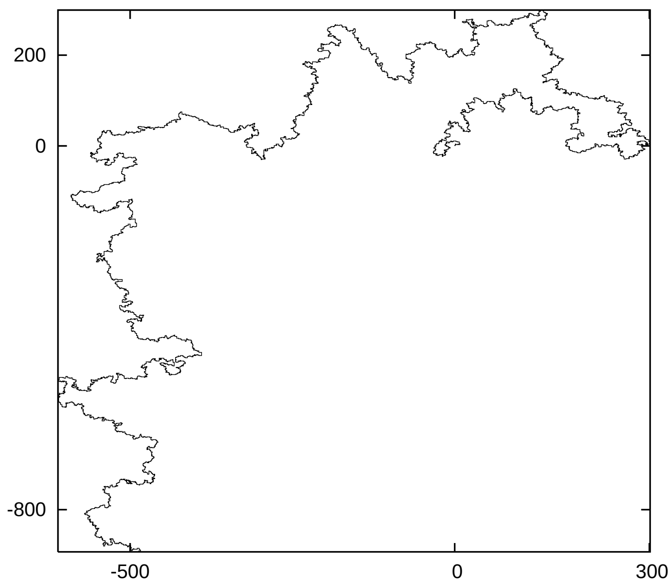


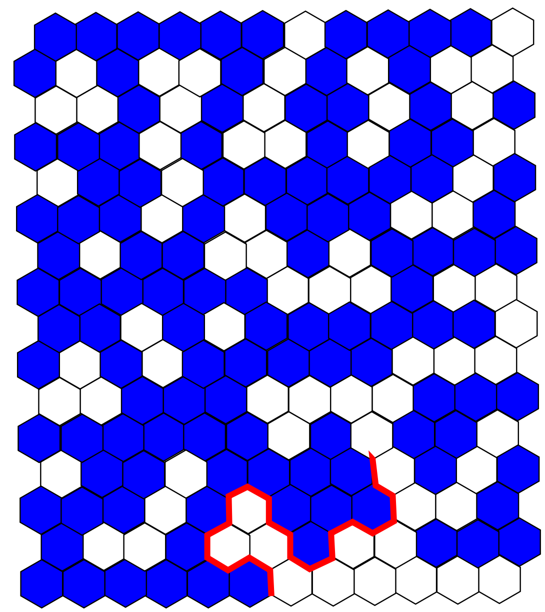
Directed animal

More or less known limits

SLE related process: limit of loop erased random walk, self avoiding random walks, contour process of percolation cluster, uniform spanning tree,...
Works of Lawler, Schramm, Werner







Convergence for the Hausdorff topology to conformally invariant distribution

Other models

Voter models, Ising models, First passage percolation, Richardson's growth model,...

That's all...
Thanks