The scaling limit of critical random graphs

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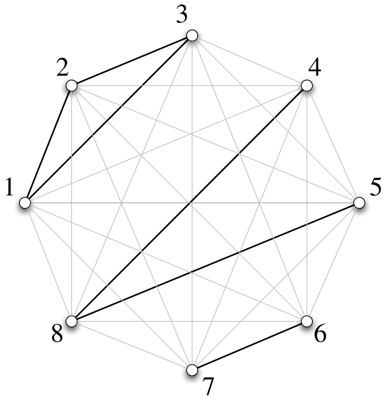
Plan

1. Random graphs
2. Exploration and branching processes
3. Large random trees
4. Comparing trees and Gromov–Hausdorff distance
5. Continuum random tree
6. Depth-first search and metric structure of large graphs

Erdős–Rényi random graphs

- n labelled vertices $\{1, 2, \ldots, n\}$
- $G_{n,p}$: each edge is present with probability $p \in [0, 1]$



Evolution of random graphs

$S_1^n \geq S_2^n \geq S_3^n \ldots$: sizes of connected components

Different regimes for sizes as $n \to \infty$:

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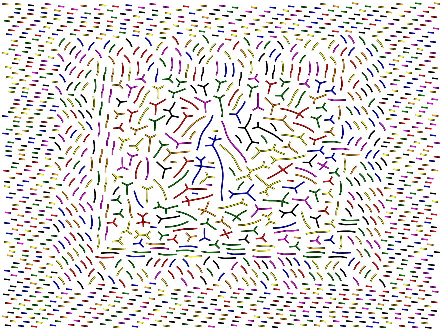

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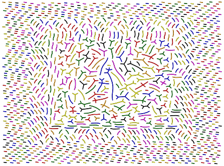
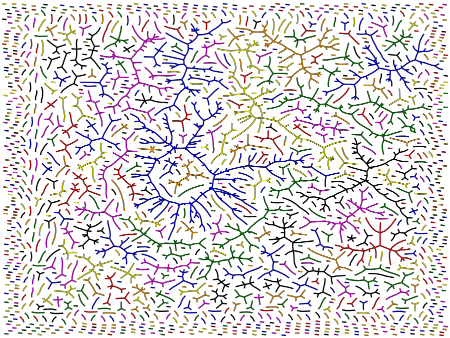


Different regimes for sizes as $n \to \infty$:

- $np = 1 - \epsilon$: $S_1^n = O(\log n)$
- $np = 1$: $\forall k \geq 1$ fixed $S_k^n = \Theta(n^{2/3})$



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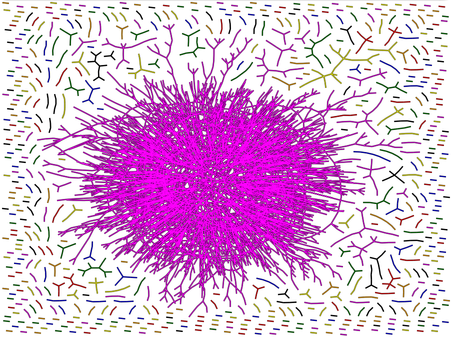

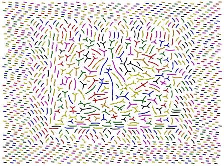

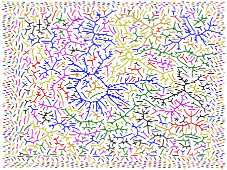

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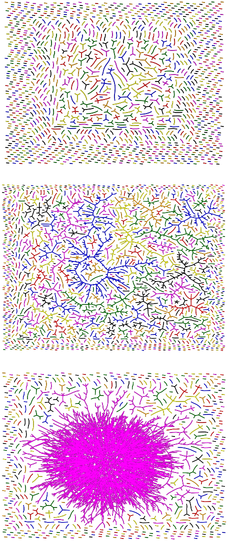


Component sizes and component structure

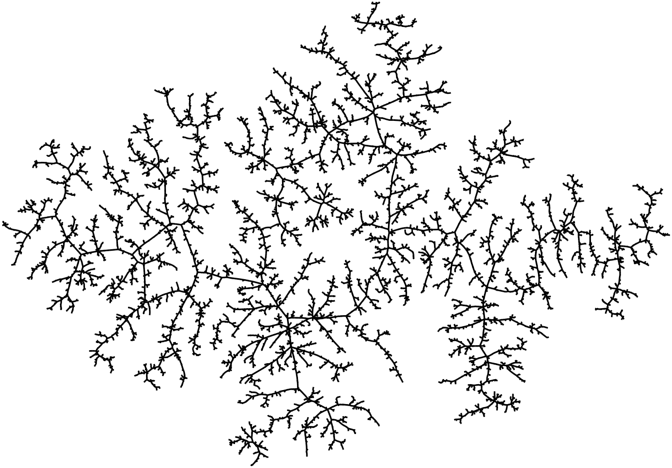
Important observation:

Given its size m, a connected component of $G(n,p)$ is distributed as $G(m,p)$ conditioned on being connected.

$\Rightarrow$ Study separately:

- sizes of connected components
- their structure given the size

What does a component look like?



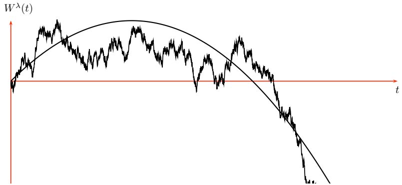
Inside the critical window: $np = 1 + \lambda n^{-1/3}$ for $\lambda \in \mathbb{R}$

Theorem (Aldous, 1997)

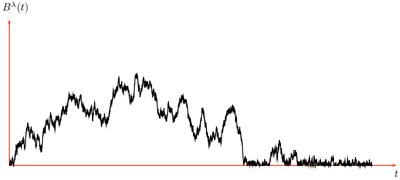
$(S_i^n; N_i^n)$ the size and surplus of the i-th largest component of $G_{n,p}$

$$((n^{-2/3} S_i^n; N_i^n) : i \geq 1) \xrightarrow[n\to\infty]{d} ((S_i; N_i) : i \geq 1)$$

as a sequence in $\ell^2_{\searrow} = \{x = (x_1, x_2, \dots) : x_1 \geq x_2 \geq \cdots \geq 0, \sum_{i\geq 1} x_i^2 < \infty\}$.



$W^\lambda(t) = W(t) + t\lambda - \frac{t^2}{2}$

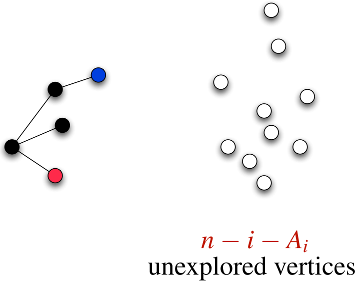


$B^\lambda(t) = W^\lambda(t) - \inf_{0<s<t} W^\lambda(s)$

Graph exploration and a branching process/random walk

Exposure vertices one after another:

- Initialization:
  one active vertex $A_0 = 1$
- At each time step:
  $A_{i+1} = A_i - 1 + Bin(n - i - A_i, p)$



The walk A_i and sizes of components

For $k \geq 0$, set $\xi_k := \inf\{i : A_i = 1 - k\}$
$\xi_k - \xi_{k-1}$ size of the k-th explored component

Asymptotics for A_i and phases in random graphs

Random walk with step distribution

$$A_{i+1} - A_i = Bin(n - i - A_i, p) - 1$$

The regimes when $np = c$

$\mathbf{E}\left[A_{i+1} - A_i\right] = \mathbf{E}Bin(n-i-A_i, p) - 1 \approx c - 1$

- $c < 1$: $A_i \to -\infty$
- $c = 1$: approx recurrent
- $c > 1$: $A_i \to \infty$

Branching process point of view

New vertices: $\mathbf{E}Bin(n - i - A_i, p) \approx c$

- $c < 1$: subcritical
- $c = 1$: critical
- $c > 1$: supercritical

The critical window

Parametrization of the probability for the critical window

$p = 1/n + \lambda n^{-4/3}$ with $\lambda \in \mathbb{R}$

$$A_{i+1} - A_i = Bin\left(n - i - A_i, \frac{1}{n} + \lambda n^{-4/3}\right) - 1$$

So for $i \sim tn^{2/3}$,

$$\mathbf{E}\left[A_{i+1} - A_i\right] \sim (\lambda - t)n^{-1/3} \quad \text{and} \quad \mathbf{Var}\left[A_{i+1} - A_i\right] \sim 1$$

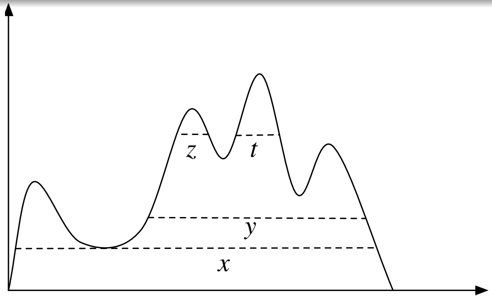
Central Limit Theorem for martingales:

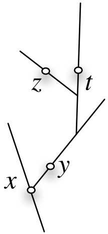
$$\left(\frac{A_{tn^{2/3}}}{n^{1/3}}\right)_{t\geq 0} \to \left(\lambda t - \frac{t^2}{2} + W(t)\right)_{t\geq 0}$$

Strategy to describe random graphs

1. Decompose into connected components
2. Extract a tree from each connected component
3. Describe the trees
4. Describe how to put back surplus edges

Real trees





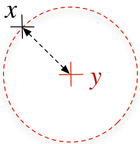
The tree in a continuous excursion

- continuous excursion $f(0) = f(1) = 0, f(s) > 0, s \in (0,1)$
- metric $d_f(x,y) = f(x) + f(y) - 2\inf_{s\in[x,y]} f(s)$.
- $x \sim_f y$ iff $d_f(x,y) = 0$
- the metric space $([0,1]/\sim_f, d_f)$ has a tree structure

Comparing subsets of a metric space: Hausdorff distance

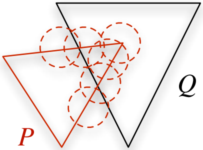
For two points $x, y \in (M, d)$

$$d(x,y) = \inf\{\epsilon > 0 : y \in B(x,\epsilon) \text{ and } x \in B(y,\epsilon)\}$$



For two subsets P, Q of a metric space (M, d): Hausdorff distance

$$d_H(P,Q) = \inf\left\{\epsilon > 0 : P \subseteq \bigcup_{x \in Q} B(x,\epsilon) \text{ and } Q \subseteq \bigcup_{y \in P} B(y,\epsilon)\right\}.$$



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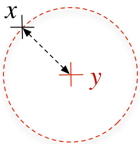

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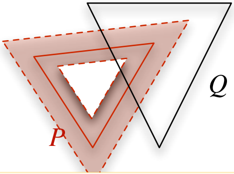


Comparing metric spaces: Gromov–Hausdorff distance

Using isometries

Embed both spaces isometrically into the same bigger space M:

$$d_{GH}((M_1, d_1); (M_2, d_2)) = \inf d_H(M_1, M_2),$$

where the infimum is over all metric spaces M containing both (M_1, d_1) and (M_2, d_2).

Using distortions of correspondences

Map each space inside the other using a *correspondence* $\mathcal{R} \subseteq M_1 \times M_2$

$$\mathrm{dist}(\mathcal{R}) = \sup_{(x_1,x_2),(y_1,y_2)\in\mathcal{R}} \{|d_1(x_1, y_1) - d_2(x_2, y_2)|\}$$

$$d_{GH}((M_1, d_1); (M_2, d_2)) = \frac{1}{2} \inf_{\mathcal{R}} \mathrm{dist}(\mathcal{R})$$

The Brownian continuum random tree (CRT)

Let $\mathcal{T}(2e)$ be the real tree encoded by twice a standard Brownian excursion e

Theorem (Aldous)

Let T_n be a Cayley tree of size n, seen as a metric space with the graph distance. Then,

$$n^{-1/2} T_n \xrightarrow[n\to\infty]{d} \mathcal{T}(2e)$$

with the Gromov–Hausdorff distance.

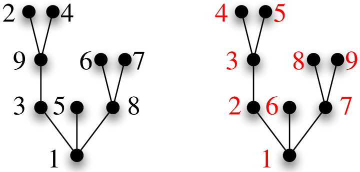
Some remarks:

- extends to all Galton–Watson trees with finite variance progeny
- the trees in random maps
- if infinite variance: Lévy trees, stable trees.

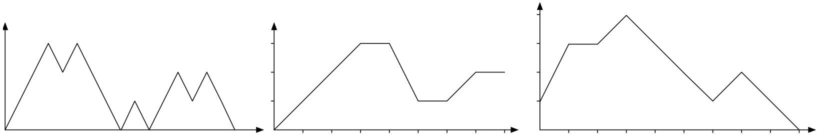
Representation of trees

A canonical order for the nodes:

- sort children by increasing label
- Depth-first order



Three different encodings of trees as nonnegative paths:



Contour process C Height process H Depth-first walk X

Walks associated with large random trees

Aldous, Le Gall, Marckert–Mokkadem

Let T_n be a Cayley tree of size n.

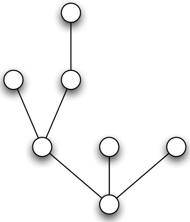
Theorem (Marckert–Mokkadem)

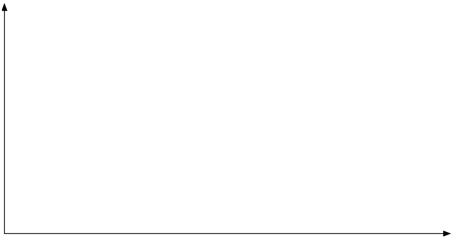
Let $e = (e(t), 0 \leq t \leq 1)$ be a standard Brownian excursion. Then,

$$\left(\frac{X(\lfloor n \cdot \rfloor)}{\sqrt{n}}; \frac{H(\lfloor n \cdot \rfloor)}{\sqrt{n}}; \frac{C(\lfloor 2n \cdot \rfloor)}{\sqrt{n}} \right) \xrightarrow[n\to\infty]{d} (e(\,\cdot\,); 2e(\,\cdot\,); 2e(\,\cdot\,))$$

Understanding Depth-first search in graphs

$X(0) = 0$ and $X(i+1) - X(i) = \#$children of i minus 1





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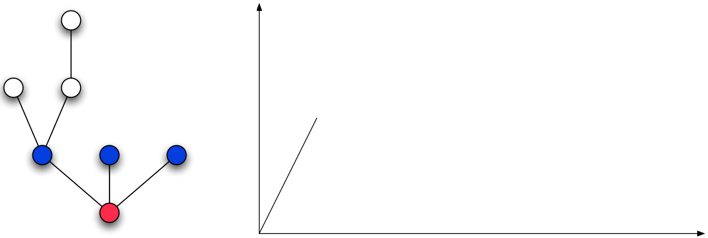

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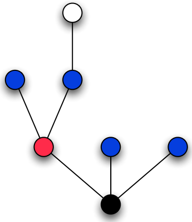

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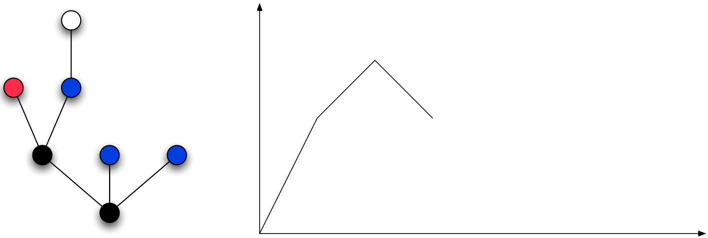

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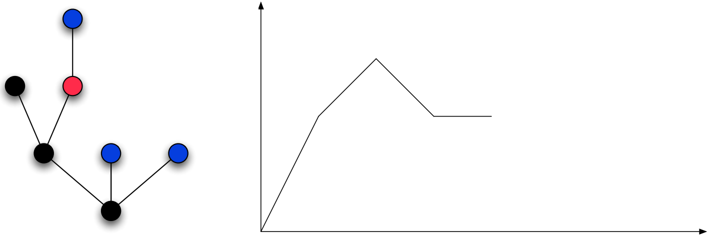

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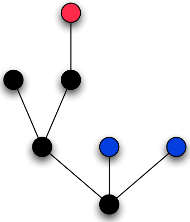

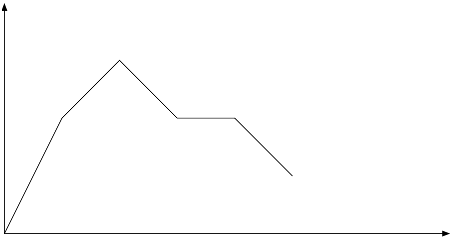

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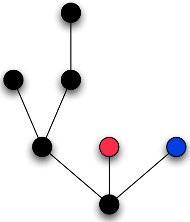

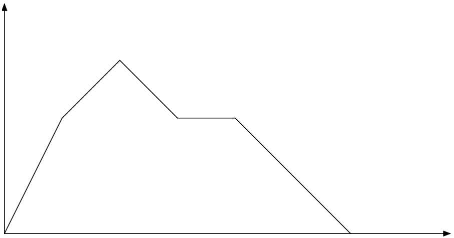

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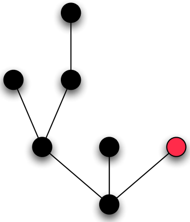

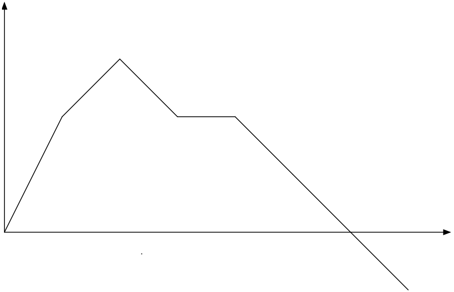

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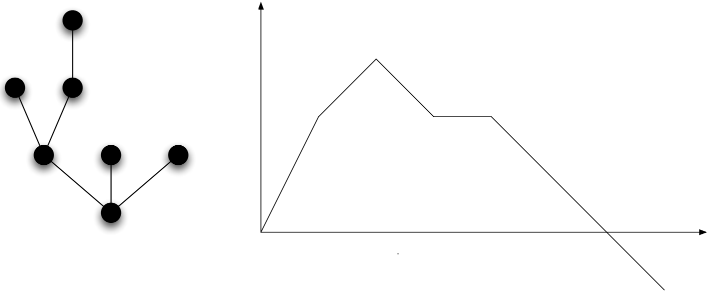

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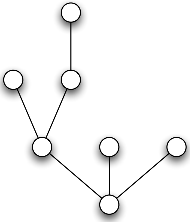

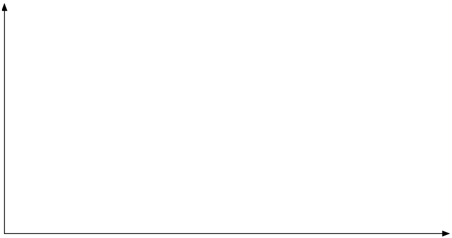

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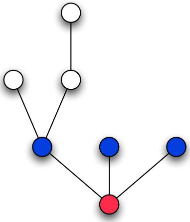

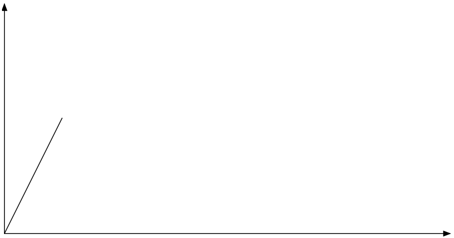

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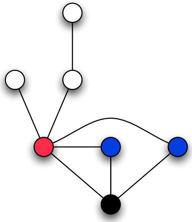

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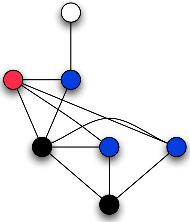

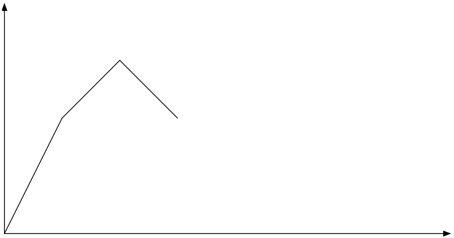

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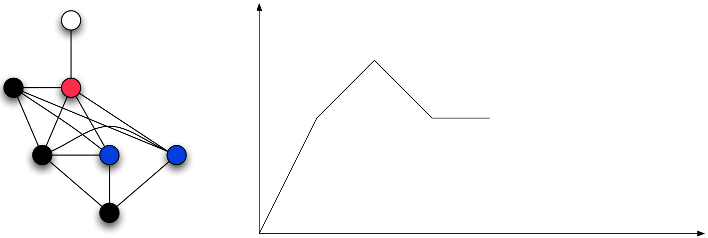

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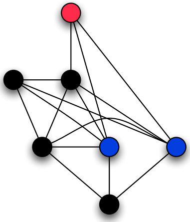

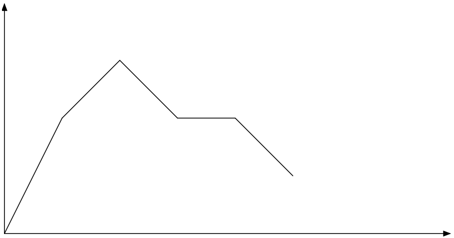

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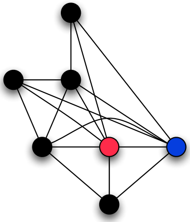

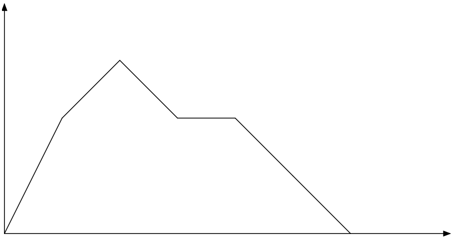

Question:

what can we change without changing the tree we obtain?

Understanding Depth-first search in graphs

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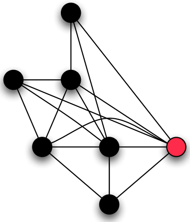

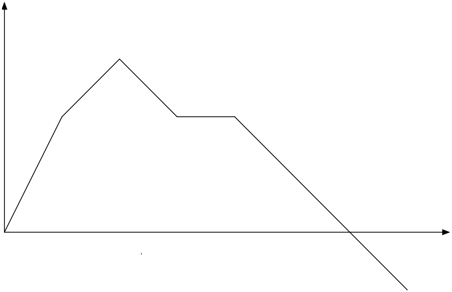

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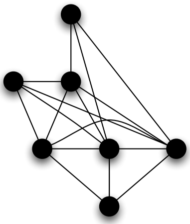
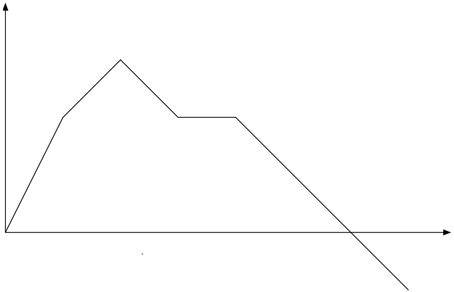




Question:

what can we change without changing the tree we obtain?

Answer:

Can add any of the $a(T)$ edges between two vertices active at the same time

Canonical tree and sampling random graphs

Partition the all the graphs G according to their canonical tree $T(G)$

$$\mathbb{G}_n = \bigcup_{T \in \mathbb{T}_n} \{G : T(G) = T\}$$

- Each graph G has a unique canonical tree $T(G)$
- Each canoninal tree T yields $2^{a(T)}$ graphs: $\#\{G : T(G) = T\} = 2^{a(T)}$

Uniform connected component of $G_{n,p}$ with m vertices

- Pick $\tilde{T}_m = T$ with probability $\propto (1-p)^{-a(T)}$
- Add each allowed edge with probability p.

Uniform connected connected graph with m vertices and $m-1+s$ edges

- Pick $\tilde{T}_m = T$ with probability $\propto \binom{a(T)}{s}$
- Add s random allowed edges.

Limit of canonical trees

In the critical regime: $p \sim 1/n$ and $m \sim n^{2/3}$:

$$(1-p)^{-a(T_m)} \sim \exp(m^{-3/2} a(T_m)) \sim \exp(\int_0^1 e(s)ds)$$

Definition (Tilted excursion)

Let e be a standard Brownian excursion:

$$\mathbb{P}(\tilde{e} \in \mathcal{B}) = \frac{\mathbf{E}\left[\mathbf{1}[e \in \mathcal{B}] \exp(\int_0^1 e(s)ds)\right]}{\mathbf{E}\left[\exp(\int_0^1 e(s)ds)\right]}.$$

Let $\tilde{T}_m$ be picked such that $\mathbb{P}\left(\tilde{T}_m = T\right) \propto (1-m^{-3/2})^{-a(T)}$

Theorem

$$\left(\frac{\tilde{X}^n(\lfloor n\cdot\rfloor)}{\sqrt{n}}; \frac{\tilde{C}^n(\lfloor 2n\cdot\rfloor)}{\sqrt{n}}; \frac{\tilde{H}^n(\lfloor n\cdot\rfloor)}{\sqrt{n}}\right) \xrightarrow[n\to\infty]{d} (\tilde{e}(\cdot); 2\tilde{e}(\cdot); 2\tilde{e}(\cdot)).$$

The limit of tilted trees

Let $\mathcal{T}(\tilde{e}^{(\sigma)})$ the real tree encoded by a tilted excursion $\tilde{e}^{(\sigma)}$ of length σ.

Theorem

Let $p \sim 1/n$ and $m \sim \sigma n^{2/3}$, $\sigma > 0$. Let G_m^p be a uniform connected component of $G_{n,p}$ on m vertices.

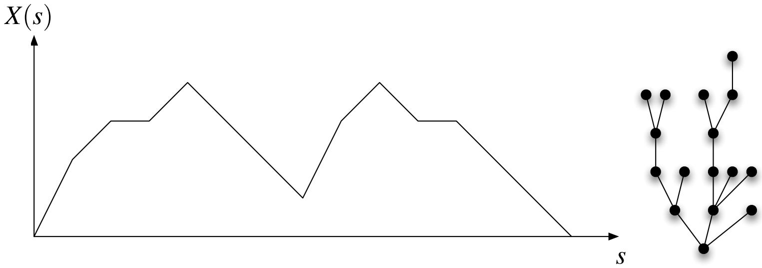
$$\frac{T(G_m^p)}{n^{1/3}} \xrightarrow[n\to\infty]{d} \mathcal{T}(2\tilde{e}^{(\sigma)}),$$

with the Gromov–Hausdorff distance.

Connected graphs as marked excursions

Bijection between:

- Excursion of length m with some points under X: $(X, \mathcal{P})$.
- Connected graphs on m vertices $G^X(X, \mathcal{P})$



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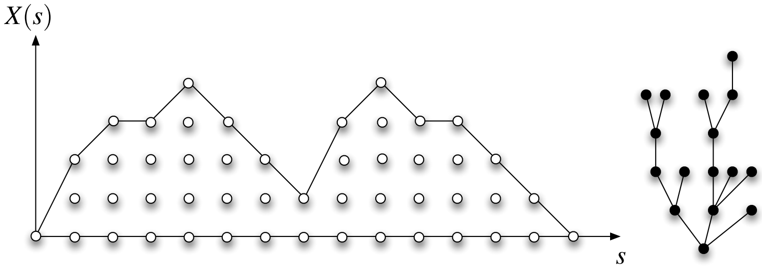

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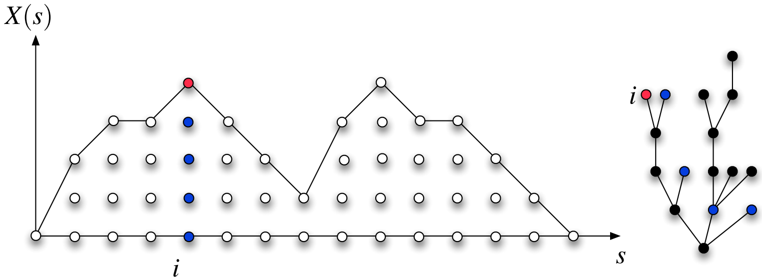

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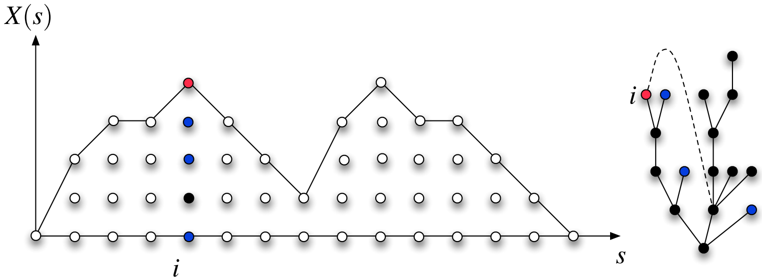

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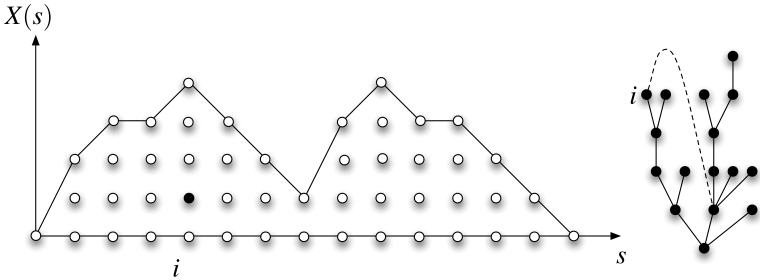
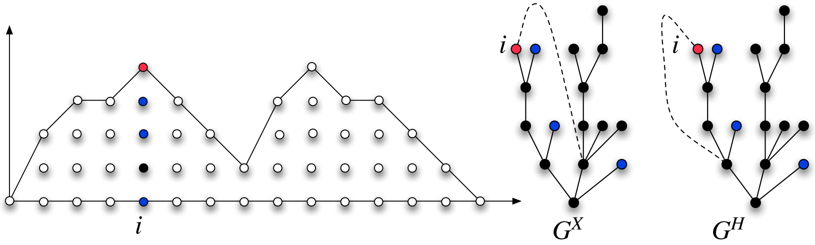


The limit of connected components: convergence

Need both X and H

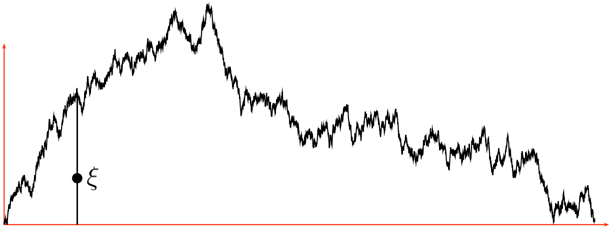
- X provides a bijection
- H provides the metric structure



Limit using the "Depth-First construction" G^X

$m^{-1/2}\tilde{X}^m(\lfloor m\cdot\rfloor) \to \tilde{e}$ and marks converge to a Poisson point process $\mathcal{P}$

Limit of connected components: construction



Metric characterization using the "height construction" G^H

$\mathcal{T}(2\tilde{e})$ where each point (ξ_x, ξ_y) of $\mathcal{P}$ identifies the leaf ξ_x with the point at distance ξ_y from the root on the path $[\![0, \xi_x]\!]$.

The limit of critical random graphs

M_i^n the i-th largest connected component of $G_{n,p}$ and S_i^n its size.
$\mathbf{M}^n = (M_1^n, M_2^n, \dots)$ as a sequence of metric spaces, with distance

$$d(\mathbf{A}, \mathbf{B}) = \left(\sum_{i \geq 1} d_{GH}(A_i, B_i)^4 \right)^{1/4}$$

$n^{-2/3}\mathbf{S}^n = (n^{-2/3} S_1^n, \dots)$ in
$\ell^2_{\searrow} = \{(x_1, x_2, \dots) : x_1 \geq 0, \dots, \sum_{i \geq 1} x_i^2 < \infty\}$.

Theorem

$$(n^{-1/3}\mathbf{M}^n, n^{-2/3}\mathbf{S}^n) \xrightarrow[n\to\infty]{d} (\mathbf{M}, \mathbf{S}) \qquad \text{where}$$

- $\mathbf{S}$ is the ordered sequence of excursion lengths of B^λ
- Given $S = (S_1, S_2, \dots)$, $(M_1, M_2, \dots)$ are independent $g(\tilde{e}^{(S_i)}, \mathcal{P}_i)$.