How to scotch a rumor in a network ?

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The SIR spreading model

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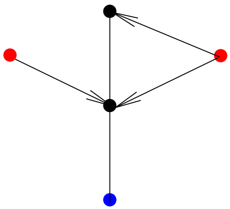
$\Longrightarrow$ *rumor spreading, epidemic, prey and predator, information dissemination ...*

Standard SIR dynamics

Consider the Markov process :

- *a (S)-vertex becomes (I) at rate λ times the number of (I)-neighbors,*
- *a (I)-vertex becomes (R) at rate 1.*



Absorbing States

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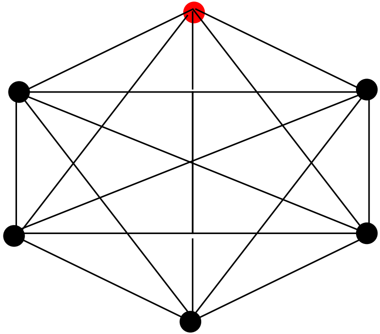
In final state : (R)-vertices $=$ vertices that have been infected.

This final absorbing state is random, what can be said about it ?

On the Complete Graph

Assume that the graph is K_n.

Infection rate is λ/n.



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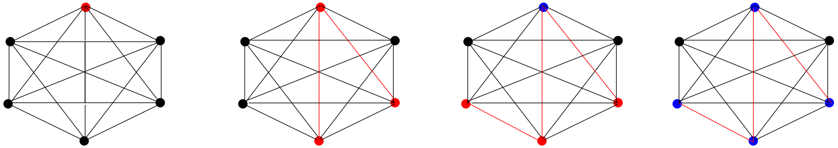


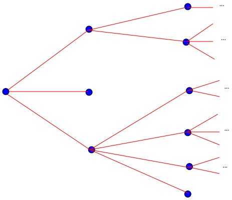
Let G_n be the graph spanned by the vertices that have been infected and $Z_n = |G_n|$.

There is a well-known scaling limit of G_n as $n \to \infty$.

Scaling limit

D_1 = number of vertices infected by the initial (I)-vertex before becoming (R).

$$\mathbb{P}(D_1 = k) \underset{n\to\infty}{\simeq} \frac{\lambda^k}{(\lambda+1)^k} = \mathrm{Geo}_{\frac{1}{\lambda+1}}(\{k\}).$$



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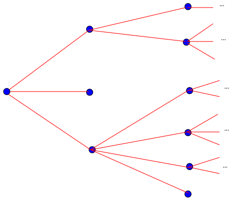

The graph G_n converges weakly to a Galton-Watson tree with Geometric offspring distribution with parameter $1/(\lambda+1)$.

Scaling limit and phase transition

$\Longrightarrow$ If $\lambda < 1$, then Z_n converges weakly to Z and

$$\lim_n \mathbb{E} Z_n = \mathbb{E} Z = 1/(1-\lambda).$$

$\Longrightarrow$ Subcritical regime : the number of infected vertices remains small.

Scaling limit and phase transition

$\Longrightarrow$ If $\lambda > 1$, then Z_n/n converges weakly to W and

$$W \stackrel{d}{=} \rho\delta_0 + (1-\rho)\delta_{1-\rho},$$

where

$$\rho = 1/\lambda = \text{probability of extinction.}$$

$\Longrightarrow$ Supercritical regime : with $0 <$ probability the number of infected vertices is macroscopic.

(there are also finer finite size estimates on Z_n*)*

Tail of distribution

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In fact : for all $n \geqslant 1$, $t \geqslant 0$, *with* $c(\lambda) = \lambda - 1 - \ln \lambda$,

$$\mathbb{P}(Z_n \geqslant t) \leqslant \lambda^{-1} e^{-c(\lambda)t}.$$

Rumor Scotching Process

We change the dynamic as follows :

- *a* (S)*-vertex becomes* (I) *at rate* λ *times the number* (I)*-neighbors,*
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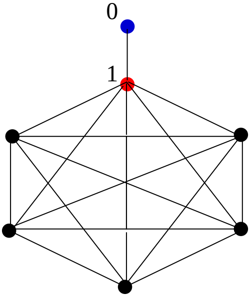
a variant :

- *a* (I)*-vertex becomes* (R) *at rate* 1 *times the number of neighboring* (R)*-vertices that have infected the vertex.*

$\Longrightarrow$ The rumor is confidential.

On the Complete Graph

Infection rate is λ/n.



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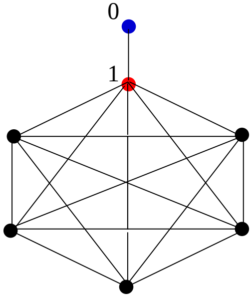

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Again, it is possible to compute the scaling limit as $n \to \infty$.

Birth and Assassination Process

(Aldous and Krebs 1990)

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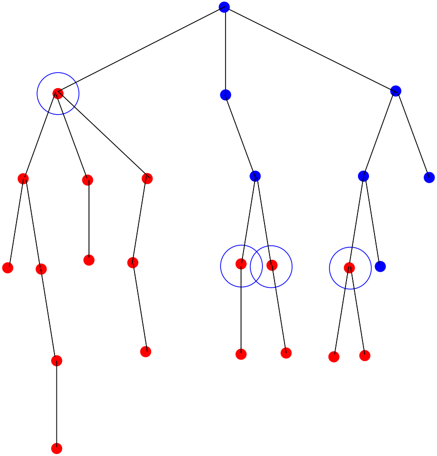
Other vertices are at risk when its ancestor dies, and dies after an independent copy of D.

Birth and Assassination Process



Phase transition

Theorem (Aldous & Krebs 1990)

If $0 < \lambda < 1/4$*, the tree is a.s. finite, if* $\lambda > 1/4$ *the process is infinite with* $0 <$ *probability.*

Phase transition

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If $0 < \lambda < 1/4$, the tree is a.s. finite, if $\lambda > 1/4$ the process is infinite with $0 <$ probability.

$\Longrightarrow$ on the complete graph, we get

Theorem
If $\lambda > 1/4$, there exists $\delta > 0$ such that

$$\liminf_n \mathbb{P}_\lambda(Z_n \geqslant \delta n) > 0.$$

A First Problem

One can guess that Z_n/n converges weakly to W with

$$W \stackrel{d}{=} \rho\delta_0 + (1-\rho)\delta_1,$$

with

$$\rho(\lambda) = \mathbb{P}_\lambda \left(\text{extinction in the BA process}\right).$$

$\Longrightarrow$ Either quick extinction or total invasion.

Subcritical phase

For $0 < \lambda < 1/4$, Z_n converges weakly to Z = total population in the BA process.

As before, we set

$$\gamma(\lambda) = \sup\{k \geqslant 0 : \mathbb{E}[Z^k] < \infty\}.$$

Total Infected Population

Theorem

(i) *For all* $0 < \lambda \leqslant 1/4$,

$$\gamma(\lambda) = \frac{1 + \sqrt{1 - 4\lambda}}{1 - \sqrt{1 - 4\lambda}}.$$

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(iii) *If* $\lambda \in (0, 2/9)$,

$$\mathbb{E}_\lambda[Z^2] = \frac{2}{3\sqrt{1 - 4\lambda} - 1}.$$

(iv) *If* $\lambda \in (0, 3/16)$,

$$\mathbb{E}_\lambda[Z^3] = \cdots$$

Recursive Distributional Equation

$Y(t)$ = the total population given that the root dies at time t.

If D is an exponential variable with parameter 1, independent of Y

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If $\{\xi_i\}_{i \geqslant 1}$ is a Poisson point process of intensity λ, independent of $(Y_i, D_i)_{i \geqslant 1}$, a sequence of independent copies of (Y, D).

$$Y(t) \quad \stackrel{d}{=} \quad 1 + \sum_{0 \leqslant \xi_i \leqslant t} Y_i(t - \xi_i + D_i)$$

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First Moment

Assume that $\mathbb{E}Y(t) < \infty$ for all $t \geqslant 0$. Taking expectation, we get

$$\begin{aligned} \mathbb{E}Y(t) &= 1 + \lambda \int_0^t \int_0^\infty \mathbb{E}Y(x+s)e^{-s} ds dx \\ &= 1 + \lambda \int_0^t e^x \int_x^\infty \mathbb{E}Y(s)e^{-s} ds dx \end{aligned}$$

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If $0 < \lambda \leqslant 1/4$, the roots of $X^2 - X + \lambda = 0$ are real $0 < \alpha \leqslant \beta \cdots$

$$\mathbb{E}Y(t) = e^{\alpha t}.$$

If $\lambda > 1/4$ no admissible solution of the integral equation.

Probability of Extinction

For $\lambda > 1/4$, can we compute the probability of extinction,

$$\rho(\lambda) = \mathbb{P}_\lambda(Z < \infty) \quad ?$$

Through

$$x(t) = -\ln \mathbb{P}_\lambda(Z < \infty | \text{root dies at time } t),$$

we get

$$x'' - x' + \lambda - \lambda e^{-x} = 0,$$

with $x(0) = 0$.

Second Problem

There is no real hope to solve the non-linear differential equation.

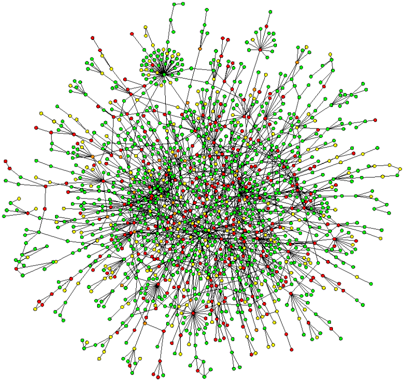
$$1-\rho(\lambda) \underset{\lambda\downarrow 1/4}{\simeq} f\left(\lambda-\frac{1}{4}\right) \quad ?$$

$\longrightarrow$ For the standard SIR dynamics, for $\lambda > 1$,

$$1-\rho(\lambda) = 1-\frac{1}{\lambda} \underset{\lambda\downarrow 1}{\simeq} (\lambda-1)\,.$$

Dynamics on graphs

Same type of results for some graph ensembles.



Graph with Prescribed Degree Distribution

Let $d_1, \cdots, d_n$ such that for some graph G on $V = \{1, \cdots, n\}$ such that

$$\deg(i; G) = d_i.$$

Define the random graph sampled uniformly over all graph with degree sequence $d_1, \cdots, d_n$.

Assume that the empirical degree distribution converges :

$$F_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{d_i} \Rightarrow F.$$

Limit of Diluted Random Graphs

Galton Watson tree with *degree distribution* F = GW branching process with

- the root has offspring distribution F,
- all other genitors have offspring distribution $\widehat{F}$ with

$$\widehat{F}(k-1) = \frac{kF(k)}{\sum_{\ell} \ell F(\ell)}.$$

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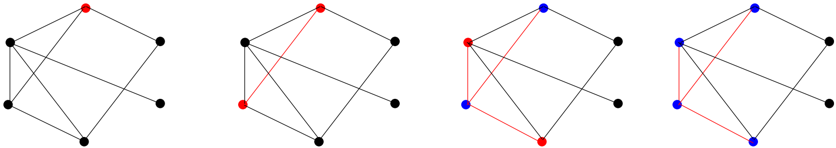
$$\widehat{F}(k-1) = \frac{kF(k)}{\sum_\ell \ell F(\ell)}.$$

$\Longrightarrow$ *The uniform graph with degree sequence* F_n *converges locally to a GWT with degree distribution* F.

Back to the SIR Dynamics



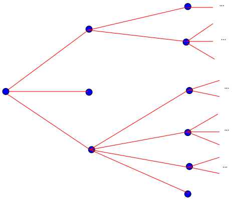
$\Longrightarrow$ As $n \to \infty$, at small time scale,

$$\text{SIR dynamic on the graph} \simeq \text{SIR dynamic on the GWT}$$

Standard SIR Dynamics

Set

$$\varphi(x) = \sum_k F(k)x^k \quad \text{and} \quad \nu = \frac{\varphi''(1)}{\varphi'(1)} = \frac{\mathbb{E}D(D-1)}{\mathbb{E}D}.$$



The graph of infected vertices G_n *converges weakly to a Galton-Watson tree with degree distribution with generating function*

$$\varphi\left(\frac{\lambda x + 1}{\lambda + 1}\right).$$

Phase transition for Standard SIR Dynamics

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$$0 < \lambda < \frac{1}{\nu - 1}$$

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In supercritical regime, probability of extinction given by $\rho = \varphi((\lambda\widehat{\rho} + 1)/(\lambda + 1))$

$$\varphi'(1)\widehat{\rho}(\lambda) = \varphi'\left(\frac{\lambda\widehat{\rho}(\lambda) + 1}{\lambda + 1}\right).$$

and

$$\frac{Z_n}{n} \Rightarrow W = \rho\delta_0 + (1 - \rho)\delta_{1-\rho}.$$

Tail Behavior

The tail behavior of Z is $\pm$ the tail behavior of

$$\widehat{F}(k-1) = \frac{kF(k)}{\sum_{\ell \geqslant 1} \ell F(\ell)}.$$

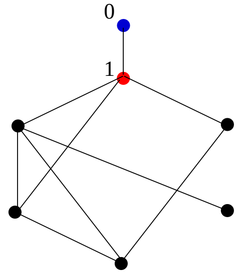
If

$$\gamma_F = \sup\{k \geqslant 0 : \sum_{\ell} k^{\ell} F(\ell) < \infty\} = \gamma_{\widehat{F}} + 1,$$

Then

$$\gamma(\lambda) = \sup\left\{k \geqslant 0 : \mathbb{E}_{\lambda}[Z^k] < \infty\right\} = \gamma_F - 1.$$

Rumor scotching process



We can also define the limit SIR dynamics on the GWT

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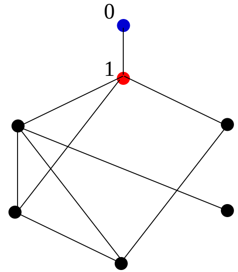

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Theorem
If

$$0 < \lambda \leqslant \lambda_1 = (2\nu - 1) - \sqrt{(2\nu-1)^2 - 1},$$

subcritical regime, if $\lambda > \lambda_1$ *supercritical regime.*

Rumor scotching process

On a GWT, again,

- explicit computation of integer moments,
- probability of extinction related to a non-linear second order differential equation.

Tail Exponent

If $0 < \lambda \leqslant \lambda_1$, let Z be the total infected population on the GWT,

$$\gamma(\lambda) = \sup\{k \geqslant 0 : \mathbb{E}[Z^k] < \infty\},$$

$$\gamma_F = \sup\{u \geqslant 0 : \sum_\ell \ell^k F(\ell) < \infty\}.$$

Theorem

$$\gamma(\lambda) = \min\left(\frac{\lambda^2 - 2\nu\lambda + 1 - (1-\lambda)\sqrt{\lambda^2 - 2\lambda(2\nu-1)+1}}{2\lambda(\nu-1)}, \gamma_F - 1\right).$$

Concluding Remarks

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- Rumor scotching process on a lattice ?