The Lefschetz Number of a Monodromy Transformation

Mihai Tibăr

Promotores: Prof. dr. D. Siersma
Prof. dr. J.H.M. Steenbrink

Faculteit der Wiskunde en Informatica

Contents

Introduction

Any germ of a holomorphic function $f : (\mathbf{X}, x) \to (\mathbf{C}, 0)$ on an analytic space germ $(\mathbf{X}, x)$ defines an (algebraic) *monodromy*:

$$h_f : H^i(F_f, \mathbf{C}) \to H^i(F_f, \mathbf{C}),$$

where F_f is the Milnor fibre of f.

We investigate throughout this thesis the Lefschetz number $\Lambda(h_f)$ of the monodromy of functions on singular germs and also the Lefschetz number of higher powers of the monodromy.

If the space germ is smooth, then the problem concerning the Lefschetz number $\Lambda(h_f)$ has a short answer: by a result of A'Campo [A'C-1], $\Lambda(h_f) = 0$ if f is singular and $\Lambda(h_f) = 1$ if f is regular.

The situation becomes more complicated if $(\mathbf{X}, x)$ is not smooth, as remarked already in [Lê-3]. The importance of this subject would come from the increasing interest for (particular) singular spaces and their slices.

In Chapter I of this thesis we prove results on the Lefschetz number and the zeta-function in full generality by using an improved version of Lê 's "carrousel" construction. Let $l : (\mathbf{X}, x) \to (\mathbf{C}, 0)$ be a sufficiently general linear function. First, we prove that, if none of the Puiseux ratios for the branches of the Cerf diagram $\Delta(l, f)$ is integer, then the Lefschetz number $\Lambda(h_f)$ is equal to the Lefschetz number of the monodromy of the restriction $f_{|\{l=0\}}$ (Theorem 3.2).

This result has some interesting consequences, stated e.g. in Corollary 3.6, Proposition 3.7. We prove, in the general case, a formula for the Lefschetz number (Theorem 3.12) and a formula for the zeta-function (Theorem 4.10). They depend on knowledge of the respective invariants for a finite number of *carrousel monodromies* (see Definition 4.4). These formulae are general but, of course, the computations involved can be very hard in particular cases.

Our construction in Chapter I yields a fine "polar decomposition" of the Milnor fibre which has some similarity with the decomposition of the Milnor fibre defined by A'Campo [A'C-2], see Chapter II, 1.2.

Chapter II of the thesis is based on the method of A'Campo [A'C-2], which involves the resolution of singularities.

In Section 1 of Chapter II we prove that, if $f : (\mathbf{X}, x) \to (\mathbf{C}, 0)$ is a *smoothing*, then the Lefschetz number $\Lambda(h_f)$ depends only on the residue class of f in $\mathbf{m}_{\mathbf{X},x}/\mathcal{F}_{\mathbf{X},x}$, where $\mathbf{m}_{\mathbf{X},x}$ is the maximal ideal of the local algebra and $\mathcal{F}_{\mathbf{X},x}$ is an ideal defined by using some resolution of the space $(\mathbf{X},x)$. It turns out that $\mathcal{F}_{\mathbf{X},x}$ does not depend on the resolution. Moreover, $\mathcal{F}_{\mathbf{X},x}$ is an intersection of a finite number of "minimal ideals" and these ideals do not depend on the resolution as well (see Proposition 1.12, Theorem 1.18 and the remark 1.20(b)). In particular, if $(\mathbf{X},0)$ is isolated, then $\mathcal{F}_{\mathbf{X},x}$ contains $\mathbf{m}^2_{\mathbf{X},x}$; hence the Lefschetz number $\Lambda(h_f)$ depends only on the residue class of f, modulo $\mathbf{m}^2_{\mathbf{X},x}$.

Beginning with Section 2 we focus on the particular case when $(\mathbf{X},0) \simeq (\mathbf{C}^n/G,0)$ is an isolated cyclic quotient singularity (where G is a finite cyclic group). In this case, the results proved in Section 1 have a particularly nice form: the Lefschetz number of a function has a sum decomposition into Lefschetz numbers of well-defined "pieces" of our function (Theorem 2.6). To each such piece there corresponds a G-invariant weighted-homogeneous polynomial, the weights depending only on the group action.

We take advantage of the finite group action and construct a toric resolution of the cyclic quotient variety together with a special diagram (see 3.1). This construction replaces a hypothetical succession of G-equivariant blowing-ups $\mathbf{Y} \to \mathbf{C}^n$ along G-stable nonsingular subvarieties such that the quotient $\mathbf{Y}/G$ is nonsingular, which, unfortunately, one cannot construct in general (see [Oda, p. 31]).

In Section 4 we prove Theorem 2.6 and some annihilation criteria which facilitate the study of the Lefschetz number.

In Section 5 a class of nondegenerate functions is defined, for which we are able to prove a more practical formula for the zeta-function. The formula is based on the previous results and on Varchenko's approach to the zeta-function [Var]. In particular, we get a formula for the zeta-function of a general linear slice.

Restricting to the Lefschetz number, we define a much larger class of functions for which we prove the corresponding formula (Proposition 5.27).

Examples which illustrate results obtained in Sections 1–5 are given in Section 6. In particular, Example 6.1 shows that the equality $\zeta_f(t) = \zeta_{\tilde{f}}(t^d)$ is not true, even for a general linear function f, where $\tilde{f}$ is the corresponding G-invariant function and $d = |G|$.

In the last Section we restrict to 2-dimensional cyclic quotients. In this situation, the previous results are more explicit and the Lefschetz number can effectively be computed. In particular, we are able to determine the range of the Lefschetz number. Finally, we obtain a splitting formula for the zeta-function (Theorem 7.20).

To give an idea about the subject, we show here an example of the computation of the Lefschetz number.

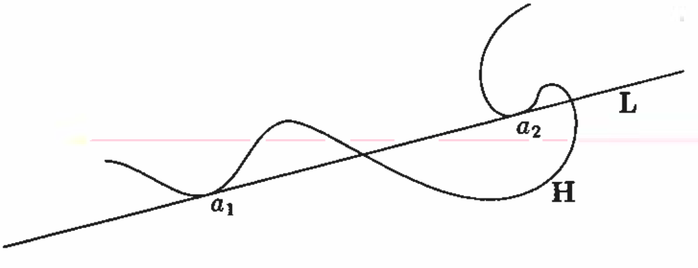
Example Let $(\mathbf{X}, 0)$ be the germ of the affine cone over a smooth projective hypersurface $\mathbf{H} := \{g = 0\} \subset \mathbf{P}^n$, $n \geq 2$. Here g is a homogeneous polynomial of degree k and we assume that $k \geq 2$. Next, consider a function $u : \mathbf{C}^{n+1} \to \mathbf{C}$, $u = l + q$, where l is linear and q contains all the higher order terms. The restriction of u to $(\mathbf{X}, 0)$ gives a function $f : (\mathbf{X}, 0) \to (\mathbf{C}, 0)$. Denote by $\mathbf{L}$ the projective hyperplane defined by l.

To compute the Lefschetz number, we first resolve the singularity of the space $(\mathbf{X}, 0)$; this is done by a single one-point-blowing-up $\pi : (\mathbf{Y}, \mathbf{E}) \to (\mathbf{X}, 0)$. Let $\{\widetilde{f = 0}\} \subset \mathbf{Y}$ be the proper transform of $\{f = 0\}$ by π.

In some small enough neighbourhood of a point $z_0 \in \mathbf{E} \setminus \{\widetilde{f = 0}\}$, the Milnor fibre of $f \circ \pi$ has equation $f \circ \pi = w$, where w is a local coordinate in $z_0 \in \mathbf{Y}$. Thus the piece of the Milnor fibre which is in the neighbourhood of $\mathbf{E} \setminus \{\widetilde{f = 0}\}$ is pointwise fixed by a geometric monodromy and counts in the computation of the Lefschetz number. By the results in Chapter II, Section 1, there are no other contributions to $\Lambda(h_f)$, hence:

$$\Lambda(h_f) = \chi(\mathbf{E} \setminus \{\widetilde{f = 0}\}).$$

The exceptional divisor $\mathbf{E}$ is isomorphic to the projective hypersurface $\mathbf{H}$ and one may also notice the isomorphism $\mathbf{E} \setminus \{\widetilde{f = 0}\} \simeq \mathbf{H} \setminus \mathbf{L}$. Moreover, $\chi(\mathbf{H} \setminus \mathbf{L}) = \chi(\mathbf{H}) - \chi(\mathbf{H} \cap \mathbf{L})$, hence our Lefschetz number $\Lambda(h_f)$ depends essentially on how $\mathbf{H}$ and $\mathbf{L}$ mutually intersect.



The space $\mathbf{H}_0 := \mathbf{H} \cap \mathbf{L}$ is a projective hypersurface in $\mathbf{L} \simeq \mathbf{P}^{n-1}$ with at most isolated singularities at some points a_i, $i \in I$, for a finite set of indices I. If $\mathbf{L}$ is a general hyperplane, then $\mathbf{H}_0$ is smooth of degree k, hence $\chi(\mathbf{H}_0) = n - [1 + (-1)^{n-1}(k-1)^n]/k$. The hypersurface $\mathbf{H} \subset \mathbf{P}^n$ is also smooth of degree k, so:

$$\Lambda(h_f) = \chi(\mathbf{H}) - \chi(\mathbf{H}_0) = 1 + (-1)^{n-1}(k-1)^n .$$

If f is not general, then $\mathbf{H}_0$ has isolated singularities and the formula for $\chi(\mathbf{H}_0)$ must be corrected: for each singular point $a_i \in \mathbf{H}_0$, $i \in I$, there is, roughly speaking, a contribution equal to the Milnor number $\mu(\mathbf{H}_0, a_i)$ of the isolated singularity germ $(\mathbf{H}_0, a_i)$. Therefore, when isolated singularities are present, the complete expression, using a result of Dimca [Di, Corollary 2.3]), is:

$$\Lambda(h_f) = \chi(\mathbf{H} \setminus \mathbf{H}_0) = 1 + (-1)^{n-1}(k-1)^n + (-1)^{n+1} \sum_{i \in I} \mu(\mathbf{H}_0, a_i).$$

Chapter I

Lefschetz number and Lê's carrousel

1 Preliminaries

1.1 Let $(\mathbf{X}, x)$ be a complex analytic set germ. We usually assume that $(\mathbf{X}, x)$ is embedded in $(\mathbf{C}^N, 0)$, for some sufficiently large $N \in \mathbf{N}$.

Let $f : (\mathbf{X}, 0) \to (\mathbf{C}, 0)$ be an analytic function germ. The same symbol f is used for a representative of the function on a small enough representative $\mathcal{X}$ of the germ $(\mathbf{X}, 0)$. Denote by B_ε the open ball of radius $\varepsilon > 0$ in $\mathbf{C}^N$, centred at 0 and by D_η the open disc of radius $\eta > 0$ in $\mathbf{C}$, centred at 0. A theorem of Lê [Lê-3, Theorem 1.1] asserts that there exists a *topological fibration* induced by f:

$$\Psi_f = \Psi_f(\varepsilon, \eta) : \mathcal{X} \cap B_\varepsilon \cap f^{-1}(D_\eta \setminus \{0\}) \to D_\eta \setminus \{0\}, \tag{1}$$

for small enough ε and $\eta \ll \varepsilon$.

The fibre F_f of this fibration is called the *Milnor fibre*, since Milnor was the first one who proved a $\mathbf{C}^\infty$-fibration theorem [Mi] in the case when $(\mathbf{X}, 0)$ is smooth.

The Milnor fibre is certainly not smooth in the general case; however, it is a CW-complex of dimension $\leq \dim(\mathbf{X}, 0) - 1$, see [Ha].

1.2 The (*algebraic*) *monodromy* h_f of the local fibration (1) is the characteristic linear endomorphism of the (co)homology of the fibre F_f. We consider cohomology with $\mathbf{C}$-coefficients.

For any integer $k \geq 0$, the *Lefschetz number* $\Lambda(h_f)$ of the monodromy is defined by:

$$\Lambda(h_f) := \sum_{i \geq 0} (-1)^i \operatorname{trace}[h_f;\ H^i(F_f, \mathbf{C})]. \tag{2}$$

We alternatively denote it by $\Lambda(f)$.

A finer invariant is the *zeta-function* $\zeta_f(t)$ of the monodromy:

$$\zeta_f(t) := \prod_{i \geq 0} \det[\mathrm{I} - t \cdot h_f;\ H^i(F_f, \mathbf{C})]^{(-1)^{i+1}}. \tag{3}$$

We alternatively denote it by $\zeta_{h_f}(t)$.

1.3 Knowledge of the zeta-function implies knowledge of the Lefschetz number $\Lambda(h_f^k)$ of any nonnegative power k of the monodromy; conversely, the zeta function is determined by the set of these Lefschetz numbers, as follows (see e.g. [Mi, p.77], [A'C-2, p.234]).

If $s_1, s_2, \ldots$ are integers defined inductively by:

$$\Lambda(h_f^k) = \sum_{i|k} s_i, \quad k \geq 1,$$

then the zeta-function of f is given by:

$$\zeta_f(t) = \prod_{i \geq 1} (1 - t^i)^{-s_i/i}.$$

1.4 Let $\mathbf{m}_{\mathbf{X},0}$ denote the maximal ideal of the local ring $\mathcal{O}_{\mathbf{X},0}$ of $(\mathbf{X}, 0)$.

The monodromy h_f need not be of finite order; however, it is *quasi-unipotent.* This result, known as the *monodromy theorem*, was proved by several authors (Grothendieck, Brieskorn, Nilsson, Landman, Lê) in the case of smooth Milnor fibre and by Lê [Lê-4] in the general case. Lê's proof is based on the interesting *carrousel* construction and does not involve the resolution of singularities. Lê's construction was published in [Lê-1] shortly after A'Campo proved the following:

Theorem [A'C-1, Théorème 1]
If $(\mathbf{X}, 0)$ *is smooth and* $f \in \mathbf{m}_{\mathbf{X},0}^2$ *then* $\Lambda(f) = 0$. □

The proof by Lê of this theorem in [Lê-1] is based on the construction of a *geometric monodromy* without fixed points (see also 3.1). We mention that there is a more general version of the theorem cited above, in the same paper [A'C-1, Théorème 1bis]; we shall discuss it in the next chapter.

1.5 If $(\mathbf{X}, 0)$ is smooth and $f \in \mathbf{m}_{\mathbf{X},0} \setminus \mathbf{m}_{\mathbf{X},0}^2$, then it is obvious that the Lefschetz number $\Lambda(f)$ is 1.

If $(\mathbf{X}, 0)$ is a singular space, then the Lefschetz number $\Lambda : \mathbf{m}_{\mathbf{X},0} \setminus \mathbf{m}_{\mathbf{X},0}^2 \to \mathbf{Z}$ may have a much larger range (even if $(\mathbf{X}, 0)$ is irreducible), as we shall see in many examples (the first one is Example 3.5). This is an important point for our study.

2 The carrousel

2.1 Here is a brief description of the carrousel construction, which follows closely [Lê-2] and [Lê-4]. We assume from now on, unless otherwise stated, that all the irreducible components of $(\mathbf{X},0)$ have dimensions greater than 1.

2.2 Let $\mathcal{S} = \{S_i\}_{i\in I}$ be a finite *Whitney stratification* of $\mathcal{X}$ such that it satisfies the *Thom condition* relative to $f : \mathcal{X} \to \mathbf{C}$ [Th], [Hi-2]. One may suppose that $\{f = 0\}$ is a union of strata; denote by I_1 the subset $\{i \in I \mid S_i \subset \mathcal{X} \setminus \{f = 0\}\}$ of I. There exists a Zariski open set Ω_f in the space of linear germs $l : (\mathbf{C}^N, 0) \to (\mathbf{C}, 0)$ such that any small enough representative of the germ of the critical locus of the restriction of the morphism:

$$\Phi := (l, f) : (\mathbf{X}, 0) \to (\mathbf{C}^2, 0)$$

to a stratum S_i, $i \in I_1$, is either a curve $\Gamma_i(l,f)$ or it is void.

Note There is one more condition imposed by Lê: the curve $\Gamma_i(l,f)$ should be reduced, for any $i \in I_1$ and $l \in \Omega_f$. This additional condition defines a proper subset $\hat{\Omega}_f$ of Ω_f, which is also Zariski-open.

As a rule, we work with the (larger) set Ω_f; the exceptions will be emphasized. The reason is that, in many examples, one might get the Lefschetz number more quickly while working with elements of Ω_f, see e.g. Example 3.5. Moreover, one can enlarge Ω_f: modify its definition by allowing also nonlinear functions. However, we work only with the first-given definition above.

Definition The closed germ $\Gamma(l,f) := \bigcup_{i\in I_1} \overline{\Gamma_i(l,f)}$ is called the *polar curve* of f with respect to l, relative to the stratification $\mathcal{S}$. The image $\Delta(l,f) := \Phi(\Gamma(l,f))$ is called the *Cerf diagram* of f with respect to l, relative to $\mathcal{S}$.

2.3 Let $l \in \Omega_f$ and let (u, λ) denote the pair of coordinates on $\mathbf{C}^2$. Hence Φ_α is given by $u = l(x)$, $\lambda = f(x)$.

There is a fundamental system of "privileged" open polydiscs in $\mathbf{C}^N$, centred at 0, of the form $(D_\alpha \times P_\alpha)_{\alpha\in A}$ and a corresponding fundamental system $(D_\alpha \times D'_\alpha)_{\alpha\in A}$ of 2-discs at 0 in $\mathbf{C}^2$, such that Φ induces, for any $\alpha \in A$, a topological fibration:

$$\Phi_\alpha : \mathcal{X} \cap (D_\alpha \times P_\alpha) \cap \Phi^{-1}(D_\alpha \times D'_\alpha \setminus (\Delta(l,f) \cup \{\lambda = 0\})) \to$$
$$\to D_\alpha \times D'_\alpha \setminus (\Delta(l,f) \cup \{\lambda = 0\}).$$

One can stratify the source and the target of Φ_α such that the stratifications satisfy the Whitney condition and that the stratification of the source satisfies the Thom condition relative to Φ_α.

Moreover, f induces a topological fibration:

$$f_\alpha : \mathcal{X} \cap (D_\alpha \times P_\alpha) \cap f^{-1}(D'_\alpha \setminus \{0\}) \to D'_\alpha \setminus \{0\},$$

respectively:

$$f'_\alpha : \mathcal{X} \cap (\{0\} \times P_\alpha) \cap f^{-1}(D'_\alpha \setminus \{0\}) \to D'_\alpha \setminus \{0\},$$

which is fibre homotopic to the fibration of f defined in (1), respectively to the fibration of $f_{|\{l=0\}}$, defined analogously. The disc D'_α has been chosen small enough such that $\Delta(l,f) \cap \partial\overline{D_\alpha} \times D'_\alpha = \emptyset$.

2.4 One can build an integrable smooth vector field on $D_\alpha \times S'_\alpha$—where S'_α is some circle in $D_{\alpha'}$ of radius sufficiently close to the radius of $\partial\overline{D_{\alpha'}}$—such that, mainly, it is tangent to $\Delta(l,f) \cap (D_\alpha \times S'_\alpha)$ and it lifts the unit vector field of S'_α by the projection $D_\alpha \times S'_\alpha \to S'_\alpha$. Lifting the former vector field by Φ_α and integrating it, one gets a characteristic homeomorphism of the fibration induced by f_α over S'_α, hence a geometric monodromy of the fibre F_f. We call it the (geometric) *carrousel monodromy.*

2.5 Lê's construction, although being very technical, has more pleasant properties: The integration of the vector field on $D_\alpha \times S'_\alpha$ produces a "carrousel" of the disc $D_\alpha \times \{\eta\}$, for some $\eta \in S'_\alpha$. Let

$$l_\alpha : \mathcal{X} \cap (D_\alpha \times \{\eta\}) \cap l^{-1}(D_\alpha) \to D_\alpha \times \{\eta\} \tag{4}$$

be the restriction of Φ_α. Notice that $F_f \simeq l_\alpha^{-1}(D_\alpha \times \{\eta\})$.

Lê gives a description of the *motions* of the "important" points of the carrousel disc, that is: the trajectory inside $D_\alpha \times S'_\alpha$ of some point $a \in D_\alpha \times \{\eta\}$ projects onto S'_α; one turn around the circle S'_α moves the point a to some other point $a' \in D_\alpha \times \{\eta\}$. By construction, the vector field restricted to $\{0\} \times S'_\alpha$ is the unit vector field of S'_α, hence the centre $(0,\eta)$ of the carrousel disc is indeed fixed; the circle $\partial\overline{D_\alpha} \times \{\eta\}$ is also pointwise fixed (if one extends the vector field on a slightly larger carrousel disc).

The distinguished points $\Delta(l,f) \cap D_\alpha \times \{\eta\}$ of the disc have a complex motion around $(0,\eta)$; these motions depend on the Puiseux parametrizations of the branches of Δ which are not included in $\{u = 0\}$. Moreover, these Puiseux parametrizations determine the motion of any "important" point in the carrousel. We briefly describe this phenomenon in the following.

The Puiseux pairs and the carrousel

2.6 We refer to the excellent textbook [BK] for the terminology and the proofs of some of the facts we use in this piece.

Let $\Delta = \Delta(l, f)$ be the Cerf diagram and let $\Delta' = \cup_{i\in\{1,\dots,r\}}\Delta_i$ be the union of those irreducible components of Δ which are not included in $\{u = 0\}$.

For $i \in \{1, \dots, r\}$, we consider a Puiseux parametrization of Δ_i (with reduced structure):

$$\begin{cases} \lambda = t^n \\ u = \sum_{j\geq m} c_j t^j, \quad \text{for some } m, n \in \mathbf{Z}_+,\ c_j \in \mathbf{C},\ c_m \neq 0. \end{cases} \tag{5}$$

Note that n is not necessarily the multiplicity of Δ_i in 0 and that m can be smaller than n.

The Puiseux parametrization enables one to write u as a function of λ, formally:

$$\begin{aligned} u = a_{k_1}\lambda^{m_1/n_1} + \sum_{l=1}^{l_1} b_{1,l}\lambda^{(m_1+l)/n_1} + a_{k_2}\lambda^{m_2/n_1n_2} + \\ + \sum_{l=1}^{l_2} b_{2,l}\lambda^{(m_2+l)/n_1n_2} + \cdots + a_{k_g}\lambda^{m_g/n_1\cdots n_g} + \sum_{l>0} b_{g,l}\lambda^{(m_g+l)/n_1\cdots n_g}, \end{aligned} \tag{6}$$

where g is a positive integer, $\gcd(m_j, n_j) = 1$, $\forall j \in \{1, \dots, , g\}$ and $n_j \neq 1$, $\forall j \in \{2, \dots, g\}$. Notice that $m_1/n_1 = m/n$ and $a_{k_1} = c_m$.

The pairs (m_j, n_j) are called the *Puiseux pairs* of Δ_i. They determine the topology of Δ_i (but not its analytic type). Each link $\Delta_i \cap (D_\alpha \times S'_\alpha)$ is an iterated torus knot. The type of the knot is described for instance in [Lê-6, p. 7].

The topology of a single knot depends only on the Puiseux pairs, but in case of more branches, the link depends on more data than just the Puiseux pairs.

Note There is an action of the group $\mu_{n_1\cdots n_g}$ of the $n_1 \cdots n_g$-roots of unity on the coefficients of the equation (6):

$$\sigma \cdot a_{k_i} := \sigma^{m_i n_{i+1}\cdots n_g} a_{k_i},$$

$$\sigma \cdot b_{i,l} := \sigma^{(m_i+l)n_{i+1}\cdots n_g} b_{i,l},$$

for any $\sigma \in \mu_{n_1 \cdots n_g}$, any $i \in \{1, \ldots, g\}$ and l as in (6).

Two Puiseux parametrizations (5), which give two equations of type (6) with coefficients $a_{k_i}, b_{i,l}$, resp. $a'_{k_i}, b'_{i,l}$, define the same curve if there is a $\sigma \in \mu_{n_1 \cdots n_g}$ such that $a'_{k_i} = \sigma \cdot a_{k_i}$, $b'_{i,l} = \sigma \cdot b'_{i,l}$.

2.7 We define two types of successive approximations of Δ_i, $i \in \{1, \ldots, r\}$. The first approximations are:

$$C_i^{(1)}: \quad u = a_{k_1} \lambda^{m_1/n_1},$$

$$\hat{C}_i^{(1)}: \quad u = a_{k_1} \lambda^{m_1/n_1} + \sum_{l=1}^{l_1} b_{1,l} \lambda^{(m_1+l)/n_1}.$$

The second ones are:

$$C_i^{(2)}: \quad u = a_{k_1} \lambda^{m_1/n_1} + \sum_{l=1}^{l_1} b_{1,l} \lambda^{(m_1+l)/n_1} + a_{k_2} \lambda^{m_2/n_1 n_2},$$

$$\hat{C}_i^{(2)}: \quad u = a_{k_1} \lambda^{m_1/n_1} + \sum_{l=1}^{l_1} b_{1,l} \lambda^{(m_1+l)/n_1} + a_{k_2} \lambda^{m_2/n_1 n_2} + \sum_{l=1}^{l_2} b_{2,l} \lambda^{(m_2+l)/n_1 n_2}$$

and so on, the last ones being $C_i^{(g)}$ and $\hat{C}_i^{(g)}$, (where $\hat{C}_i^{(g)} = \Delta_i$).

The curve $C_i^{(1)}$ intersects the carrousel disc $D_\alpha \times \{\eta\}$ in n_1 points situated on a circle and their carrousel movement is a rotation of angle $2\pi m_1/n_1$. If we take $\hat{C}_i^{(1)}$ instead, we get also n_1 intersection points but their position is a slight perturbation of the previous one.

Each of the points $C_i^{(1)} \cap (D_\alpha \times \{\eta\})$ is the centre of a small disc which contains just one point from the set $\hat{C}_i^{(1)} \cap (D_\alpha \times \{\eta\})$. This latter point, called a *distinguished point*, becomes the centre of a *new (smaller) carrousel.*

2.8 Definition We call (*smaller*) *carrousel disc of order* k a sufficiently small open disc centred at some point $c \in \hat{C}_i^{(k)} \cap (D_\alpha \times \{\eta\})$, $i \in \{1, \ldots, r\}$. This disc is supposed to contain all the points $\hat{C}_j^{(k+l)} \cap (D_\alpha \times \{\eta\})$, $\forall l > 0$, $\forall j \in \{1, \ldots, r\}$ such that $\hat{C}_j^{(k)} = \hat{C}_i^{(k)}$, which are close enough ("satellites") to c. By definition, if δ_1, δ_2 are two smaller carrousel discs (not necessarily of the same order), then either one is included in the other, or they are disjoint.

We may and do assume that the carrousel discs of order k centred in the points $\hat{C}_i^{(k)} \cap (D_\alpha \times \{\eta\})$ are of equal radii.

Remark It is possible that a small carrousel disc of order k, say with centre at some $c \in \hat{C}_i^{(k)} \cap (D_\alpha \times \{\eta\})$, contains other carrousel discs of the same order. Here is an example:

$$\Delta_1\colon \quad u_1 = \lambda^{3/2} + \lambda^{17/2}, \qquad C_1^{(1)} \neq \hat{C}_1^{(1)},\ \Delta_1 = \hat{C}_1^{(1)},$$
$$\Delta_2\colon \quad u_1 = \lambda^{3/2} + \lambda^{7/4}, \qquad C_2^{(1)} = \hat{C}_2^{(1)} = C_1^{(1)},\ \Delta_2 = C_2^{(2)}.$$

A carrousel disc of order 1 corresponding to Δ_2 contains a carrousel disc of order 1 corresponding to Δ_1.

2.9 Finally, a simultaneous parametrization of all the analytic branches of Δ':

$$\begin{cases} \lambda = t^n, \\ u_1 = \sum_{j \geq m_1} a_{1,j} t^j, \\ \vdots \\ u_r = \sum_{j \geq m_r} a_{r,j} t^j \end{cases}$$

leads to the construction of the full carrousel.

If we (formally) define the "essential" curve associated to Δ_i by:

$$\Delta_i^{\mathrm{es}}\colon \quad u = a_{k_1} \lambda^{m_1/n_1} + a_{k_2} \lambda^{m_2/n_1 n_2} + \cdots + a_{k_g} \lambda^{m_g/n_1 \cdots n_g},$$

then the carrousel associated to $\Delta^{\mathrm{es}} = \bigcup_{i \in \{1,\ldots,r\}} \Delta_i^{\mathrm{es}}$ is a kind of "idealization" of the carrousel defined by Δ'.

2.10 Denote by $(m_{i,j}, n_{i,j})_{j \in \{1,\ldots,g_i\}}$ the Puiseux pairs of Δ_i, $\forall i \in \{1,\ldots,r\}$. Suppose that we have the following ordering among the first Puiseux pairs (eventually after some permutation of indices):

$$\frac{m_{1,1}}{n_{1,1}} \geq \frac{m_{2,1}}{n_{2,1}} \geq \cdots \geq \frac{m_{r,1}}{n_{r,1}}.$$

To each branch Δ_i there corresponds an annulus A_i—with central symmetry in $(0,\eta)$—inside the carrousel disc, such that A_i contains $\Delta_i \cap (D_\alpha \times \{\eta\})$, see [Lê-2]. We define also A_0 to be an arbitrarily small open disc centred in $(0,\eta)$. By definition, $A_i = A_j$ if and only if $m_{i,1}/n_{i,1} = m_{j,1}/n_{j,1}$; moreover, the set of annuli give a partition of the carrousel disc.

For any $i \in \{1,\ldots,r\}$, there are $n_{i,1}$ carrousel discs $\delta_{i,j}$, $j \in \{1,\ldots,n_{i,1}\}$, of order 1, centred at the $n_{i,1}$ points $\hat{C}_i^{(1)} \cap (D_\alpha \times \{\eta\})$. The annulus A_i contains all the carrousel discs $\delta_{s,j}$ such that $C_s^{(1)} = C_i^{(1)}$. Each point of the annulus A_i, which is not inside some disc $\delta_{i,j}$, is fixed by the $n_{i,1}$th power of the carrousel;

in the case of the "ideal" carrousel (i.e. the one associated to Δ^{cs}), these points have a carrousel motion which is a rotation of angle $2\pi\frac{m_{i,1}}{n_{i,1}}$. The disc A_0 is just pointwise fixed.

Of course, one needs a continuous transition between two successive annuli, but this is something easy to do. In fact, two successive annuli are separated by a circle (centered at $(0,\eta)$). We may replace this circle by a sufficiently thin annulus, which we call "transition zone"; because here the continuous transition between rotations of two different angles takes place.

2.11 Example Let $(\mathbf{X},0)$ be a 2-dimensional isolated cyclic quotient singularity, where $\mathbf{X}$ is the algebraic quotient of $\mathbf{C}^2$ by a cyclic group of order 5, denoted by $\mathbf{X}_{5,2}$ and defined in Chapter II, Section 7.

Let $\tilde{f} : (\mathbf{C}^2,0) \to (\mathbf{C},0)$, $\tilde{f} = y^{10} + x^2y^4$ and let $f : (\mathbf{X},0) \to (\mathbf{C},0)$ be the induced function on the quotient.

Take a function $\tilde{l} : (\mathbf{C}^2,0) \to (\mathbf{C},0)$, $\tilde{l} = x^5$ and let l be the induced one on $(\mathbf{X},0)$. Then l defines a linear slice of $(\mathbf{X},0)$. The polar curve $\Gamma(l,f)$ is covered by the curve $\tilde{\Gamma}(\tilde{l},\tilde{f}) := \{x^4 = 0\} \cup \{x - i\sqrt{5/2}y^3 = 0\} \cup \{x + i\sqrt{5/2}y^3 = 0\}$.

Notice that $\Gamma(l,f)$ has a nonreduced structure, because of the component $\{x^4 = 0\}$ of $\tilde{\Gamma}(\tilde{l},\tilde{f})$. It follows that l is not in $\hat{\Omega}_f$, but it is still an element of Ω_f.

The Cerf diagram $\Delta = \Delta(l,f)$ has only 2 analytic branches which are parametrized as follows:

$$\Delta_0: \quad \begin{cases} \lambda = t, \\ u = 0, \quad \Delta_0 \not\subset \Delta'; \end{cases}$$

$$\Delta_1: \quad \begin{cases} \lambda = t^2, \\ u = \gamma t^3, \quad n_{1,1} = 2,\ m_{1,1} = 3. \end{cases}$$

One can find the nonzero constant γ by some easy computations. (Notice that $-\gamma$ is good as well, see Note 2.6). The important thing is that we get just a 1-term Puiseux expansion, i.e. $C_1^{(1)} = \Delta_1$.

However, Δ_1 is not tangent to the $\{\lambda = 0\}$ axis and, for any $x \in \Delta_1 \setminus \{0\}$, sufficiently close to 0, there are two points on the polar curve $\Gamma(l,f)$ which project to x; the fibre $\Phi_\alpha^{-1}(x,\eta)$ has two singularities of type A_1.

3 The Lefschetz number via the carrousel

3.1 The carrousel construction can be used to get information on the Lefschetz number. This was the original idea of Lê, who showed that, if $f \in \mathbf{m}^2_{\mathbf{X},0}$, then the carrousel monodromy has no fixed points outside the slice $\{l = 0\}$, so $\Lambda(f) = 0$, by induction. As far as we know, he explicitly stated this result only in the case when f is a smoothing [Lê-2].

We presume that, in the general case, the result (see Corollary 3.6) might have been evident to him after he proved the existence of the fibration (1). The main point is that, as in the smooth case, the Cerf diagram is tangent to the $\{\lambda = 0\}$-axis and this implies that no point of the carrousel disc is fixed (except its origin).

We extend this result by studying the set of fixed points, in the case $f \in \mathbf{m}_{\mathbf{X},0} \setminus \mathbf{m}^2_{\mathbf{X},0}$. We start with the particular case: none of the *Puiseux ratios* $p_{i,1} := m_{i,1}/n_{i,1}$, $i \in \{1, \ldots, r\}$, is integral (see Theorem 3.2). At the end, we give a general result (Theorem 3.12).

We keep the previous notations. We recall that, for some fixed $l \in \Omega_f$, $(m_{i,1}, n_{i,1})$ denotes the first Puiseux pair of the analytic branch Δ_i, where $i \in \{1, \ldots, r\}$, of the Cerf diagram $\Delta := \Delta(l, f)$; by definition (see 2.6) the collection $\Delta' = \bigcup_{i \in \{1,\ldots,r\}} \Delta_i$ does not contain $\{u = 0\}$.

3.2 Theorem *Let all the irreducible components of* $(\mathbf{X}, 0)$ *have dimensions greater than* 1. *If* $n_{i,1} > 1$, $\forall i \in \{1, \ldots, r\}$, *then we have the equality of Lefschetz numbers:*

$$\Lambda(f) = \Lambda(f_{|\{l=0\}}).$$

Proof If $\Delta \subset \{u = 0\}$ then the Milnor fibration of f is homotopy equivalent to the Milnor fibration of $f_{|\{l=0\}}$, hence $\Lambda(f) = \Lambda(f_{|\{l=0\}})$.

Assume that $\Delta \not\subset \{u = 0\}$. Since $n_{i,1} > 1$, the carrousel construction tells us that the discs $\delta_{s,j}$ (defined in 2.10), where $n_{s,1} = n_{i,1}$, are cyclically permuted (by a cycle of length $n_{i,1}$). Moreover, the hypothesis in our statement implies that $p_{i,1} \notin \mathbf{Z}$, $\forall i \in \{1, \ldots, r\}$.

Using the facts in 2.10 we may conclude that no point in the carrousel disc is fixed, except the centre and, possibly, some subsets in the transition zones. In the following, we show that the latter have no contribution to the Lefschetz number.

If the interval $(p_{i,1}, p_{i+1,1})$ contains some integer, then there are $s_i := \#\mathbf{Z} \cap (p_{i,1}, p_{i+1,1})$ concentric circles of fixed points inside the transition zone between the annuli A_i and A_{i+1}. Something similar is happening inside the transition zones between A_0 and A_1, resp. A_r and the circle $\partial\overline{D}_\alpha \times \{\eta\}$.

Let S be such a circle of fixed points. There is an annular neighbourhood $\mathcal{A}_S$ of S which is globally fixed by the carrousel. Since the fibration (defined in (4)):

$$l_\alpha: \quad \mathcal{X} \cap (D_\alpha \times \{\eta\}) \cap l^{-1}(D_\alpha) \to D_\alpha \times \{\eta\}$$

is locally trivial over $\mathcal{A}_S$, we have $H^\bullet(l_\alpha^{-1}(S)) \simeq H^\bullet(l_\alpha^{-1}(\mathcal{A}_S))$, hence:

$$\Lambda(h_f;\ l_\alpha^{-1}(\mathcal{A}_S)) = \Lambda(h_f;\ l_\alpha^{-1}(S)).$$

One can decompose the Milnor fibre into suitable pieces on which the geometric monodromy acts and such that the Mayer–Vietoris argument can be applied. Actually, we first cover the carrousel disc by some annuli like those defined in 2.10, but slightly "thicker", such that two "consecutive" ones intersect over a transition zone; then lift this patching to the Milnor fibre.

We may conclude: $\Lambda(f) = \Lambda(f_{|\{l=0\}})$, provided that the Lefschetz number of the restriction of the monodromy on any piece of F_f which is the lift (by l_α) of some circle of fixed points is zero. But this fact is proved by the next Lemma 3.3, hence we are through. □

3.3 Definition Let $a \in (D_\alpha \setminus 0) \times \{\eta\}$ and let F'_a be the fibre of l_α over a. If a is fixed by the carrousel, then the monodromy h_f restricts to an action on $H^\bullet(F'_a)$, denoted by h'_a.

Lemma *If the carrousel disc $D_\alpha \times \{\eta\}$ contains a circle S of fixed points, all of them regular values for the map l_α, then $\Lambda(h_f;\ H^\bullet(l_\alpha^{-1}(S))) = 0$.*

Proof The smooth space $E := l_\alpha^{-1}(S)$ is the total space of a fibration over the circle S, with F'_a as fibre, where a is some point in S. Let h_1 be the algebraic monodromy of this fibration; one has the Wang exact sequence:

$$\cdots \to H^j(E) \to H^j(F'_a) \xrightarrow{h_1 - \mathbf{I}} H^j(F'_a) \to H^{j+1}(E) \to \cdots$$

on which h_f acts. Indeed, the two monodromies h_f and h_1 commute, since they are defined over a meridian, resp. a longitude, of a torus $S^1 \times S^1$ consisting of regular points of Φ_α.

We may apply the Lefschetz functor $\Lambda(h_f)$ to the Wang sequence above and get:

$$\Lambda(h_f;\ H^\bullet(E)) = \Lambda(h'_a) - \Lambda(h'_a) = 0.$$

□

3.4 Example Consider again the data in Example 2.11. Since $n_{1,1}$ is greater than 1, we get that $\Lambda(f) = \Lambda(f_{|\{l=0\}})$.

It is easy to see that $\Lambda(f_{|\{l=0\}}) = 0$, hence $\Lambda(f) = 0$. This last equality follows also by the next Corollary 3.6, since $f \in \mathbf{m}^2_{\mathbf{X},0}$.

3.5 Example Let's modify the functions in Example 2.11: instead of $\tilde{f}$ and $\tilde{l}$, consider $\tilde{f}_1 := x^5 + y^5$ and $\tilde{l}_1 := xy^2$. Then $l_1 \notin \hat{\Omega}_{f_1}$, but $l_1 \in \Omega_{f_1}$. Notice that $f_1 \in \mathbf{m}_{\mathbf{X},0} \setminus \mathbf{m}^2_{\mathbf{X},0}$.

We get that $\Delta'(l_1, f_1)$ is irreducible and has a 1-term Puiseux parametrization; the (first) Puiseux pair is $(3,5)$. It follows that $\Lambda(f_1) = \Lambda(f_{1|\{l_1=0\}})$.

The Milnor fibre of $f_{1|\{l_1=0\}}$ has two components: each of them is the Milnor fibre of a linear function on $(\mathbf{C},0)$. This implies that $\Lambda(f_{1|\{l_1=0\}}) = 2$, hence $\Lambda(f_1) = 2$.

3.6 Corollary *Let* $(\mathbf{X},0)$ *be an analytic germ of dimension* ≥ 1.
If $f \in \mathbf{m}^2_{\mathbf{X},0}$ *then* $\Lambda(f) = 0$.

Proof Let $(\mathbf{X},0) = (\mathbf{X}_1,0) \cup (\mathbf{X}_2,0)$, where $(\mathbf{X}_2,0)$ is the union of the irreducible components of $(\mathbf{X},0)$ which are of dimension ≥ 2 and $(\mathbf{X}_1,0)$ is the union of the 1-dimensional branches of $(\mathbf{X},0)$.

We slice $(\mathbf{X}_2,0)$ by a general hyperplane defined by some $l \in \Omega_f$ and treat separately (later) the 1-dimensional components of the slice. If $f \in \mathbf{m}^2_{\mathbf{X}_2,0}$ then each component of the Cerf diagram $\Delta(l,f)$ is tangent to the axis $\{\lambda = 0\}$, provided that l is *general enough.* (It is possible that not every $l \in \Omega_f$ has this property, see also Note 2.2 and Example 3.5.) The proof of this fact is similar to the proof of [Lê-5, Proposition 1.2], but one must be more careful, since our underlying space is not smooth; it goes as follows. Let $(\Gamma_j,0)$ be a component of the polar curve $\Gamma(l,f)$. Let $p_j : (\mathbf{C},0) \to (\Gamma_j,0)$ be a parametrization i.e., if $(\mathbf{X},0)$ is embedded in $(\mathbf{C}^N,0)$, then some point on Γ_j in the neighbourhood of 0 has coordinates $x_1(p_j(\tau)), \ldots, x_N(p_j(\tau))$.

Next, for any extension $F : (\mathbf{C}^N,0) \to (\mathbf{C},0)$ of f (i.e. $F_{|(\mathbf{X},0)} = f$), we have $F \in \mathbf{m}^2_{\mathbf{C}^N,0}$, which means that $\operatorname{ord} F \geq 2$. Let $k_j := \min\{\operatorname{ord}(x_i \circ p_j) \mid i \in \{1,\ldots,N\}\}$. Then $\operatorname{ord}(F \circ p_j) \geq 2k_j$.

If l is the restriction on $(\mathbf{X},0)$ of a general linear function $L : (\mathbf{C}^N,0) \to (\mathbf{C},0)$, then $\operatorname{ord}(L \circ p_j) = k_j$. Hence:

$$\lim_{\tau\to 0} \frac{f(p_j(\tau))}{l(p_j(\tau))} = \lim_{\tau\to 0} \frac{F(p_j(\tau))}{L(p_j(\tau))} = 0.$$

The tangency to $\{\lambda = 0\}$ means exactly that $m_{i,1}/n_{i,1} < 1$, in particular $n_{i,1} > 1$, $\forall i \in \{1, \dots, r\}$. Thus, our proof relays on a decreasing induction: at each step, we may apply Theorem 3.2.

The only thing we have to do more is to prove the assertion for 1-dimensional branches. This is done in the next lemma. □

Lemma *If* $(\mathbf{X}, 0)$ *is* 1*-dimensional, irreducible and if* $f \in \mathbf{m}^2_{\mathbf{X},0}$ *then there is a geometric monodromy of* f *without fixed points.*

Proof If $(\mathbf{X}, 0)$ is smooth, then it is clear that f has a fixed-point-free geometric monodromy. If $(\mathbf{X}, 0)$ is not smooth, let $n : (\tilde{\mathbf{X}}, a) \to (\mathbf{X}, 0)$ be its normalization. It follows that $f \circ n \in \mathbf{m}^2_{\tilde{\mathbf{X}},a}$, hence we may conclude as above, since $(\tilde{\mathbf{X}}, a)$ is smooth. □

As a complement to Theorem 3.2, for $\dim(\mathbf{X}, 0) = 1$, we have the following precise determination of the Lefschetz number:

3.7 Proposition *If* $(\mathbf{X}, 0) = \bigcup_{i \in R}(C_i, 0)$ *is a curve and its decomposition into irreducible components, then, for any* $f \in \mathbf{m}_{\mathbf{X},0} \setminus \mathbf{m}^2_{\mathbf{X},0}$, *we have:*

$$\Lambda(f) = \#\{i \in R \mid (C_i, 0) \text{ is smooth and } f \in \mathbf{m}_{C_i,0} \setminus \mathbf{m}^2_{C_i,0}\}.$$

Proof Let $f_i := f_{|(C_i,0)}$. Then the Milnor fibre of f is a finite set, the disjoint union of the Milnor fibres of f_i, $i \in R$. Hence, $\Lambda(f) = \sum_{i \in R} \Lambda(f_i)$.

If $(C_i, 0)$ is smooth, then one has: $\Lambda(f_i) = 1$ if and only if $f_i \in \mathbf{m}_{C_i,0} \setminus \mathbf{m}^2_{C_i,0}$.

If $(C_i, 0)$ is not smooth, let $n : (\tilde{C}_i, a_i) \to (C_i, 0)$ be its normalization. Then $f_i \circ n_i \in \mathbf{m}^2_{\tilde{C}_i,a_i}$, since the Milnor fibre of $f_i \circ n_i$ contains more than one point. It follows that we have a fixed-point-free monodromy of f_i and $\Lambda(f_i) = 0$. □

3.8 In the following, we focus on the determination of the fixed points of the carrousel. For this problem, the important Puiseux pairs are those $(m_{i,1}, n_{i,1})$, $i \in \{1, \dots, r\}$ such that $n_{i,1} = 1$.

Consider the partition of the set of indices $\{1, \dots, r\}$ of the components of Δ', into the following two subsets:

$$P^{(1)} := \{i \mid n_{i,1} = 1\}, \qquad P^{(2)} := \{i \mid n_{i,1} > 1\}.$$

The set of fixed points in the carrousel disc may be not a union of concentric circles any more.

3.9 We identify the fixed points inside a given annulus A_i, $i \in P^{(1)}$:

Let B_i be the union of all carrousel discs of order 1 included in A_i. Then the carrousel construction tells us that the set $A_i \setminus B_i$ is pointwise fixed.

Further, let $\delta(i) \subset A_i$ be a carrousel disc of order 1 defined as in the next 3.11. If there are no carrousel discs of order 1 included in $\delta(i)$, then the only fixed point of $\delta(i)$ is its centre.

If $\delta(i)$ contains some carrousel disc of order 1, then we decompose $\delta(i)$ into annuli, since $\delta(i)$ is itself a carrousel. For any such annulus, we may adapt the present argument, from the beginning of 3.9.

3.10 It is easily seen that the set $A_i \setminus B_i$, as in 3.9, has as deformation retract the set:

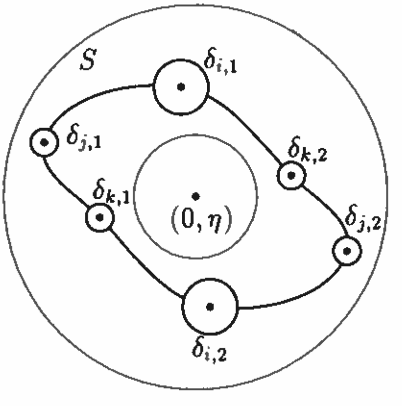
$$(S_i \setminus \bigcup_{\delta \in \mathcal{K}_i^{(1)}} \delta) \cup \bigcup_{\delta \in \mathcal{K}_i^{(1)}} \partial\bar{\delta}, \tag{7}$$

where:

$$\mathcal{K}_i^{(1)} := \left\{ \delta \subset A_i \,\middle|\, \begin{array}{l} \delta \text{ is a carrousel disc of order 1 which is} \\ \text{not included (strictly) in any other carrousel disc} \end{array} \right\}, \tag{8}$$

and S_i is a closed curve homotopic to a circle which intersects δ, $\forall \delta \in \mathcal{K}_i^{(1)}$.

The picture shows a possible shape of the retract of the set of fixed points inside A_i: the "thick" curves and the "fat" points are fixed.



Then a neighbourhood of the set of fixed points after one turn of the (big) carrousel retracts to a set such that each connected component of it is either:

(a) a circle centred in $(0, \eta)$ or in a centre of some carrousel disc of order 1, or

(b) a set having a definition similar to the one in (7), or

(c) the centre of a carrousel of order 1 inside A_i, for some $i \in P^{(1)}$, or

(d) the centre $(0, \eta)$ of the big carrousel.

3.11 Definition Let $\mathcal{I}^{(1)}$ be a maximal set of indices $i \in P^{(1)}$ such that, if $i_1, i_2 \in \mathcal{I}^{(1)}$, then $\hat{C}_{i_1}^{(1)} \neq \hat{C}_{i_2}^{(1)}$.

For any $i \in \mathcal{I}^{(1)}$, denote by $\delta(i)$ the carrousel disc of order 1 centred at a fixed (arbitrarily chosen) point of the set $\hat{C}_i^{(1)} \cap (D_\alpha \times \{\eta\})$; denote by $c(i)$ the centre of $\delta(i)$.

Let $a(i)$ be an arbitrarily chosen point on $\partial\overline{\delta}(i)$. It is, by definition, a regular value for l_α.

With the notations in 3.3 and 3.11, we have the next:

3.12 Theorem *If $f \in \mathbf{m}_{\mathbf{X},0}$ and $l \in \Omega_f$, then:*

$$\Lambda(f) = \Lambda(f_{|\{l=0\}}) + \sum_{i \in \mathcal{I}^{(1)}} \left[\Lambda(h'_{c(i)}) - \Lambda(h'_{a(i)})\right].$$

Proof The Lefschetz number $\Lambda(f)$ splits into a sum, following the decomposition of the set of fixed points into connected components, see 3.10. We define a suitable open covering of a set defined as in (7) and then apply the Mayer–Victoris exact sequence. By a straightforward computation, using also Lemma 3.3, we get the formula above. □

Notice that, by Theorem 3.12 above, only the first "hat" approximations $\hat{C}_i^{(1)}$ count for the Lefschetz number $\Lambda(f)$. In general, the Lefschetz numbers $\Lambda(h'_{c(i)})$, $\Lambda(h'_{a(i)})$ are not easy to compute; we refer to Remarks 4.11. At the end of the next section, there is an example where the computations can be carried out.

4 The zeta-function and the carrousel monodromies

4.1 We show that, within the present point of view, there is just a little step from the formula for the Lefschetz number (Theorem 3.12) to a formula for the zeta-function $\zeta_f(t)$. We mention first some related and similar results.

A theorem of Eisenbud and Neumann [EN, Theorem 4.3] asserts that the zeta-function of a *fibred multilink* **L** is the product of the zeta-functions over all *splice components* of **L**. If the multilink is defined by some Cerf diagram $\Delta(l, f)$, then the zeta-function $\zeta_f(t)$ becomes the zeta-function, this time with coefficients in a local system, of the multilink **L**. This observation leads to a formula of Némethi [Ne]; compare his formula to the one of Eisenbud and Neumann, as cited by Schrauwen [Sch, p.16].

Our approach is rather elementary: it uses only the carrousel method and not the deep results in the theory of links. However, we mention that in the paper [EN], the authors worked and, in particular, defined explicitly their *splicing diagram*, only when the Puiseux expansion of any component of the link is equal to the *essential* one (see 2.9). This was noticed also by Schrauwen in his thesis [Sch, p.8]. In general, as we have more than one knot in the link, the *actual Puiseux expansion* (that is, also the nonessential terms) effectively count for the understanding of the link. This well-known fact has an evident reflection in our approach.

4.2 One of the most important properties of the carrousel construction is that, loosely speaking, each "important point" of the carrousel disc is fixed after a finite number of turns of the carrousel.

We have seen that the set of fixed points after one turn is the only important set for the Lefschetz number $\Lambda(h_f)$. One can easily figure out that the set of fixed points after k turns is the one responsible for the number $\Lambda(h_f^k)$; we make this more clear in the following.

The set of fixed points of some power k of the carrousel motion may contain a finite number of circles consisting of regular values for the map l_α. Actually, these circles do not count in the computation of $\Lambda(h_f^k)$, as shown by Lemma 3.3 (where h_f has to be replaced by h_f^k).

4.3 The Lefschetz numbers $\Lambda(h_f^k)$ are topological invariants of f, they do not depend on the slice $l \in \Omega_f$. However, preserving the notations before, there is the following:

Proposition *If $n_{i,1} \nmid k$, $\forall i \in \{1, \ldots, r\}$, then:*

$$\Lambda(h_f^k) = \Lambda(h^k_{f_{|\{l=0\}}}).$$

Proof If $n_{i,1}$ is not a divisor of k then the annulus A_i contains no fixed points of the k^{th} power of the carrousel. There may be circles of fixed points in some "transition zones", but they give no contribution in $\Lambda(h_f^k)$. □

Note that Theorem 3.2 becomes *a posteriori* a special case of Proposition 4.3, for $k = 1$.

4.4 If a point $x \in D_\alpha \times \{\eta\}$ is fixed by a certain power of the carrousel motion, then a whole (small enough) disc around x is globally (maybe not pointwise) fixed.

Definition Let $U \subset D_\alpha \times \{\eta\}$ and let $k_U := \min\{k \mid U \text{ is globally fixed by the } k^{\text{th}} \text{ power of the carrousel}\}$. Then $h_f^{k_U}$ restricts to an action on $H^\bullet(l_\alpha^{-1}(U))$, which we denote by h'_U. We call such actions *carrousel monodromies.*

4.5 The zeta-function is determined by the set of Lefschetz numbers $\Lambda(h_f^k)$, $k \geq 1$, see 1.3.

On the other hand, if $\mathcal{B}^{(k)}$ denotes some neighbourhood of the set of fixed points of the k^{th} power of the carrousel, then h_f^k acts on the cohomology $H^\bullet(l_\alpha^{-1}(\mathcal{B}^{(k)}))$ and, with the definition above, we get:

$$\Lambda(h_f^k) = \Lambda(h'_{\mathcal{B}^{(k)}}). \tag{9}$$

4.6 Let's consider the annulus A_i , as before, in the big carrousel disc. By the proof of Proposition 4.3, if $x \in A_i$ is fixed by some power k of the carrousel, then this power has to be a multiple of $n_{i,1}$.

Denote by h_{A_i} the restriction of h_f to $H^\bullet(l_\alpha^{-1}(A_i))$. Then, by the formulae in 1.3, we get:

$$[\zeta_{h_{A_i}}(t)]^{n_{i,1}} = \zeta_{h_{A_i}^{n_{i,1}}}(t^{n_{i,1}}). \tag{10}$$

Definition For any $i \in \{1, \ldots, r\}$, denote by $\delta(i)^{(1)}$ the carrousel disc of order 1 centred in an arbitrarily chosen point of $\hat{C}_i^{(1)} \cap (D_\alpha \times \{\eta\})$, but fixed once and for all.

Let $\mathcal{J}^{(1)} := \{\delta = \delta(i)^{(1)} \mid i \in \{1, \ldots, r\},\ \delta(i)^{(1)} \text{ is not strictly contained in any other carrousel disc of order } 1\}$.

Finally, let for $\delta \in \mathcal{J}^{(1)}$, $a(\delta)$ be an arbitrarily chosen point on the circle $\partial\delta$.

Then we have the next:

4.7 Theorem

$$\zeta_f(t) = \zeta_{f|\{l=0\}}(t) \cdot \prod_{\delta \in \mathcal{J}^{(1)}} \zeta_{h'_\delta}(t^{n_{i,1}}) \cdot \zeta^{-1}_{h'_{\alpha(\delta)}}(t^{n_{i,1}}).$$

Proof We apply the Mayer–Vietoris exact sequences to the covering by annuli A_i described before. Since the fixed circles do not count for the Lefschetz numbers, we get that the zeta-function is a product over all different annuli, each factor being of the form $\zeta_{h_{A_i}}(t)$.

To get $\zeta_{h_{A_i}}(t)$, we compute $\zeta_{h^{n_{i,1}}_{A_i}}(t)$, in view of (10). Here the situation is similar to the one in 3.9: the carrousel discs of order 1 included in A_i are globally fixed and the complement of their union in A_i is pointwise fixed.

It follows that A_i retracts to the set:

$$\mathcal{R}_i := S_i \cup \bigcup_{\delta \in \mathcal{K}_i^{(1)}} \delta,$$

where S_i is homotopic (in A_i) to a circle and $S_i \cap \delta \neq \emptyset$, $\forall \delta \in \mathcal{K}_i^{(1)}$. The set $\mathcal{K}_i^{(1)}$ has been defined in (8).

Moreover we get that:

$$\zeta_{h^{n_{i,1}}_{A_i}}(t) = \zeta_{h'_{\mathcal{R}_i}}(t).$$

Let $\mathcal{J}_i^{(1)} := \{\delta \in \mathcal{J}^{(1)}, \ \delta \subset A_i\}$. If $\delta \in \mathcal{J}_i^{(1)}$, then notice that there are $n_{i,1}$ carrousel discs in A_i of the same radius as δ; if δ_1, δ_2 are any two of them, then $\zeta_{h'_{\delta_1}}(t) = \zeta_{h'_{\delta_2}}(t)$.

An open covering of $\mathcal{R}_i$ and a Mayer–Vietoris argument lead to the conclusion:

$$\zeta_{h'_{\mathcal{R}_i}}(t) = \prod_{\delta \in \mathcal{J}_i^{(1)}} [\zeta_{h'_\delta}(t)]^{n_{i,1}} \cdot [\zeta_{h'_{\alpha(\delta)}}(t)]^{n_{i,1}}.$$

Using (10), the formula in our theorem gets its proof. Notice that the factor $\zeta_{f|\{l=0\}}(t)$ corresponds to the disc A_0, defined in 2.10. □

4.8 It is relatively easy to figure out how the process started in the proof above may continue: each $\delta \in \mathcal{J}^{(1)}$ is a small carrousel (of order 1); the algebraic monodromy on $H^\bullet(l_\alpha^{-1}(\delta))$ is h'_δ. We apply Theorem 4.7 with h_f replaced by h'_δ and get a formula for the zeta-function $\zeta_{h'_\delta}(t)$, for any $\delta \in \mathcal{J}^{(1)}$.

In a finite number of steps, going through the carrousel discs of order $1, 2, \ldots, m$, where $m := \max\{g_i \mid i \in \{1, \ldots, r\}\}$, we get a formula for $\zeta_f(t)$. (We recall that g_i is the number of Puiseux pairs in our parametrization of Δ_i). To write it down, we just need some more notation.

4.9 Definition Let $\delta(i)^{(k)}$ denote the carrousel disc of order k centred in a fixed (arbitrarily chosen) point of the set $\hat{C}_i^{(k)} \cap (D_\alpha \times \{\eta\})$. (Note that this later set contains exactly $n_{i,1} \cdots n_{i,k}$ points). Denote by:

$$\mathcal{C}(\Delta') := \{\delta(i)^{(k)} \mid i \in \{1, \ldots, r\},\ k \in \{1, \ldots, m\}\}.$$

For any $\delta \in \mathcal{C}(\Delta')$, denote by $c(\delta)$ its centre and by $a(\delta)$ an arbitrarily chosen point on $\partial\overline{\delta}$.

Let $\delta \in \mathcal{C}(\Delta')$, where $\delta = \delta(i)^{(k)}$, for some indices i and k as above. Then define:

$$n(\delta) := n_{i,1} \cdots n_{i,k}.$$

We get the following conclusion:

4.10 Theorem

$$\zeta_f(t) = \zeta_{f_{|\{l=0\}}}(t) \cdot \prod_{\delta \in \mathcal{C}(\Delta')} \zeta_{h'_{c(\delta)}}(t^{n(\delta)}) \cdot \zeta^{-1}_{h'_{a(\delta)}}(t^{n(\delta)}).$$

□

4.11 Remarks (a) The points $a(\delta)$, $\delta \in \mathcal{C}(\Delta')$ may be defined as follows: Let $\delta = \delta(i)^{(k)}$ and let $\hat{C}_i^{(k)}$ be (formally) defined by the equation (see (6)):

$$u_i = a_{k_i} \lambda^{m_{i,1}/n_{i,1}} + \cdots + \sum_{l=1}^{l_k} b_{k,l} \lambda^{(m_k+l)/n_{i,1} \cdots n_{i,k}}. \tag{11}$$

Then define a curve $G_{i,k}$, by slightly perturbing in (11) just the last coefficient b_{k,l_k}, such that $G_{i,k} \neq \hat{C}_j^{(k)}$, $\forall j \in \{1, \ldots, r\}$. For $k = g_i$, we cut the Puiseux expansion at a sufficiently high power of λ and modify the last coefficient.

It follows that $a(\delta(i)^{(k)})$ may be identified to the point in $G_{i,k} \cap (D_\alpha \times \{\eta\})$ which is in the closest neighbourhood of $c(\delta(i)^{(k)})$.

(b) Let $\delta := \delta(i)^{(k)}$. Then $c(\delta)$ is a regular value for the map l_α if and only if, for any $j \in \{1, \ldots, r\}$ such that $\hat{C}_j^{(k)} = \hat{C}_i^{(k)}$, we have $g_j > k$.

It is possible that $a(\delta)$ cannot be chosen arbitrarily close to $c(\delta)$, see also Remark 2.8.

(c) The carrousel monodromies $h'_{c(\delta)}, h'_{a(\delta)}$ may be defined as monodromies of functions. For instance, if $\delta = \delta(i)^{(k)}$ and $(u_i^{(k)}(t), \lambda(t))$ is the parametrization of $\hat{C}_i^{(k)}$ defined in 2.7, then the pull-back diagram:

$$\begin{array}{ccc} (\mathbf{X}_i^{(k)}, 0) & \longrightarrow & (\mathbf{X}, 0) \\ f_i^{(k)} \downarrow & & \downarrow \Phi \\ (\mathbf{C}, 0) & \xrightarrow{(u_i^{(k)}, \lambda)} & (\mathbf{C}^2, 0) \end{array} \tag{12}$$

defines a space $(\mathbf{X}_i^{(k)}, 0)$ and a function $f_i^{(k)}$ on it. Then $h'_{c(\delta)}$ is just the monodromy of $f_i^{(k)}$.

A pull-back diagram similar to (12) was used by Lê in his proof of the Monodromy Theorem [Lê-2].

4.12 In general, it is difficult to compute the zeta-function by the formula 4.10. One of the reasons is that the Cerf diagram and its Puiseux parametrization are hard jobs in practice.

4.13 We rather illustrate the formula on a simple particular case: any component Δ_i has just one Puiseux pair, i.e. $g_i = 1$, $\forall i \in \{1, \ldots, r\}$.

In this case, we have $\hat{C}_i^{(1)} = \Delta_i$ and a carrousel disc $\delta(i)^{(1)}$ is just an arbitrarily small disc centred in $c(\delta(i)^{(1)}) \in \Delta_i \cap (D_\alpha \times \{\eta\})$, which is pointwise fixed by the $n_{i,1}{}^{\text{th}}$ power of the big carrousel.

It follows that the point $a(\delta(i)^{(1)})$ can be chosen arbitrarily close to the point $c(\delta(i)^{(1)})$. The centres $c(\delta)$, $\delta \in \mathcal{C}(\Delta')$ are, of course, critical values for the map l_α.

The fibre $F'_{a(\delta)}$ is also topologically equivalent to the fibre of the fibration defined by l_α over $U_{c(\delta)} \setminus \{c(\delta)\}$, where $U_{c(\delta)}$ is a small enough neighbourhood of $c(\delta)$ in $D_\alpha^{(1)} \times \{\eta\}$.

It follows that, after attaching a finite number of cells to $F'_{a(\delta)}$, one gets a space which is homotopy equivalent to $F'_{c(\delta)}$.

There is a long exact sequence corresponding to the pair of fibres

$$(F'_{c(\delta)}, F'_{a(\delta)}),$$

on which the restrictions of the monodromy h_f are acting:

$$\begin{array}{ccccccccc} \to & H^k(F'_{c(\delta)}) & \to & H^k(F'_{a(\delta)}) & \xrightarrow{\partial} & H^{k+1}(F'_{c(\delta)}, F'_{a(\delta)}) & \to & H^{k+1}(F'_{c(\delta)}) & \to \\ & \uparrow h'_{c(\delta)} & & \uparrow h'_{a(\delta)} & & \uparrow h^{\text{rel}}_{c(\delta)} & & \uparrow h'_{c(\delta)} & \end{array} \tag{13}$$

There is a well defined relative monodromy:

$$h^{\mathrm{rel}}_{c(\delta)} : H^\bullet(F'_{c(\delta)}, F'_{a(\delta)}) \to H^\bullet(F'_{c(\delta)}, F'_{a(\delta)}).$$

We get:

$$\zeta_{h^{\mathrm{rel}}_{c(\delta)}}(t) = \zeta_{h'_{c(\delta)}}(t) \cdot \zeta^{-1}_{h'_{a(\delta)}}(t).$$

Let $c(i)$ denote an arbitrarily chosen point of the set $\Delta_i \cap (D_\alpha \times \{\eta\})$. Then $C(\Delta')$ can be identified to the set $\{c(i) \mid i \in \{1, \ldots, r\}\}$. With these notations, the zeta-function formula becomes:

$$\zeta_f(t) = \zeta_{f|\{l=0\}}(t) \cdot \prod_{i \in \{1,\ldots,r\}} \zeta_{h^{\mathrm{rel}}_{c(i)}}(t^{n_{i,1}}). \tag{14}$$

The Lefschetz number formula is also easier:

$$\Lambda(f) = \Lambda(f_{|\{l=0\}}) + \sum_{n_{i,1}=1,\ i \in \{1,\ldots,r\}} \Lambda(h^{\mathrm{rel}}_{c(i)}).$$

In particular cases, these relative monodromies may have interesting interpretations, as in the paper [Si] of Siersma (which, actually, was at the origin of this investigation).

We end by an example:

4.14 Example Let $\mathbf{X} := \{x^3 + y^4 + z^3 = 0\} \subset \mathbf{C}^3$ and let $f \in \mathbf{m}_{\mathbf{X},0}$ be the function induced by $\hat{f} \in \mathbf{m}_{\mathbf{C}^3,0}$, $\hat{f} = x$.

Consider the linear function l induced by $\hat{l} = y$. Then $l \in \Omega_f$. We get that $\Delta(l, f)$ is irreducible and has the Puiseux parametrization: $l = \alpha v^3$, $\lambda = v^4$, where α is a nonzero constant, easy to determine.

Let $c \in \Delta(l, f) \cap (D_\alpha \times \{\eta\})$ and let $a \notin \Delta(l, f) \cap (D_\alpha \times \{\eta\})$ be a neighbour point of c.

By Remark 4.11(c), in our case h'_c can be identified to the monodromy of the function: $f_c : (\mathbf{X}_c, 0) \to (\mathbf{C}, 0)$ induced by $\hat{f}_c = v$, where $\mathbf{X}_c := \{x = v^4, y = \alpha v^3\}$. Thus $\zeta_{h'_c}(t) = (1-t)^{-1}$.

Next, the monodromy h'_a can be identified to the monodromy of the function $f_a : (\mathbf{X}_a, 0) \to (\mathbf{C}, 0)$ induced by $\hat{f}_a = v$, where $\mathbf{X}_a := \{x = v^4, y = v^3, z = \sqrt[3]{2}\gamma v^4\}$ and γ is a 3-root of -1. We get $\zeta_{h'_a}(t) = (1-t)^{-3}$, hence $\zeta_{h^{\mathrm{rel}}_c} = (1-t)^2$.

By using (14), the final result is:

$$\zeta_f(t) = (1-t)^{-3} \cdot (1-t^4)^2.$$

We also get: $\Lambda(f) = 3$.

Notice that there is another way of computing the zeta function in this example, by using the usual $\mathbf{C}^*$-action on $\mathbf{X}$, which fixes the zero set $\{\hat{f} = 0\}$. It follows that the monodromy h_f of f is equal to the 3^{rd} power of the monodromy h_g of the function $g : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$, $g = y^4 + z^3$ and $\zeta_{h_g^3}(t)$ can be computed from the eigenvalues of h_g.

If we change the above function $\hat{f}$ into $\hat{f}_1 := x + y$, then the set $\{\hat{f}_1 = 0\}$ is no more invariant under the above-mentioned $\mathbf{C}^*$-action. The computations for the zeta-function of h_{f_1} are slightly more complicated, since we get two Puiseux pairs, with $n_{1,1} = 1$, $n_{1,2} = 3$. This time, the final result is:

$$\zeta_{f_1}(t) = (1-t)^{-1}(1-t^3)^{-1}(1-t^9).$$

Chapter II

Lefschetz number and isolated cyclic quotient singularities

1 Leftschetz number and resolution of singularities

1.1 Let $(\mathbf{X}, x)$ be a reduced analytic germ and let $i : (\mathbf{X}, x) \hookrightarrow (\mathbf{C}^N, 0)$ be an embedding, for some large enough $N \in \mathbf{N}$. Denote by $(\mathbf{X}_{\mathrm{sing}}, 0)$ the germ of the singular locus of $(\mathbf{X}, 0)$. Let H be a divisor, defined as the germ in 0 of the zero set of a function $f : (\mathbf{X}, 0) \to (\mathbf{C}, 0)$. Denote by $\mathcal{O} := \mathcal{O}_{\mathbf{X},0}$ the local algebra of $(\mathbf{X}, 0)$ and by $\mathbf{m} := \mathbf{m}_{\mathbf{X},0}$ its maximal ideal.

Let $\mathcal{X}$ denote some small enough representative of the germ $(\mathbf{X}, 0)$ and let $\mathcal{H}$ denote the zero locus of some representative of f on $\mathcal{X}$.

By the work of Hironaka [Hi-1], there is a *local resolution*:

$$r : (\mathcal{W}, \mathcal{W}_0) \to (\mathcal{X}, \mathcal{H})$$

such that: $\mathcal{W}_0 := r^{-1}(\mathcal{H})$ is a *normal crossings divisor* (abbreviated *n.c. divisor*) in the smooth space $\mathcal{W}$.

The set $\mathbf{S} := r^{-1}(0)$ is a compact subvariety of $\mathcal{W}$, since r is proper (by definition). Also by definition, r induces an isomorphism:

$$r_| : \mathcal{W} \setminus (r^{-1}(\mathcal{X}_{\mathrm{sing}}) \cup \mathcal{W}_0) \to \mathcal{X} \setminus (\mathcal{X}_{\mathrm{sing}} \cup \mathcal{H}).$$

1.2 In general, the Milnor fibre F_f of f is not smooth and $r^{-1}(F_f)$ is not homotopy equivalent to F_f. Nevertheless, if f is a *smoothing* of H then F_f is smooth and $r^{-1}(F_f) \simeq F_f$.

The construction of A'Campo [A'C-2] yields a *model* for the Milnor fibre F_f, in the case when f is a smoothing, that is when:

$$(\mathbf{X}_{\mathrm{sing}}, 0) \subset (H, 0). \tag{1}$$

Let $\mathbf{I}_{\mathbf{X}_{\mathrm{sing}},0} \subset \mathcal{O}_{\mathbf{X},0}$ be the reduced ideal of $(\mathbf{X}_{\mathrm{sing}}, 0)$.

The model constructed by A'Campo also provides a decomposition of the Milnor fibre $r^{-1}(F_f)$ of $f \circ r$ into certain "pieces", as we roughly show in the following:

Let $\check{\mathbf{E}} := \cup_{j \in J_r} \check{\mathbf{E}}_j$ be the exceptional divisor of r (which is not compact, unless $\mathbf{X}_{\text{sing}} = \{0\}$) and its decomposition into irreducible components. Denote by $\tilde{\mathcal{H}}$ the proper transform of $\mathcal{H}$ by r. Then we have the equality of divisors:

$$\mathcal{W}_0 = \tilde{\mathcal{H}} + \sum_{j \in J_r} n_f(\check{\mathbf{E}}_j)\check{\mathbf{E}}_j, \tag{2}$$

where $n_f(\check{\mathbf{E}}_j)$ is a positive integer, for any $j \in J_r$, by the condition (1).

Let U be a small enough tubular neighbourhood of $\mathcal{W}_0$; then $r^{-1}(F_f) \subset U$. Since $\mathcal{W}_0$ is a germ in $\mathbf{S}$ of a n.c. divisor, this neighbourhood U can be retracted to a neighbourhood U' of $\mathbf{S}$ and the retraction is compatible with the Milnor fibration of $f \circ r$. Next, $\mathbf{S}$ intersects all the exceptional divisors $\check{\mathbf{E}}_j$ and also $\tilde{\mathcal{H}}$; we consider the set of all the points where a fixed divisor $\check{\mathbf{E}}_{j_0}$ intersects $\mathbf{S}$ but no other component of $\mathcal{W}_0$ does. Take a small tubular neighbourhood U_{j_0} of this smooth set and consider the intersection of $r^{-1}(F_f)$ with U_{j_0}. This is one of the pieces in the decomposition of F_f produced by A'Campo.

The decomposition behaves nicely with respect to the monodromy: each piece has a well defined geometric monodromy which makes the computation of the Lefschetz number of any power of the algebraic monodromy easy.

It follows that the zeta-function is a product over all the pieces in the decomposition. Actually, A'Campo defines:

$$\mathbf{S}_i := \left\{ s \in \mathbf{S} \;\middle|\; \begin{array}{c} \text{the equation of } \mathcal{W}_0 \text{ in } s \text{ is of the form } z^i = 0, \\ \text{for a local coordinate } z \text{ in the point } s \in \mathcal{W}. \end{array} \right\}$$

and is able to prove, under the condition (1)—which is, however, not explicitly stated in [A'C-2]—the formula:

1.3 Theorem [A'C-2, Théorème 3].

$$\zeta_f(t) = \prod_{i \geq 1} (1 - t^i)^{-\chi(\mathbf{S}_i)}.$$

□

This is a consequence of the formula for the Lefschetz number of a power of the algebraic monodromy h_f:

1.4 Theorem [A'C-2, Théorème 1].

(a) $\Lambda(h_f^k) = \sum_{i|k} i \cdot \chi(\mathbf{S}_i)$, *for* $k \geq 1$,

(b) $\Lambda(h_f^0) = \chi(F_f) = \sum_{i\geq 1} i \cdot \chi(\mathbf{S}_i)$. □

1.5 Remark In [A'C-2], A'Campo applies his nice construction to functions on smooth spaces. An attempt to make it work beyond the limit given in our condition (1) is by replacing the resolution r by a modification over $\mathcal{H}$. We do not intend to discuss about this aspect here.

We assume from now on that f is a smoothing, i.e. $f \in \mathbf{I}_{\mathbf{X}_{sing},0}$.

1.6 For any component $\check{\mathbf{E}}_j$, there is a well defined valuation $\mathrm{ord}_{\check{\mathbf{E}}_j}$ on $\mathcal{O}_{\mathcal{W}|\check{\mathbf{E}}_j}$. For any $f \in \mathbf{I}_{\mathbf{X}_{sing},0}$, we say that $\mathrm{ord}_{\check{\mathbf{E}}_j}(f \circ r)$ is the *multiplicity* of $f \circ r$ along the component $\check{\mathbf{E}}_j$. According to the relation (2), we have:

$$\mathrm{ord}_{\check{\mathbf{E}}_j}(f \circ r) = n_f(\check{\mathbf{E}}_j).$$

These multiplicities enter in the zeta-function formula of A'Campo (Theorem 1.3) together with the Euler characteristic of each component of $\check{\mathbf{E}}$ minus its intersections with other components of the set $\{f \circ r = 0\}$.

The Lefschetz number formula, as part of the zeta-function formula, requires only the data for those components of $\check{\mathbf{E}}$ for which the multiplicity (in the sense above) is 1. Our first aim is to identify them.

1.7 We hope not to confuse the reader by using sometimes the same notation for a (space) germ and for some small enough representative of it. We find the following way convenient to produce a resolution r as in 1.1:

(i) resolve the space germ $(\mathbf{X}, 0)$ by $\pi : (\mathbf{X}', \mathbf{E}) \to (\mathbf{X}, \mathbf{X}_{\mathrm{sing}})$ such that $\mathbf{E}$ is a n.c. divisor and then

(ii) resolve the divisor $\{f \circ \pi = 0\}$ on the smooth space germ $(\mathbf{X}', \pi^{-1}(0))$ into a normal crossings divisor.

The second stage (ii) is necessary since $\{f \circ \pi = 0\}$ may be not a n.c. divisor; hence one has to blow-up further. By [Hi-1], there is indeed a sequence of blowing-ups which leads to the final resolution r with the desired properties:

$$r: \quad \mathbf{X}_k \xrightarrow{r_k} \mathbf{X}_{k-1} \to \cdots \to \mathbf{X}_1 \xrightarrow{r_1} \mathbf{X}' \xrightarrow{\pi} \mathbf{X}.$$

Each step $r_i : \mathbf{X}_i \to \mathbf{X}_{i-1}$ is a blowing-up along a smooth subvariety V_{i-1} included in the locus where $D_{i-1} := \{f \circ \pi \circ r_1 \circ \cdots \circ r_{i-1} = 0\}$ is equimultiple and is not a n.c. divisor. It follows that D_{i-1} has multiplicity ≥ 2 along V_{i-1} and, consequently, the new exceptional divisor introduced by r_i gets multiplicity ≥ 2 in the total transform D_i.

The immediate consequence is that *the divisors created by further blowing-ups do not count for the Lefschetz number.*

Moreover, we show in the next two easy lemmas that the further blowing-ups do not influence the data which are necessary for the formula of the Lefschetz number.

1.8 Let $\mathbf{E} = \bigcup_{j \in J_\pi} \mathbf{E}_j$ be the decomposition of the exceptional divisor of π into irreducible components and let $\{\widetilde{f = 0}\}$ denote the proper transform of $\{f = 0\}$ by π.

Lemma *Denote by $\tilde{\mathbf{E}}_j^{(i)}$ the proper transform of the component $\mathbf{E}_j$, $j \in J_\pi$ by the composition of blowing-ups $(r_1 \circ \cdots \circ r_i)$. For any $i \in \{1, \ldots, k\}$ and any $j \in J_\pi$ we have:*

$$\mathrm{ord}_{\tilde{\mathbf{E}}_j^{(i)}}(f \circ \pi \circ r_1 \circ \cdots \circ r_i) = \mathrm{ord}_{\mathbf{E}_j}(f \circ \pi).$$

Proof It is a step-by-step proof; we make explicit the first step. The first blowing-up r_1 is along a smooth subvariety of the ambient space in which $\mathbf{E}_j$ is a smooth subvariety itself. The intersection of these two subvarieties has codimension in $\mathbf{E}_j$ at least 1. The local equation of $\{f \circ \pi = 0\}$ in some generic point of $\mathbf{E}_j$ is: $y^m = 0$, where y is some local coordinate and $m := \mathrm{ord}_{\mathbf{E}_j}(f \circ \pi)$. It is clear that the function $f \circ \pi \circ r_1$ will have the same multiplicity m along the proper transform of $\{y = 0\}$ by r_1. □

1.9 Definition For any $j \in J_\pi$, define the open subset of $\mathbf{E}_j$:

$$\mathbf{E}_j^*(f) := \mathbf{E}_j \setminus (\{\widetilde{f = 0}\} \cup \bigcup_{i \in J_\pi,\ i \neq j} \mathbf{E}_i).$$

We usually write $\mathbf{E}_j^*$ instead of $\mathbf{E}_j^*(f)$, if no confusion may arise.

Lemma *For any $f \in \mathbf{I}_{\mathbf{X}_{sing},0}$, any $j \in J_\pi$ and any $i \in \{1, \ldots, k\}$ we have the analytic isomorphisms:*

$$(r_i \circ \cdots \circ r_1)^{-1}(\mathbf{E}_j^*(f)) \simeq \mathbf{E}_j^*(f).$$

Proof By definition, the locus of $\mathbf{X}'$ where $\{f \circ \pi = 0\}$ is not a n.c. divisor does not intersect $\mathbf{E}_j^*$, $\forall j \in J_\pi$. Hence, any modification in a subvariety V_1 contained in this locus (in particular our blowing-up r_1) is an isomorphism over $\mathbf{E}_j^*$.

The repetition of this argument in each step (i.e. for $r_1, r_2, \ldots, r_i$) gives the proof of our assertion. □

1.10 To each component $\mathbf{E}_j$, one associates a filtration $\mathcal{F}(\mathbf{E}_j)^\bullet$ on $\mathcal{O}$ as follows:

$$\mathbf{I}_{\mathbf{X}_{sing},0} = \mathcal{F}(\mathbf{E}_j)^1 \supset \mathcal{F}(\mathbf{E}_j)^2 \supset \cdots,$$

where $\mathcal{F}(\mathbf{E}_j)^k := \{f \in \mathbf{I}_{\mathbf{X}_{sing},0} \mid \mathrm{ord}_{\mathbf{E}_j}(f \circ \pi) \geq k\}$ and $\mathcal{F}(\mathbf{E}_j)^0 := \mathcal{O}$.

Hence, to any component $\mathbf{E}_j$, there corresponds a graded ring:

$$\mathrm{Gr}(\mathbf{E}_j)_\bullet \mathcal{O} := \bigoplus_{i=0}^{\infty} \mathcal{F}(\mathbf{E}_j)^i / \mathcal{F}(\mathbf{E}_j)^{i+1}.$$

1.11 It is easy to see that the method of A'Campo gives a proof of the:

Proposition *If $f \in \bigcap_{j \in J_\pi} \mathcal{F}(\mathbf{E}_j)^2$, then $\Lambda(f) = 0$.* □

1.12 By Lemmas 1.8 and 1.9, we have proved that, if we fix a resolution $\pi : (\mathbf{X}', \mathbf{E}) \to (\mathbf{X}, \mathbf{X}_{sing})$, then the Lefschetz number of some $f \in \mathbf{I}_{\mathbf{X}_{sing},0}$ is determined only by the Euler characteristics $\chi(\mathbf{E}_j^*)$, for $j \in J_\pi$ such that $\mathrm{ord}_{\mathbf{E}_j}(f \circ \pi) = 1$. Moreover:

Proposition *The Lefschetz number $\Lambda(f)$ depends only on the residue classes of f in $\mathrm{Gr}(\mathbf{E}_j)_1 \mathcal{O}$, for $j \in J_\pi$.*

Proof Suppose that $f \notin \mathcal{F}(\mathbf{E}_j)^2$, $f = f_1 + f_2$, where $f_2 \in \mathcal{F}(\mathbf{E}_j)^2$, for a fixed $j \in J_\pi$.

Let $a \in \mathbf{E}_j$ be a generic point of $\mathbf{E}_j$ and let $y = 0$ be the reduced local equation of $\mathbf{E}_j$ in the point a. Then the function $f \circ \pi$ in local coordinates around the point a becomes $y \cdot f_1' + y^2 \cdot f_2'$, for some holomorphic functions f_1', f_2'. Hence, we get the equality of sets:

$$\{f \circ \widetilde{\pi = 0}\} \cap \mathbf{E}_j = \{f_1 \circ \widetilde{\pi = 0}\} \cap \mathbf{E}_j.$$

This proves: $\chi(\mathbf{E}_j^*(f)) = \chi(\mathbf{E}_j^*(f_1))$, $\forall j \in J_\pi$ such that $\mathrm{ord}_{\mathbf{E}_j}(f \circ \pi) = 1$. □

1.13 In the case $(\mathbf{X}, 0)$ is an isolated singularity, any nonzero function $f : (\mathbf{X}, 0) \to (\mathbf{C}, 0)$ is a smoothing and we can prove:

Lemma *If* $(\mathbf{X}, 0)$ *is an isolated singularity then, for any* $f \in \mathbf{m}^2$, *we have:* $\mathrm{ord}_{\mathbf{E}_j}(f \circ \pi) \geq 2$, $\forall j \in J_\pi$. *In particular:*

$$\mathbf{m}^2 \subset \cap_{j \in J_\pi} \mathcal{F}(\mathbf{E}_j)^2.$$

Proof The morphism π induces a morphism of local algebras $\mathcal{O}_{\mathbf{X},0} \to \mathcal{O}_{\mathbf{X}',\mathbf{E}_j}$, $\forall j \in J_\pi$. Hence $\mathbf{m}^2_{\mathbf{X},0}$ is mapped to $\mathbf{m}^2_{\mathbf{X}',\mathbf{E}_j}$ and this is all we have to prove. □

Remarks

(a) The statement [A'C-1, Théorème 1] becomes a consequence of Lemma 1.13 and Proposition 1.11 above.

(b) If $(\mathbf{X}, 0)$ is isolated then, by Proposition 1.12, the Lefschetz number depends only on the residue class of f in $\mathbf{m}/\mathbf{m}^2$. Apparently, this result cannot be derived by the carrousel method.

(c) Example 6.3 shows that the inclusion in Lemma 1.13 can be strict.

The uniqueness of the minimal filtrations

1.14 The Lefschetz number does not depend on the chosen resolution of $\mathcal{X}$, since it is a topological invariant of the germ $(\mathbf{X}, 0)$. On the other hand, we have shown that the Lefschetz number is a well defined function:

$$\Lambda : \mathbf{I}_{\mathbf{X}_{sing},0} / \cap_{j \in J_\pi} \mathcal{F}(\mathbf{E}_j)^2 \to \mathbf{Z}.$$

We show in the following that the intersection of ideals $\cap_{j\in J_\pi}\mathcal{F}(\mathbf{E}_j)^2$ does not depend on the resolution π, hence we may denote it by $\mathcal{F}_{\mathbf{X},0}$. Actually, we prove a stronger result (Theorem 1.18), which implies that even some of the ideals $\mathcal{F}(\mathbf{E}_j)^2$, $j \in J_\pi$, namely the minimal ones, are uniquely determined by the underlying space.

A similar, but weaker result was proved by Fine [Fi, Theorem 1]. As he did, we use Hironaka's *theorem on the elimination of poins of indeterminacy* [Hi-1, p. 140] as a basis for the proofs.

1.15 Let $\pi_1 : (\mathbf{X}^{(1)}, \mathbf{E}^{(1)}) \to (\mathbf{X}, \mathbf{X}_{sing})$ and $\pi_2 : (\mathbf{X}^{(2)}, \mathbf{E}^{(2)}) \to (\mathbf{X}, \mathbf{X}_{sing})$ be two resolutions of the space germ $(\mathbf{X}, 0)$. Thus, the germs $(\mathbf{X}^{(1)}, \mathbf{E}^{(1)})$ and $(\mathbf{X}^{(2)}, \mathbf{E}^{(2)})$ are bimeromorphically equivalent.

According to the above cited theorem of Hironaka, there are a smooth germ $(\hat{\mathbf{X}}, \hat{\mathbf{E}})$ and two proper morphisms $\rho_1 : (\hat{\mathbf{X}}, \hat{\mathbf{E}}) \to (\mathbf{X}^{(1)}, \mathbf{E}^{(1)})$, $\rho_2 : (\hat{\mathbf{X}}, \hat{\mathbf{E}}) \to (\mathbf{X}^{(2)}, \mathbf{E}^{(2)})$, which are isomorphisms over $\mathbf{X}^{(1)} \setminus \mathbf{E}^{(1)}$, resp. $\mathbf{X}^{(2)} \setminus \mathbf{E}^{(2)}$ and which fit into the commutative diagram:

$$\begin{array}{ccc} & (\hat{\mathbf{X}}, \hat{\mathbf{E}}) & \\ {}^{\rho_1}\swarrow & & \searrow^{\rho_2} \\ (\mathbf{X}^{(1)}, \mathbf{E}^{(1)}) & & (\mathbf{X}^{(2)}, \mathbf{E}^{(2)}) \\ {}_{\pi_1}\searrow & & \swarrow_{\pi_2} \\ & (\mathbf{X}, \mathbf{X}_{sing}) & \end{array} \qquad (3)$$

such that ρ_2 is a finite succession of blowing-ups.

1.16 Definition Let $\pi : (\mathbf{X}', \mathbf{E}) \to (\mathbf{X}, \mathbf{X}_{sing})$ be a resolution. Define the set $L(\pi)$ of *filtrations associated to* π by:

$$L(\pi) := \{\mathcal{F}(\mathbf{E}_j)^\bullet \mid j \in J_\pi\}$$

and order this set by the inclusion relation "$\subseteq$", that is:

$$\mathcal{F}(\mathbf{E}_j)^\bullet \subseteq \mathcal{F}(\mathbf{E}_i)^\bullet \quad \text{if and only if} \quad \mathcal{F}(\mathbf{E}_j)^k \subseteq \mathcal{F}(\mathbf{E}_i)^k, \quad \forall k \in \mathbf{N}.$$

1.17 Definition The subset $\mu(\pi)$ of *minimal* filtrations associated to π is defined as follows:

$$\mu(\pi) := \{\mathcal{F}(\mathbf{E}_j)^\bullet \in L(\pi) \mid \mathcal{F}(\mathbf{E}_j)^\bullet \text{ is minimal with respect to the order “}\subseteq\text{”}\}.$$

Define also the subset M_π of J_π by: $M_\pi := \{j \in J_\pi \mid \mathcal{F}(\mathbf{E}_j)^\bullet \in \mu(\pi)\}$.

With these notations at hand, we prove the following:

1.18 Theorem *The set of filtrations $\mu(\pi)$ is an invariant of the space $(\mathbf{X},0)$.*

Proof Let π_1, π_2 be any two resolutions of $(\mathbf{X},\mathbf{X}_{\mathrm{sing}})$ as in 1.15; they fit into the diagram (3). Take $j \in M_{\pi_2}$ and $f \in \mathbf{I}_{\mathbf{X}_{\mathrm{sing}},0}$ such that $\mathrm{ord}_{\mathbf{E}_j}(f \circ \pi_2) = k$, $k \in \mathbf{Z}_+$. Denote by $\tilde{\mathbf{E}}_j$ the proper transform of $\mathbf{E}_j$ by ρ_2. Then, by Lemma 1.8, we have $\mathrm{ord}_{\tilde{\mathbf{E}}_j}(f \circ \pi_2 \circ \rho_2) = k$.

There exists $i \in J_{\pi_1}$ such that ρ_1 induces a morphism of germs $\rho_{1|}$: $(\hat{\mathbf{X}},\tilde{\mathbf{E}}_j) \to (\mathbf{X}^{(1)},\mathbf{E}_i^{(1)})$, thus a morphism of local algebras: $\mathcal{O}_{\mathbf{X}^{(1)},\mathbf{E}_i^{(1)}} \to \mathcal{O}_{\hat{\mathbf{X}},\tilde{\mathbf{E}}_j}$. Consequently, $\mathbf{m}^l_{\mathbf{X}^{(1)},\mathbf{E}_i^{(1)}}$ is mapped to $\mathbf{m}^l_{\hat{\mathbf{X}},\tilde{\mathbf{E}}_j}$, $\forall l \in \mathbf{N}$.

Suppose that $\mathrm{ord}_{\mathbf{E}_i^{(1)}}(f \circ \pi_1) > k$. Then $f \circ \pi_1 \in \mathbf{m}^{k+l}_{\mathbf{X}^{(1)},\mathbf{E}_i^{(1)}}$, for some $l \geq 1$, hence $f \circ \pi_1 \circ \rho_1 \in \mathbf{m}^{k+l}_{\hat{\mathbf{X}},\tilde{\mathbf{E}}_j}$. Since $f \circ \pi_1 \circ \rho_1 = f \circ \pi_2 \circ \rho_2$, we get a contradiction.

We have proved that $\mathcal{F}(\mathbf{E}_i^{(1)})^\bullet \subseteq \mathcal{F}(\mathbf{E}_j^{(2)})^\bullet$.

To get the converse inclusion, we interchange $\mathbf{X}^{(2)}$ with $\mathbf{X}^{(1)}$ in the diagram (3) and reason once again as above. We get that there is a $j_1 \in J_{\pi_2}$ such that:

$$\mathcal{F}(\mathbf{E}_{j_1}^{(2)})^\bullet \subseteq \mathcal{F}(\mathbf{E}_i^{(1)})^\bullet \subseteq \mathcal{F}(\mathbf{E}_j^{(2)})^\bullet.$$

By the minimality, all these inclusions are equalities and $j_1 \in M_{\pi_2}$. It also follows that $i \in M_{\pi_1}$, by the same minimality principle. The conclusion, $\mu(\pi_1) = \mu(\pi_2)$, is now evident. □

1.19 We show that, actually, the sets M_π and $\mu(\pi)$ have the same number of elements. The reason was explained to us by Steenbrink.

Proposition $\# M_\pi = \# \mu(\pi)$.

Proof If we slice $(\mathbf{X},0)$ by a general hyperplane H, we reduce the dimension of the singular locus. A resolution $\pi : (\mathbf{X}',\mathbf{E}) \to (\mathbf{X},\mathbf{X}_{\mathrm{sing}})$ induces a modification:

$$\pi_| : \tilde{H} \cap \mathbf{X}' \to H \cap \mathbf{X}.$$

If two valuations $\mathrm{ord}_{\mathbf{E}_i}, \mathrm{ord}_{\mathbf{E}_j}$ are equal on $\mathbf{m}_{\mathbf{X}',\mathbf{E}}$, then $\mathrm{ord}_{\mathbf{E}_i \cap \tilde{H}}$ and $\mathrm{ord}_{\mathbf{E}_j \cap \tilde{H}}$ are equal on $\mathbf{m}_{\tilde{H}, \tilde{H} \cap \mathbf{E}}$. Thus, it is sufficient to prove the assertion in the case: $(\mathbf{X}, 0)$ is an isolated singularity.

By results of Samuel [Sam] and Artin [Ar-1], [Ar-2] we have that any isolated singularity is algebraic. Hence $(\mathbf{X}, 0)$ is isomorphic to a germ $(\mathcal{U}, x)$, where $\mathcal{U}$ is an affine algebraic variety. We embed $\mathcal{U}$ in a projective variety $\mathcal{V}$ and resolve all the singularities of $\mathcal{V}$, except the one at the point $x \in \mathcal{U}$. We get a projective space $\mathcal{Y}$ with just one singular point x, where the germ $(\mathcal{Y}, x)$ is isomorphic to our initial $(\mathbf{X}, 0)$.

There is a resolution $\rho : \mathcal{Z} \to \mathcal{Y}$ of the singularity of $\mathcal{Y}$ such that $\mathcal{Z}$ is projective; let $\mathcal{E} := \rho^{-1}(x)$ be the exceptional divisor. Two different components $\mathcal{E}_i$, $\mathcal{E}_j$ of it give two different valuations $\mathrm{ord}_{\mathcal{E}_i}$, $\mathrm{ord}_{\mathcal{E}_j}$ on the function field $C(\mathcal{Y})$.

On the other hand, $C(\mathcal{Y})$ is also the field of fractions of the local ring A of $\mathcal{Y}$ in the point x. It follows that the two valuations are different on A and that's all we need. □

1.20 Some immediate consequences of the Theorem 1.18 are:

(a) The set of graded $\mathcal{O}$-modules:

$$G(\pi) := \{\mathrm{Gr}(\mathbf{E}_j)_{\bullet}\mathcal{O} \mid j \in M_\pi\}$$

is an invariant of $(\mathbf{X}, 0)$.

(b) The set of minimal, order two ideals $\mu^{(2)}(\pi) := \{\mathcal{F}(\mathbf{E}_j)^2 \mid j \in M_\pi\}$ is an invariant of $(\mathbf{X}, 0)$. This implies that the ideal $\mathcal{F}_{\mathbf{X},0} = \cap_{j \in J_\pi} \mathcal{F}(\mathbf{E}_j)^2$ does not depend on the resolution π.

1.21 Definition A resolution $\pi : \mathbf{X}' \to \mathbf{X}$ is called $\mu_{\mathbf{X}}$-*minimal* if and only if $L(\pi) = \mu_{\mathbf{X}}$, where $\mu_{\mathbf{X}} := \mu(\pi)$.

We refer to Remark 6.6 for examples of $\mu_{\mathbf{X}}$-minimal resolutions.

2 Isolated cyclic quotients; introduction to the results

2.1 Let μ_d denote the finite cyclic group of the d-roots of unity, for $d \geq 2$; it is a subgroup of $\mathbf{C}^*$. Consider a *diagonal* representation of μ_d in $GL(n, \mathbf{C})$, for some integer $n \geq 2$, and denote by G the image of μ_d. Hence a generator of the group G can be represented as a vector:

$$(p_0, p_1, \ldots, p_{n-1}), \quad \text{with } 0 \leq p_i \leq d-1, \forall i \in \{0, \ldots, n-1\},$$

meaning that, if ξ is a fixed primitive d-root of 1, then our generator acts as the multiplication of each coordinate x_i by ξ^{p_i}, respectively.

We impose from now on the condition that p_i and d are relatively prime, for all indices i. Consequently, a generator of our action can be represented as:

$$(1, p_1, \ldots, p_{n-1}), \quad \text{with } \gcd(p_i, d) = 1, \forall i \in \{1, \ldots, n-1\}. \tag{4}$$

This implies that the quotient space germ $(\mathbf{X}, 0) := (\mathbf{C}^n, 0)/G$ is an *isolated singularity*. Any isolated cyclic quotient singularity can be obtained as the quotient with respect to an action as above which has the property (4).

We focus on the Lefschetz number $\Lambda(f)$ and the zeta-function $\zeta_f(t)$ of the local monodromy of a function:

$$f : (\mathbf{X}, 0) \to (\mathbf{C}, 0).$$

2.2 We need some notations in order to state one of the main results we want to prove.

Let $\mathcal{A}$ be the set of integral vectors of the form $(a, \alpha_1, \ldots, \alpha_{n-1})$, with $1 \leq a \leq d-1$ and $\alpha_i \equiv ap_i$ (modulo d), where $1 \leq \alpha_i \leq d-1$.

Denote by $\mathcal{A}'$ the subset of $\mathcal{A}$ which is left after excluding the vectors that are linear combinations with positive integral coefficients of some other vectors in $\mathcal{A}$. It turns out easily that $\mathcal{A}'$ is included in the subset of the *primitive elements of* $\mathcal{A}$. (An element u_0 is called a primitive of $\mathcal{A}$ if the equality $u_0 = k \cdot u$, for some $k \in \mathbf{Z}$ and $u \in \mathcal{A}$, implies $k = 1$.) It is important to stress that some elements $v \in \mathcal{A}'$ might be not primitive in the lattice $\mathbf{Z}^n$ (see e.g. Example 2.7).

We define certain filtrations on the maximal ideal $\mathbf{m} := \mathbf{m}_{\mathbf{X},0}$ of the local ring $\mathcal{O} := \mathcal{O}_{\mathbf{X},0}$, which depend only on the group G. For any $f \in \mathbf{m}$, we denote by $\tilde{f}$ the corresponding G-invariant function on $(\mathbf{C}^n, 0)$.

2.3 Definition Let S denote the polynomial algebra $\mathbf{C}[x_1 \dots, x_n]$. For any $v \in (\mathbf{Z}_+)^n$, denote by $S(v)$ the same algebra, but this time graded: the variables have the weights given by the respective components of the strictly positive integral vector v. For any $g \in \mathbf{C}\{x_1, \dots, x_n\}$, denote by $o_v(g)$ the *order* of g with respect to the weights given by v.

2.4 One has the following identification:

$$\mathcal{O} \simeq \mathbf{C}\{x_1, \dots, x_n\}^G.$$

Define a filtration $\mathcal{G}(v)^\bullet$ on $\mathcal{O}$ by:

$$\mathcal{G}(v)^k := \{f \in \mathcal{O} \mid o_v(\tilde{f}) \geq kd\}.$$

This gives the graded $\mathbf{C}$-algebra:

$$\mathrm{Gr}(v)_\bullet \mathcal{O} := \mathbf{C} \oplus \bigoplus_{k \geq 1} \mathcal{G}(v)^k / \mathcal{G}(v)^{k+1}.$$

If $v \in \mathcal{A}'$ then one identifies this to another graded $\mathbf{C}$-algebra:

$$\mathrm{Gr}(v)_\bullet S^G = \mathbf{C} \oplus \bigoplus_{k \geq 1} S(v)_{kd}.$$

For any $k \geq 1$, $v \in \mathcal{A}'$ and any residue class $[f] \in \mathrm{Gr}(v)_k \mathcal{O}$, let $\tilde{f}_{v,k} \in S(v)_{kd}$ be the (canonical) polynomial *representative* of $[f]$. Let $f_{v,k}$ denote the same representative, but viewed as a function on $(\mathbf{X}, 0)$.

2.5 Definition Let $v \in \mathcal{A}'$, $f \in \mathfrak{m}$ and let $\tilde{f}_v := \tilde{f}_{v,1}$ and $f_v := f_{v,1}$, be the representatives of $[f]$ defined in 2.4 above.

Then, for any $v \in \mathcal{A}'$, define the "partial Lefschetz numbers" by:

$$\Lambda_v(f) := \Lambda(f_v).$$

By definition, $\Lambda(0) = 0$, hence we also get $\Lambda_v(0) = 0$, $\forall v \in \mathcal{A}'$. Note that $\tilde{f}_v$ is either 0 or a quasihomogeneous polynomial of degree d with respect to the weights v.

One of the main results we want to prove is the following:

2.6 Theorem *Let $f \in \mathfrak{m}$. Then:*

(a) The Lefschetz number $\Lambda(f)$ depends only on the functions f_v (defined above), for $v \in \mathcal{A}'$.

(b) $\Lambda(f) = \sum_{v \in \mathcal{A}'} \Lambda_v(f)$.

As an illustration of this theorem, we present the following example; more others are given in Section 6.

2.7 Example Consider the following data for the G-action: $n = 2$, $d = 8$ and denote by $(1, 5)$ a generator, as in 2.1. We get:

$$\begin{aligned} \mathcal{A} &= \{(1,5),(2,2),(3,7),(4,4),(5,1),(6,6),(7,3)\}, \\ \mathcal{A}' &= \{(1,5),(2,2),(5,1)\}. \end{aligned}$$

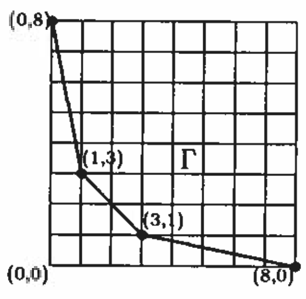
Notice that $(2,2) \in \mathcal{A}$ is not primitive in the lattice $\mathbf{Z}^n$.

The G-invariant monomials of "lowest" degrees are:

$$x^8, \quad x^3y, \quad xy^3, \quad y^8,$$

and they are on a polygonal line Γ consisting of three segments, as shown below (see Definition 4.13). These monomials generate the algebra of G-invariant polynomials $\mathbf{C}[x,y]^G$ and the embedding dimension of $\mathbf{C}^2/G$ is 4.

Note that x^2y^2 is not G-invariant, although the point $(2,2)$ is on Γ.



We have:

$$o_v(\tilde{f}) \geq 2 \cdot 8, \quad \forall f \in \mathrm{m}, \forall v \in \mathcal{A} \setminus \mathcal{A}'.$$

The subset $\mathcal{A}'$ of $\mathcal{A}$ is also connected to the *minimal resolution* of the quotient $\mathbf{C}^2/G$ (which has a $(-2,-3,-2)$ dual graph). We refer to Section 7, where the case $n = 2$ is presented in more detail.

One way of computing the partial Lefschetz numbers $\Lambda_v(f)$ is by using the minimal resolution of the 2-dimensional cyclic singularity. For details about this resolution, we refer to the nice book of Lamotke [Lam].

Take the following polynomial function:

$$\tilde{f} = x^8 + x^3y + xy^3 + y^8.$$

The Newton polygon associated to $\tilde{f}$ (see [Ku]) is just our Γ. We have:

$$\tilde{f}_{(1,5)} = x^8 + x^3y, \quad \tilde{f}_{(2,2)} = x^3y + xy^3, \quad \tilde{f}_{(5,1)} = xy^3 + y^8.$$

Then:

$$\Lambda_{(1,5)}(f) = 0, \quad \Lambda_{(2,2)}(f) = -1, \quad \Lambda_{(5,1)}(f) = 0,$$

hence $\Lambda(f) = -1$.

For the following two other functions:

$$\tilde{f}' = x^8 + xy^3 + y^{16}, \qquad \tilde{f}'' = x^8 + y^8,$$

we get:

$$\Lambda_{(1,5)}(f') = 1, \quad \Lambda_{(2,2)}(f') = 0, \quad \Lambda_{(5,1)}(f') = 0,$$

hence $\Lambda(f') = 1$, and:

$$\Lambda_{(1,5)}(f'') = 1, \quad \Lambda_{(2,2)}(f'') = 0, \quad \Lambda_{(5,1)}(f'') = 1,$$

hence $\Lambda(f'') = 2$.

It turns out that the number 2 is the maximum of the possible Lefschetz numbers, over all functions in $\mathbf{m}$. Actually, for all the computations above, one can use the forthcoming formula (26), since it gives the answer much more quickly.

We prove Theorem 2.6 using also results from the preceding section. In Section 3 we describe a "nice" resolution of $\mathbf{X}$ to work with and in Section 4 we complete the proof.

3 A toric resolution of the cyclic quotient singularity

3.1 We construct a resolution of the reduced algebraic variety $\mathbf{X} = \mathbf{C}^n/G$, which provides us, of course, with a resolution of the analytic germ $(\mathbf{X}, 0)$.

We build up a *smooth toric variety* $\mathbf{Y}$ together with a morphism $\tilde{\pi} : \mathbf{Y} \to \mathbf{C}^n$ and a commutative diagram:

$$\begin{array}{ccc} \mathbf{Y} & \xrightarrow{\tilde{\pi}} & \mathbf{C}^n \\ {\scriptstyle p'} \downarrow & & \downarrow {\scriptstyle p} \\ \mathbf{X}' & \xrightarrow{\pi} & \mathbf{X} \end{array}$$

such that $\mathbf{X}'$ is a smooth space and $\pi : \mathbf{X}' \to \mathbf{X}$ is a resolution with normal crossings. It was Ehlers [Eh] who constructed resolutions of cyclic quotient singularities, using toric methods. Our construction is related to his, but contains more information, since we want moreover:

(i) a finite group G' and a surjective morphism of groups $\rho : G' \to G$,

(ii) the group G' acting on $\mathbf{Y}$ such that in local charts this is an action of a *reflection group*,

(iii) $\mathbf{X}' = \mathbf{Y}/G'$,

(iv) the morphism $\tilde{\pi}$ be equivariant with respect to ρ.

We refer to [Oda] for some of the terminology we use in the following.

3.2 Definition Let τ be a *strongly convex rational polyhedral cone* in $\mathbf{R}^n$, with vertex at the origin 0, (we shall say only *cone*, for short) and let $\mathcal{P}$ be a finite set of integral vectors contained in τ. We say that a decomposition of τ into polyhedral cones is *generated* by $\mathcal{P}$ if it satisfies the condition: *a* 1*-dimensional cone inside* τ *is a* 1-face *of some polyhedral cone in the decomposition if and only if it is generated by a vector* $v \in \mathcal{P}$.

Let $\mathcal{V} := \mathcal{A}' \cup \{(d, 0, \ldots, 0), (0, d, 0, \ldots, 0), \ldots, (0, \ldots, 0, d)\}$. For n linearly independent vectors $v_i \in \mathbf{Z}^n$ we say that the cone $\mathbf{R}_{\geq 0}\langle v_1, \ldots, v_n\rangle$ is a *cell.* We call moreover such a cell a *k-cell* if the determinant of the $n \times n$ matrix $(v_1, \ldots, v_n)$ is equal to k.

3.3 Proposition *There is a finite family $\mathcal{W}$ of positive integral vectors such that :*

(a) $\mathcal{V} \subset \mathcal{W}$

(b) There is a decomposition of $(\mathbf{R}_{\geq 0})^n$ *into* d^{n-1}*-cells generated by* $\mathcal{W}$.

(c) Any vector $v \in \mathcal{W}$ *is congruent, modulo* d, *to some vector in* $\mathcal{A}$ *or to* 0.

Note For $n = 2$, one can show that there is a (unique) d-cell decomposition generated by $\mathcal{V}$ and it is the most "economical" one (see Section 7 for the precise definitions and statements).

For $n = 3$, many examples show that $\mathcal{V}$ is just sufficient for a d^2-cell decomposition.

Question Is this last fact true in general (at least for $n = 3$)?

Proof (of Proposition 3.3) The idea is the following:
Find a cone $\tau \subset \mathbf{R}^n$ and a matrix $\mathbf{T}$ with integral entries, such that:

i) $\det \mathbf{T} = d^{n-1}$,

ii) $\mathbf{T}$ transforms τ into the positive orthant $(\mathbf{R}_{\geq 0})^n$.

Then find a finite family of integral vectors inside the cone τ so that to generate a 1-cell decomposition. Taking the image by $\mathbf{T}$, we get a d^{n-1}-cell decomposition of $(\mathbf{R}_{\geq 0})^n$.

But this is not enough, since the conditions (a) and (c) are not automatically satisfied; hence τ and $\mathbf{T}$ must be more carefully chosen.

A good candidate for τ is the cone of Ehlers:

$$\tau_E = \mathbf{R}_{\geq 0}\langle w, e_2, \ldots, e_n \rangle,$$

where $w = d \cdot e_1 - \sum_{i=1}^{n-1} p_i e_{i+1}$.

Denote by $\mathbf{I}$ the $(n-1) \times (n-1)$ identity matrix. Consider the matrix

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & . & . & . & 0 \\ p_1 & & & & & \\ . & & & & & \\ . & & & d \cdot \mathbf{I} & & \\ . & & & & & \\ p_{n-1} & & & & & \end{bmatrix}.$$

We have: $\det \mathbf{T} = d^{n-1}$ and $\mathbf{T}(w, e_2, \ldots, e_n) = (d \cdot e_1, \ldots, d \cdot e_n)$, hence conditions i) and ii) are satisfied.

Due to the special form of the matrix $\mathbf{T}$, all the inverse images $\mathbf{T}^{-1}(v)$, for $v \in \mathcal{V}$, are integral vectors inside the cone τ_E.

We choose a decomposition of τ_E generated by the set of vectors $\mathbf{T}^{-1}(\mathcal{V})$ and make a *subdivision* of it into *nonsingular* cones (i.e. 1-cells).

There is an algorithm for that (see [KKMS], but short proofs can be found in [Eh, p. 130] or [Oka, p. 410]). At each step, one introduces a new integral vector which subdivides one of the cones. After a finite number of such operations, one gets a 1-cell decomposition of τ_E together with a set, say $\mathcal{Q}$, of *primitive* vectors which generates it.

Moreover, $\mathcal{Q}$ contains $\mathbf{T}^{-1}(\mathcal{V})$, since $\mathbf{T}^{-1}(\mathcal{V})$ is a set of primitive vectors, as shown by the next lemma.

The conclusion is that $\mathbf{T}(\mathcal{Q})$ is just a family $\mathcal{W}$ we were looking for and a d^{n-1}-cell decomposition of $(\mathbf{R}_{\geq 0})^n$ is the image of the 1-cell decomposition of τ_E as constructed above. Point (c) follows from the definition of $\mathbf{T}$. □

3.4 Since $\mathcal{A}'$ may contain nonprimitive vectors, we need to prove the following:

Lemma *The set $\mathbf{T}^{-1}(\mathcal{V})$ is a set of primitive $\mathbf{Z}_+$-independent elements in $\mathbf{Z}^n$.*

Proof It is sufficient to prove the statement for the subset $\mathbf{T}^{-1}(\mathcal{A}')$ of $\mathbf{T}^{-1}(\mathcal{V})$. Since $\mathbf{T}$ is linear, the $\mathbf{Z}_+$-independence is clear from the definition of $\mathcal{A}'$. Actually, the only thing to be proven is the primitivity. We have:

$$\mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 & . & . & . & 0 \\ -p_1/d & & & & & \\ . & & & & & \\ . & & & 1/d \cdot \mathbf{I} & & \\ . & & & & & \\ -p_{n-1}/d & & & & & \end{bmatrix}.$$

Take $v := (a, ap_1 - i_1 d, \ldots, ap_{n-1} - i_{n-1} d)$ a vector in $\mathcal{A}'$ and suppose that $\mathbf{T}^{-1}(v)$ is not primitive. This means that there is an integral vector v' such that:

i) $v = k \cdot v'$, for some $k > 1$, $k \in \mathbf{N}$, and

ii) $\mathbf{T}^{-1}(v')$ is an integral vector.

Since we have the equality:

$$\mathbf{T}^{-1}(v') = \mathbf{T}^{-1}(\frac{a}{k}, \frac{ap_1}{k} - \frac{i_1 d}{k}, \ldots, \frac{ap_{n-1}}{k} - \frac{i_{n-1}d}{k})^t = (\frac{a}{k}, -\frac{i_1}{k}, \ldots, -\frac{i_{n-1}}{k})^t,$$

we can write the following equalities of integers:

$$a = k \cdot a',\ i_1 = k \cdot i'_1,\ \ldots,\ i_{n-1} = k \cdot i'_{n-1}.$$

Hence the vector v is as follows:

$$v = (k \cdot a',\ k \cdot a'p_1 - k \cdot i'_1 d,\ \ldots,\ k \cdot a'p_{n-1} - k \cdot i'_{n-1}d),$$

which means that

$$\frac{1}{k} \cdot v = (a',\ a'p_1 - i'_1 d,\ \ldots,\ a'p_{n-1} - i'_{n-1}d)$$

is a vector in $\mathcal{A}$. This contradicts the fact that $v \in \mathcal{A}'$. □

3.5 Remarks

(a) For any n vectors from $\mathcal{W}$ the determinant of their $n \times n$ matrix is a multiple of d^{n-1} or is equal to 0.

(b) The image of e_1 by $\mathbf{T}$ (where $e_1 \in \tau_E$) is the vector $(1, p_1, \ldots, p_{n-1}) \in \mathcal{V}$.

3.6 We work only with families $\mathcal{W}$ subject to the following supplementary condition:

($^+$) All the vectors from $\mathcal{W}$, except those along the coordinate axes, are *strictly positive.*

Lemma *There exist families $\mathcal{W}$ which satisfy condition ($^+$) above.*

Proof For any $j \in \{1, \ldots, n\}$ there is exactly one vector $v(j) \in \mathcal{A}$ which has 1 on the position j. This correspondence (which may be not injective) is due to the condition (1): $\gcd(p_i, d) = 1$, $\forall i \in \{1, \ldots, n-1\}$.
Then, for any such j, we have a d^{n-1}-cell:

$$\sigma_j := \mathbf{R}_{\geq 0}\langle d \cdot e_1, \ldots, d \cdot e_{j-1}, v(j), d \cdot e_{j+1}, \ldots, d \cdot e_n\rangle.$$

Since the inverse images by $\mathbf{T}$ of the generators of σ_j generate a 1-cell in τ_E, one can complete a 1-cell decomposition of τ_E starting with the cells $\mathbf{T}^{-1}(\sigma_j)$, $j \in \{1, \ldots, n\}$.

Hence, we have proved that there is a decomposition of $(\mathbf{R}_{\geq 0})^n$ into d^{n-1}-cells, as required by Proposition 3.3, which contains the cells σ_j. It is obvious that the subsequent family $\mathcal{W}$ has the property ($^+$). □

3.7 Let $\mathcal{C} = \mathcal{C}_{\mathcal{W}}$ be the family of the d^{n-1}-cells in a decomposition provided by the proof of the above Lemma 3.6 (where $\mathcal{W}$ has the property (+)) and denote by $\Sigma = \Sigma_{\mathcal{W}}$ the *fan* defined by the same decomposition. Hence we consider each cone from Σ as *generated by vectors from our specific set* $\mathcal{W}$; we shall steadily keep this convention from now on.

Let $\sigma := \mathbf{R}_{\geq 0}\langle v_1, \ldots, v_n\rangle$, $\sigma' := \mathbf{R}_{\geq 0}\langle v'_1, \ldots, v'_n\rangle$ be two elements in $\mathcal{C}$ and define a matrix $\mathbf{G}_{\sigma',\sigma}$ by:

$$\mathbf{G}_{\sigma',\sigma}(v_1, \ldots, v_n)^t = (v'_1, \ldots, v'_n)^t. \tag{5}$$

Lemma *The matrices* $\mathbf{G}_{\sigma',\sigma}$ *have integral entries and* $\det \mathbf{G}_{\sigma',\sigma} = \pm 1$, *for any* $\sigma, \sigma' \in \mathcal{C}$.

Proof It is enough to prove the statement for "neighbour" $\sigma, \sigma' \in \mathcal{C}$. So suppose $\sigma = \mathbf{R}_{\geq 0}\langle v_1, v_2, \ldots, v_n\rangle$, $\sigma' = \mathbf{R}_{\geq 0}\langle v'_1, v_2, \ldots, v_n\rangle$. Then $\mathbf{G}_{\sigma',\sigma}$ looks as follows:

$$\begin{bmatrix} \alpha_1 & \cdot & \cdot & \cdot & \alpha_n \\ 0 & & & & \\ \cdot & & & & \\ \cdot & & \mathbf{I} & & \\ \cdot & & & & \\ 0 & & & & \end{bmatrix}.$$

Hence

$$\alpha_1 = \frac{\det(v'_1, v_2, \ldots, v_n)}{\det(v_1, v_2, \ldots, v_n)} = -1$$

and $\alpha_i = \det(v_1, \ldots, v_{i-1}, v'_1, v_{i+1}, \ldots, v_n)/\det(v_1, \ldots, v_n)$, for $i \in \{2, \ldots, n\}$, have integral values, as a consequence of our Remark 3.5(a). The sign of α_1 is indeed minus, since the two *simplices* $(v_1, v_2, \ldots, v_n)$ and $(v'_1, v_2, \ldots, v_n)$ have opposite orientations. Note also that the matrix $\mathbf{G}_{\sigma',\sigma}$ is equal to its inverse. □

3.8 We use the family $\mathcal{C}$ to construct a *smooth* variety in a similar way one constructs a resolution space starting with a Newton polyhedron of a sufficiently general function, see [Var], [Oka]. For some of the following notations we refer to [Eh].

For $\sigma, \sigma' \in \mathcal{C}$, denote by $\mathbf{C}^n(\sigma)$, $\mathbf{C}^n(\sigma')$ two copies of $\mathbf{C}^n$ and define a morphism $g_{\sigma',\sigma}$ from an open subset of $\mathbf{C}^n(\sigma)$ to $\mathbf{C}^n(\sigma')$ by the rule:

$$g_{\sigma',\sigma} : z \mapsto z^{\mathbf{G}^*_{\sigma',\sigma}},$$

where $\mathbf{G}^*_{\sigma',\sigma} := (\mathbf{G}^{-1}_{\sigma',\sigma})^t$.

The domain of $g_{\sigma',\sigma}$ is:

$$\operatorname{def}(g_{\sigma',\sigma}) := \{(z_1,\ldots,z_n) \in \mathbf{C}^n(\sigma) \mid z_k \neq 0 \text{ for some } k \text{ if } (\mathbf{G}^*_{\sigma^2,\sigma^1})_{ik} < 0 \text{ for some } i\}.$$

We define a manifold $\mathbf{Y} := \mathbf{Y}_{\mathcal{C}}$ with local charts $\mathbf{C}^n(\sigma)$, for $\sigma \in \mathcal{C}$, and the transition functions $g_{\sigma',\sigma}$ as above. We show that $\mathbf{Y}_{\mathcal{C}}$ is the needed variety $\mathbf{Y}$, as denoted in 3.1.

3.9 Define a projection:

$$\tilde{\pi} : \mathbf{Y}_{\mathcal{C}} \to \mathbf{C}^n,$$

which in local charts looks as follows:

$$\tilde{\pi}_\sigma : \mathbf{C}^n(\sigma) \to \mathbf{C}^n, \qquad z \to z^{[\sigma]},$$

that is:

$$\begin{cases} x_1 = z_1^{v_1^1} \cdot z_2^{v_2^1} \cdots z_n^{v_n^1} \\ \vdots \\ x_n = z_1^{v_1^n} \cdot z_2^{v_2^n} \cdots z_n^{v_n^n} \end{cases} \tag{6}$$

where $\sigma = \mathbf{R}_{\geq 0}\langle v_1, v_2, \ldots, v_n\rangle$.

One can see that:

(a) Outside the union of the coordinate hyperplanes in $\mathbf{C}^n$, $\tilde{\pi}$ is a topological covering of degree d^{n-1}. Moreover, this union is precisely the *discriminant* of $\tilde{\pi}$.

The restriction of $\tilde{\pi}$ to $\tilde{\pi}^{-1}(\mathbf{C}^n \setminus \{0\})$ is an analytic ramified covering.

(b) The projection $\tilde{\pi}$ is a proper algebraic morphism. The only fact which is still to prove is that the fibre $\tilde{\pi}^{-1}(0)$, *the exceptional divisor of* $\tilde{\pi}$, is a compact divisor in $\mathbf{Y}_{\mathcal{C}}$.

This follows from similar arguments as those in the theory of toric varieties, see e.g. [Oda, p.16].

(c) The group $G' := \underbrace{\mu_d \times \cdots \times \mu_d}_{n} \subset (\mathbf{C}^*)^n$ acts on $\mathbf{Y}_{\mathcal{C}}$ such that there is a natural commutative diagram:

$$\begin{array}{ccc} \mathbf{Y}_{\mathcal{C}} & \xrightarrow{\tilde{\pi}} & \mathbf{C}^n \\ \downarrow & & \downarrow \\ \mathbf{Y}_{\mathcal{C}}/G' & \xrightarrow{\pi} & \mathbf{C}^n/G. \end{array}$$

The quotient variety is smooth since the group $\mu_d \times \cdots \times \mu_d$ acts as a *reflection group* in each coordinate chart:

$$\begin{array}{ccc} \mathbf{C}^n(\sigma) & \xrightarrow{\tilde{\pi}_\sigma} & \mathbf{C}^n \\ \downarrow & & \downarrow \\ \mathbf{C}^n(\sigma)/\mu_d \times \cdots \times \mu_d & \xrightarrow{\pi_\sigma} & \mathbf{C}^n/G, \end{array} \tag{7}$$

where the projection from the left is equivalent to the morphism:

$$\mathbf{C}^n \to \mathbf{C}^n$$

$$(z_1, \ldots, z_n) \mapsto (z_1^d, \ldots, z_n^d) \tag{8}$$

Proof of (c). We take any cone $\sigma \in C$ and show that the map $\tilde{\pi}_\sigma$ is equivariant with respect to a certain canonical surjective morphism of groups $\rho_\sigma : \mu_d \times \cdots \times \mu_d \to G$.

In the following we identify μ_d with $\mathbf{Z}_d$ by fixing a primitive d-root of unity ξ, i.e. ξ^a is identified to a, for $0 \leq a \leq d-1$. We use also the identification defined at 1.1 (1).

Let $\sigma = \mathbf{R}_{\geq 0}\langle v_1, \ldots, v_n\rangle$. If $(a_1, \ldots, a_n) \in \mathbf{Z}_d \times \cdots \times \mathbf{Z}_d$ then, according to the relation (6), we define $\rho_\sigma(a_1, \ldots, a_n) = a_1 v_1 + \cdots + a_n v_n$ and this last vector is congruent (modulo d) to some vector in $\mathcal{A}$ or is zero, hence it is an element of G. As ρ_σ is obviously a morphism, we have only to see that it is onto. It will be sufficient, by Lemma 3.7, to prove this for a particular cone, say $\sigma_0 = \mathbf{R}_{\geq 0}\langle g,\ d \cdot e_2, \ldots,\ d \cdot e_n\rangle$, where $g = (1, p_1, \ldots, p_{n-1})$. Then $\rho_{\sigma_0}(1, 0, \ldots, 0) = g$, hence ρ_σ is onto, since g is a generator of our G-action.

All other details (like the gluing of ρ_σ's) are straightforward.

Since the morphism $\tilde{\pi}$ is equivariant with respect to ρ, it induces indeed a morphism π which completes the diagram above. □

Define $\mathbf{X}' := \mathbf{Y}_C/G'$. Then we have:

3.10 Theorem *The morphism $\pi : \mathbf{X}' \longrightarrow \mathbf{X}$ is a resolution with normal crossings.*

Proof The space $\mathbf{Y}_C$ is smooth and, by 3.9(b), the morphism π is proper. One can check in coordinate charts, like in the proof of (2.8)(c), that π is one-to-one over $\mathbf{C}^n \setminus \{0\}/G$. The normal crossings property follows from the fact that $\tilde{\pi}^{-1}(0)$ is itself a divisor with normal crossings. Locally, $\tilde{\pi}^{-1}(0)$ is the union of some coordinate hyperplanes and the projection (8) preserves this property. □

3.11 Let $\mathbf{E} := (\pi)^{-1}(0)$ be the exceptional divisor of the resolution π and let $\mathbf{E} = \bigcup_{i \in I_\pi} \mathbf{E}_i$ be its decomposition into irreducible compact components, where I_π is a finite set of indices.

Denote by $\mathcal{W}_+$ the subset of $\mathcal{W}$ of strictly positive vectors. Then:

$$\mathcal{W} = \mathcal{W}_+ \cup (\mathcal{V} \setminus \mathcal{A}').$$

It is a standard fact that there is a one-to-one correspondence among strictly positive vectors $v \in \mathcal{W}_+$ and compact components $\mathbf{E}_i$, $i \in I_\pi$; hence:

$$|I_\pi| = |\mathcal{W}_+| \geq |\mathcal{A}'|.$$

From now on we replace I_π by $\mathcal{W}_+$ and write $\mathbf{E}_v$ instead $\mathbf{E}_i$.

If $p' : \mathbf{Y}_\mathcal{C} \to \mathbf{X}' = \mathbf{Y}_\mathcal{C}/G'$ denotes the quotient projection then denote by $\tilde{\mathbf{E}}_v$ the irreducible component $(p')^{-1}(\mathbf{E}_v)$ of the exceptional divisor $\tilde{\mathbf{E}} := \tilde{\pi}^{-1}(0)$. It follows from 3.9(c) that, indeed, any irreducible component of $\tilde{\mathbf{E}}$ is the inverse image by p' of some $\mathbf{E}_v$, for $v \in \mathcal{W}_+$.

3.12 Note The toric varieties $\mathbf{X}'$ and $\mathbf{Y}_\mathcal{C}$ are in fact two avatars of the same space.

The proof of this fact goes as follows. We use the notations from [Oda]. If $\mathbf{Y}_\mathcal{C} \simeq T_N \operatorname{emb}(\Delta)$, i.e. the toric variety associated to a lattice $N = \mathbf{Z}\langle e_1, \ldots, e_n \rangle$ and a fan Δ, then, by our construction, $\mathbf{X}' \simeq T_{\frac{1}{d}N} \operatorname{emb}(\Delta)$, where $\frac{1}{d}N$ is the overlattice of N generated by $\frac{1}{d}e_1, \ldots, \frac{1}{d}e_n$. The projection $T_N \operatorname{emb}(\Delta) \to T_{\frac{1}{d}N} \operatorname{emb}(\Delta)$ is induced by the inclusion $N \subset \frac{1}{d}N$.

On the other hand, the natural linear isomorphism $N \simeq \frac{1}{d}N$ (which sends e_i to $\frac{1}{d}e_i$) extends to an isomorphism of fans $\phi : (N, \Delta) \to (\frac{1}{d}N, \Delta)$.

This gives rise to an isomorphism:

$$\phi_* : T_N \operatorname{emb}(\Delta) \xrightarrow{\sim} T_{\frac{1}{d}N} \operatorname{emb}(\Delta). \tag{9}$$

4 The Lefschetz number of functions on isolated cyclic quotients

4.1 We keep the notations from Section 3. The decomposition $\mathcal{C}$ and the family $\mathcal{W}$ are fixed, such that $\mathcal{W}$ satisfies the condition $(^+)$ in 3.6.

Let $\mathbf{E} = \bigcup_{v \in \mathcal{W}_+} \mathbf{E}_v$ be the exceptional divisor of π and $\tilde{\mathbf{E}} = \cup_{v \in \mathcal{W}_+} \tilde{\mathbf{E}}_v$ the one of $\tilde{\pi}$, as defined in 3.11. The construction of the "nice" resolution in Section 3 plays here an important role; actually, it provides all the data we need for the Lefschetz number.

4.2 Lemma *If $v \in \mathcal{W} \setminus \mathcal{V}$ then at least one of the components of v is greater than d.*

Proof The vector v is the image by $\mathbf{T}$ of some integral vector $(a_1, \ldots, a_n)$ with $a_1 > 0$. This means:

$$v = (a_1, p_1 a_1 + d a_2, \ldots, p_{n-1} a_1 + d a_n).$$

Since v is positive and congruent (modulo d) to some vector in $\mathcal{A}$ (see Proposition 3.3(c)), the conclusion follows. □

4.3 To each component $\mathbf{E}_v$, one associates a filtration $\mathcal{F}(\mathbf{E}_v)^\bullet$ as in 1.10. We use also the valuation $\mathrm{ord}_{\tilde{\mathbf{E}}_v}$ on $\mathcal{O}_{\mathbf{Y}_{\mathcal{C}} | \tilde{\mathbf{E}}_v}$ (see 1.6).

On the other hand, we have defined in 2.4 the filtration $\mathcal{G}(v)^\bullet$ associated to a strictly positive vector v.

Lemma *If $v \in \mathcal{W}_+$ then:*

$$\mathcal{F}(\mathbf{E}_v)^\bullet = \mathcal{G}(v)^\bullet.$$

Proof We can prove our equality by considering the morphism $\tilde{\pi}$ in local charts (7), where, for $f \in \mathbf{m}$, we have almost by definition:

$$o_v(\tilde{f}) = \mathrm{ord}_{\tilde{\mathbf{E}}_v}(\tilde{f} \circ \tilde{\pi}).$$

Passing to π means to divide by d each member of this last equality. □

4.4 Proof of Theorem 2.6(a)
We use here Proposition 1.12. As we need those exceptional divisors for which the multiplicity is 1, we may restrict our attention to the subfamily $\mathcal{A}' \subset \mathcal{W}$. Indeed, by Lemmas 4.2 and 4.3, if $v \in \mathcal{W} \setminus \mathcal{V}$ then $\mathrm{ord}_{\mathbf{E}_v}(f \circ \pi) \geq 2$. This proves the part (a) of our theorem.

Moreover, Proposition 1.12 together with the observation just above prove that the Lefschetz number formula (Theorem 1.4(a)) becomes:

$$\Lambda(f) = \sum_{v \in \mathcal{A}',\ o_v(\tilde{f})=d} \chi(\mathbf{E}_v^*). \tag{10}$$

□

4.5 The "nice" resolution π enables us to get more information about the Euler characteristics $\chi(\mathbf{E}_v^*)$, hence about $\Lambda(f)$.

A general fact is that a toric variety is a disjoint union of torus orbits. In other words, there is a stratification of the variety such that the strata are tori of various dimensions, (see [KKMS], [Oda], [Var]). One of them has dimension n and the others have smaller dimensions.

There is a bijective correspondence between l-dimensional tori and $(n-l)$-dimensional cones of the *fan* which defines the toric variety. We describe this correspondence for $l = n-1$ and $l = n-2$, for details consult *loc. cit.*

Let $v \in \mathcal{W}$ and $\sigma \in \mathcal{C}$ such that $v \in \sigma$. Suppose $\sigma = \mathbf{R}_{\geq 0}\langle v_1, \dots, v_n\rangle$, with $v = v_1$. Then the torus $T_v \simeq (\mathbf{C}^*)^{n-1}$ which corresponds to v is, in the local chart $\mathbf{C}^n(\sigma)$:

$$T_v := \{z \in \mathbf{C}^n(\sigma) \mid z_1 = 0,\ z_i \neq 0 \text{ for } i = 2, \dots, n\}.$$

The torus $T_{\langle v_1, v_2\rangle} \simeq (\mathbf{C}^*)^{n-2}$ which corresponds to the cone $\mathbf{R}_{\geq 0}\langle v_1, v_2\rangle \subset \sigma$ will be:

$$T_{\langle v_1, v_2\rangle} := \{z \in \mathbf{C}^n(\sigma) \mid z_1 = z_2 = 0,\ z_i \neq 0 \text{ for } i = 3, \dots, n\}.$$

One shows that the tori do not depend on the local chart.

4.6 Let $\mathcal{S}$ denote the family of all the toric strata in the stratification of $\mathbf{Y}_{\mathcal{C}}$ as above. The following result enables us to compute $\chi(\tilde{\mathbf{E}}_v^*)$:

Lemma *For any $v \in \mathcal{W}_+$ we have:*

$$\chi(\tilde{\mathbf{E}}_v^*) = \sum_{T \in \mathcal{S}} \chi(T \cap \tilde{\mathbf{E}}_v^*). \tag{11}$$

Proof This follows from a general fact about Whitney stratifications. One computes the Euler characteristic of a compact, Whitney stratified space $\mathcal{Z}$, by using a compatible triangulation of it (see [GM] for the references about the Whitney stratifications and triangulability).

Since the strata have even dimensions (we are in the complex case), the Euler characteristic of $\mathcal{Z}$ is the sum of the Euler characteristics of all its strata. □

4.7 The space $\mathbf{X}'$ is also a toric variety and its decomposition into tori follows the one of $\mathbf{Y}_{\mathcal{C}}$, because of the diagram (7).

For instance, the torus T_v goes, by the quotient-projection $p' : \mathbf{Y}_{\mathcal{C}} \to \mathbf{X}'$, to the torus $\hat{T}_v$, where:

$$\hat{T}_v \simeq (\mathbf{C}^*)^{n-1}/\mu_d \times \cdots \times \mu_d \simeq (\mathbf{C}^*)^{n-1}.$$

Of course, the analogue of Lemma 4.6 (for $\mathbf{X}'$, $\mathbf{E}_v^* \ldots$) is true; we get the equality:

$$\chi(\mathbf{E}_v^*) = \sum_{T \in \mathcal{S}} \chi(\hat{T} \cap \mathbf{E}_v^*). \tag{12}$$

To compare $\chi(\mathbf{E}_v^*)$ with $\chi(\tilde{\mathbf{E}}_v^*)$, one can use the formulae (11) and (12) together with the following important:

Lemma *Let T be a torus stratum in the stratification of $\mathbf{Y}_{\mathcal{C}}$ and $\hat{T} := p'(T)$ be its projection. Let $v \in \mathcal{W}_+$. Then:*

$$\chi(\hat{T} \cap \mathbf{E}_v) = d^{-\dim T} \cdot \chi(T \cap \tilde{\mathbf{E}}_v).$$

Proof Let $T = T_{\sigma_0}$, for some $\sigma_0 \in \Sigma$ and let $k := \dim T$. If $v \notin \sigma_0$ then the Euler numbers are zero.

If $v \in \sigma_0$, let $\sigma \in \mathcal{C}$ be some cone such that $\sigma_0 \subset \sigma$. In the local chart $\mathbf{C}^n(\sigma)$, the projection p' is equivalent to the morphism (see 3.9(c)):

$$\mathbf{C}^n \to \mathbf{C}^n, \quad (z_1, \ldots, z_n) \mapsto (z_1^d, \ldots, z_n^d)$$

and its restriction to the torus $T \subset \mathbf{C}^n(\sigma)$ is (up to some permutation of coordinates):

$$T = (\mathbf{C}^*)^k \to (\mathbf{C}^*)^k, \quad (z_1, \ldots, z_k) \mapsto (z_1^d, \ldots, z_k^d).$$

The conclusion follows, since this last morphism is a topological d^k-covering. □

A consequence of the formula (10) and Lemma 4.7 above is the following:

4.8 Corollary *For any $f \in \mathbf{m}$, the Lefschetz number of the monodromy of f is given by:*

$$\Lambda(f) = \sum_{T \in \mathcal{S}} \sum_{v \in \mathcal{A}'} d^{-\dim T} \cdot \chi(T \cap \tilde{\mathbf{E}}_v^*). \tag{13}$$

□

We prove some criteria for the annihilation of the Euler characteristic, useful not only to simplify the computation of the Lefschetz number, but also for the proof of Theorem 2.6(b).

4.9 Each vector $v \in \mathcal{A}'$ is normal to a corresponding hyperplane H_v in $\mathbf{R}^n$, defined by the equation:

$$v \cdot (x_1, \ldots, x_n)^t = d.$$

Let f_v be the canonical representative (see 2.4) for the residue class of $f \in \mathbf{m}$ in $\mathrm{Gr}_1\, \mathcal{G}(v)$. This might be zero, of course. If it is not, then $\tilde{f}_v$ is a linear combination of G-invariant monomials x^m, where m is represented by an integral lattice point in the hyperplane H_v.

4.10 Let $v \in \mathcal{W}_+$. The decomposition of $\tilde{\mathbf{E}}_v$ into tori induces a decomposition of $\tilde{\mathbf{E}}_v^0 := \tilde{\mathbf{E}}_v \setminus \bigcup_{w \in \mathcal{W}_+,\, w \neq v} \tilde{\mathbf{E}}_w$ into tori and these tori are exactly the ones of the form (see [Var]):

(*) $\quad T_{\langle v_1, \ldots, v_k \rangle}, \quad k \geq 1$, where one of the vectors, say v_1, is equal to v and the others are of the form de_i, for some indices i.

Keeping the previous convention, we denote by $\hat{T}_{\langle v_1, \ldots, v_k \rangle}$ the corresponding torus in the decomposition of $\mathbf{E}_v^0$. We recall that, by definition, the cone $\mathbf{R}_{\geq 0}\langle v_1, \ldots, v_k \rangle$ must be contained in some $\sigma \in \mathcal{C}$.

Let $m_1, \ldots, m_n$ be the coordinates on $\mathbf{R}^n$ and let $\Gamma_{\tilde{f}}$ be the Newton polygon of $\tilde{f}$.

4.11 Proposition *Let $T = T_{\langle v, de_{j_1}, \ldots, de_{j_{k-1}} \rangle}$ be an $(n-k)$-dimensional torus of the form (*), $k \geq 1$. If the dimension of the set $\Gamma_{\tilde{f}} \cap H_v \cap \{m_{j_i} = 0 \mid i \in \{1, \ldots, k-1\}\}$ is less than $(n-k)$ then:*

$$\chi(\hat{T} \setminus \{\widetilde{f_v = 0}\}) = 0.$$

In particular, if $\dim(\Gamma \cap H_v \cap \{m_{j_i} = 0 \mid i \in \{1, \ldots, k-1\}\}) < n - k$ *then* $\chi(\hat{T}_v \setminus \{\widetilde{f_v = 0}\}) = 0$*, for any $f \in \mathbf{m}$.*

Proof The equality we have to prove is equivalent, by Lemma 4.7, to:

$$\chi(T \setminus \{\widetilde{\tilde{f}_v = 0}\}) = 0.$$

It is sufficient to prove this relation in some local chart $\mathbf{C}^n(\sigma)$, where $\sigma \in \mathcal{C}$ such that $\sigma = \mathbf{R}_{\geq 0}\langle v, de_{j_1}, \ldots, de_{j_{k-1}}, v_k, \ldots, v_n\rangle$. We use here the notations from 4.5.

If $\dim(\Gamma_{\tilde{f}} \cap H_v \cap \{m_{j_i} = 0 \mid i \in \{1, \ldots, k-1\}\}) < n-k$ then $\Gamma_{\tilde{f}_v} \cap H_v \cap \{m_{j_i} = 0 \mid i \in \{1, \ldots, k-1\}\}$ is contained in some affine subspace of $H_v \cap \{m_{j_i} = 0 \mid i \in \{1, \ldots, k-1\}\}$. This implies that the pull-back by $\tilde{\pi}_\sigma$ induces a $\mathbf{C}^*$-action on $\{\widetilde{\tilde{f}_v = 0}\} \cap \{z_1 = \cdots = z_k = 0\}$. More concretely, the polynomial $\tilde{f}_{v,\sigma}$ defined as in 5.12 has support on an affine subspace of $\mathbf{C}^n(\sigma) \cap \{z_1 = \cdots = z_k = 0\} \simeq \mathbf{C}^{n-k}$; the coefficients (we may assume that all of them are integral) of the equation of this affine subspace give the weights with respect to which $\tilde{f}_{v,\sigma}$ is a quasihomogeneous polynomial.

Moreover, the $\mathbf{C}^*$-action on $\{\widetilde{\tilde{f}_v = 0}\} \cap \{z_1 = \cdots = z_k = 0\}$ induces a $\mathbf{C}^*$-action on the complement $(\mathbf{C}^n(\sigma) \cap \{z_1 = \cdots = z_k = 0\}) \setminus \{\widetilde{\tilde{f}_v = 0}\}$. This one restricts to a free $\mathbf{C}^*$-action on $T \setminus \{\widetilde{\tilde{f}_v = 0}\}$.

The proposition follows, since the Euler characteristic of a $\mathbf{C}^*$-bundle is zero. □

An easy consequence, sometimes useful in computations, is the following (the statement involves the Newton polyhedron Γ defined in 4.13):

Corollary *Let $v \in \mathcal{A}'$. If $\Gamma_{\tilde{f}} \cap H_v$ is not a maximal face of Γ and $\Gamma_{\tilde{f}} \cap H_v$ does not intersect any coordinate hyperplane of $\mathbf{R}^n$ then $\Lambda_v(f) = 0$.*

Proof If $\Gamma_{\tilde{f}} \cap H_v$ is not a maximal face of Γ then $\chi(\hat{T}_w \setminus \{\widetilde{f_v = 0}\}) = 0$, $\forall w \in \mathcal{A}'$, by Proposition 4.11. If $\Gamma_{\tilde{f}} \cap H_v$ does not intersect any coordinate hyperplane then, for any torus T of the form (*), $\dim T < n-1$, we have $\hat{T} \cap \{\widetilde{f_v = 0}\} = \hat{T}$, hence the Euler number is zero. The conclusion follows from (12) and (10), where we replace f by f_v. □

4.12 Proof of Theorem 2.6(b)

We start from the formula (10):

$$\Lambda(f) = \sum_{v \in \mathcal{A}',\ o_v(\tilde{f}) = d} \chi(\mathbf{E}_v^*)$$

and prove that:

$$\Lambda_v(f) = \chi(\mathbf{E}_v^*), \tag{14}$$

for any $v \in \mathcal{A}'$ such that $f_v \neq 0$.

Denote by $T^{(v)}$ some torus in the decomposition of $\tilde{\mathbf{E}}_v^0$ described in 4.10.

For any such $T^{(v)}$ we have obviously: $\chi(T^{(v)} \setminus \{\widetilde{\tilde{f} = 0}\}) = \chi(T^{(v)} \setminus \{\widetilde{\tilde{f}_v = 0}\})$, hence:

$$\chi(\tilde{\mathbf{E}}_v^*) = \chi(\tilde{\mathbf{E}}_v^0 \setminus \{\widetilde{\tilde{f}_v = 0}\}).$$

Next, we prove that, for any $w \in \mathcal{W}_+$, $w \neq v$ such that $o_w(\tilde{f}_v) = d$ and any torus $T^{(w)}$ in the decomposition of $\tilde{\mathbf{E}}_w^0$, we have:

$$\chi(T^{(w)} \setminus \{\widetilde{\tilde{f}_v = 0}\}) = 0, \quad \text{if } \dim T^{(w)} \geq 1. \tag{15}$$

Let $T^{(w)} := T_{(w, de_{i_1}, \ldots, de_{i_k})}$, for some $k \in \mathbf{N}$, $0 \leq k < n-1$.

If $\dim(\Gamma_{\tilde{f}_v} \cap H_w \cap \{m_j = 0 \mid j \in \{i_1, \ldots, i_k\}\})$ is less than $\dim T^{(w)}$, then the equality (15) is proved by Proposition 4.11. We prove that the situation $\dim(\Gamma_{\tilde{f}_v} \cap H_w \cap \{m_j = 0 \mid j \in \{i_1, \ldots, i_k\}\}) = \dim T^{(w)}$ cannot occur.

The vectors v and w are both normal to the face $\Gamma_{\tilde{f}_v} \cap H_w \cap \{m_j = 0 \mid j \in \{i_1, \ldots, i_k\}\}$ of $\Gamma_{\tilde{f}_v}$. Then the orthogonal projections v', resp. w', of the vectors v, resp. w, to the subspace $\{m_j = 0 \mid j \in \{i_1, \ldots, i_k\}\} \subset \mathbf{R}^n$ define the same direction; hence either $v' = l_1 w'$ for some integer $l_1 \geq 2$, or $w' = l_2 v'$ for some integer $l_2 \geq 2$. Any of the two equalities contradicts the assumption that $o_v(\tilde{f}_v) = o_w(\tilde{f}_v) = d$.

There is one case left: $T^{(w)}$ is one point. This can happen only if $w = v(i)$, for some $i \in \{1, \ldots, n\}$ (see the definition of $v(i)$ in 3.6), hence $T^{(w)} = T_{(v(i), de_1, \ldots, \hat{de_i}, \ldots, de_n)}$. Then $T^{(w)} \cap \{\widetilde{\tilde{f}_v = 0}\} \neq \emptyset$ if and only if x_i^d is one of the monomials in the polynomial $\tilde{f}_v$; this can occur only if $v = v(i)$. Since $w \neq v$, we get again $\chi(T^{(w)} \setminus \{\widetilde{\tilde{f}_v = 0}\}) = 0$.

We apply Lemma 4.7 to translate all the achieved information to the space $\mathbf{X}'$; we have proved by now the equality (14).

Finally, if $f_v = 0$, then $\Lambda_v(f_v) = 0$, by definition. On the other hand, $\chi(\mathbf{E}_v^*)$ does not enter in the sum (10), since $o_v(\tilde{f}) > d$. This completes the proof of our theorem.

□

We describe an immediate consequence of Theorem 2.6.

4.13 Definition Let $S := \mathbf{C}[x_1, \ldots, x_n]$ and consider the convex hull Γ_+ of all the points $m \in (\mathbf{Z}_{\geq 0})^n$, $m \neq 0$, such that $x^m \in S$ is a G-invariant monomial (where x^m stays for $x_1^{m_1} \cdots x_n^{m_n}$). Let Γ be the union of all compact faces of Γ_+. We say that Γ is the *Newton polyhedron* associated to the action of the group G.

Then Γ is also the Newton polyhedron of $\tilde{f}_0$, where the residue class of $f_0 \in \mathbf{m}$ in $\mathbf{m}/\mathbf{m}^2$ is a linear combination, with nonzero coefficients, of the elements of the canonical basis.

4.14 Definition Let $f \in \mathbf{m}$ and let $\tilde{f}$ be the corresponding G-invariant function, where $\tilde{f} = \sum_{m \in \mathbf{N}^n} a_m x^m$. We say that $\tilde{f}_\Gamma$ is the Γ-*principal part* of $\tilde{f}$, where:

$$\tilde{f}_\Gamma := \sum_{m \in \Gamma \cap \mathbf{N}^n} a_m x^m.$$

Let f_Γ be the corresponding function on $\mathbf{X}$. We also say that f_Γ is the Γ-principal part of f.

4.15 Let $\mathcal{N}^\bullet$ be the *Newton filtration* (see [Ku, p.10]) with respect to Γ.

Corollary *The Lefschetz number* $\Lambda(f)$ *depends only on* f_Γ. *In particular, if* $\tilde{f} \in \mathcal{N}^2$ *then* $\Lambda(f) = 0$.

Proof This follows from Theorem 2.6 and the fact that:

$$\mathcal{N}^2 \subseteq \cap_{v \in \mathcal{W}_+} \mathcal{G}(v)^2. \tag{16}$$

□

The inclusion (16) can be strict, as shown in the next:

Example Take $n = 3$, $d = 11$ and generator $(1, 5, 7)$. The monomial x^3y^2z belongs to $\bigcap_{v \in \mathcal{W}_+} \mathcal{G}(v)^2$ but not to $\mathcal{N}^2$. See Example 6.3.

4.16 Remarks

(a) For any $v \in \mathcal{W}_+$ we have:

$$\mathbf{m}^2 \subseteq \mathcal{G}(v)^2. \tag{17}$$

This follows from Lemma 1.13 and Lemma 4.3. It also follows from (16) and the observation that: $\mathbf{m}^2 \subseteq \mathcal{N}^2$. In particular, we see again that the Lefschetz number depends only on the residue class of f in $\mathbf{m}/\mathbf{m}^2$.

(b) Let $v \in \mathcal{A}'$. For any $g \in \mathbf{m}$ such that $g - f_v \in \mathbf{m}^2$, we have: $\Lambda_v(f) = \Lambda(g)$. This follows from Theorem 2.6(a) and the remark (a) above.

5 Nondegenerate functions

5.1 We keep the previous notations. Let $f \in \mathbf{m}$ and its corresponding G-invariant function $\tilde{f}$. Let h_f, resp. $h_{\tilde{f}}$, denote the algebraic monodromy of f, resp. $\tilde{f}$.

One would like to relate the invariants of $h_{\tilde{f}}$ to the corresponding invariants of h_f by as nice as possible formulae. The projection p induces a topological d-covering from the Milnor fibre $F_{\tilde{f}}$ of $\tilde{f}$ to the Milnor fibre F_f of f. This leads to the conclusion that the Euler numbers are related by the formula:

$$\chi(F_f) = \frac{1}{d} \cdot \chi(F_{\tilde{f}}).$$

If we consider the Lefschetz number or the zeta-function of the monodromy then the possible relations might be more complicated. Our aim is to define a "good" class of functions for which we would be able to give explicit and computable formulae for these two invariants. This class should include the class of *general functions*, defined below (Definition 5.2).

It will turn out that an equality like $\zeta_{\tilde{f}_0}(t) = \zeta_{f_0}(t^d)$ is not true for any group G, even if f_0 is a general function in $\mathbf{m}$.

5.2 Let g be a function on $(\mathbf{C}^n, 0)$ and denote by Γ_g its Newton polyhedron. Kushnirenko and other authors considered functions which have *nondegenerate Newtonian principal parts* [Ku]. We shall call such functions only *nondegenerate*, for short.

Our situation here is different, since a Newton polyhedron of a function $f \in \mathbf{m}$ is not defined. However, we have, on the one hand, the cyclic quotient space with its associated Newton polyhedron Γ (cf. Definition 4.13) and, on the other hand, the G-invariant function $\tilde{f}$ and its Newton polyhedron $\Gamma_{\tilde{f}}$.

To define a sufficiently good class of functions on $(\mathbf{X}, 0)$, somehow similar to the previous class of nondegenerate functions, we have to impose additional conditions involving the action of the group G.

Definition We say that a linear function $f_0 \in \mathbf{m}$ — in the sense that f_0 is the restriction of a linear germ $(\mathbf{C}^N, 0) \to (\mathbf{C}, 0)$ — is *general* if $\Gamma_{\tilde{f}_0} = \Gamma$ and $\tilde{f}_0$ is nondegenerate.

Notice that the general functions yield a Zariski-open subset in the space of linear functions on $(\mathbf{X}, 0)$.

We recall that, for any $f \in \mathbf{m}$, we have the inclusion $\Gamma_{\tilde{f}} \subset \Gamma_+$ and that, by Corollary 4.15, only the Γ-principal part $\tilde{f}_\Gamma$ of $\tilde{f}$ counts for the Lefschetz number $\Lambda(f)$. We may start from this observation and from the definition of nondegeneracy used by Kushnirenko [Ku]:

5.3 Definition Let $\Delta \subset (\mathbf{R}_{\geq 0})^n$ be a compact subspace of $\mathbf{R}^n$. Let $g \in \mathbf{C}\{x_1, \ldots, x_n\}$ be a function on $(\mathbf{C}^n, 0)$ with Taylor series expansion $g = \sum_{m \in \mathbf{N}^n} a_m x^m$ and define $g_\Delta = \sum_{m \in \Delta \cap \mathbf{N}^n} a_m x^m$.

One says that g is *nondegenerate on* Δ if the polynomials:

$$(x_1 \frac{\partial g_\Delta}{\partial x_1}), \ldots, (x_n \frac{\partial g_\Delta}{\partial x_n})$$

have no common zero in $\{x \in \mathbf{C}^n \mid x_1 \cdots x_n \neq 0\}$.

5.4 Fix once and for all a decomposition $\mathcal{C}$ and a family $\mathcal{W}$ as in Lemma 3.6. Let $f \in \mathbf{m}$ and define for each $v \in \mathcal{W}_+$ a hyperplane $H_v(f)$ by the equation:

$$v \cdot (x_1, \ldots, x_n)^t = o_v(\tilde{f}). \tag{18}$$

Denote by $H_v(f)_+$ the half-space in $\mathbf{R}^n$ defined by:

$$v \cdot (x_1, \ldots, x_n)^t \geq o_v(\tilde{f}).$$

We have defined in 4.9 a hyperplane H_v, for any $v \in \mathcal{A}'$. Denote by $(H_v)_+$ the half-space defined by:

$$v \cdot (x_1, \ldots, x_n)^t \geq d.$$

We define two new polyhedra, associated to $(\Gamma, \Gamma_{\tilde{f}})$, respectively to Γ:

5.5 Definition Let $\Gamma_{\tilde{f},\mathcal{W}}$, respectively $\Gamma_{\mathcal{A}'}$, be the union of the compact faces of the infinite convex body:

$$(\Gamma_{\tilde{f},\mathcal{W}})_+ := (\mathbf{R}_{\geq 0})^n \cap \bigcap_{v \in \mathcal{W}_+} H_v(f)_+,$$

respectively:

$$(\Gamma_{\mathcal{A}'})_+ := (\mathbf{R}_{\geq 0})^n \cap \bigcap_{v \in \mathcal{A}'} (H_v)_+.$$

Note that $\Gamma_{\tilde{f}} \subset (\Gamma_{\tilde{f},\mathcal{W}})_+ \subset (\Gamma_{\mathcal{A}'})_+$ for any $f \in \mathbf{m}$ (see also Example 6.3, where $\Gamma_{\mathcal{A}'}$ is different from Γ).

5.6 Definition Define the class of $\mathcal{W}$-*nondegenerate functions* by:

$$\mathcal{M}_\mathcal{W} := \{f \in \mathrm{m} \mid \Gamma_{\tilde{f}} \subseteq \Gamma_{\tilde{f},\mathcal{W}} \text{ and } \tilde{f} \text{ is nondegenerate on any face of } \Gamma_{\tilde{f}}\}.$$

5.7 Remark Denote by $\mathcal{N}_G$ the class of nondegenerate G-invariant functions (in the sense of Kushnirenko) and define $\tilde{\mathcal{M}}_\mathcal{W} := \{\tilde{f} \mid f \in \mathcal{M}_\mathcal{W}\}$. It follows that:

$$\mathcal{N}_G \supset \tilde{\mathcal{M}}_\mathcal{W}.$$

The inclusion above cannot be an equality, as shown by the next example. However, for any $\tilde{f} \in \mathcal{N}_G$, there exists $\mathcal{W}$ such that $f \in \mathcal{M}_\mathcal{W}$.

Example Let $n = 2$, $d = 5$ and consider the generator of G which is represented by $(1, 2)$, as in 2.1. Then:

$$\mathcal{A} = \{(1,2),(2,4),(3,1),(4,3)\}, \qquad \mathcal{A}' = \{(1,2),(3,1)\}.$$

A d-cell decomposition $\mathcal{C}$ is generated by the set $\mathcal{W} := \mathcal{A}' \cup \{(5,0),(0,5)\}$. The polyhedron $\Gamma_{\tilde{f},\mathcal{W}}$ may have one or two maximal faces (depending on $\tilde{f}$), with only two possible normal directions given by the vectors in $\mathcal{A}'$. Consider the polynomial:

$$\tilde{f} = x^{20} + x^6y^2 + y^{35}.$$

Then $\Gamma_{\tilde{f},\mathcal{W}}$ has two maximal faces and $\Gamma_{\tilde{f},\mathcal{W}} \cap \Gamma_{\tilde{f}} = \{(6,2)\} \in \mathbf{R}^2$.

Hence $\tilde{f}$ is nondegenerate with respect to $\Gamma_{\tilde{f}}$, but not $\mathcal{W}$-nondegenerate.

5.8 Let $f \in \mathrm{m}$ and let $\tilde{\pi}$ and π be the morphisms defined in Section 3. The idea which might emerge from Sections 3 and 4 is that we have rather good control over the situation where $\{f \circ \pi = 0\}$ is already a normal crossings divisor (hence where we do not need to resolve any further). The following assertion shows that, to obtain the data we need for the zeta-function of f, we have to collect the corresponding data for the divisor $\{\tilde{f} \circ \tilde{\pi} = 0\} \subset \mathbf{Y}_\mathcal{C}$ and modify them by a certain algorithm.

We recall that $\mathcal{S}$ is the family of all the strata in the stratification of $\mathbf{Y}_\mathcal{C}$ considered in 4.5.

5.9 Theorem *Let $f \in \mathrm{m}$ be a germ such that the divisor $\{\tilde{f} \circ \tilde{\pi} = 0\}$ is a normal crossings divisor in $\mathbf{Y}_\mathcal{C}$. Then the zeta-function of the monodromy of f is:*

$$\zeta_f(t) = \prod_{T\in\mathcal{S}} \prod_{v\in\mathcal{W}_+} (1 - t^{\mathrm{o}_v(\tilde{f})/d})^{-d^{-\dim T}\cdot\chi(T\cap\tilde{\mathbf{E}}_v^*)}. \tag{19}$$

Proof We first note that $\{\tilde{f} \circ \tilde{\pi} = 0\}$ is a normal crossings divisor if and only if $\{f \circ \pi = 0\}$ is a normal crossings divisor. One can easily prove this using 3.9(c).

Consequently, we may use the formula of A'Campo (Theorem 1.3), which, in our case, takes the following form:

$$\zeta_f(t) = \prod_{v \in \mathcal{W}_+} (1 - t^{o_v(\tilde{f})/d})^{-\chi(\mathbf{E}_v^*)}.$$

The exponent of t is indeed $o_v(\tilde{f})/d$, by the proof of the Lemma 4.3, since

$$\mathrm{ord}_{\mathbf{E}_v}(f \circ \pi) = \frac{1}{d} \mathrm{ord}_{\tilde{\mathbf{E}}_v}(\tilde{f} \circ \tilde{\pi}) = \frac{1}{d} o_v(\tilde{f}).$$

Next, by the equality (12), we get:

$$\zeta_f(t) = \prod_{T \in \mathcal{S}} \prod_{v \in \mathcal{W}_+} (1 - t^{o_v(\tilde{f})/d})^{-\chi(\hat{T} \cap \mathbf{E}_v^*)}.$$

Finally, the Euler characteristic $\chi(\hat{T} \cap \mathbf{E}_v^*)$ is equal to $d^{-\dim T} \cdot \chi(T \cap \tilde{\mathbf{E}}_v^*)$ by the Lemma 4.7. □

5.10 Remark For a fixed $v \in \mathcal{W}_+$, some factors in the zeta-function $\zeta_f(t)$ may disappear, for instance due to the annihilation criterion Proposition 4.11.

5.11 Proposition *If $f \in \mathcal{M}_{\mathcal{W}}$ then $\{f \circ \pi = 0\}$ is a normal crossings divisor. In particular, the formula* (19) *is true for $f \in \mathcal{M}_{\mathcal{W}}$.*

Before giving the proof, we need some preparation.

5.12 For any $\sigma \in \mathcal{C}$, define a function $\tilde{f}_\sigma$ on $\mathbf{C}^n(\sigma)$, which is a germ at $\tilde{\pi}_\sigma^{-1}(0)$, by the following (compare with [Var, Lemma 10.2]):

$$(\tilde{f} \circ \tilde{\pi}_\sigma)(z_1, \ldots, z_n) = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \cdot \tilde{f}_\sigma(z_1, \ldots, z_n), \quad \text{where } \tilde{f}_\sigma \text{ is not divisible by } z_i, \forall i \in \{1, \ldots, n\}. \tag{20}$$

Note that the integers $\alpha = \alpha_i(\sigma)$ are nonnegative multiples of d, that $\{\tilde{f}_\sigma = 0\}$ is not necessarily the proper transform of $\{\tilde{f} = 0\}$ in the local chart $\mathbf{C}^n(\sigma)$ and that $\tilde{f}_\sigma(0)$ can be zero.

Define $\delta : \mathbf{Z}_{\geq 0} \to \{0, 1\}$ by $\delta(0) = 0$ and $\delta(k) = 1$ for any integer $k > 0$.

Then we have the following criterion:

5.13 Lemma *The divisor* $\{(x_1 \cdots x_n \cdot \tilde{f}) \circ \tilde{\pi} = 0\}$ *is a normal crossings divisor (as a germ in* $\tilde{\pi}^{-1}(0)$*) if for any* $\sigma \in \mathcal{C}$ *the functions:*

$$\tilde{f}_\sigma, \quad z_1^{\delta(\alpha_1)} \frac{\partial \tilde{f}_\sigma}{\partial z_1}, \quad \ldots, \quad z_n^{\delta(\alpha_n)} \frac{\partial \tilde{f}_\sigma}{\partial z_n}$$

have no common zero in some neighbourhood of $\tilde{\pi}_\sigma^{-1}(0) \subset \mathbf{C}^n(\sigma)$.

Proof Having normal crossings is a local property; we have to check it in each chart $\mathbf{C}^n(\sigma)$, namely at points $a \in \mathbf{C}^n(\sigma)$ such that $\tilde{f}_\sigma(a) = 0$ and $a \in \tilde{\pi}_\sigma^{-1}(0)$.

A sufficient condition is that the ideal $(\tilde{f}_\sigma,\ z_i \mid i \in I(a))$, defines a germ in a of a smooth complete intersection, where $I(a) := \{i \mid a_i = 0\}$. But this is equivalent to the condition in the statement. □

5.14 Proof of Proposition 5.11

Let $f \in \mathcal{M}_\mathcal{W}$. The condition $\Gamma_{\tilde{f}} \subseteq \Gamma_{\tilde{f},\mathcal{W}}$ insures that $\tilde{f}_\sigma(0) \neq 0$, $\forall \sigma \in \mathcal{C}$.

Suppose that $a \in \tilde{\pi}_\sigma^{-1}(0)$ would be a common zero of the functions in Lemma 5.13. Assume (without loss of generality) that $a = (0, \ldots, 0, a_{k+1}, \ldots, a_n)$ with $a_i \neq 0$, $\forall i \in \{k+1, \ldots, n\}$, for some $k < n$.

If $\sigma = \mathbf{R}_{\geq 0}\langle v_1, \cdots, v_n\rangle$ then let $\sigma_0 := \mathbf{R}_{\geq 0}\langle v_1, \ldots, v_k\rangle$ and denote by Δ the corresponding face of $\Gamma_{\tilde{f},\mathcal{W}}$ defined by the intersection of hyperplanes $\bigcap_{i \in \{1,\ldots,k\}} H_{v_i}(f)$.

We have, by the definition of $\tilde{f}_\Delta$ (see Definition 5.3 and equation (20)):

$$\tilde{f}_\Delta \circ \tilde{\pi}_\sigma = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \cdot \tilde{f}_\sigma(0, \ldots, 0, z_{k+1}, \ldots, z_n).$$

Define a function $\tilde{f}_{\sigma_0}$ by $\tilde{f}_{\sigma_0}(z_{k+1}, \ldots, z_n) := \tilde{f}_\sigma(0, \ldots, 0, z_{k+1}, \ldots, z_n)$. Then:

$$\frac{\partial(\tilde{f}_\Delta \circ \tilde{\pi}_\sigma)}{\partial z_i} = \begin{cases} z_1^{\alpha_1} \cdots z_i^{\alpha_i - 1} \cdots z_n^{\alpha_n}[\alpha_i \tilde{f}_{\sigma_0} + z_i \frac{\partial \tilde{f}_{\sigma_0}}{\partial z_i}] & \text{if } \alpha_i \neq 0, \\ z_1^{\alpha_1} \cdots \hat{z}_i \cdots z_n^{\alpha_n} \cdot \frac{\partial \tilde{f}_{\sigma_0}}{\partial z_i} & \text{if } \alpha_i = 0, \end{cases}$$

hence the value in a of this function is zero, by our assumption.

Denote by a' the point $(1, \ldots, 1, a_{k+1}, \ldots, a_n)$. Then we get moreover:

$$\frac{\partial(\tilde{f}_\Delta \circ \tilde{\pi}_\sigma)}{\partial z_i}(a') = 0 \quad \forall i \in \{1, \ldots, n\}.$$

On the other hand:

$$\begin{aligned} \frac{\partial(\tilde{f}_\Delta \circ \tilde{\pi}_\sigma)}{\partial z_i}(a') &= \sum_{j \in \{1,\ldots,n\}} \frac{\partial \tilde{f}_\Delta}{\partial x_j}(\tilde{\pi}_\sigma(a')) \cdot \frac{\partial x_j}{\partial z_i}(a') \\ &= \sum_{j \in \{1,\ldots,n\}} [x_j \frac{\partial \tilde{f}_\Delta}{\partial x_j}](\tilde{\pi}_\sigma(a')) \cdot [\sigma]_{ij} = 0, \end{aligned}$$

$\forall i \in \{1, \dots, n\}$ and the matrix $[\sigma]$ is nonsingular.

Hence:

$$[x_j \cdot \frac{\partial \tilde{f}_\Delta}{\partial x_j}](\tilde{\pi}_\sigma(a')) = 0, \quad \forall i \in \{1, \dots, n\},$$

which means that $\tilde{f}$ is degenerate on Δ. This contradicts the initial supposition. □

5.15 The Euler characteristics which appear in the formula (19) can sometimes be computed in terms of volumes of polyhedra; some nondegeneracy conditions must be assumed. The strategy is based, as one can expect, on a well-known result of Bernstein, Hovansky and Kushnirenko, see [Var, Theorem 7.1]. But note that, since $\tilde{\pi} : \mathbf{Y}_C \to \mathbf{C}^n$ is *not a resolution* of some subvariety of $\mathbf{C}^n$, the computations in [Var] cannot be applied *ad literam.*

To give the formulae, we need some more notations.

5.16 Let Δ be a $(n-i-1)$-dimensional face of some compact convex polyhedron. We denote by $\mathrm{Vol}_{n-i}(\Delta)$ the $(n-i)$-volume of the finite cone with base Δ and vertex the origin. The 0-volume is, by definition, equal to 0.

Any torus $T \in \mathcal{S}$ is defined by a certain cone $\sigma \in \Sigma$ and σ defines uniquely a face of $\Gamma_{\tilde{f},\mathcal{W}}$, which will be denoted by Δ_T. We have:

$$\dim T \geq \dim \Delta_T \geq \dim \Delta_T \cap \Gamma_{\tilde{f}}.$$

5.17 The "interesting" tori $T \in \mathcal{S}$ are those of the form (see also 4.10(*)):

(*) $\quad T = T_{(v, de_{j_1}, \dots, de_{j_{k-1}})}, \quad k \geq 1$, where v is strictly positive.

Note that $\dim T \in \{0, \dots, n-1\}$.

Definition Let $\mathcal{S}_0$ be the family of the tori (*). If $T \in \mathcal{S}_0$ then denote by v_T the strictly positive vector v from the definition (*) above.

5.18 Proposition *Let $T \in \mathcal{S}_0$. Then, for any $f \in \mathcal{M}_\mathcal{W}$ we have:*

$$d^{-\dim T} \cdot \chi(T \cap \tilde{\mathbf{E}}^*_{v_T}) = (-1)^{\dim T}(\dim T + 1)!\, o_{v_T}(\tilde{f})^{-1} \cdot \mathrm{Vol}_{\dim T+1}(\Delta_T \cap \Gamma_{\tilde{f}}).$$

Proof Let T be defined by $\sigma \in \Sigma$ and let $\sigma' \in \mathcal{C}$ such that $\sigma \subset \sigma'$. The polynomial $\tilde{f}_{\Delta_T} \circ \tilde{\pi}$ has a Newton boundary, denoted by $\Gamma_{\tilde{f}_{\Delta_T} \circ \tilde{\pi}}$. By [Var, Theorem 7.1] or [Ku, Theorem IV] and by the remark [Var, p.255], we have:

$$\chi(T \cap \tilde{\mathbf{E}}^*_{v_T}) = (-1)^{\dim T}(\dim T + 1)!\, o_{v_T}(\tilde{f})^{-1} \cdot \mathrm{Vol}_{\dim T+1}(\Gamma_{\tilde{f}_{\Delta_T} \circ \tilde{\pi}_{\sigma'}}).$$

Assume that $\sigma' = \mathbf{R}_{\geq 0}\langle v_1, \ldots, v_n\rangle$ and $\sigma = \mathbf{R}_{\geq 0}\langle v_1, \ldots, v_k\rangle$, with $T := T_{\langle v_1,\ldots,v_k\rangle}$ and $v_k = v_T$.

Let L_σ denote the coordinate subspace

$$\{m \in \mathbf{R}^n \mid m_i = 0,\ \forall i \in \{1, \ldots, k-1\}\} \subset \mathbf{R}^n.$$

Then the linear morphism $[\sigma']^t \in GL(n, \mathbf{R})$ restricted to L_σ defines an automorphism of L_σ, say γ, which is, as a matrix, a $(n-k+1) \times (n-k+1)$ minor of $[\sigma']^t$.

It follows that $\det \gamma = d^{n-k}$, since $\det[\sigma'] = d^{n-1}$. This implies the relation:

$$\mathrm{Vol}_{n-k+1}(\Gamma_{\tilde{f}_{\Delta_T} \circ \tilde{\pi}_{\sigma'}}) = d^{n-k} \cdot \mathrm{Vol}_{n-k+1}(\Delta_T \cap \Gamma_{\tilde{f}}),$$

which completes the proof. □

5.19 Corollary *If $f \in \mathcal{M}_{\mathcal{W}}$ then:*

$$\zeta_f(t) = \prod_{T \in \mathcal{S}_0} (1 - t^{o_{v_T}(\tilde{f})/d})^{(-1)^{\dim T+1}(\dim T+1)! \cdot o_{v_T}(\tilde{f})^{-1} \cdot \mathrm{Vol}_{\dim T+1}(\Delta_T \cap \Gamma_{\tilde{f}})}. \qquad (21)$$

Proof The corollary follows from Proposition 5.18 and formula (19). □

5.20 Remark If $f \in \mathcal{M}_{\mathcal{W}}$ then only the Newton boundary $\Gamma_{\tilde{f}}$ counts in the zeta-function formula.

If $\dim T > \dim(\Delta_T \cap \Gamma_{\tilde{f}})$ then the factor corresponding to T in formula (21) is equal to 1, since the volume $\mathrm{Vol}_{\dim T+1}(\Delta_T \cap \Gamma_{\tilde{f}})$ is zero.

Since $\tilde{\mathcal{M}}_{\mathcal{W}} \subset \mathcal{N}_G$, one can write the Varchenko's formula for the zeta-function (there is a change of sign to be made) of some $\tilde{f} \in \tilde{\mathcal{M}}_{\mathcal{W}}$ and then compare it to formula (21). We do this for any general function $f_0 \in \mathbf{m}$ (see Definition 5.2).

5.21 We note that there is a decomposition $\mathcal{C}_{\mathcal{W}}$ as in 3.6 with the additional property that any maximal face of Γ has a normal vector v belonging to $\mathcal{W}_+$. We assume this property in the following. The first implication is that $\Gamma_{\tilde{f}_0, \mathcal{W}} = \Gamma$, hence general functions are $\mathcal{W}$-nondegenerate.

If f_0 is a general function then $\{f_0 = 0\}$ is a general hyperplane slice of $(\mathbf{X}, 0)$ and the divisor $\{\tilde{f}_0 \circ \tilde{\pi} = 0\}$ is a normal crossings divisor (cf. Proposition 5.11).

Denote by $\max(\Gamma)$ the set of maximal faces of Γ. Denote by v_Δ the vector in $\mathcal{W}_+$ which is normal to the face $\Delta \in \max(\Gamma)$. The hyperplane H_Δ which contains Δ has equation:

$$v_\Delta \cdot (x_1, \ldots, x_n)^t = m(v_\Delta), \quad \text{where } m(v_\Delta) := o_{v_\Delta}(\tilde{f}_0). \tag{22}$$

We may replace $o_{v_\Delta}(\tilde{f}_0)$ in formula (21) by $m(v_\Delta)$ but, in order to replace all $o_{v_T}(\tilde{f}_0)$ from the cited formula by something similar, we need more definitions.

5.22 Definition Let $I \subset \{1, \ldots, n\}$, $0 \leq |I| < n$ and define $L_I := \{m \in \mathbf{R}^n \mid m_j = 0,\ \forall j \in I\}$.

For any $v \in \mathcal{W}_+$ and I as above, let $v(I) \in (\mathbf{Z}_+)^n \cap L_I$ be the orthogonal projection of the vector v to the subspace L_I.

Let $\mathcal{W}_+(I) := \{v(I) \mid v \in \mathcal{W}_+\}$ and let $\mathcal{W}_+(I)'$ denote the subset of primitive vectors in $\mathcal{W}_+(I)$.

Definition For any $v \in \mathcal{W}_+$ and any I as above, denote by v^I the vector $w \in \mathcal{W}_+$ such that $w(I)$ is the primitive of $v(I)$ in $\mathcal{W}_+(I)$.

By definition, we have ${v_\Delta}^\emptyset = v_\Delta$. Define $m({v_\Delta}^I) := o_{{v_\Delta}^I}(\tilde{f}_0)$ and notice that $m({v_\Delta}^I)$ may be different from $m(v_\Delta)$ if $|I| \geq 1$; see Example 6.3.

With these notations and a moment's thought, we may replace in (21) a vector v_T, for $T \in \mathcal{S}_0$ such that $\Delta_T = \Delta \cap L_I$, $\Delta \in \max(\Gamma)$, by the vector ${v_\Delta}^I$, where T is defined as in 5.17(*) and then $I := \{j_1, \ldots, j_{k-1}\}$. It turns out that only these vectors contribute to the formula, since for the others, we have $\mathrm{Vol}_{n-|I|}(\Delta \cap L_I) = 0$. Note that, if $\Delta \cap L_I$ is a maximal face (equivalently: $\mathrm{Vol}_{n-|I|}(\Delta \cap L_I) \neq 0$), then $v_\Delta(I)$ is normal to it and, of course, ${v_\Delta}^I$ is normal to it as well.

Thus, formula (21) becomes:

$$\zeta_{f_0}(t) = \prod_{\Delta \in \max(\Gamma)} \prod_{I \in \{1,\ldots,n\}} \left(1 - t^{m({v_\Delta}^I)/d}\right)^{(-1)^{n-|I|}(n-|I|)!\, m({v_\Delta}^I)^{-1} \cdot \mathrm{Vol}_{n-|I|}(\Delta \cap L_I)}. \tag{23}$$

For any $\Delta \in \max(\Gamma)$ and $I \subset \{1, \ldots, n\}$, we consider the face $\Delta \cap L_I$ of Δ (which may be also void). The hyperplane $H_\Delta \cap L_I$ of L_I which contains $\Delta \cap L_I$ has equation:

$$\sum_{i \in \{1,\ldots,n\} \setminus I} a_i x_i = m(\Delta \cap L_I), \text{ where } a_i, m(\Delta \cap L_I) \in \mathbf{N} \text{ and } \gcd\{a_i \mid i \notin I\} = 1. \tag{24}$$

The nonnegative integers $m(\Delta \cap L_I)$ enter in the zeta-function formula of Varchenko [Var] (with a change of the signs of the exponents, to agree with the formula of A'Campo), which takes the form:

$$\zeta_{\tilde{f}_0}(t) = \prod_{\Delta \in \max(\Gamma)} \prod_{I \in \{1,\dots,n\}} (1 - t^{m(\Delta \cap L_I)})^{(-1)^{n-|I|}(n-|I|)!\, m(\Delta \cap L_I)^{-1} \cdot \mathrm{Vol}_{n-|I|}(\Delta \cap L_I)}. \tag{25}$$

The two formulae (23) and (25) have much similarity, in particular they suggest the forthcoming Corollary 5.28. However, for some $\Delta \in \max(\Gamma)$, the two integers $m(v_\Delta{}^I)$ and $m(\Delta \cap L_I)$ may be different, even if $I = \emptyset$. We illustrate this behaviour in Examples 6.1 and 6.5.

5.23 There is a criterion for the monodromy h_{f_0} of a general function f_0 to be *unipotent*, which follows from the spectrum formula of M. Saito [Sa]:

> *The monodromy h_{f_0} is unipotent if and only if the Newton degree of any G-invariant monomial $x_1^{s_1} \cdots x_n^{s_n}$, $s_i > 0$, $\forall i \in \{1, \dots, n\}$ is integral.*

5.24 In the case $n = 2$, there is no need for a criterion since, from the forthcoming remark 7.7(30) and formula (21), it becomes clear that:

Proposition *The monodromy h_{f_0} of a general function f_0 on a 2-dimensional cyclic quotient is unipotent.* □

5.25 For $n = 3$, Example 6.3 shows that h_{f_0} need not be unipotent. Steenbrink showed in [St] that, if the generator of the G-action is $(1, 1, k)$, with $\gcd(k, d) = 1$, then the monodromy of the general function f_0 is unipotent.

We prove a criterion, different from the one above (but, of course, equivalent to that), which follows from the formula of the zeta-function (23):

Theorem *The monodromy h_{f_0} of a general function f_0 on a 3-dimensional cyclic quotient is unipotent if and only if $m(v_\Delta) = d$, $\forall \Delta \in \max(\Gamma)$.*

Proof "$\Leftarrow$" The integers $m(v_\Delta{}^I)$are divisors of $m(v_\Delta)$, but also multiples of d. If $m(v_\Delta) = d$, then $m(v_\Delta{}^I) = d$, $\forall I \subset \{1,2,3\}$, $|I| < 3$. Hence our claim follows indeed at once from (23).

"$\Rightarrow$" Suppose that there is a $\Delta \in \max(\Gamma)$ such that $m(v_\Delta) = kd$, for some $k > 1$. We have to prove that the factor of the form $(1-t^k)^{-i}$, for some positive integer i, which corresponds to Δ and $I = \emptyset$ in (23), cannot be completely cancelled in the product.

Hypothetically, it could be cancelled by a product of factors. At least one of them must be of the form $(1-t^k)^j$, for some $j > 0$ and corresponds to some I with $|I| = 1$, i.e. $m(v_\Delta{}^I) = kd$. On the other hand, $m(v_\Delta{}^I) = d$, $\forall \Delta \in \max(\Gamma)$ and $\forall I \in \{1,2,3\}$ with $|I| = 1$ such that $\dim \Delta \cap L_I = 1$, since we are now in the 2-dimensional case; see the remark 7.7(30). This is the contradiction we need. □

5.26 The Lefschetz number again

For this weaker invariant we may enview a larger class of functions, the $\mathcal{A}'$-*nondegenerate* functions (recall Definition 5.5):

$$\mathcal{M}_{\mathcal{A}'} := \left\{ f \in \mathrm{m} \;\middle|\; \begin{array}{r} \text{for any } I \subset \{1,\ldots,n\} \text{ and any face } \Delta \subset \Gamma_{\mathcal{A}'} \cap L_I, \\ \text{the function } \tilde{f} \text{ is either nondegenerate on } \Delta \\ \text{or } \dim \Delta \cap \Gamma_{\tilde{f}} < n - |I| - 1 \end{array} \right\}.$$

5.27 Proposition *If $f \in \mathcal{M}_{\mathcal{A}'}$ then:*

$$\Lambda(f) = -\frac{1}{d} \cdot \sum_{\substack{I \subset \{1,\ldots,n\} \\ \Delta \in \max(\Gamma_{\mathcal{A}'})}} (-1)^{n-|I|}(n-|I|)! \cdot \mathrm{Vol}_{n-|I|}(\Delta \cap L_I \cap \Gamma_{\tilde{f}}). \tag{26}$$

Proof

Note again that $\mathrm{Vol}_{n-|I|}(\Delta \cap L_I \cap \Gamma_{\tilde{f}}) = 0$ if $\dim \Delta \cap L_I \cap \Gamma_{\tilde{f}} < n - |I| - 1$. The formula (26)—which is a formal consequence of (23)—follows from (13) and the $\mathcal{A}'$-nondegeneracy condition, in the same way one proves Proposition 5.18. The main difference is that this time we need the nondegeneracy condition only to convert Euler numbers into volumes (and not for the normal crossings property, which is not important for the Lefschetz number formula). Hence the $\mathcal{A}'$-nondegeneracy condition is just sufficient. □

We recall that $h_{\tilde{f}}$ denotes the algebraic monodromy of $\tilde{f}$, $\Lambda(h_{\tilde{f}}^k)$ is the Lefschetz number of the k-th power of this monodromy and $\Lambda(h_f)$ is another notation for $\Lambda(f)$.

5.28 Corollary *If $f_0 \in \mathrm{m}$ is a general function, then:*

$$\Lambda(h_{\tilde{f}_0}) = d \cdot \Lambda(f_0).$$

Proof A general function f_0 is obviously $\mathcal{A}'$-nondegenerate; we also have $\Gamma_{\tilde{f}_0} = \Gamma$, by Definition 5.2. The following equality is a consequence of formula (25):

$$\Lambda(h^d_{\tilde{f}_0}) = - \sum_{\substack{I \subset \{1,\dots,n\} \\ \Delta \in \max(\Gamma_{\mathcal{A}'})}} (-1)^{n-|I|}(n-|I|)! \cdot \mathrm{Vol}_{n-|I|}(\Delta \cap L_I \cap \Gamma).$$

If we compare it to formula (26), the result becomes obvious. □

6 Examples

We show a few significative computations of the Lefschetz number and zeta-function. They also illustrate some statements proved in the sections before.

We consider three examples of isolated cyclic quotient singularities (dimensions 2 and 3). We end by examples (due to Wahl) of 3-dimensional cyclic quotients for which a general function has Milnor number equal to zero.

6.1 Let the data for the G-action be those in Example 2.7, i.e.:

$$n = 2,\ d = 8, \text{ and } (1,5) \text{ represents a generator.}$$

Consider the general function $f_0 \in \mathbf{m}$, where:

$$\tilde{f}_0 = x^8 + x^3y + xy^3 + y^8.$$

The zeta-function of the monodromy $h_{\tilde{f}_0}$ can be computed in at least two ways:

(a) by resolving $\{\tilde{f}_0 = 0\}$ (this amounts to a simple blowing-up of the origin of $\mathbf{C}^2$) and applying the formula of A'Campo;

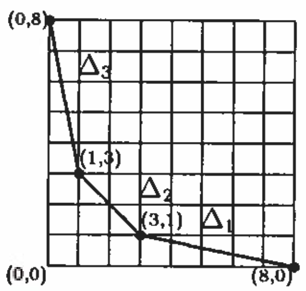
(b) by the formula of Varchenko (with the change of sign we have considered).

In both cases we get:

$$\zeta_{\tilde{f}_0}(t) = (1-t^4)^2.$$

Denote by Δ_1, Δ_2, Δ_3 the faces of the Newton polygon $\Gamma_{\tilde{f}_0}$, as in the figure below. Using the definition relations (22) and (24), we get:

$$m(v_{\Delta_1}) = m(\Delta_1) = 8,$$
$$m(v_{\Delta_2}) = 8,\ m(\Delta_2) = 4,$$
$$m(v_{\Delta_3}) = m(\Delta_3) = 8.$$



Since $m(v_{\Delta_2}) \neq m(\Delta_2)$, we might expect $\zeta_{\tilde{f}_0}(t) \neq \zeta_{f_0}(t^8)$. It is so, indeed:

$$\zeta_{f_0}(t) = (1-t)^3(1-t)^{-2} = (1-t), \qquad \text{by (23)},$$

hence $\zeta_{f_0}(t^8) = 1 - t^8$.

The Lefschetz numbers are:

$$\Lambda(f_0) = -1, \qquad \Lambda(h^8_{\tilde{f}_0}) = 4 \cdot (-2) = -8,$$

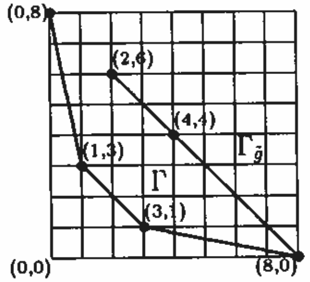
hence we get the equality: $\Lambda(h^8_{\tilde{f}_0}) = 8 \cdot \Lambda(f_0)$, which was predicted by Corollary 5.28.

6.2 The next example shows that, if $f \in \mathfrak{m}$ is not general, then the equality $\Lambda(h^d_{\tilde{f}}) = d \cdot \Lambda(f)$ might be false.

For the same data as in 6.1, let's consider the G-invariant function (with its Newton polygon as in the picture below):

$$\tilde{g} = x^8 - 3x^4y^4 + 2x^2y^6,$$

($\tilde{g}$ is not isolated).



The resolution of $\{\tilde{g} = 0\}$ yields an irreducible exceptional divisor $\mathbf{E} \simeq \mathbf{P}^1$ with multiplicity 8, which intersects the proper transform $\{\widetilde{\tilde{g} = 0}\}$ in 5 points.

Since $\chi(\mathbf{P}^1 \setminus 5 \text{ points}) = -3$, we get:

$$\Lambda(h^8_{\tilde{g}}) = 8 \cdot (-3) = -24.$$

On the other hand: $\Lambda(g) = \Lambda_{(1,5)}(g) = 1$, as in Example 2.7.

6.3 Consider the case: $n = 3$, $d = 11$, and a generator is given by $(1, 7, 5)$. This example is refered to several times in this chapter and was suggested to us by Steenbrink (see also [St]). It is interesting also because the monodromy of the general function is not unipotent. We have:

$$\mathcal{A} = \{v_1, \ldots, v_{10}\},$$

where $v_1 = (1,7,5)$, $v_2 = (2,3,10)$, $v_3 = (3,10,4)$, $v_4 = (4,6,9)$, $v_5 = (5,2,3)$, $v_6 = (6,9,8)$, $v_7 = (7,5,2)$, $v_8 = (8,1,7)$, $v_9 = (9,8,1)$, $v_{10} = (10,4,6)$.

The vector v_{10} is the only one not primitive in $\mathcal{A}$ and $v_6 = v_1 + v_5$ is the only $\mathbf{Z}_+$-relation. Hence:

$$\mathcal{A}' = \{v_1, v_2, v_3, v_4, v_5, v_7, v_8, v_9\}.$$

We subdivide the positive orthant as in the picture below. Our 2-dimensional picture is obtained by intersecting the positive orthant with the hyperplane $\{(m_1, m_2, m_3) \in \mathbf{R}^3 \mid m_1 + m_2 + m_3 = 1\}$.

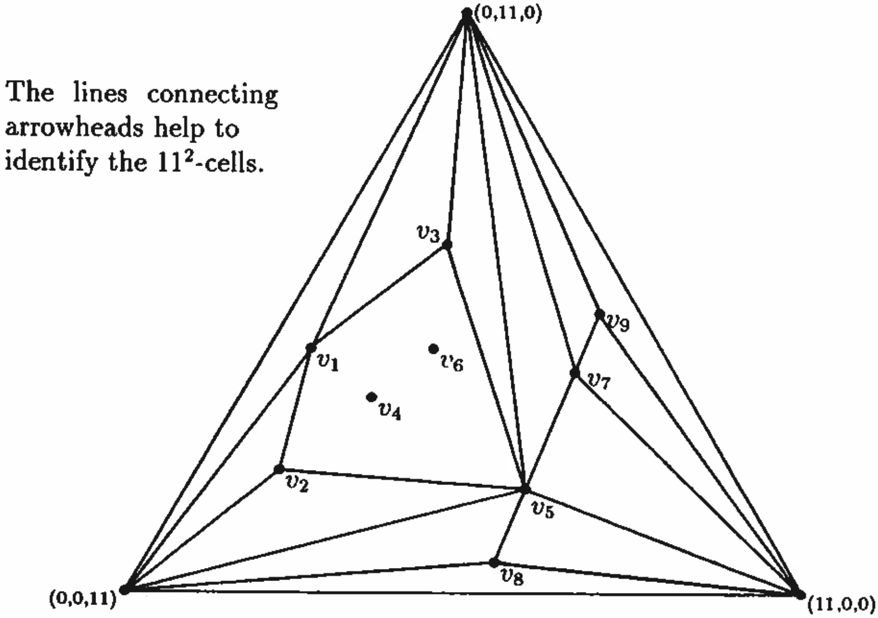
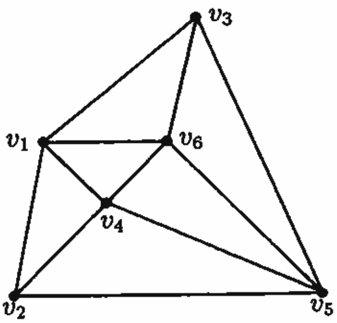


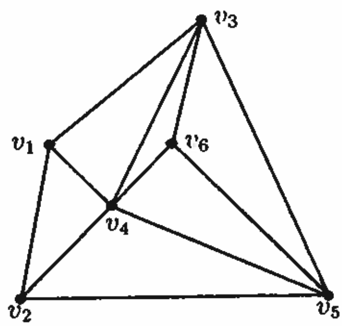
Figure 1

The picture does not contain the complete decomposition; there is still a cone that has to be subdivided, namely the cone generated by the vectors v_1, v_2, v_5, v_3. The vector v_4 is contained in the interior of this cone. We have figured also the vector $v_6 \in \mathcal{A} \setminus \mathcal{A}'$, which is included in the interior of the named cone as well.

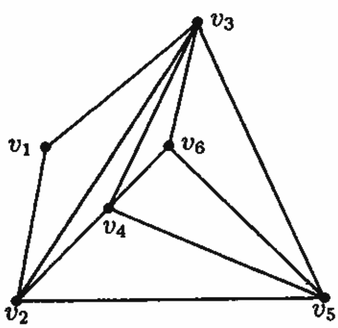
One can decompose the remaining cone into 11^2-cells in several ways, as the following pictures show:



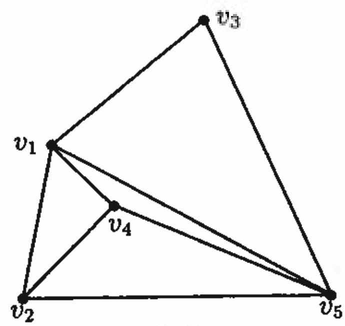
(i)



(ii)



(iii)



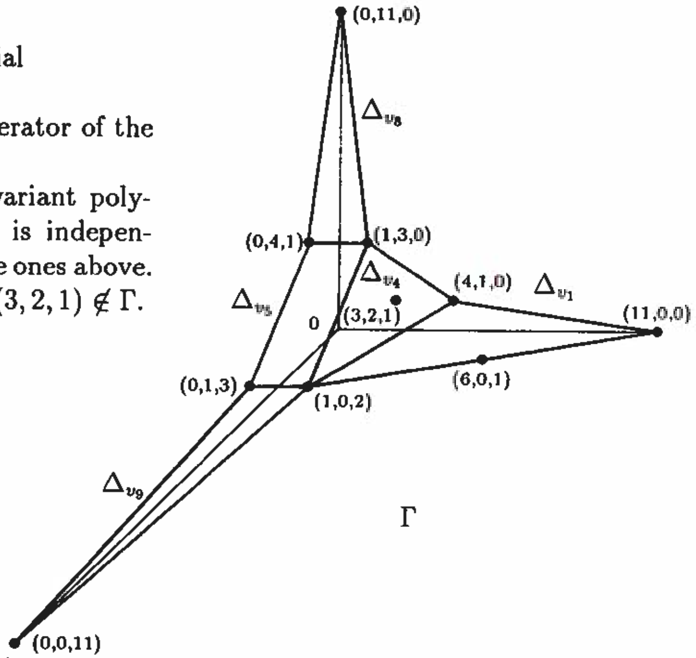
(iv)

The first conclusion is that there is a 11^2-cell decomposition (as in Figure (*iv*)) generated by the set of vectors $\mathcal{V}$ (in the notations of 3.2), that is: $\mathcal{W}_+ = \mathcal{A}'$.

The Newton polyhedron Γ associated to our G comes in the next picture. To draw it, we need the G-invariant monomials of "lowest" degrees, and they are:

$$x^{11},\ x^4y,\ xy^3,\ y^{11},\ y^4z,\ yz^3,\ z^{11},\ xz^2,\ x^6z.$$

The monomial
x^3y^2z
is also a generator of the algebra
of the G-invariant polynomials and is independent from the ones above.
Notice that $(3,2,1) \notin \Gamma$.



We note the following:

(a) Each maximal face of Γ has as normal vector one from $\mathcal{A}'$; the maximal faces are: Δ_{v_1}, Δ_{v_4}, Δ_{v_5}, Δ_{v_8}, Δ_{v_9} (with the notations from 5.16).

(b) For any $f \in \mathbf{m}$, we have:

$$o_{v_4}(\tilde{f}) \geq 2 \cdot 11, \qquad o_{v_6}(\tilde{f}) \geq 2 \cdot 11,$$

with equalities in the case of a general f, for instance.

(c) The vectors v_2, v_7 are normal to 1-faces of Γ: v_2 to the one connecting the point $(4,1,0)$ to the point $(1,3,0)$ and v_7 to the one connecting the point $(1,0,2)$ to the point $(0,1,3)$.

(d) The vector v_3 is not normal to any 2- or 1-face of Γ.

We have to fix a decomposition $\mathcal{C}$; we choose the one described by Figure 1 and Figure (iv). The set of generators of $\mathcal{C}$ is $\mathcal{W} := \mathcal{A}' \cup \{11 \cdot e_1, 11 \cdot e_2, 11 \cdot e_3\}$.
Let $\tilde{\pi} : \mathbf{Y}_{\mathcal{C}} \to \mathbf{C}^3$, $\pi : \mathbf{X}' \to \mathbf{X}$ be the morphisms constructed in Section 3, and let $\tilde{\mathbf{E}}_{v_i}$, resp. $\mathbf{E}_{v_i}$ be the prime exceptional divisors, where $v_i \in \mathcal{A}'$. It

follows from (c) and (d) above and from Proposition 4.11 that:

$$\Lambda_{v_3}(f) = \Lambda_{v_7}(f) = 0, \quad \forall f \in \mathbf{m}.$$

Hence: $\Lambda(f) = \Lambda_{v_1}(f) + \Lambda_{v_8}(f) + \Lambda_{v_9}(f) + \Lambda_{v_2}(f) + \Lambda_{v_5}(f)$.

6.4 Let us compute the Lefschetz number and the zeta-function for a general $f_0 \in \mathbf{m}$; take:

$$\tilde{f}_0 := x^{11} + x^4 + xy^3 + y^{11} + y^4z + yz^3 + z^{11} + xz^2 + x^6z + x^3y^2z.$$

Consider the following 11^2-cells in our decomposition:

$$\sigma_1 := \mathbf{R}_{\geq 0}\langle v_1, 11e_2, 11e_3\rangle, \qquad \sigma_8 := \mathbf{R}_{\geq 0}\langle v_8, 11e_3, 11e_1\rangle,$$

$$\sigma_9 := \mathbf{R}_{\geq 0}\langle v_9, 11e_1, 11e_2\rangle, \qquad \sigma_2 := \mathbf{R}_{\geq 0}\langle v_2, 11e_3, v_1\rangle,$$

$$\sigma_5 := \mathbf{R}_{\geq 0}\langle v_5, 11e_1, v_8\rangle.$$

Using the formulae (10) and (11) and Proposition 5.18, we get:

$$\begin{aligned}
\Lambda_{v_1}(f_0) &= \chi(\mathbf{E}^*_{v_1} \cap \hat{T}_{v_1}) + \chi(\mathbf{E}^*_{v_1} \cap \hat{T}_{\langle v_1, 11e_2\rangle}) + \chi(\mathbf{E}^*_{v_1} \cap \hat{T}_{\langle v_1, 11e_3\rangle}) + \\
&\quad + \chi(\mathbf{E}^*_{v_1} \cap \hat{T}_{\langle v_1, 11e_2, 11e_3\rangle}) = 2 - 2 - 1 + 1 = 0, \\
\Lambda_{v_8}(f_0) &= \chi(\mathbf{E}^*_{v_8} \cap \hat{T}_{v_8}) + \chi(\mathbf{E}^*_{v_8} \cap \hat{T}_{\langle v_8, 11e_3\rangle}) + \chi(\mathbf{E}^*_{v_8} \cap \hat{T}_{\langle v_8, 11e_1\rangle}) + \\
&\quad + \chi(\mathbf{E}^*_{v_8} \cap \hat{T}_{\langle v_8, 11e_3, 11e_1\rangle}) = 1 - 1 - 1 + 1 = 0, \\
\Lambda_{v_9}(f_0) &= \chi(\mathbf{E}^*_{v_9} \cap \hat{T}_{v_9}) + \chi(\mathbf{E}^*_{v_9} \cap \hat{T}_{\langle v, 11e_1\rangle}) + \chi(\mathbf{E}^*_{v_9} \cap \hat{T}_{\langle v_9, 11e_2\rangle}) + \\
&\quad + \chi(\mathbf{E}^*_{v_9} \cap \hat{T}_{\langle v_9, 11e_1, 11e_2\rangle}) = 1 - 1 - 1 + 1 = 0, \\
\Lambda_{v_2}(f_0) &= \chi(\mathbf{E}^*_{v_2} \cap \hat{T}_{v_2}) + \chi(\mathbf{E}^*_{v_2} \cap \hat{T}_{\langle v_2, 11e_3\rangle}) = 0 - 1 = -1, \\
\Lambda_{v_5}(f_0) &= \chi(\mathbf{E}^*_{v_5} \cap \hat{T}_{v_5}) + \chi(\mathbf{E}^*_{v_5} \cap \hat{T}_{\langle v_5, 11e_1\rangle}) = 2 - 1 = 1.
\end{aligned}$$

The result is:

$$\Lambda(f_0) = 0 + 0 + 0 - 1 + 1 = 0.$$

To compute the zeta-function, we use formula (23). The vector v_4 comes into the computations:

$$\chi(\mathbf{E}^*_{v_4}) = \chi(\mathbf{E}^*_{v_4} \cap \hat{T}_{v_4}) = 1.$$

We get:

$$\zeta_{f_0}(t) = (1-t)^{-\Lambda(f_0)} \cdot (1-t^2)^{-\chi(\mathbf{E}^*_{v_4})} = (1-t^2)^{-1}.$$

It also follows that the Milnor number of f_0 is 1 and the monodromy h_{f_0} on $H^2(F_{f_0}, \mathbf{C})$ has eigenvalue -1.

During the last computation, we have used implicitly the equalities:

$$m(v_i{}^I) = m(\Delta_{v_i} \cap L_I) = 11, \quad \text{for } i \in \{1,5,8,9\},$$

$$m(v_4) = m(\Delta_{v_4}) = 2 \cdot 11,$$
$$m(v_4{}^{\{3\}}) = m(\Delta_{v_4} \cap \{m_3 = 0\}) = 11.$$

These equalities show that the formulae (23) and (25) yield:

$$\zeta_{f_0}(t^{11}) = \zeta_{\tilde{f}_0}(t).$$

An interesting observation is that the monomial x^3y^2z plays no role in our computation of the zeta-function.

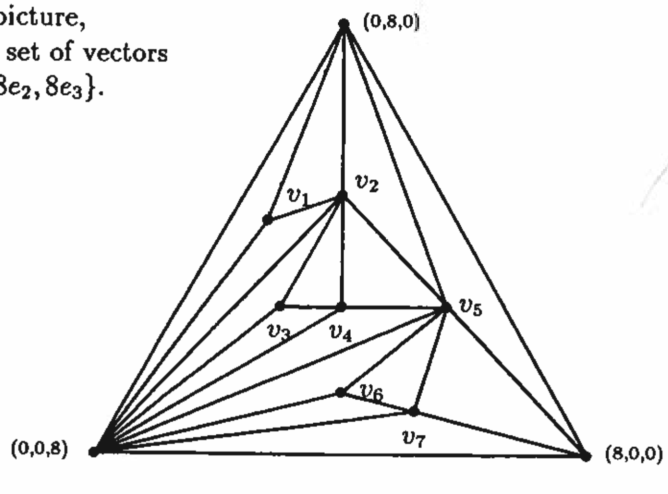
There is another point of view from which exactly this monomial x^3y^2z is the one responsible for the eigenvalue -1: the computation of the *spectrum* of $\tilde{f}_0$ using the Newton polyhedron (see [Sa]).

6.5 Consider the case: $n = 3$, $d = 8$ and a generator is defined by $(1,7,5)$. We have $\mathcal{A} = \{v_1, \ldots, v_7\}$, where:

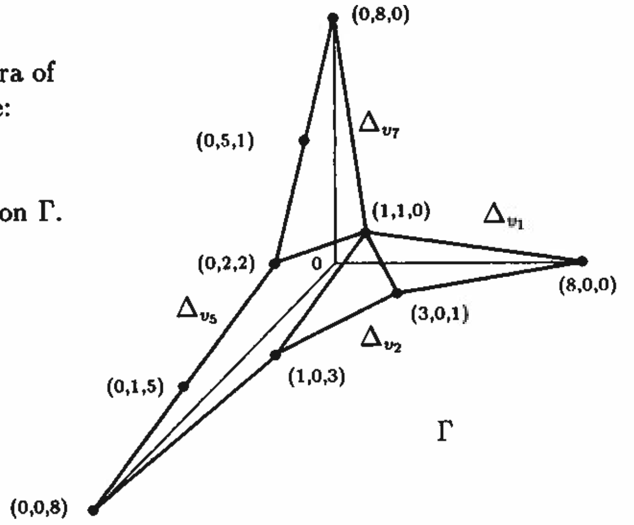
$$v_1 = (1,7,5),\ v_2 = (2,6,2),\ v_3 = (3,5,7),\ v_4 = (4,4,4),$$
$$v_5 = (5,3,1),\ v_6 = (6,2,6),\ v_7 = (7,1,3),$$

hence $\mathcal{A}' = \mathcal{A}$ (despite the fact that several vectors are not primitive in $\mathbf{Z}^3$).

There is a 8^2-cell decomposition, as shown in the picture, generated by the set of vectors $\mathcal{W} := \mathcal{A}' \cup \{8e_1, 8e_2, 8e_3\}$.



The generators of the algebra of G-invariant polynomials are:
x^8, xy, y^8, y^5z, y^2z^2,
yz^5, z^8, xz^3, x^3z,
all of them having support on Γ.



We observe that:

(a) the vectors v_1, v_7, v_5, v_2 are normal to the four maximal faces, respectively, as shown in the picture;

(b) the point $(1,1,0)$ is contained in all the maximal faces of Γ and $o_v(xy) = 8$, $\forall v \in \mathcal{A}'$;

(c) using the notations in 4.9, we have:

$$H_{v_i} \cap \Gamma = \{(1,1,0)\}, \quad \forall i \in \{3,4,6\}.$$

It follows from (c) above and Corollary 4.11 that:

$$\Lambda_{v_3}(f) = \Lambda_{v_4}(f) = \Lambda_{v_6}(f) = 0, \quad \forall f \in \mathbf{m}.$$

A general function $f_0 \in \mathbf{m}$ has Milnor number 0, hence $\Lambda(f_0) = 1$, $\zeta_{f_0}(t) = (1-t)^{-1}$.

If we compute $\zeta_{\tilde{f}_0}(t)$ by the formula (25) we get:

$$\zeta_{\tilde{f}_0}(t) = (1-t^8)^{-1}, \quad \text{hence} \quad \zeta_{f_0}(t^8) = \zeta_{\tilde{f}_0}(t),$$

in spite of the fact that: $m(v_2) = 8$, $m(\Delta_{v_2}) = 4$, $m(\Delta_{v_2} \cap \{y = 0\}) = 4$.

6.6 Remark It follows that, in all our examples above, the resolutions we consider are $\mu_{\mathbf{X}}$-minimal (see Definition 1.21 and Lemma 4.3).

6.7 Milnor number equal to zero and Wahl's examples [Wahl, p. 240]

(a) Let $n = 3$, $d, k \in \mathbf{Z}_+$ such that $0 < k < d$ and $\gcd(k,d) = 1$. Let $(1, k, d-1)$ define a generator of the cyclic group action. Then a general function f_0 on $(\mathbf{X}, 0)$ has Milnor number equal to zero, hence $\Lambda(f_0) = 1$. A linear function f with Milnor number zero can be obtained, for instance, from the G-invariant A_{d-1}-singularity $\tilde{f} := xz + y^d$.

(b) Let $n = 3$, $d = pqr + 1$, where $p, q, r \in \mathbf{Z}_+$ and let $(1, qr, d-r)$ define a generator of the cyclic group action. A general function f_0 has again Milnor number zero. An explicit example of a (nongeneral) linear function f with Milnor number zero is given by the triple-infinite family $\tilde{f} := xy^p + yz^q + zx^r$.

These two classes of examples are, in fact, *the only 3-dimensional isolated cyclic quotient singularities for which the general linear slices have Milnor number equal to zero.*

We are able to prove the assertion by some (tedious, but rather easy) computations involving Newton polyhedra, which we do not reproduce here.

Wahl did not make explicit this remark, but he told us that he was convinced of its truth and that, probably, his computations done at the time he found these examples would have lead to the above conclusion.

7 Functions on 2-dimensional cyclic quotients

7.1 Keeping the previous notations, we restrict our attention to the case $n = 2$. The general results are more explicit and we can prove some interesting statements about the Lefschetz number and the zeta-function, most of them being true only for $n = 2$.

We would like to recall first a few well-known facts about two approaches to the resolution of surface cyclic quotients: the "classical" one (Hirzebruch, Jung, Brieskorn) and the toric resolution. Both of them lead to the *minimal resolution*.

Let $G \subset GL(2, \mathbf{C})$ be a cyclic diagonal group of order d and let $(1, k)$ represent a generator, as in 2.1 (4).

7.2 We refer for the following to [Lam]. The quotient space $\mathbf{X} := \mathbf{C}^2/G$ is usually denoted by $\mathbf{X}_{d,k}$. The minimal resolution of $\mathbf{X}_{d,k}$ is obtained as a result of iterated modifications:

$$\mathbf{X}'' \xrightarrow{\pi_{s+1}} \mathbf{Y}_{r_{s-1},r_s} \xrightarrow{\pi_s} \cdots \xrightarrow{\pi_2} \mathbf{Y}_{k,r_1} \xrightarrow{\pi_1} \mathbf{X}_{d,k}, \tag{27}$$

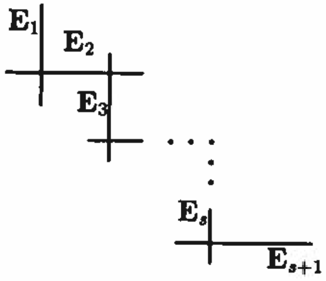
where $\mathbf{X}''$ is a smooth space and each intermediate space contains one cyclic quotient singularity of the type indicated by the pair of indices. These indices come from the repeated division with negative remainder, as shown in the next sequence of $\mathbf{Z}_+$-equations:

$$\begin{aligned} d &= b_1 \cdot k - r_1, \quad r_1 < k, \\ k &= b_2 \cdot r_1 - r_2, \quad r_2 < r_1, \\ r_1 &= b_3 \cdot r_2 - r_3, \quad r_3 < r_2, \\ &\vdots \\ r_{s-1} &= b_{s+1} \cdot r_s, \quad \text{where } r_s = 1. \end{aligned} \tag{28}$$

The positive integers b_i characterize the type of the quotient and they can be defined also by the fraction expansion:

$$\frac{d}{k} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \frac{1}{b_{s+1}}}}.$$

Each modification π_i introduces a new exceptional divisor $\mathbf{E}_i \simeq \mathbf{P}^1$, which is the zero section of $\mathcal{O}_{\mathbf{P}^1}(-b_i)$. Finally, the irreducible exceptional divisors will intersect each other in a stairs shape:



and the dual graph of the resolution will be:

The resolution of the surface cyclic quotient singularities plays an essential role in the resolution of an arbitrary surface quotient.

A resolution as above can be obtained equally in the way we have described in Section 3. We do this in the following. One can find a related approach in [Oda].

7.3 To produce a d-cell decomposition of the positive quadrant is much easier. Define $\mathcal{W} := \mathcal{A}' \cup \{d \cdot e_1, d \cdot e_2\}$ and order the vectors by their angle to e_1, increasingly. Each subset $\mathcal{W}' \subset \mathcal{W}$ has an induced order.

Denote by $\operatorname{conv}(B)$ the convex hull of a set $B \subset \mathbf{R}^2$.

Lemma *If $v_1, v_2 \in \mathcal{W}$ are two consecutive vectors, then they generate a d-cell.*

Proof We consider the cone τ_E defined in 3.3; in our case: $\tau_E = \mathbf{R}_{\geq 0}\langle de_1 - ke_2, e_2\rangle$. Denote by $\Delta(\tau_E)$ the area delimited by the triangle with vertices $(0,0)$, $(0,1)$, $(d,-k)$. Denote by Γ_{τ_E} the union of the compact faces of the infinite polygonal surface:

$$(\Gamma_{\tau_E})_+ := \operatorname{conv}(\cup_{m \in \mathbf{Z}^2 \cap \Delta(\tau_E) \setminus (0,0)} \{m + (\mathbf{R}_{\geq 0})^2\}). \tag{29}$$

Note that $(1,0) \in \Gamma_{\tau_E}$. It is a standard fact that the lattice points on Γ_{τ_E} are generating a nonsingular decomposition of the cone τ_E. (This is equivalent to saying that the semi-group $\mathbf{Z}^2 \cap \tau_E$ is generated by $\mathbf{Z}^2 \cap \Gamma_{\tau_E}$). We transport the 1-cell decomposition by the linear map $\mathbf{T}$ (defined in 3.3) to the positive quadrant and get a d-cell decomposition generated by the set of vectors $\mathcal{W}_{d,k} := \mathbf{T}(\mathbf{Z}^2 \cap \Gamma_{\tau_E})$. □

7.4 Corollary

(a) The polygonal line $\mathbf{T}(\Gamma_{\tau_E})$ is convex.

(b) $\mathcal{A}' = \mathcal{W}_{d,k} \cap (\mathbf{Z}_+)^2$.

(c) $\mathcal{A}' = \{(n_1, n_2) \in \mathcal{A} \mid n_2 = r_i, \text{ for some } i \in \{0, \ldots, s\}\}$, *where* $r_0 := k$.

(d) $|\mathcal{A}'| = s + 1$.

Proof (a) and (b) follow easily from the proof of Lemma 7.3 and the definition of $\mathcal{A}'$ in 2.2. Then (c) is just the number-theoretic version of the definition of $\mathcal{A}'$ and (d) is an obvious consequence of (c). □

The decomposition $\mathcal{C}$ generated by $\mathcal{W}_{d,k}$ described above produces a resolution $\pi : \mathbf{X}' \to \mathbf{X}_{d,k}$, as shown in Section 3.

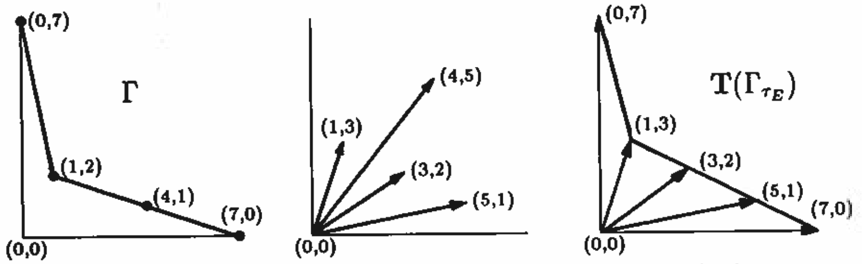
7.5 Example $d = 7$, $k = 3$. We have:

$$b_1 = 3, \quad b_2 = 2, \quad b_3 = 2,$$
$$r_0 = 3, \quad r_1 = 2, \quad r_2 = 1,$$
$$\mathcal{A} = \{(1,3), (2,6), (3,2), (4,5), (5,1), (6,4)\},$$

hence:

$$\mathcal{A}' = \{(1,3),\ (3,2),\ (5,1)\}.$$

The Newton polygon Γ is in Figure (i) below, the primitive vectors of $\mathcal{A}$ are in Figure (ii) and Figure (iii) indicates the 7-cell decomposition and shows the convex polygonal line $\mathbf{T}(\Gamma_{\tau_E})$.



(i) (ii) (iii)

The vector $(4,5)$ is "too long" and does not belong to $\mathcal{A}'$, since $(4,5) = (1,3) + (3,2)$.

7.6 Remarks

(a) There is an order relation on $\mathcal{A}'$ defined in 7.3 and another order relation on $\mathcal{A}'$ induced by the natural (increasing) order of the indices i in the equality 7.4(c). One can easily show that the latter one is the reversed of the former one.

(b) One can show that the resolution $\pi : \mathbf{X}' \to \mathbf{X}_{d,k}$ just described is also minimal (see [Oda, section 1.6]), hence the two resolutions above do, in fact, coincide.

(c) There is also the following way to get the minimal resolution. Consider the toric variety which corresponds to the Newton diagram Γ (see [Oka] for the definition). This variety is not smooth, but has only A_n-singularities; by blowing them up one obtains the minimal resolution of $\mathbf{X}_{d,k}$. See Koelman's thesis [Ko] for details.

A consequence is that the number of 1-faces of Γ is at most equal to $s+1$ and that, for each 1-face of Γ, there is one vector in $\mathcal{A}'$ which is normal to it.

(d) Conversely, consider the minimal resolution of $\mathbf{X}_{d,k}$ and its dual graph. By contracting the chains of (-2)-points, except the end points, we get a graph which corresponds to the toric variety from (c) above. It also follows that the number of 1-faces of Γ is in fact equal to the number of vertices of this contracted graph.

Furthermore, the variety from (c) above is the total space of a *rational double point resolution*, but not the minimal one in case at least one end of the dual graph (mentioned before) is a (-2)-point.

7.7 Due to the 2-dimensionality, if $f_0 \in \mathbf{m}$ is general, then:

$$o_v(\tilde{f}_0) = d, \quad \forall v \in \mathcal{A}', \tag{30}$$

hence: $m(v_\Delta) = d$, $\forall \Delta \in \max(\Gamma)$. One can prove this remark, for instance by induction along the sequence of blowing-ups (27).

7.8 We focus in the following on the determination of the range of the Lefschetz number, over all functions in $\mathbf{m}$.

Lemma *Let $\mathbf{X}_{d,k}$ be a cyclic quotient singularity as above. Then:*

(a) for $k > 1$, $\max\{\Lambda(f) \mid f \in \mathbf{m}\} = 2$ *and* $\Lambda(x^d + y^d) = 2$*;*

(b) for $k = 1$, $\max\{\Lambda(f) \mid f \in \mathbf{m}\} = 1$ *and* $\Lambda(x^d) = 1$.

Proof We use the formula (10) in 4.4. Let $k > 1$ and take an arbitrary $f \in \mathbf{m}$. Since $\mathbf{E}_i \simeq \mathbf{P}^1$, $\forall i \in \{1, \ldots, s+1\}$, we have $\chi(\mathbf{E}_i^0) = 1$, for $i \in \{1, s+1\}$ and $\chi(\mathbf{E}_i^0) = 0$, for $i \in \{2, \ldots, s\}$.

Moreover: $\chi(\mathbf{E}_i^*) \leq \chi(\mathbf{E}_i^0)$, $\forall i \in \{1, \ldots, s+1\}$ and if $f = x^d + y^d$, then all these become equalities.

To prove (b), we use the same arguments; the difference is that $s = 0$ and there is only one irreducible exceptional divisor $\mathbf{E}_1$. □

7.9 We prove that the Lefschetz number $\Lambda(f_0)$, for a general $f_0 \in \mathbf{m}$, is the minimum over all $f \in \mathbf{m}$. It turns out that $\Lambda(f_0) = 3 - e$, where e denotes the *embedding dimension* of $\mathbf{X}_{d,k}$. This is equivalent to $\zeta_{f_0}(t) = (1-t)^{e-3}$ and also to $\mu_{f_0} = e - 2$, by Proposition 5.24. These equalities can be proven in several ways, for instance:

(a) using Kushnirenko's formula for the Milnor number [Ku, Théorème I], as C.T.C. Wall did in [Wall, Theorem 5.1], to prove $\mu_{f_0} = e - 2$,

(b) using the computations for the spectrum from [Sa] and the forthcoming observation 7.10(a),

(c) using Remark 7.18.

7.10 We enumerate several known facts which we make use of. They are essentially due to the 2-dimensional situation. We recall that our group G is cyclic, isomorphic to $\mathbf{Z}_d$. Let $\mathbf{C}[x,y]^G$ be the algebra of G-invariant polynomials. If $L^G := \{(i,j) \mid x^i y^j \in \mathbf{C}[x,y]^G\}$ denotes the subsemigroup of $\mathbf{N}^2$ defined by $\mathbf{C}[x,y]^G$ and M^G the submodule of $\mathbf{Z}^2$ generated by L^G, then $\mathbf{Z}^2/M^G \simeq \mathbf{Z}_d$.

(a) The subset $L^G \cap \Gamma$ generates L^G. (Note that this is not true for $n \geq 3$, as shown in Example 6.3).

(b) Order the set $L^G \cap \Gamma$ by the encountering of its points along the polygonal line Γ, starting from one of its ends. If $L^G \cap \Gamma = \{\alpha_1, \ldots, \alpha_m\}$, then $\mathbf{Z}^2/\mathbf{Z}\langle\alpha_i, \alpha_{i+1}\rangle \simeq \mathbf{Z}_d$, $\forall i \in \{1, \ldots, m-1\}$.

(c) Let $\sigma := \mathbf{R}_{\geq 0}\langle v, u\rangle$ be a d-cell in our decomposition $\mathcal{C}$. Suppose that u is normal to a 1-face Δ_u of Γ. Let $\alpha_1, \ldots, \alpha_l$ be all the G-invariant monomials which have support on Δ_u. Then we have the equality:

$$\{o_v(\alpha_i) \mid i \in \{1, \ldots, l\}\} = \{d, \ldots, l \cdot d\}.$$

7.11 The Newton polygon Γ associated to G has a number of 1-faces equal to:

$$\#\{b_i \neq 2 \mid i \in \{2, \ldots, s\}\} + 2, \quad \text{if } k > 1 \quad \text{(see Remark 7.6(c))},$$

and has just one face, if $k = 1$. Denote by $\Delta(i)$ the 1-face of Γ which has the vector $(*, r_{i-1}) \in \mathcal{A}'$ as normal (see 7.4(c)); the index i must be equal to 1, to $s+1$, or such that $b_i \neq 2$, otherwise $\Delta(i)$ is not maximal. Each $\Delta(i) \in \max(\Gamma)$ supports a number of G-invariant monomials, say $n(i)$.

Lemma *If $k > 1$ then:*

(a) $n(i) = b_i - 1$, *for* $i \neq 1, s+1$.

(b) $n(1) = b_1$, $n(s+1) = b_{s+1}$.

If $k = 1$ then $n(1) = d + 1$.

Proof For $k = 1$, it is obvious. If $k > 1$, the "first" face $\Delta(1)$ contains indeed b_i points from L^G, by the first equality in (28). We prove the statement inductively, along the sequence (27), going backwards. While passing from $\mathbf{X}_{r_i, r_{i-1}}$ to $\mathbf{X}_{r_{i+1}, r_i}$, the two Newton polygons Γ^{i-1}, Γ^i correspond as follows: the first 1-face of the initial polygon Γ^{i-1} is off but the others correspond one-to-one to the 1-faces of Γ^i (in their natural order) and the numbers of invariant monomials supported by corresponding faces is the same. There is only one exception, namely in the last step (i.e. $i = s$): the single 1-face of Γ^s supports one invariant monomial more than its corresponding face of Γ^{s-1}. □

7.12 Let C be the d-cell decomposition generated by $\mathcal{W}_{d,k} = \mathcal{A}' \cup \{de_1, de_2\}$ and let $\mathbf{Y}_C$ be the corresponding toric variety, as in Section 3, with its exceptional divisor $\tilde{\mathbf{E}} := \cup_{v \in \mathcal{A}'} \tilde{\mathbf{E}}_v$. We have: $\tilde{\mathbf{E}}_v \simeq \mathbf{P}^1$, $\forall v \in \mathcal{A}'$. There are two privileged components: $\tilde{\mathbf{E}}_{v_1}$ and $\tilde{\mathbf{E}}_{v_p}$, where $v_1 = (1, k)$ and $v_p = (p, 1)$, the positive integer p verifying: $pk \equiv 1$, (modulo d), $1 \leq p < d$. That is because those curves are the two "ends" in the chain of exceptional components, similar to the one figured in 7.2. We get: $\chi(\tilde{\mathbf{E}}^0_{v_1}) = \chi(\tilde{\mathbf{E}}^0_{v_p}) = 1$.

In our case, because of the remark (30) and Corollary 4.11, the formula 2.6(b) takes the form:

$$\Lambda(f) = \sum_{\Delta \in \max(\Gamma)} \Lambda(f_\Delta). \tag{31}$$

To any $f \in \mathfrak{m}$ and $\Delta \in \max(\Gamma)$ one may associate a polynomial $q_{f,\Delta} \in \mathbf{C}[t]$, as follows.

Definition Let $L^G \cap \Delta = \{(a_0, b_0), \ldots, (a_{i(\Delta)}, b_{i(\Delta)})\}$, where $a_0 < a_1 < \ldots < a_{i(\Delta)}$. If $\tilde{f}_\Delta = \sum_{j \in \{0,\ldots,i(\Delta)\}} \alpha_j x^{a_j} y^{b_j}$, for some coefficients $\alpha_j \in \mathbf{C}$, then define:

$$q_{f,\Delta}(t) := \sum_{j \in \{0,\ldots,i(\Delta)\}} \alpha_j t^j. \tag{32}$$

7.13 Proposition *Let* $f \in \mathbf{m}$ *and* $\Delta \in \max(\Gamma)$. *Then:*

$$-\chi(\mathbf{E}^*_{v_\Delta} \cap \hat{\mathbf{T}}_{v_\Delta}) = \#\{t \in \mathbf{C}^* \mid q_{f,\Delta}(t) = 0\}.$$

Proof Let $\sigma \in C$ such that $\sigma = \mathbf{R}_{\geq 0} < u, v_\Delta >$ and

$$\det \begin{bmatrix} u^1 & v^1_\Delta \\ u^2 & v^2_\Delta \end{bmatrix} = d.$$

By the definition (20) of $\tilde{f}_\sigma$ and the definition (32) of $q_{f,\Delta}$ we get:

$$\tilde{f}_\sigma(t,0) = q_{f,\Delta}(t^d).$$

The result follows, since $-\chi(\mathbf{E}^*_{v_\Delta} \cap \hat{\mathbf{T}}_{v_\Delta}) = \#\{t \in \mathbf{C}^* \mid \tilde{f}_\sigma(t,0) = 0\}$. □

7.14 Corollary *Let* $n(f,\Gamma) := \sum_{\Delta \in \max(\Gamma)} \#\{t \in \mathbf{C}^* \mid q_{f,\Delta}(t) = 0\}$. *Then:*

(a) $\Lambda(f) = -n(f,\Gamma)$, *if* $(d,0) \notin \Gamma_{\tilde{f}}$ *and* $(0,d) \notin \Gamma_{\tilde{f}}$.

(b) $\Lambda(f) = -n(f,\Gamma) + 1$, *if* $(d,0) \in \Gamma_{\tilde{f}}$ *and* $(0,d) \notin \Gamma_{\tilde{f}}$ *or* $(d,0) \notin \Gamma_{\tilde{f}}$ *and* $(0,d) \in \Gamma_{\tilde{f}}$.

(c) $\Lambda(f) = -n(f,\Gamma) + 2$, *if* $(d,0), (0,d) \in \Gamma_{\tilde{f}}$.

Proof It follows easily from the preceding Proposition 7.13 and the formula (12) in 4.7. □

7.15 If $\Delta = \Delta(i) \in \max(\Gamma)$, then $q_{f,\Delta}(t)$ is a polynomial of degree at most $n(i) - 1$. This follows from Lemma 7.11 and the definition of $q_{f,\Delta}$.

Corollary *If* $f_0 \in \mathbf{m}$ *is general, then:* $\Lambda(f_0) = 3 - e$.

Proof If f_0 is general (see Definition 5.2), then the equation $q_{f_0,\Delta(i)}(t) = 0$ has $n(i) - 1$ distinct solutions, for $i \in \{0, \ldots, s+1\}$.

On the other hand, the embedding dimension of $\mathbf{X}_{d,k}$ is equal to the number $\# L^G \cap \Gamma$, hence:

$$e = 1 + \sum_{i \in \{0,\ldots,s+1\}} [n(i) - 1], \tag{33}$$

and our formula follows from Corollary 7.14(c). □

7.16 We have seen that the partial Lefschetz number Λ_{v_Δ}, for $\Delta \in \max(\Gamma)$ is equal to $\Lambda(f_\Delta)$ and is computed from the data provided by the polynomial $q_{f,\Delta}$.

There is an isomorphism of **C**-vector spaces:

$$\mathrm{Gr}(v_{\Delta(i)})_1\mathcal{O} \simeq \mathbf{C}^{n(i)}, \quad i \in \{1,\ldots,s+1\}$$

and one may consider $\Lambda_{v_{\Delta(i)}}$ as a function:

$$\Lambda_{v_{\Delta(i)}} : \mathbf{C}^{n(i)} \longrightarrow \mathbf{Z}.$$

We have, by Proposition 7.13 and Corollary 7.14:

$$\mathrm{Im}\,\Lambda_{v_{\Delta(i)}} = \begin{cases} \mathbf{Z} \cap [-b_i+2, 0] & \text{if } i \notin \{1, s+1\}. \\ \mathbf{Z} \cap [-b_i+1, 1] & \text{if } i \in \{1, s+1\}. \end{cases}$$

A small variation of the argument cannot increase the value of $\Lambda_{v_{\Delta(i)}}$: this comes from the same behaviour of the zeroes of the polynomial $q_{*,\Delta(i)}$ while the coefficients are varying. Some special attention should be payed to the end faces Δ_{v_1}, Δ_{v_p}, see 7.12. The conclusion is that:

Proposition *For any $i \in \{1,\ldots,s+1\}$, the partial Lefschetz number $\Lambda_{v_{\Delta(i)}}$ is an integral-valued, semicontinuous function on $\mathbf{C}^{n(i)}$.* □

It follows that $\Lambda_{v_{\Delta(i)}}$ defines a stratification $\mathcal{S}_i$ on $\mathbf{C}^{n(i)}$, the *coarsest* stratification such that $\Lambda_{v_{\Delta(i)}}$ is constant along a stratum.

7.17 The partial Lefschetz numbers are not independent, since each 0-face of the Newton polygon Γ (except the two ends) belongs to two 1-faces.

Moreover, the full Lefschetz number cannot be semicontinuous as a function:

$$\Lambda : \mathbf{m}/\mathbf{m}^2 \simeq \mathbf{C}^e \to \mathbf{Z},$$

because of the trouble at the "ends": $\Lambda(x^d) = 1$, but $\Lambda(x^d + ty^d) = 2$, $\forall t \in \mathbf{C}^*$, which show that the specialization $t \to 0$ decreases the value of Λ. But, if we exclude the two generators of $\mathbf{m}/\mathbf{m}^2$ corresponding to x^d and y^d, the semicontinuity of the restriction:

$$\Lambda_| : \mathbf{C}^{e-2} \to \mathbf{Z}$$

is saved. However, we can prove:

Proposition

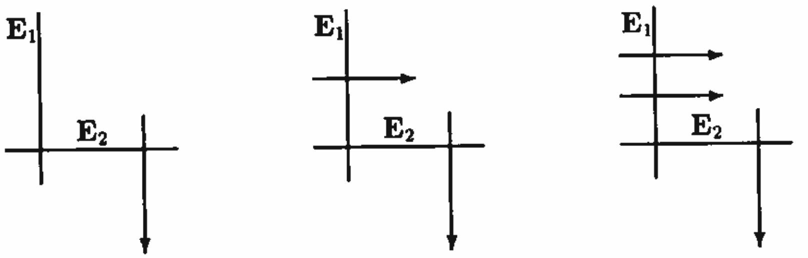
- *If* $k > 1$, *then* $\operatorname{Im}\Lambda = \mathbf{Z} \cap [3-e, 2]$.
- *If* $k = 1$, *then* $\operatorname{Im}\Lambda = \mathbf{Z} \cap [3-e, 1]$.

Proof It is clear, from Corollary 7.14 and Corrolary 7.15, that $3-e$ is the minimum of Λ and from Lemma 7.8, that the maximum is 2 (if $k > 1$) or 1 (if $k = 1$).

We order the G-invariant monomials supported by Γ as follows: $x^{i_1}y^{j_1} \leq x^{i_2}y^{j_2}$ if and only if $i_1 \leq i_2$. The monomial y^d is the first one in our ordering.

The strategy of obtaining all the intermediate values can be the following: Start with $\tilde{f}_1 := y^d$, which gives $\Lambda = 1$, then add the next monomial with a suitable coefficient such that the new function $\tilde{f}_2$ is $\mathcal{W}_{d,k}$-nondegenerate, and so on. Construct in this way a sequence of functions $\tilde{f}_1, \ldots, \tilde{f}_{e-1}$. At each step a new arrow appears in the resolution graph of the function (the $\mathcal{W}_{d,k}$-nondegeneracy insures that $\{f_j \circ \pi = 0\}$ is a n.c. divisor, for any index j). Hence we get: $\Lambda(f_i) = -i + 2$, and that's all we need to prove. □

Example For $n = 2$, $d = 5$, $k = 2$ we get:



y^5 $\qquad$ $y^5 + xy^2$ $\qquad$ $y^5 + xy^2 + x^3y$

7.18 Remark Another way of studying the behaviour of Λ is by starting from a general function f_0 and specialize as possible.

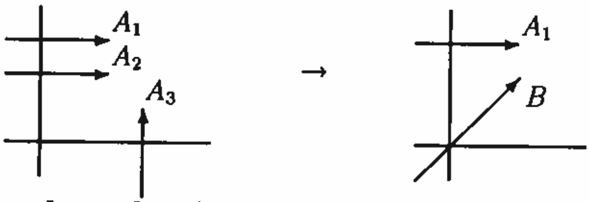
By a theorem of Artin [Ar-3, Theorem 4], if Z_π is the *fundamental cycle*, then any divisor $D > Z_\pi$ on $\mathbf{X}'$ such that $D \cdot \mathbf{E}_i = 0$, $\forall i \in \{1, \ldots, s+1\}$, is the divisor of $f \circ \pi$, for some $f \in \mathfrak{m}$.

Since, in our case, the fundamental cycle is equal to the sum $\sum_{i\in\{1,\ldots,s+1\}} \mathbf{E}_i$, we get the intersection numbers of $\{f \circ \pi = 0\}$ with each $\mathbf{E}_i$, hence the Lefschetz numbers of a general function and of all the possible specializations of it.

As explained to us by Van Straten, in this way one produces an alternative proof of Proposition 7.17. Moreover, one can get some information about the zeta-function of germs in the class $\mathcal{M}_{\mathcal{W}_{d,k}}$.

The theorem of Artin works for isolated rational surface singularities. In this higher generality, if the resolution graph is given, then one would be able to determine the range of the Lefschetz number.

7.19 Specializing the general function is equally possible by our method (in the cyclic quotient case). We can observe during the specializations why the partial Λ's are dependent. A short illustration (with the Example 7.17 above) is the following:



$y^5 + txy^2 + x^3y$ $\qquad\qquad$ $y^5 + x^3y$

The two arrows A_2 and A_3 come from a single noncompact component with multiplicity 1 and they specialize to a component which corresponds to the arrow B. Note that B has multiplicity 1.

7.20 The zeta-function

For $\mathcal{W}_{d,k}$-nondegenerate functions, the zeta-function can be computed by the formula (21), but for some other functions, it depends not only on the data from the resolution $\pi : \mathbf{X}' \to \mathbf{X}$ (since one has to resolve the divisor $\{f \circ \pi = 0\} \subset \mathbf{X}'$ at a finite number of points $\{a_1, \ldots, a_{\eta(f)}\} \subset \mathbf{X}'$ to get a n.c. situation), hence not only on the Newton polyhedron $\Gamma_{\tilde{f}}$.

However, one can separate the data from the resolution π such that the zeta-function splits into two products:

Theorem

$$\zeta_f(t) = [\prod_{T \in \mathcal{S},\ v \in \mathcal{A}'} (1 - t^{o_v(\tilde{f})/d})^{-d^{-\dim T} \cdot \chi(T \cap \check{\mathbf{E}}_v^*)}] \cdot [\prod_{i \in \{1, \ldots, \eta(f)\}} \zeta_{g_i}(t)],$$

where $g_i = 0$ is the equation of the (reducible, nonreduced) germ of the divisor $\{f \circ \pi = 0\}$ at the point $a_i \in \mathbf{X}'$.

Proof The formula (19) gives the first part of the above formula (see Remark 5.10). The second part is, in our case, an easy consequence of the definition of the germs g_i. □

References

[Ar-1] M. Artin, *On the solutions of analytic equations*, Inventiones Math., **5** (1966), pp.129–136.

[Ar-2] M. Artin, *The implicit function theorem in algebraic geometry*, Algebraic Geometry, International Colloq. Bombay 1968, Oxford Univ. Press, London 1969, pp. 13–34.

[Ar-3] M. Artin, *On isolated rational singularities of surfaces*, Amer. J. Math., **88** (1966), pp. 129–136.

[BK] E. Brieskorn, H. Knörer, *Ebene algebraische Kurven*, Birkhäuser, 1981.

[A'C-1] N. A'Campo, *Le nombre de Lefschetz d'une monodromie*, Indag. Math., **35** (1973), pp.113–118.

[A'C-2] N. A'Campo, *La fonction zêta d'une monodromie*, Comment. Math. Helvetici, **50** (1975), pp. 233–248.

[Di] A. Dimca, *On the homology and cohomology of complete intersections with isolated singularities*, Compositio Math., **58** (1986), pp. 321–339.

[Eh] F. Ehlers, *Eine Klasse komplexer Mannigfaltigkeiten und die Auflösung einiger isolierter Singularitäten*, Math. Annalen, **218** (1975), pp. 127–156.

[EN] D. Eisenbud, W. Neumann, *Three-dimensional link theory and invariants of plane curve singularities*, Annals of Math. Studies, **110** (1985), Princeton Univ. Press.

[Fi] J. Fine, *Some essential valuations in the resolution of singularities*, A. M. S. Proc. of Symp. in Pure Math. **40**, part 1 (1983), pp.381–383.

[GM] M. Goresky, R. MacPherson, *Stratified Morse Theory*, Springer-Verlag, Berlin-Heidelberg-New York 1988.

[Ha] H. Hamm, *Zum Homotopietyp Steinische Räume*, J. Reine Angew. Math. **364** (1986), pp. 1–9.

[Hi-1] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Annals of Math.,(2) **79** (1964), pp. 109–326.

[Hi-2] H. Hironaka, *Stratification and flattness*, Real and Complex Singularities Oslo 1976, Sijhoff en Nordhoff, Alphen a.d. Rijn 1977, pp. 199–266.

[KKMS] G. Kempf, F. Knudsen, D. Mumford, B. Saint-Donat, *Toroidal Embeddings I*, Lecture Notes in Mathemetics **339**, Springer-Verlag, Berlin-Heidelberg-New York 1973 .

[Ko] R.J. Koelman, *The number of moduli of families of curves on toric surfaces*, Ph.D. thesis, Nijmegen 1991.

[Ku] A.G. Kushnirenko, *Polyhèdres de Newton et nombres de Milnor*, Inventiones Math., **32** (1976), pp. 1–31.

[Lam] K. Lamotke, *Regular Solids and Isolated Singularities*, Vieweg, Braunschweig; Wiesbaden, 1986.

[Lê-1] D.T. Lê, *La monodromie n'a pas de points fixes*, J. of Fac. Sc., Univ. Tokyo, sec.1A, **22** (1973), pp. 409–427.

[Lê-2] D.T. Lê, *The geometry of the monodromy theorem*, C.P. Ramanujam – a tribute, Tata Institute, Springer-Verlag 1978.

[Lê-3] D.T. Lê, *Some remarks on the relative monodromy*, Real and Complex Singularities Oslo 1976, Sijhoff en Nordhoff, Alphen a.d. Rijn 1977, pp. 397–403.

[Lê-4] D.T. Lê, *La théorème de monodromie singuliére*, C.R. Acad. Sci. Paris t **288**, III (1979), pp. 985–988.

[Lê-5] D.T. Lê, *Calcul du nombre de cycles évanouissants d'une hypersurface complexe*, Ann. Inst. Fourier, Grenoble, **23**, 4 (1973), pp. 261–270.

[Lê-6] D.T. Lê, *Sur les noeuds algebriques*, Compositio Math., **25** (1972), pp. 281–321.

[Mi] J. Milnor, *Singular points of complex hypersurfaces*, Ann. of Math. Studies 61, Princeton 1968.

[Ne] A. Némethi, *The zeta-function of singularities*, to appear in Jour. of Algebraic Geometry, (1992).

[Oda] T. Oda, *Convex Bodies and Algebraic Geometry*, Springer Verlag, Berlin-Heidelberg-New York 1988.

[Oka] M. Oka, *On the Resolution of the Hypersurface Singularities*, Advanced Studies in Pure Math., **8** (1986), pp. 405–436.

[Sa] M. Saito, *Exponents and Newton Polyhedra of Isolated Hypersurface Singularities*, Math. Ann., **281** (1988), pp. 411–417.

[Sam] P. Samuel, *Algebricité de certains points singuliers algebroides*, Jour. Math. Pures et Appl. **35** (1956), pp. 1–6.

[Sch] R. Schrauwen, *Series of Singularities and Their Topology*, Ph. D. thesis, Utrecht 1991.

[Si] D. Siersma, *The monodromy of a series of hypersurface singularities*, Comment. Math. Helvetici, **65** (1990), pp. 181–197.

[St] J.H.M. Steenbrink, *On the Picard group of certain smooth surfaces in weighted projective spaces*, Algebraic Geometry, Procedings La Rabida 1981, Lect. Notes in Mathematics **961** (1982).

[Th] R. Thom, *Ensembles et morphismes stratifiés*, Bull. A.M.S., **75** (1969), pp. 240–284.

[Var] A.N. Varchenko, *Zeta-Function of Monodromy and Newton's Diagram*, Inventiones Math., **37** (1976), pp. 253–262.

[Wahl] J. Wahl, *Smoothings of normal surface singularities*, Topology, **20** (1981), pp. 219–246.

[Wall] C.T.C. Wall, *Functions on quotient singularities*, Phil. Trans. R. Soc. Lond., A **324** (1987), pp. 1–45.

Acknowledgements

I am especially grateful to my advisors Dirk Siersma and Joseph Steenbrink, for their continuous, stimulating support. Their ideas and suggestions have been of great help for me.

I am much indebted to Alexandru Dimca, who introduced me some time ago to the field of Singularities and from whose comments I have learned a lot ever since.

I wish to thank Frans Oort and Jan Stevens for some useful comments on the manuscript and Rob Schrauwen and Rutger Noot for considerably improving the final layout.

I also wish to thank the Mathematical Institute and the University of Utrecht for their support and hospitality.

Special thanks go to my wife Teodorina and all my family for their loving understanding and patience.

Samenvatting in het Nederlands

Elke kiem van een holomorfe functie $f : (\mathbf{X}, x) \to (\mathbf{C}, 0)$ op een analytische ruimte $(\mathbf{X}, x)$ definieert een (algebraische) *monodromie*

$$h_f : H^i(F_f, \mathbf{C}) \to H^i(F_f, \mathbf{C}),$$

waar F_f de Milnorvezel van f is.

In dit proefschrift bestuderen we het Lefschetzgetal $\Lambda(h_f)$ van de monodromie—en machten van de monodromie—van functies op singuliere kiemen.

Over een gladde ruimte kunnen we kort zijn: A'Campo [A'C-1] bewees dat $\Lambda(h_f) = 0$ als f singulier is en $\Lambda(h_f) = 1$ als f regulier is.

Lê merkte al op in [Lê-3] dat de zaak veel ingewikkelder wordt als $(\mathbf{X}, x)$ niet glad is. Dat geval is van groter belang nu de belangstelling voor (bijzondere) singuliere ruimten en snedes daarvan, toeneemt.

In Hoofstuk I bewijzen we algemene resultaten over het Lefschetzgetal en de zetafunctie. Hierbij maken we gebruik van een verbeterde versie van de befaamde *draaimolen-* (caroussel-) constructie van Lê.

Zij $l : (\mathbf{X}, x) \to (\mathbf{C}, 0)$ een voldoende algemene lineaire functie. Eerst bewijzen we dat als de Puiseuxverhoudingen van de takken van het Cerfdiagram $\Delta(l, f)$ alle geheel zijn, het Lefschetzgetal $\Lambda(h_f)$ gelijk is aan het Lefschetzgetal van de monodromie van de beperking $f_{|\{l=0\}}$. Dit resultaat blijkt een aantal interessante gevolgen te hebben.

We bewijzen een formule voor het Lefschetzgetal en een formule voor de zetafunctie in termen van het Lefschetzgetal resp. de zetafunctie van een eindig aantal *carrouselmonodromieën*. Deze formules, die zeer algemeen zijn, kunnen in bepaalde gevallen lastige berekeningen vereisen.

Onze constructie in Hoofdstuk I levert een fijne *polaire decompositie* van de Milnorvezel op, die gelijkenis vertoont met A'Campo's decompositie [A'C-2], zie Hoofdstuk II, 1.2.

Hoofdstuk II is gebaseerd op methode van A'Campo [A'C-2], waarin de resolutie van singulariteiten een rol speelt.

In Sectie 1 van Hoofdstuk II bewijzen we dat als $f : (\mathbf{X}, x) \to (\mathbf{C}, 0)$ een zgn. *smoothing* is, het Lefschetzgetal alleen afhangt van de residuklasse van f in $\mathbf{m}_{\mathbf{X},x}/\mathcal{F}_{\mathbf{X},x}$, waar $\mathbf{m}_{\mathbf{X},x}$ het maximale ideaal van de locale algebra is en $\mathcal{F}_{\mathbf{X},x}$ een ideaal dat niet van de resolutie afhangt. $\mathcal{F}_{\mathbf{X},x}$ is bovendien de doorsnede van een eindig aantal "minimale" idealen die ook niet van de resolutie afhangen. In het bijzonder, als $(\mathbf{X},0)$ geïsoleerd is, volgt uit het feit dat $\mathcal{F}_{\mathbf{X},x}$ het ideaal $\mathbf{m}^2_{\mathbf{X},x}$ bevat, dat het Lefschetzgetal alleen afhangt van de klasse van f modulo $\mathbf{m}^2_{\mathbf{X},x}$.

Vanaf Sectie 2 richten we de aandacht op het speciale geval dat $(\mathbf{X},x) \simeq (\mathbf{C}^n/G,0)$ een geïsoleerde cyclische quotiëntsingulariteit is (waar G een eindige cyclische groep is). In dit geval blijken de resultaten van Sectie 1 een bijzonder mooie vorm te hebben: het Lefschetzgetal van een functie is de som van goed gedefinieerde "delen" van de functie. Bij elk zo'n deel behoort een G-invariante gewogen-homogene veelterm, waarvan de gewichten alleen van de groepsactie afhangen.

Gebruikmakend van de eindige groepsactie construeren we een torische resolutie van de cyclische quotiëntsingulariteit samen met een speciaal diagram (zie 3.1). Deze constructie komt in de plaats van een hypothetische opeenvolging van G-invariante opblaasacties $\mathbf{Y} \to \mathbf{C}^n$ langs G-stabiele niet-singuliere deelvarieteiten zo, dat het quotiënt $\mathbf{Y}/G$ niet-singulier is—iets wat men in het algemeen helaas niet kan doen (zie [Oda, p. 31]).

In Sectie 5 definiëren we een klasse van niet-gedegenereerde functies waarvoor we een wat praktischer formule voor de zetafunctie kunnen bewijzen, die gebaseerd is op de voorgaande resultaten en Varchenko's aanpak van de zetafunctie [Var]. In het bijzonder verkrijgen we de formule voor de zetafunctie van een algemene lineaire snede. Als we ons tot het Lefschetzgetal beperken, krijgen we een veel grotere klasse van functies waarvoor de corresponderende formule toepasbaar is.

In Sectie 6 verduidelijken we de resultaten met een aantal voorbeelden, waarvan er één laat zien dat $\zeta_f(t) = \zeta_{\tilde{f}}(t^d)$ niet geldt, zelfs niet voor een lineaire functie f (met $\tilde{f}$ de bijbehorende G-invariante functie en $d = |G|$).

In de laatste sectie kunnen we nog explicieter zijn door ons te beperken tot tweedimensionale quotiënten. Zo bewijzen we wat het bereik van het Lefschetzgetal is, en een splijtingsformule voor de zetafunctie.

Rezumat în limba română

Un germene de funcţie holomorfă $f : (\mathbf{X}, x) \to (\mathbf{C}, 0)$ pe un germene de spaţiu analitic $(\mathbf{X}, x)$ defineşte o *monodromie (algebrică)*:

$$h_f : H^i(F_f, \mathbf{C}) \to H^i(F_f, \mathbf{C}),$$

unde F_f este fibra Milnor a lui f.

Numărul Lefschetz $\Lambda(h_f) := \sum_{i \geq 0} (-1)^i \text{trace}[h_f \; ; \; H^i(F_f, \mathbf{C})]$ al monodromiei (mai general, numărul Lefschetz al unei puteri întregi a monodromici) face obiectul investigaţiei din această teză.

Dacă spaţiul $(\mathbf{X}, x)$ este neted, atunci, datorită unui rezultat al lui A'Campo [A'C-1], se cunoaşte că $\Lambda(h_f) = 0$ dacă f este singular, respectiv $\Lambda(h_f) = 1$ dacă f este regulat. Situaţia se complică în cazul cînd $(\mathbf{X}, x)$ nu mai e neted.

În primul capitol al tezei producem o rafinare a „caruselului" lui Lê pentru a demonstra rezultate despre numărul Lefschetz şi—mai general—funcţia zeta ale monodromiei, oricare ar fi spaţiul de bază $(\mathbf{X}, x)$.

În Capitolul II folosim metoda lui A'Campo [A'C-2], bazată pe rezoluţia singularităţilor. Un loc central îl ocupă cazul $(\mathbf{X}, x) \simeq (\mathbf{C}^n/G, 0)$, unde G este un grup ciclic finit.

Construim o anumită rezoluţie torică a spaţiului-cît $\mathbf{C}^n/G$, care face posibil studiul funcţiei zeta a lui f in legătură cu acţiunea grupului G pe spaţiul $\mathbf{C}^n$. Demonstrăm apoi cîteva rezultate importante bazate pe această construcţie.

În Secţiunea 5 definim o clasă de funcţii „nedegenerate" pentru care putem demonstra o formulă mai „practică" de calcul al funcţiei zeta. Rezultatul se bazează pe metoda lui Varchenko [Var] de abordare a funcţiei zeta, în termeni de poliedre Newton.

Secţiunea 6 conţine exemple semnificative care ilustrează o parte din rezultatele Capitolului II.

În ultima secţiune ne ocupăm de cazul particular în care spaţiul-cît are dimensiune 2; aici rezultatele obţinute anterior se pot explicita mai mult. Spre exemplu, găsim codomeniul lui $\Lambda(h_f)$—ca funcţie de f; el depinde de dimensiunea de scufundare a spaţiului $\mathbf{C}^2/G$.

Curriculum Vitae

The author was born on March 15, 1960 at Braşov, Romania. Between 1975 and 1979 he followed the high-school "I. Meşotă" in Braşov. He was a student at the Faculty of Mathematics, University of Bucharest, beginning with 1980. In June 1984 he took his licence and then followed one year special courses in Algebra and Geometry at the same university.

After two years (1985-1987) of teaching in a high-school at Prejmer, Braşov, he became a junior researcher at the Institute of Mathematics of the Romanian Academy, Bucharest.

From September 1990 until September 1992 he prepared his Ph.D. thesis in the field of Singularities at the Mathematical Institute, Utrecht University, with Prof. dr. D. Siersma and Prof. dr. J.H.M. Steenbrink as supervisors.