REPRODUCING KERNEL HILBERT SPACES AND RANDOM MEASURES

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We show how to use Guilbart's embedding of signed measures into a R.K.H.S. to study some limit theorems for random measures and stochastic processes.

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1. R.K.H.S. and metrics on signed measures

In the late seventies, C. Guilbart [4, 5] introduced an embedding into a reproducing kernel Hilbert space (R.K.H.S.) $\mathcal{H}$ of the space $\mathcal{M}$ of signed measures on some topological space $\mathfrak{X}$. He characterized the inner products on $\mathcal{M}$ inducing the weak topology on the subspace $\mathcal{M}^+$ of bounded positive measures and established in this setting a Glivenko-Cantelli theorem with applications to estimation and hypothesis testing. In this contribution we present a constructive approach of Guilbart's embedding following [20]. This embedding provides a Hilbertian framework for signed random measures. We shall discuss some applications of this construction to limit theorems for random measures and partial sums processes.

Let $\mathfrak{X}$ be a metric space and let $\mathcal{M}$ denote the space of *signed measures* on the Borel σ-field of $\mathfrak{X}$. A signed measure μ is the difference of two positive bounded measures. We denote by (μ^+, μ^-) its Hahn-Jordan decomposition and by $|\mu| = \mu^+ + \mu^-$ its total variation measure. We consider the class of reproducing kernels having the following representation

$$K(x,y) = \int_{\mathbb{U}} r(x,u)\overline{r(y,u)}\rho(\mathrm{d}u), \quad x,y \in \mathfrak{X}, \tag{1}$$

where ρ is a positive measure on some measurable space $(\mathbb{U}, \mathcal{U})$ and the function $r : \mathfrak{X} \times \mathbb{U} \to \mathbb{C}$ satisfies

$$\sup_{x\in\mathfrak{X}} \|r(x,\,.\,)\|_{L^2(\rho)} < \infty. \tag{2}$$

We denote by $\mathcal{H}$ the reproducing kernel Hilbert space associated with K. It is easily checked (Prop.2 in [20]) that under (2), $r(.,u)$ is μ-integrable over $\mathfrak{X}$ for ρ-almost

$u \in \mathbb{U}$. We assume moreover that

$$\text{if } \mu \in \mathcal{M} \text{ and } \int_{\mathfrak{X}} r(x,u)\mu(\mathrm{d}x) = 0 \text{ for } \rho\text{-almost } u, \text{ then } \mu = 0. \tag{3}$$

The essential facts about the embeddings of $\mathcal{M}$ into $\mathcal{H}$ and $L^2(\rho)$ are gathered in the following theorem which is proved in [20].

Theorem 1.1. *Under (1), (2) and (3), the following properties hold.*

a) Let E be the closed subspace of $L^2(\rho)$ spanned by $\{r(x,.),\ x \in \mathfrak{X}\}$. A function $h : \mathfrak{X} \to \mathbb{C}$ belongs to $\mathcal{H}$ if and only if there is a unique $g \in L^2(\rho)$ such that

$$h(x) = \int_{\mathbb{U}} g(u)\overline{r(x,u)}\rho(\mathrm{d}u), \quad x \in \mathfrak{X}. \tag{4}$$

The representation (4) defines an isometry of Hilbert spaces $\Psi : \mathcal{H} \to E$, $h \mapsto g$.

b) K induces an inner product on $\mathcal{M}$ by the formula

$$\langle \mu, \nu \rangle_K := \int_{\mathfrak{X}^2} K(x,y)\mu \otimes \nu(\mathrm{d}x, \mathrm{d}y), \quad \mu, \nu \in \mathcal{M}. \tag{5}$$

c) $(\mathcal{M}, \langle .,. \rangle_K)$ is isometric to a dense subspace of $\mathcal{H}$ by

$$\mathcal{I} : \mathcal{M} \to \mathcal{H}, \quad \mu \longmapsto \mathcal{I}_\mu := \int_{\mathfrak{X}} K(x,.)\mu(\mathrm{d}x). \tag{6}$$

Moreover we have

$$\langle h, \mathcal{I}_\mu \rangle = \int_{\mathfrak{X}} h \,\mathrm{d}\mu, \quad \langle \mathcal{I}_\mu, h \rangle = \int_{\mathfrak{X}} \overline{h} \,\mathrm{d}\mu, \quad h \in \mathcal{H}, \mu \in \mathcal{M}. \tag{7}$$

d) The isometric embedding $\zeta = \Psi \circ \mathcal{I} : \mu \mapsto \zeta_\mu$ of $\mathcal{M}$ into $L^2(\rho)$ satisfies

$$\zeta_\mu(u) = \int_{\mathfrak{X}} r(x,u)\mu(\mathrm{d}x), \quad u \in \mathbb{U}. \tag{8}$$

Let us examine some examples where Theorem 1.1 applies.

Example 1.1. Take for ρ the counting measure on $\mathbb{U} = \mathbb{N}$ and define r by $r(x,i) := f_i(x)$, $x \in \mathfrak{X}$, $i \in \mathbb{N}$, where the sequence of functions $f_i : \mathfrak{X} \to \mathbb{R}$ separates the measures, i.e. the only $\mu \in \mathcal{M}$ such that $\int_{\mathfrak{X}} f_i \,\mathrm{d}\mu = 0$ for all $i \in \mathbb{N}$ is the null measure. To have a bounded kernel we also assume that $\sum_{i \in \mathbb{N}} \|f_i\|_\infty^2 < \infty$. Then

$$K(x,y) = \sum_{i \in \mathbb{N}} f_i(x) f_i(y), \quad x, y \in \mathfrak{X}^2.$$

μ is represented in $\ell^2(\mathbb{N})$ by $\zeta_\mu = \left(\int_{\mathfrak{X}} f_i \,\mathrm{d}\mu\right)_{i \in \mathbb{N}}$ and in $\mathcal{H}$ by $\mathcal{I}_\mu = \sum_{i \in \mathbb{N}} \left(\int_{\mathfrak{X}} f_i \,\mathrm{d}\mu\right) f_i$. It easily follows from (4) that every f_i belongs to $\mathcal{H}$.

Example 1.2. Take $\mathfrak{X} = \mathbb{U} = \mathbb{R}^d$, with $r(x,u) := \exp(i\langle x, u \rangle)$, $x, u \in \mathbb{R}^d$ and choose ρ as a bounded positive measure on $\mathbb{R}^d$. This gives the continuous *stationary* kernels

$$K(x,y) = \int_{\mathbb{R}^d} \exp(i\langle x - y, u \rangle)\rho(\mathrm{d}u), \quad x, y \in \mathbb{R}^d.$$

Here $\zeta_\mu(u) = \int_{\mathbb{R}^d} \exp(i\langle x, u\rangle)\mu(\mathrm{d}x) =: \hat{\mu}(u)$, is the characteristic function of μ and $\mathfrak{I}_\mu(x) = \int_{\mathbb{R}^d} \exp(-i\langle x, u\rangle)\hat{\mu}(u)\rho(\mathrm{d}u)$. These kernels are used in [20] to study the convergence rate in the CLT.

Example 1.3. Take $\mathfrak{X} = \mathbb{U} = [0,1]$, $\rho = \lambda + \delta_1$, where λ is the Lebesgue measure and δ_1 the Dirac mass at the point 1. With $r(x,u) := \mathbf{1}_{[x,1]}(u)$, we obtain $K(x,y) = 2 - \max(x,y)$ and $\zeta_\mu(u) = \mu([0,u])$.

Remark 1.1. The usual topologies on $\mathcal{M}$ are generated by functionals $f \mapsto \int_{\mathfrak{X}} f\,\mathrm{d}\mu$, $f \in F$, where F is some family of continuous functions defined on $\mathfrak{X}$. When $\mathfrak{X}$ is locally compact, $F = C(\mathfrak{X})$, the space of all bounded continuous functions on $\mathfrak{X}$ gives the weak topology while restricting to $F = C_0(\mathfrak{X})$ the space of continuous function converging to zero at infinity gives the vague topology. By convergence to zero at infinity we mean that for every positive ε there is a compact subset A of $\mathfrak{X}$ such that $|f(x)| < \varepsilon$ for every $x \in \mathfrak{X} \setminus A$. In the special case where $\mathfrak{X}$ is compact, $C(\mathfrak{X}) = C_0(\mathfrak{X})$. Endowed with the supremum norm, $C_0(\mathfrak{X})$ is a Banach space with topological dual $\mathcal{M}$ (Riesz's theorem). Now if we choose in Example 1.1 the f_i's in $C_0(\mathfrak{X})$, a simple Hahn-Banach argument gives the density of $\mathcal{H}$ in $C_0(\mathfrak{X})$. In this setting, let $(\mu_n)_{n\geq 1}$ be a sequence in $\mathcal{M}$ such that $\sup_{n\geq 1} |\mu_n|(\mathfrak{X}) < \infty$. Then weak and strong convergence in $\mathcal{H}$ of $\mathfrak{I}_{\mu_n}$ to $\mathfrak{I}_\mu$ are equivalent to the weak convergence in $\mathcal{M}$ of μ_n to μ.

2. Some limit theorems for random measures

2.1. *Random measures*

A random measure $\mu^\bullet$ is a random element in a set $\mathfrak{M}$ of measures equipped with some σ-field $\mathcal{G}$, i.e. a measurable mapping

$$\mu^\bullet : (\Omega, \mathcal{F}, P) \longrightarrow (\mathfrak{M}, \mathcal{G}), \quad \omega \mapsto \mu^\omega.$$

Here $(\Omega, \mathcal{F}, P)$ is a probability space and the law or distribution of $\mu^\bullet$ (under P) is the image measure $P \circ (\mu^\bullet)^{-1}$ on $\mathcal{G}$. Among the well known examples of random measures let us mention the empirical process $\mu_n^\bullet = n^{-1}\sum_{i=1}^n \delta_{X_i}$, where the X_i's are random elements in the space $\mathfrak{X}$ and the point processes $\sum_{i=1}^N \delta_{Y_i}$, where N and the Y_i's are random. In the classical theory, e.g. Kallenberg [7], $\mathfrak{X}$ is locally compact with a countable basis of neighborhoods, $\mathfrak{M}$ is the set of *positive* Radon measures on the Borel σ-field of $\mathfrak{X}$ and $\mathfrak{M}$ is endowed with the Borel σ-field $\mathcal{G}$ of the vague topology. This framework of positive measures is sufficient to the classical study of point processes and positive random measures. But the above setting does not cover the case of signed measures. Still random signed measures appear naturally by centering of positive ones [6]. Guilbart's embedding of $\mathcal{M}$ in an R.K.H.S. $\mathcal{H}$ provides the background for a Hilbertian theory of *signed* random measures. This way we can exploit the nice probabilistic properties of Hilbert spaces and obtain useful limit theorems like CLT or FCLT.

From now on, we assume for simplicity that $\mathfrak{X}$ is metric locally compact and that K is as in Example 1.1 with the f_i's in $C_0(\mathfrak{X})$. Identifying $\mathcal{H}$ with a completion of $\mathcal{M}$, we call random measure a random element $\mu^\bullet$ in $\mathcal{H}$ such that $P(\mu^\bullet \in \mathcal{M}) = 1$. The *observations* of such a random measure are the random variables $\langle h, \mu^\bullet\rangle_K = \int_{\mathfrak{X}} h \, \mathrm{d}\mu^\bullet$, $h \in \mathcal{H}$, accounting (7). Some natural measurability questions raised by our definition of random measures are positively answered in [19]: $\mathcal{M}$ is a Borel subset of $\mathcal{H}$, $|\mu^\bullet|$ is also a random measure, the $\int_{\mathfrak{X}} f \, \mathrm{d}\mu^\bullet$'s, $f \in C_0(\mathfrak{X})$, and $|\mu^\bullet|(\mathfrak{X})$ are random variables.

2.2. *Strong law of large numbers*

If $\mathbf{E}\|\mu^\bullet\|_K$ is finite, the random measure $\mu^\bullet$ is Bochner integrable and $\mathbf{E}\mu^\bullet$ is defined as a deterministic element of $\mathcal{H}$. Then $\mu^\bullet$ is also Pettis integrable, when

$$\mathbf{E}\langle h, \mu^\bullet\rangle_K = \langle h, \mathbf{E}\mu^\bullet\rangle_K, \quad h \in \mathcal{H}. \tag{9}$$

The following theorem is an immediate application of the strong law of large numbers in separable Banach spaces, see e.g. [9].

Theorem 2.1. *Let $\mu_1^\bullet, \dots, \mu_n^\bullet, \dots$ be independent identically distributed copies of $\mu^\bullet$. If $\mathbf{E}\|\mu^\bullet\|_K$ is finite, then*

$$\nu_n^\bullet := \frac{1}{n}\sum_{i=1}^{n} \mu_i^\bullet \xrightarrow[\text{a.s.}]{\mathcal{H}} \mathbf{E}\mu^\bullet. \tag{10}$$

Conversely, if $\nu_n^\bullet$ converges almost surely in $\mathcal{H}$ to some limit ℓ, this limit is deterministic, $\mathbf{E}\|\mu^\bullet\|_K$ is finite and $\ell = \mathbf{E}\mu^\bullet$.

Although $\nu_n^\bullet$ is obviously a random measure, it is not clear that the same holds true for its a.s. limit $\mathbf{E}\mu^\bullet$. When $\mathbf{E}\mu^\bullet$ belongs to $\mathcal{M}$, we call it the *mean measure of* $\mu^\bullet$. In this case, (9) can be recast as

$$\mathbf{E}\langle h, \mu^\bullet\rangle_K = \int_{\mathfrak{X}} h \, \mathrm{d}(\mathbf{E}\mu^\bullet), \quad h \in \mathcal{H}. \tag{11}$$

Here is a simple sufficient condition for the existence of the mean measure.

Proposition 2.1. *The membership of $\mathbf{E}\mu^\bullet$ in $\mathcal{M}$ follows from the finiteness of $\mathbf{E}|\mu^\bullet|(\mathcal{X})$ if $\mathfrak{X}$ is locally compact, K is continuous on $\mathfrak{X}^2$ and $K(x, .) \in C_0(\mathfrak{X})$ for every $x \in \mathfrak{X}$.*

The proof (cf. Prop. XI.1.2 in [17]) relies on the characterization of measures in $\mathcal{H}$ by

$$g \in \mathcal{I}(\mathcal{M}) \quad \text{iff} \quad \sup_{f \in \mathcal{H}, \|f\|_\infty \le 1} |\langle f, g\rangle| < \infty, \tag{12}$$

using the fact that when finite, the supremum in (12) equals $|\mu|(\mathfrak{X})$, where $\mu := \mathcal{I}^{-1}(g)$, together with the elementary estimate

$$\|\mu\|_K \le \big(\sup_{\mathfrak{X}^2} K\big)^{1/2} |\mu|(\mathfrak{X}), \quad \mu \in \mathcal{M}. \tag{13}$$

Corollary 2.1. *If* $\mathbf{E}|\mu^\bullet|(\mathcal{X}) < \infty$, *let* μ *be the mean measure of* $\mu^\bullet$. *Then the a.s. convergence of* $\nu_n^\bullet$ *to* μ *holds both in* $\mathcal{H}$ *and in the weak topology on* $\mathcal{M}$.

The a.s. convergence in $\mathcal{H}$ obviously follows from Theorem 2.1 by applying (13) to $\mu^\bullet$. By Remark 1.1, (10) implies the a.s. weak convergence in $\mathcal{M}$ of $\nu_n^\bullet$ to μ provided that $\sup_{n\geq 1}|\nu_n^\bullet|(\mathfrak{X}) < \infty$. This uniform boundedness follows from the estimate $|\nu_n^\bullet|(\mathfrak{X}) \leq n^{-1}\sum_{i=1}^n |\mu_i^\bullet|(\mathfrak{X})$ and of the a.s. convergence of this upper bound to $\mathbf{E}|\mu^\bullet|(\mathcal{X})$ by the strong law of large numbers applied to the i.i.d. random variables $|\mu_i^\bullet|(\mathfrak{X})$.

2.3. *Central limit theorem for i.i.d. summands*

In any separable Hilbert space H, the central limit theorem for a sum of i.i.d. random elements is equivalent to the square integrability of the summands. This nice property does not extend to general Banach spaces, because the CLT is deeply connected to the geometry of the space [9]. A square integrable random element X in H is always *pregaussian*, i.e. there is a Gaussian random element in H with the same covariance structure as X.

Theorem 2.2. *Let* $\mu_1^\bullet,\dots,\mu_n^\bullet,\dots$ *be i.i.d. copies of* $\mu^\bullet$. *If* $\mathbf{E}\|\mu^\bullet\|_K^2 < \infty$, *then*

$$S_n^* := \frac{1}{\sqrt{n}}\sum_{i=1}^n(\mu_i^\bullet - \mathbf{E}\mu^\bullet) \xrightarrow[\textit{in law}]{\mathcal{H}} \gamma^\bullet, \tag{14}$$

where $\gamma^\bullet$ *is a Gaussian random element in* $\mathcal{H}$ *with* $\mathbf{E}\gamma^\bullet = 0$ *and covariance given by*

$$\operatorname{Cov}(\gamma^\bullet)(f,g) = \mathbf{E}\left(\int_{\mathfrak{X}} f\,\mathrm{d}\mu^\bullet \int_{\mathfrak{X}} g\,\mathrm{d}\mu^\bullet\right) - \left(\mathbf{E}\int_{\mathfrak{X}} f\,\mathrm{d}\mu^\bullet\right)\left(\mathbf{E}\int_{\mathfrak{X}} g\,\mathrm{d}\mu^\bullet\right), \tag{15}$$

for every $f,g\in\mathcal{H}$.

Conversely, if S_n^* *converges in law in* $\mathcal{H}$, *its limit is Gaussian and* $\mathbf{E}\|\mu^\bullet\|_K^2 < \infty$.

Corollary 2.2. *If* $\mathfrak{X}$ *is locally compact and* $\mathbf{E}|\mu^\bullet|(\mathfrak{X})^2 < \infty$, *then both* $\mu^\bullet$ *and* $\mu^\bullet\otimes\mu^\bullet$ *have mean measures, say* μ *and* ν *and (14) holds. In this case, (15) can be recast as*

$$\operatorname{Cov}(\gamma^\bullet)(f,g) = \int_{\mathfrak{X}^2} f\otimes g\,\mathrm{d}\nu - \left(\int_{\mathfrak{X}} f\,\mathrm{d}\mu\right)\left(\int_{\mathfrak{X}} g\,\mathrm{d}\mu\right).$$

Example 2.1. (CLT for empirical measure) Let X be a random element $(\Omega,\mathcal{F},P)\to(\mathfrak{X},\mathcal{B}_{\mathfrak{X}})$ with unknown distribution $\mu = P\circ X^{-1}$. Denote by $X_1,\dots,X_n$, i.i.d. copies of X and put $\mu_i^\bullet := \delta_{X_i}$, $i=1,\dots,n$. Then $n^{-1}\sum_{i=1}^n\delta_{X_i}$ is the *empirical measure* associated with the sample $X_1,\dots,X_n$. The CLT in $\mathcal{H}$ for the empirical measure was obtained by Berlinet [2] by a direct approach. It can also be seen as a special case of Corollary 2.2. Indeed here $\mu^\bullet = \delta_X$, so $|\mu^\bullet|(\mathfrak{X}) = 1$, $\mathbf{E}\mu^\bullet = \mu = P\circ X^{-1}$ and $\mathbf{E}(\mu^\bullet\otimes\mu^\bullet) =: \nu$ is the image measure of $P\circ X^{-1}$ by the mapping $x\mapsto(x,x)$. Hence

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n\delta_{X_i} - \mu\right) \xrightarrow[\text{in law}]{\mathcal{H}} \gamma^\bullet,$$

where the covariance of the Gaussian centered random element $\gamma^\bullet$ is given by

$$\mathrm{Cov}(\gamma^\bullet)(f,g) = \int_{\mathfrak{X}} fg \,\mathrm{d}\mu - \left(\int_{\mathfrak{X}} f \,\mathrm{d}\mu\right)\left(\int_{\mathfrak{X}} g \,\mathrm{d}\mu\right).$$

2.4. *CLT for Donsker random measure and FCLT in* $L^2[0,1]$

It is also possible to obtain central limit theorems for sums of non i.i.d. random measures, like the Donsker random measure

$$\nu_n^\bullet := \frac{1}{s_n}\sum_{i=1}^{n} X_i \delta_{\frac{i}{n}}, \quad n \geq 1, \tag{16}$$

where the X_i's are mean zero random variables, possibly dependent, with $s_n^2 := \mathbf{E}S_n^2$ and $S_n = \sum_{i=1}^n X_i$. An application of such CLT is a functional central limit theorem (FCLT) in $L^2[0,1]$ for the partial sums processes

$$W_n(t) := s_n^{-1} S_{[nt]}, \quad t \in [0,1]. \tag{17}$$

This application was suggested by P. Jacob to P.E. Oliveira and the author. The weak convergence of W_n is classically studied in the Skorohod space $D(0,1)$ which is continuously embedded in $L^2[0,1]$. As many test statistics are functionals continuous in $L^2[0,1]$ sense of W_n or of the empirical process, see [12] and [10], the weaker topological framework of $L^2[0,1]$ has its own interest. This way we can hope to relax the assumptions on the dependence structure of the underlying variables X_i's. Here we just sketch the method and refer to [11, 12] for more precise results.

Let us choose $\mathfrak{X} = [0,1]$ with the kernel of Example 1.3. Then

$$\zeta_{\nu_n^\bullet}(t) = \nu_n^\bullet([0,t]) = s_n^{-1} S_{[nt]} = W_n(t), \quad t \in [0,1]. \tag{18}$$

Hence by the isometry between the Hilbert spaces $\mathcal{H}$ and $L^2[0,1]$,

$$\nu_n^\bullet \xrightarrow[\text{in law}]{\mathcal{H}} \gamma^\bullet \iff W_n \xrightarrow[\text{in law}]{L^2[0,1]} W, \tag{19}$$

where under mild assumptions, the limiting process W is identified as a Brownian motion by a simple covariance computation. Now the relevant CLT for $\nu_n^\bullet$ may be established by checking the following conditions.

a) The inner products $\langle h, \nu_n^\bullet\rangle_K$ converge in law to $\langle h, \gamma^\bullet\rangle_K$ for any fixed $h \in \mathcal{H}$.
b) The sequence $(\nu_n^\bullet)_{n\geq 1}$ is tight in $\mathcal{H}$, i.e. for any positive ε, there is a compact subset C_ε of $\mathcal{H}$ such that $\inf_{n\geq 1} P(\nu_n^\bullet \in C_\varepsilon) \geq 1-\varepsilon$.

The first condition reduces to a CLT in $\mathbb{R}$ for triangular arrays because

$$\langle h, \nu_n^\bullet\rangle_K = \frac{1}{s_n}\sum_{i=1}^{n} X_i \langle h, \delta_{\frac{i}{n}}\rangle_K = \frac{1}{s_n}\sum_{i=1}^{n} h\Big(\frac{i}{n}\Big) X_i. \tag{20}$$

By an adaptation of a classical Prohorov's result (Th.1.13 in [14]), sufficient conditions for the tightness of $(\nu_n^\bullet)_{n\geq 1}$ are

$$\sup_{n\geq 1} \mathbf{E}\|\nu_n^\bullet\|_K^2 < \infty, \tag{21}$$

$$\lim_{n\to\infty} \sup_{n\geq 1} \mathbf{E} \sum_{i\geq N} |\langle f_i, \nu_n^\bullet\rangle_K|^2 = 0, \tag{22}$$

for some Hilbertian basis $(f_i)_{i\in\mathbb{N}}$ of $\mathcal{H}$. Concerning (21) which does not come from Th.1.13 in [14], see the remark after Theorem 5 in [21].

Now the heart of the matter is in the following elementary estimate.

$$\begin{aligned}
\mathbf{E} \sum_{i\geq N} |\langle f_i, \nu_n^\bullet\rangle_K|^2 &= \sum_{i\geq N} \mathbf{E} \left(\int f_i \, \mathrm{d}\nu_n^\bullet \right)^2 \\
&= \sum_{i\geq N} \frac{1}{s_n^2} \sum_{j,k=1}^{n} \mathbf{E}(X_j X_k) f_i\left(\frac{j}{n}\right) f_i\left(\frac{k}{n}\right) \\
&\leq \left(\frac{1}{s_n^2} \sum_{j,k=1}^{n} |\mathbf{E}(X_j X_k)| \right) \sup_{x\in[0,1]} \sum_{i\geq N} f_i(x)^2. \qquad (23)
\end{aligned}$$

The first factor in (23) may be bounded uniformly in n, subject to good covariance estimates for the X_j's. The second factor goes to zero due to Dini's theorem (the f_i's being continous like any element of $\mathcal{H}$). Moreover (21) obviously follows from (23) with $N = 0$ in the same setting.

To sum up, the FCLT in $L^2[0,1]$ for the partial sums process W_n based on some dependent sequence $(X_j)_{j\geq 1}$ is obtained under the estimate $\sum_{j,k=1}^n |\mathbf{E}(X_j X_k)| = O(s_n^2)$ and a one-dimensional CLT for the triangular arrays (20).

2.5. *Functional central limit theorems*

We discuss now the extension to random measures of the classical FCLT for random variables. First note that polygonal lines in $\mathcal{M}$ make sense, due to $\mathcal{M}$'s vector space structure. Let $\mu^\bullet$ be a signed random measure and the $\mu_i^\bullet$'s be i.i.d. copies of $\mu^\bullet$. We denote by $\xi_n^\bullet$ the $\mathcal{M}$-valued stochastic process indexed by $[0,1]$, whose paths are polygonal lines with vertices $(k/n, n^{-1/2}S_k)$, $k = 0, 1, \ldots, n$, $S_k := \mu_1^\bullet + \cdots + \mu_k^\bullet$.

Combining Theorem 2.2 with Kuelbs FCLT [8], we immediately obtain the FCLT for $\xi_n^\bullet$ in the space $\mathcal{C}([0,1], \mathcal{H})$ of continuous functions $[0,1] \to \mathcal{H}$.

Theorem 2.3. *The following statements are equivalent.*

a) $\mathbf{E}\|\mu^\bullet\|_K^2 < \infty$ *and* $\mathbf{E}\mu^\bullet = 0$,

b) $\xi_n^\bullet$ *converges in law in* $\mathcal{C}([0,1], \mathcal{H})$ *to some* $\mathcal{H}$*-valued Brownian motion* W*, i.e. a Gaussian process with independent increments such that* $W(t) - W(s)$ *has the same distribution as* $|t-s|^{1/2}\gamma^\bullet$*, where* $\gamma^\bullet$ *is a Gaussian random element in* $\mathcal{H}$ *with null expectation and same covariance structure as* $\mu^\bullet$*.*

As the paths of $\xi_n^\bullet$ are Lipschitz $\mathcal{H}$-valued functions, it is natural to look for a stronger topological framework than $\mathcal{C}([0,1],\mathcal{H})$ for the FCLT. A clear limitation in this quest comes from the modulus of uniform continuity of the limiting process, $\omega(W,u) := \sup_{0\le t-s\le u} \|W(t)-W(s)\|_{\mathcal{H}}$. Indeed by a simple projection argument and Lévy's well known result, $\omega(W,u)$ cannot be better than $u^{1/2}\ln(1/u)$. This forbids any weak convergence of $\xi_n^\bullet$ in some Hölder topology based on a weight function stronger than $u^{1/2}\ln(1/u)$. Introduce the separable Hölder spaces $\mathrm{H}_\rho^o([0,1],\mathcal{H})$ of functions $f:[0,1]\to\mathcal{H}$, such that

$$\|f\|_\rho := \|f(0)\|_{\mathcal{H}} + \omega_\rho(f,1) < \infty \quad \text{and} \quad \lim_{u\to 0}\omega_\rho(f,u) = 0,$$

where

$$\omega_\rho(f,u) := \sup_{0<t-s\le u} \frac{\|f(t)-f(s)\|_{\mathcal{H}}}{\rho(t-s)}.$$

We assume moreover that the weight functions ρ are of the form $\rho(u) = u^\alpha L(1/u)$, $0<\alpha\le 1/2$, where L is continuous normalized slowly varying at infinity. The $\mathrm{H}_\rho^o([0,1],\mathcal{H})$ weak convergence of $\xi_n^\bullet$ to W requires stronger integrability of $\mu^\bullet$ than Condition a) in Theorem 2.3. Combining Theorem 2.2 with the Hölderian FCLT in [15], leads to the FCLT for $\xi_n^\bullet$ in the space $\mathrm{H}_\rho^o([0,1],\mathcal{H})$.

Theorem 2.4. *Assume that there is a $\beta > 1/2$ such that*

$$t^{1/2}\rho(1/t)\ln^{-\beta}(t) \text{ is non decreasing on some } [a,\infty). \tag{24}$$

Then the following statements are equivalent.

a) $\mathbf{E}\mu^\bullet = 0$ *and*

$$\text{for every } A>0, \quad \lim_{t\to\infty} t\,P\big(\|\mu^\bullet\|_K \ge At^{1/2}\rho(1/t)\big) = 0. \tag{25}$$

b) $\xi_n^\bullet$ *converges in law in* $\mathrm{H}_\rho^o([0,1],\mathcal{H})$ *to the* $\mathcal{H}$*-valued Brownian motion* W *of Th. 2.3.*

When $\alpha < 1/2$, Condition (24) is automatically satisfied and it is enough to take $A=1$ in (25). To clarify Condition (25), let us consider two important special cases. When $\rho(t) = t^\alpha$ for some $0<\alpha<1/2$, (25) reduces to $P\big(\|\mu^\bullet\|_K \ge t\big) = o(t^{-p(\alpha)})$, with $p(\alpha) := (1/2-\alpha)^{-1}$ and this is slightly weaker than $\mathbf{E}\|\mu^\bullet\|_K^{p(\alpha)} < \infty$. When $\rho(t) = t^{1/2}\ln^\beta(c/t)$ for some $\beta > 1/2$, then (25) is equivalent to the finiteness of $\mathbf{E}\exp(d\|\mu^\bullet\|_K^{1/\beta})$ for each $d>0$.

Following [16], we present briefly a statistical application of Theorem 2.4 to the detection of epidemic change in the expectation of a random measure. In what follows, $\mu_k^\bullet$, $k=1,\dots,n$ are always i.i.d. copies of the *mean zero* random measure $\mu^\bullet$. Based on the observation of the random measures $\nu_1^\bullet,\dots,\nu_n^\bullet$, we want to test the null hypothesis

(H_0): $\nu_k^\bullet = \mu_k^\bullet$, $k=1,\dots,n$,

against the so called epidemic alternative

$$(H_A) \qquad \nu_k^\bullet = \begin{cases} \mu_c + \mu_k^\bullet & \text{if } k \in \mathbb{I}_n := \{k^*+1,\dots,m^*\} \\ \mu_k^\bullet & \text{if } k \in \mathbb{I}_n^c := \{1,\dots,n\} \setminus \mathbb{I}_n \end{cases}$$

where $\mu_c \neq 0$ is some deterministic signed measure which may depend on n. To achieve this goal, we use some weighted dyadic increments statistics which behave like continuous functionals of $\xi_n^\bullet$ in Hölder topology. Consider partial sums

$$S_n(a,b) = \sum_{na<k\le nb} \nu_k^\bullet, \quad 0 \le a < b \le 1.$$

Let us denote by D_j the set of dyadic numbers in $[0,1]$ of level j, i.e. $\mathrm{D}_0 = \{0,1\}$, and $\mathrm{D}_j = \{(2l-1)2^{-j};\ 1 \le l \le 2^{j-1}\}$, $j \ge 1$. Write for $r \in \mathrm{D}_j$, $j \ge 0$, $r^- := r - 2^{-j}$ and $r^+ := r + 2^{-j}$. Then define the dyadic increments statistics $\mathrm{DI}(n,\rho)$ by

$$\mathrm{DI}(n,\rho) := \frac{1}{2} \max_{1\le j\le \log n} \frac{1}{\rho(2^{-j})} \max_{r\in \mathrm{D}_j} \left\| S_n(r^-,r) - S_n(r,r^+) \right\|_K . \tag{26}$$

Here "log" stand for the logarithm with basis 2 ($\log(2^j) = j$) while "ln" denotes the natural logarithm ($\ln(\mathrm{e}^t) = t$).

Theorem 2.5. *Assume that the weight function ρ satisfies (24) and that the mean zero random measure $\mu^\bullet$ satisfies (25). Then under (H_0), $n^{-1/2}\mathrm{DI}(n,\rho)$ converges in law to a non negative random variable Z with distribution function*

$$P(Z \le z) = \prod_{j=1}^{\infty} \Big(P(\|\gamma^\bullet\|_K \le 2^{(j+1)/2}\rho(2^{-j})z \Big)^{2^{j-1}}, \quad z \ge 0, \tag{27}$$

where $\gamma^\bullet$ is a mean zero Gaussian random element in $\mathcal{H}$ with the same covariance as $\mu^\bullet$. The convergence of the product (27) is uniform on any interval $[\varepsilon,\infty)$, $\varepsilon > 0$.

Theorem 2.5 is easily obtained from Theorem 2.2 and from [16] Th. 2 and Prop. 3. For general estimates on the convergence rate in (27), see Prop. 4 in [16]. The consistency of the sequence of test statistics $n^{-1/2}\mathrm{DI}(n,\rho)$ follows from the next result which is an easy adaptation of Th. 5 in [16].

Theorem 2.6. *Let ρ satisfying (24). Under (H_A), write $l^* := m^* - k^*$ for the length of epidemics and assume that*

$$\lim_{n\to\infty} n^{1/2} \frac{u_n\|\mu_c\|_K}{\rho(u_n)} = \infty, \quad \textit{where} \quad u_n := \min\Big\{\frac{l^*}{n}; 1 - \frac{l^*}{n}\Big\}. \tag{28}$$

Then

$$n^{-1/2}\mathrm{DI}(n,\rho) \xrightarrow[n\to\infty]{\mathrm{pr}} \infty.$$

To discuss Condition (28), assume for simplicity that μ_c does not depend on n. When $\rho(t) = t^\alpha$, (28) allows us to detect *short epidemics* such that $l^* = o(n)$ and $l^* n^{-\delta} \to \infty$, where $\delta = (1-2\alpha)(2-2\alpha)^{-1}$. When $\rho(t) = t^{1/2}\ln^\beta(c/t)$ with $\beta > 1/2$, (28) is satisfied provided that $u_n = n^{-1}\ln^\gamma n$, with $\gamma > 2\beta$. This leads to detection of short epidemics such that $l^* = o(n)$ and $l^*\ln^{-\gamma} n \to \infty$. In both cases one can detect symmetrically *long epidemics* such that $n - l^* = o(n)$.

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