Random fields and central limit theorem in some generalized Hölder spaces

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Abstract. For rather general moduli of smoothness ρ (like e.g. $\rho(h) = h^\alpha \ln^\beta(c/h)$) the Hölder spaces $H_\rho([0,1]^d)$, are characterized by the rate of coefficients in the skew pyramidal basis. With this analytical tool, we study in terms of *second differences* the existence of a version in H_ρ for a given random field. In the same spirit, central limit theorems are obtained both for i.i.d. and martingale differences sequences of random elements in H_ρ.

1. INTRODUCTION

In many situations, stochastic processes and random fields have a smoothness intermediate between the continuity and differentiability. The scale of Hölder spaces is then a natural functional framework to investigate the regularity of such processes and fields. And weak convergence in this setting is a stronger result than in the space of continuous functions. In this paper we consider the scale of generalized Hölder spaces $H_\rho([0,1]^d)$, where ρ is a modulus of smoothness (precise definition is given in Section 1 below) and discuss two questions:

(I) For a given random field indexed by $[0,1]^d$, find sufficient conditions for the existence of a version with sample paths in H_ρ.

(II) Find sufficient conditions for a sequence of random elements in H_ρ to satisfy the central limit theorem in this space.

The earliest result for the problem (I) goes back to the Kolmogorov sufficient condition for the existence of a sample continuous version of a given stochastic process ξ on $[0,1]$, namely:

$$P\{|\xi(t+h) - \xi(t)| > \lambda\} \leq c\lambda^{-\gamma} h^{1+\delta},$$

where c, $\delta > 0$ and $\gamma > 1$ are constants. In fact the same condition is sufficient for ξ to have a version with sample paths in the Hölder space with the parameter α (i.e. in H_ρ with $\rho(h) = |h|^\alpha$ in our notation) for any $0 < \alpha < \delta/\gamma$. Ciesielski (1961) gave sufficient conditions for a Gaussian process to have a version with α-Hölderian paths, Ibragimov (1984) and Nobelis (1981) studied the problem (I) for general ρ and $d \geq 1$.

As for the central limit theorem, to our best knowledge, the only results available in the literature concern two invariance principles for partial sums and for empirical processes. The weak Hölder convergence of Donsker Prokhorov's partial sums process (polygonal lines) was investigated by Lamperti (1962) in the case where $d = 1$, $\rho(h) = |h|^\alpha$ and later by Erickson (1981) for $d \geq 1$ and

The usefulness of $\Delta_h^2\xi(t)$ in the problem of sample paths differentiability is known (see Cramér-Leadbetter (1967)). From an analytical point of view, there is no loss in working with $\Delta_h^2 f$ to study the Hölder regularity of a non-random function f. This observation goes back to Zygmund (1945) who noticed that a necessary and sufficient condition that a continuous and periodic function $f(x)$ should satisfy a Hölder condition of order α, $0 < \alpha < 1$, is that

$$\Delta_h^2 f(x) = O(h^\alpha), \quad \text{as} \quad h \to +0,$$

uniformly in x. The role of Δ_h^2 is now well understood in the more general context of Besov spaces (see Peetre (1976)).

From a probabilistic point of view, it is clear that any control in probability on $\Delta_h^1\xi(t)$ provides similar type of control on $\Delta_h^2\xi(t)$, but the converse is false in general. So the use of $\Delta_h^2\xi(t)$, brings more flexibility in our basic assumptions. Moreover the second difference appear very naturally in the discretization procedure corresponding to the decomposition of a function in the Faber Schauder basis of triangular functions (those obtained by means of the affine interpolation between dyadic points).

The present contribution extends our previous results (Račkauskas ans Suquet (1998)) in the following three directions

- use of more general moduli of smoothness ρ;
- multidimensional parameter space $[0,1]^d$;
- central limit theorem for martingale differences.

To this end we follow the procedure already used in the dimension 1, replacing the basis of triangular functions by some special basis for pyramidal functions. It turns out that the coefficients in this basis are the second differences and we have a very convenient Banach isomorphism between H_ρ and appropriate sequences space. This analytical background is detailed in Section 2. In Section 3 we obtain sufficient conditions for the existence of a H_ρ version of a given random field. In Section 4 we give central limit theorems in H_ρ in the i.i.d. case and also for triangular arrays of martingale differences.

2. ANALYTICAL BACKGROUND

Throughout $T = [0,1]^d$ and $\mathbb{R}^d$ is endowed with the norm

$$|t| := \max_{1\le i\le d} |t_i|\,, \quad t = (t_1, \dots, t_d) \in \mathbb{R}^d.$$

Denote by H_ρ the set of real valued continuous functions $x : T \to \mathbb{R}$ such that $w_\rho(x,1) < \infty$, where

$$w_\rho(x,\delta) := \sup_{t,s\in T, 0<|t-s|<\delta} \frac{|x(t)-x(s)|}{\rho(|s-t|)}$$

and ρ is a modulus of smoothness satisfying conditions (1) to (5) below where c_1, c_2 and c_3 are positive constants:

$$\rho(0) = 0,\ \rho(\delta) > 0,\ 0 < \delta \le 1; \tag{1}$$

$$\rho \text{ is non decreasing on } [0,1]; \tag{2}$$

$$\rho(2\delta) \le c_1\rho(\delta), \quad 0 \le \delta \le 1/2; \tag{3}$$

$$\int_0^\delta \frac{\rho(u)}{u}\, du \le c_2\rho(\delta), \quad 0 < \delta \le 1; \tag{4}$$

$$\delta\int_\delta^1 \frac{\rho(u)}{u^2}\, du \le c_3\rho(\delta), \quad 0 < \delta \le 1. \tag{5}$$

For instance, elementary computations show that the functions

$$\rho(\delta) := \delta^\alpha \ln^\beta\big(\frac{c}{\delta}\big), \quad 0 < \alpha < 1,\ \beta \in \mathbb{R},$$

satisfy conditions (1) to (5), for a suitable choice of the constant c, namely $c \ge \exp(\beta/\alpha)$ if $\beta > 0$ and $c > \exp(-\beta/(1-\alpha))$ if $\beta < 0$.

The set H_ρ is a Banach space when endowed with the norm

$$\|x\|_\rho := |x(0)| + w_\rho(x,1).$$

Obviously an equivalent norm is obtained replacing $|x(0)|$ in the above formula by $\|x\|_\infty := \sup\{|x(t)|\,;\ t \in T\}$.

Define

$$H_\rho^o = \{x \in H_\rho; \lim_{\delta\to 0} w_\rho(x,\delta) = 0\}.$$

Then H_ρ^o is a closed subspace of H_ρ. Now let us remark that for any function ρ satisfying (1)–(5), there is a positive constant c_4 such that

$$\rho(\delta) \ge c_4\delta, \quad 0 \le \delta \le 1. \tag{6}$$

Hence the spaces H_ρ^o always contain all the Lipschitz functions and in particular the (continuous) piecewise affine functions. The separability of the spaces H_ρ^o follows by standard interpolation arguments.

Since we are interested in the analysis of these spaces in terms of second differences of the functions x, our first task is to establish the equivalence of the norm $\|x\|_\rho$ with some sequential norm involving the dyadic second differences of x. To this aim, we shall use some Schauder basis of pyramidal functions. Our main reference for this part is Semadeni (1982). The so called *skew pyramidal basis* was introduced by Bonic Frampton and Tromba (1969) and independently by Ciesielski and Geba (see the historical notes in

Semadeni (1982) p. 72). This choice, which is not the only possible, leads to rather simple formulas for the Schauder coefficients in terms of second differences.

To explain the construction of the skew pyramidal basis, consider first a cube

$$Q = s + aT = \{s + \sum_{1 \le i \le d} u_i e_i;\ 0 \le u_i \le a\},$$

where the e_i's denote the vectors of the canonical basis of $\mathbb{R}^d$. The *standard triangulation* of Q is the family $T(Q)$ of simplexes defined as follows. Write Π_d for the set of permutations of the indexes $1, \ldots, d$. For any $\pi = (i_1, \ldots, i_d) \in \Pi_d$, let $\Delta_\pi(Q)$ be the convex hull of the $d+1$ points

$$s,\ s + ae_{i_1},\ s + a(e_{i_1} + e_{i_2}), \ldots, s + a \sum_{k=1}^{d} e_{i_k}.$$

So, each simplex $\Delta_\pi(Q)$ corresponds to one path from s to $s' = s + a(1, \ldots, 1)$ via vertices of Q and such that along each segment of the path, only one coordinate increases while the others remain constants. Thus Q is divided into $d!$ simplexes with disjoint interiors. Next let H_i be the hyperplane perpendicular to e_i and passing through the middle of the edge $[s, s + ae_i]$. The hyperplanes $H_1, \ldots, H_d$ divide the cube Q into 2^d cubes, say, $\Gamma_k(Q)$, $k = 0, \ldots, 2^d - 1$. More precisely, if $k = \varepsilon_1 + \varepsilon_2 2^1 + \ldots + \varepsilon_d 2^{d-1}$ is the binary representation of k,

$$\Gamma_k(Q) := s + \frac{1}{2} \sum_{i=1}^{d} \varepsilon_i e_i + \frac{a}{2} T.$$

By lemma 3.4.2 in Semadeni (1982), each simplex $\Delta_\pi(\Gamma_k(Q))$ of the standard triangulation of $\Gamma_k(Q)$ is contained in an unique simplex $\Delta_{\pi'}(Q)$ of the standard triangulation of Q.

Consider now the sequence $(P_j)_{j \ge 0}$ of partitions of T defined by

$$P_0 := \{T\}, \quad P_j := \{\Gamma_k(Q); Q \in P_{j-1}, 0 \le k < 2^d\}.$$

In other words, P_j is composed of the 2^{jd} cubes obtained by dividing each edge of the cube $[0,1]^d$ into 2^j segments of length 2^{-j}. Finally we define the triangulation T_j as the union of the standard triangulations of the cubes in P_j.

$$T_j := \{\Delta_\pi(Q); Q \in P_j, \pi \in \Pi_d\}, \quad j = 0, 1, \ldots$$

Clearly the set $W_j := \mathrm{vert}(T_j)$ of vertices of the simplexes in T_j is the set of vertices of the cubes in P_j, whence

$$W_j = \mathrm{vert}(T_j) = \{k 2^{-j};\ 0 \le k \le 2^j\}^d.$$

In what follows we put $V_0 := W_0$ and $V_j := W_j \setminus W_{j-1}$ for $j \geq 1$. So V_j is the set of new vertices born with the triangulation T_j. More explicitly, V_j is the set of dyadic points $v = (k_1 2^{-j}, \ldots, k_d 2^{-j})$ in W_j with at least one k_i odd.

The T_j-pyramidal function $\Lambda_{j,v}$ with peak vertex $v \in V_j$ is defined on T by the three conditions

i) $\Lambda_{j,v}(v) = 1$;

ii) $\Lambda_{j,v}(w) = 0$ if $w \in \mathrm{vert}(T_j)$ and $w \neq v$;

iii) $\Lambda_{j,v}$ is affine on each simplex Δ in T_j, i.e., if the w_i are the vertices of Δ,

$$\Lambda_{j,v}\Big(\sum_{i=0}^{d} r_i w_i\Big) = \sum_{i=0}^{d} r_i \Lambda_{j,v}(w_i), \qquad r_i \geq 0,\ \sum_{i=0}^{d} r_i = 1.$$

From iii) it follows clearly that the support of $\Lambda_{j,v}$ is the union of all simplexes in T_j containing the peak vertex v. By Proposition 3.4.5 in Semadeni (1982), the functions $\Lambda_{j,v}$ are obtained by dyadic translations and changes of scale:

$$\Lambda_{j,v}(t) = \Lambda(2^j(t - v)), \quad t \in T, v \in V_j$$

from the same function Λ with support included in $[-1, 1]^d$:

$$\Lambda(t) := \max\big(0,\ 1 - \max_{t_i<0} |t_i| - \max_{t_i>0} t_i\big), \quad t = (t_1, \ldots, t_d) \in [-1, 1]^d.$$

But this apparent simplicity is misleading. The edges effects due to the restriction to $t \in T$ give different shapes for the supports of the $\Lambda_{j,v}$'s. For instance when $d = 2$, the support of Λ is hexagonal, but among the five functions $\Lambda_{1,v}$, only one has hexagonal support (corresponding to the peak vertex $v = (1/2, 1/2)$), the four others having pentagonal supports.

The *skew pyramidal basis* is the family $\mathcal{L} := \{\Lambda_{j,v};\ j \geq 0,\ v \in V_j\}$ lexicographically ordered. As a special case of the Proposition 3.1.6. in Semadeni (1982), $\mathcal{L}$ is a Schauder basis of the Banach space $C(T)$ of real valued continuous functions on T. Hence any $x \in C(T)$ admits the unique uniformly convergent series expansion:

$$x(t) = \sum_{j=0}^{\infty} \sum_{v \in V_j} \lambda_{j,v}(x) \Lambda_{j,v}(t), \quad t \in T.$$

The Schauder coefficients $\lambda_{j,v}(x)$ are given by:

$$\begin{aligned} \lambda_{0,v}(x) &= x(v), \quad v \in V_0; \\ \lambda_{j,v}(x) &= x(v) - \frac{1}{2}\big(x(v^-) + x(v^+)\big), \quad v \in V_j,\ j \geq 1. \end{aligned}$$

For a detailed derivation of these formulas, the reader is referred to 3.1.5 and 3.4.9 in Semadeni (1982). We just need to explain the definition of v^- and v^+. Each $v \in V_j$ admits a unique representation $v = (v_1, \dots, v_d)$ with $v_i = k_i/2^j$, $(1 \le i \le d)$. The points $v^- = (v_1^-, \dots, v_d^-)$ and $v^+ = (v_1^+, \dots, v_d^+)$ are defined by

$$v_i^- = \begin{cases} v_i - 2^{-j} & \text{if } k_i \text{ is odd;} \\ v_i & \text{if } k_i \text{ is even;} \end{cases} \qquad v_i^+ = \begin{cases} v_i + 2^{-j}, & \text{if } k_i \text{ is odd;} \\ v_i & \text{if } k_i \text{ is even.} \end{cases}$$

Since v is in V_j, at least one of the k_i's is odd, so v^-, v and v^+ are really three distinct points of T. Moreover we can write

$$v^- = v - 2^{-j} e(v), \quad v^+ = v + 2^{-j} e(v) \quad \text{with} \quad e(v) := \sum_{k_i \text{ odd}} e_i,$$

so $\lambda_{j,v}(x)$ is a *second difference* directed by the vector $e(v)$.
Define the projectors E_j $(j \ge 0)$ by

$$E_j x := \sum_{i=0}^{j} \sum_{v \in V_i} \lambda_{i,v}(x) \Lambda_{i,v}, \quad x \in C(T).$$

The function $E_j x$ is affine on each simplex of T_j and such that $E_j x(w) = x(w)$ for each $w \in W_j$. In other words E_j is the operator of affine interpolation at the vertices of T_j.

Now we are able to obtain the equivalence of norms we were looking for.

Proposition 1. *The norm $\|x\|_\rho$ is equivalent to the sequential norm*

$$\|x\|_\rho^{\text{seq}} := \sup_{j \ge 0} \frac{1}{\rho(2^{-j})} \max_{v \in V_j} |\lambda_{j,v}(x)| .$$

Proof. The main arguments of the proof are already in Semadeni (1982). But this equivalence being a key point in the present contribution, it seems preferable to give a rather detailed proof adapted to our purpose.
Noting that for $v \in V_j$, we have $|v - v^-| = |v - v^+| = 2^{-j}$, it is easily seen that

$$\|x\|_\rho^{\text{seq}} \le \max\Big(1, \frac{1}{\rho(1)}\Big) \|x\|_\rho .$$

To prove the reverse inequality we need some technical lemmas.

Lemma 2. *For each $t \in T$ and each $j \ge 0$, $0 \le \sum_{v \in V_j} \Lambda_{j,v}(t) \le 1$.*

Proof of Lemma 2. By continuity it suffices to check the result for t interior in some simplex Δ of T_j. Then for each $v \in V_j$ which is not a vertex of Δ, $\Lambda_{j,v}(t) = 0$. Remarking also that some vertices of Δ may not belong to V_j, let us denote by $V_{j,\Delta}$ the set of vertices of Δ which are in V_j. So we have

$$0 \leq \sum_{v \in V_j} \Lambda_{j,v}(t) \leq \sum_{v \in V_{j,\Delta}} \Lambda_{j,v}(t).$$

Now Δ being the convex hull of its $(d+1)$ vertices, we get the barycentric representation $t = \sum_{w \in \mathrm{vert}(\Delta)} r_w w$ with $r_w \geq 0$ and $\sum r_w = 1$. Since each $\Lambda_{j,v}$ is affine on Δ and vanishes at every vertex of T_j except at v, we obtain

$$\sum_{v \in V_{j,\Delta}} \Lambda_{j,v}(t) = \sum_{v \in V_{j,\Delta}} \sum_{w \in \mathrm{vert}(\Delta)} r_w \Lambda_{j,v}(w) = \sum_{v \in V_{j,\Delta}} r_v \leq 1,$$

where $\mathrm{vert}(\Delta)$ denotes the set of vertices of Δ.

As a tool for chaining arguments, we collect here the following estimates. There is a constant b depending only of the dimension d such that for each $j \geq 0$ and each simplex Δ in T_j,

$$\mathrm{diam}(\Delta) \leq 2^{-j} \quad \text{and} \quad \alpha(\Delta) \geq b 2^{-j}, \tag{7}$$

where the diameter is in the sense of the sup norm metric of $\mathbb{R}^d$ and

$$\alpha(\Delta) := \inf_{|u|=1} \sup\{|t-s|\,;\, s,t \in \Delta,\ t-s = cu, c \in \mathbb{R}\}.$$

By change of scale it suffices to consider the case $j = 0$ and to remark that each Δ has a non empty interior.

LEMMA 3. *There is a constant c_0 depending only on d such that for each triangulation T_j, for each $n = 1, 2, \ldots$ and each pair s, t in T such that $|t-s| \leq 2^{-j}$, there exist a finite sequence $s = z_0, z_1, \ldots, z_k = t$ and simplexes Δ_l in T_j $(l = 1, \ldots, k)$ such that $k \leq c_0$ and for each l the successive points z_{l-1}, z_l belong to the same simplex Δ_l.*

Lemma 3 is simply a rewriting in our setting of Lemma 3.5.4 in Semadeni (1982).

LEMMA 4. *For each $j \geq 1$, $\|x - E_{j-1}x\|_\infty \leq 2c_1c_2 \|x\|_\rho^{\mathrm{seq}} \rho(2^{-j})$.*

Proof of Lemma 4. Using successively Lemma 2 and assumptions (4) and (3), we get

$$|x(t) - E_{j-1}x(t)| \quad \leq \quad \sum_{i=j}^{\infty} \max_{v \in V_i} |\lambda_{i,v}(x)| \sum_{v \in V_i} \Lambda_{i,v}(t)$$

$$
\begin{aligned}
&\le \sum_{i=j}^{\infty} \|x\|_\rho^{\mathrm{seq}} \rho(2^{-i}) \\
&\le 2\|x\|_\rho^{\mathrm{seq}} \int_0^{2^{-j+1}} \frac{\rho(u)}{u}\, du \\
&\le 2c_1c_2 \|x\|_\rho^{\mathrm{seq}} \rho(2^{-j}).
\end{aligned}
$$

To bound $\|x\|_\rho$ by $\|x\|_\rho^{\mathrm{seq}}$, we have to estimate $|x(t) - x(s)|$. From now on, the letter c denotes a positive constant whose explicit value may differ at each occurrence.

Consider first the special case where for some fixed j, s and t belong to *the same simplex* Δ in T_j. For each $i \le j$, there is an unique simplex Δ_i in T_i containing Δ. By (7), $|t - s| \le 2^{-j}$ and there exist points $s^{(i)}$, $t^{(i)}$ in Δ_i such that $t^{(i)} - s^{(i)}$ is parallel to $t - s$ and $\left|t^{(i)} - s^{(i)}\right| \ge b2^{-i}$. For each $v \in V_i$, $\Lambda_{i,v}$ is affine on Δ_i, so

$$
|\Lambda_{i,v}(t) - \Lambda_{i,v}(s)| = \frac{|t-s|}{\left|t^{(i)} - s^{(i)}\right|} \left|\Lambda_{i,v}(t^{(i)}) - \Lambda_{i,v}(s^{(i)})\right| \le \frac{2^{i-j}}{b}.
$$

These estimates together with Lemma 4 lead to

$$
\begin{aligned}
|x(t) - x(s)| &\le \sum_{i=0}^{j-1} \sum_{v \in V_i} |\lambda_{i,v}(x)| \, |\Lambda_{i,v}(t) - \Lambda_{i,v}(s)| + 2\|x - E_{j-1}x\|_\infty \\
&\le \sum_{i=0}^{j-1} \|x\|_\rho^{\mathrm{seq}} \rho(2^{-i}) \sum_{v \in V_i \cap \mathrm{vert}(\Delta_i)} |\Lambda_{i,v}(t) - \Lambda_{i,v}(s)| \\
&\quad + 4c_1c_2 \|x\|_\rho^{\mathrm{seq}} \rho(2^{-j}) \\
&\le c\|x\|_\rho^{\mathrm{seq}} 2^{-j} \sum_{i=0}^{j} 2^i \rho(2^{-i}).
\end{aligned}
$$

Next consider the more general case where s and t are any two distinct points of T such that $|t - s| \le 1/2$. Then there is an integer $j \ge 1$ such that $2^{-j-1} < |t - s| \le 2^{-j}$. By Lemma 3 (with the same notations) we obtain

$$
\begin{aligned}
|x(t) - x(s)| &\le \sum_{l=1}^{k} |x(z_l) - x(z_{l-1})| \\
&\le c_0 c \|x\|_\rho^{\mathrm{seq}} 2^{-j} \sum_{i=0}^{j} 2^i \rho(2^{-i}) \\
&\le c\|x\|_\rho^{\mathrm{seq}} |t - s| \int_{2^{-j}}^{1} \frac{\rho(u)}{u^2}\, du.
\end{aligned}
$$

Using (5) this gives

$$|x(t)-x(s)| \le c\,\|x\|_\rho^{\mathrm{seq}}\,\rho(|t-s|), \qquad s,t\in T,\ |t-s|\le 1/2.$$

To complete the proof of Proposition 1, note that if $1/2<|t-s|\le 1$, writing $z:=(s+t)/2$, we have

$$\frac{|x(t)-x(s)|}{\rho(|t-s|)} \le \frac{|x(t)-x(z)|}{\rho(|t-z|)} + \frac{|x(z)-x(s)|}{\rho(|z-s|)} \le 2c\,\|x\|_\rho^{\mathrm{seq}}.$$

3. RANDOM FIELDS WITH VERSION IN H_ρ^o

We consider now a continuous random field $\xi=\{\xi(t),\ t\in T\}$ and the problem of existence of a version of ξ with almost all paths in H_ρ^o. The replacement of the initial norm $\|x\|_\rho$ by $\|x\|_\rho^{\mathrm{seq}}$ reduces the problem to the control of maxima of the Schauder coefficients $\lambda_{j,v}(\xi)$ which are dyadic second differences of ξ. We define here the second difference $\Delta_h^2\xi(t)$ in a symmetrical form by

$$\Delta_h^2\xi(t) := \xi(t+h)+\xi(t-h)-2\xi(t), \quad t\in T,\ h\in C_t,$$

where

$$C_t := \{h=(h_1,\dots,h_d);\ 0\le h_i\le \min(t_i,1-t_i),\ 1\le i\le d\}.$$

Recall that a Young function ϕ is a convex increasing function on $\mathbb{R}^+$ such that $\phi(0)=0$ and $\lim_{t\to\infty}\phi(t)=\infty$. If Z is a random variable such that $E\phi(Z/c)<\infty$ for some $c>0$ then its ϕ-Orlicz norm is

$$\|Z\|_\phi := \inf\{c>0 : \mathbf{E}\,\phi(Z/c)\le 1\}.$$

Throughout σ will denote an increasing function on $[0,\infty)$ such that $\sigma(0)=0$. For convenience's sake, we recall here the basic inequalities used throughout the paper to handle the maxima of random variables. The first of them is Lemma 11.3 p. 303 in Ledoux and Talagrand (1995).

LEMMA 5. *Let (X_i) be positive random variables on some probability space $(\Omega,\mathcal{F},P)$ such that for all $1\le i\le n$ and all $A\in\mathcal{F}$*

$$\int_A X_i\,dP \le a_i P(A)\phi^{-1}\Big(\frac{1}{P(A)}\Big), \tag{8}$$

where ϕ is some Young function and a_i a constant. Then, for every set $A\in\mathcal{F}$

$$\int_A \max_{1\le i\le n} X_i\,dP \le aP(A)\phi^{-1}\Big(\frac{n}{P(A)}\Big),$$

with $a=\max_{1\le i\le n} a_i$.

The two following lemmas provide practical conditions to verify (8), in terms of Orlicz norms or weak moments. The proofs can be found for instance in Račkauskas and Suquet (1998).

LEMMA 6. *Let* (X_i) *be positive random variables on some probability space* $(\Omega, \mathcal{F}, P)$ *and* ϕ *some Young function. Then* X_i *satisfies (8) with* $a_i = \|X_i\|_\phi$. *In particular,*

$$\mathbf{E} \max_{1\le i\le n} X_i \le \phi^{-1}(n) \max_{1\le i\le n} \|X_i\|_\phi .$$

LEMMA 7. *Let* (X_i) *be positive random variables on some probability space* $(\Omega, \mathcal{F}, P)$ *satisfying for some constant* $1 < p < \infty$ *and each* $i = 1, \dots, n$

$$b_i = \sup_{t>0} t P^{1/p}(X_i > t) < \infty.$$

Then X_i *satisfies (8) with* $a_i = q b_i$ *where* $q = p/(p-1)$. *In particular,*

$$\mathbf{E} \max_{1\le i\le n} X_i \le q \max_{1\le i\le n} b_i\, n^{1/p}.$$

THEOREM 8. *Assume the random field* $\xi = \{\xi(t),\ t \in T\}$ *is defined on the probability space* $(\Omega, \mathcal{F}, P)$ *and satisfies: for each set* $A \in \mathcal{F}$ *and all* $t \in T,\ h \in C_t$

$$\int_A |\Delta_h^2 \xi(t)|\, dP \le \sigma(|h|) P(A) \phi^{-1}\Big(\frac{1}{P(A)}\Big). \tag{9}$$

Then for each $A \in \mathcal{F}$, *and integers* $K > J \ge 0$,

$$\int_A \|E_K\xi - E_J\xi\|_\rho^{\mathrm{seq}}\, dP \le 4^d c_1 P(A) \int_{2^{-K}}^{2^{-J}} \frac{\sigma(u)}{u\rho(u)} \phi^{-1}\Big(\frac{1}{P(A)u^d}\Big)\, du. \tag{10}$$

Proof. Clearly $\|E_K\xi - E_J\xi\|_\rho^{\mathrm{seq}} = \max_{J<j\le K} 1/\rho(2^{-j}) \max_{v\in V_j} |\lambda_{j,v}(\xi)|$. Since $j \ge 1$ in this formula, the $\lambda_{j,v}$ involved are really second differences:

$$\lambda_{j,v}(\xi) = -\frac{1}{2}\Delta_h^2 \xi(t), \quad \text{with} \quad h = v^+ - v = v - v^-.$$

By the condition (9), for each measurable set A,

$$\int_A |\lambda_{j,v}(\xi)|\, dP \le \frac{1}{2}\sigma(2^{-j}) P(A) \phi^{-1}\Big(\frac{1}{P(A)}\Big).$$

Now Lemma 5 yields

$$\int_A \max_{v\in V_j} |\lambda_{j,v}(\xi)|\, dP \le \frac{1}{2} P(A) \sigma(2^{-j}) \phi^{-1}\Big(\frac{\mathrm{card} V_j}{P(A)}\Big). \tag{11}$$

The cardinality cardV_j of V_j is asymptotically $(1-2^{-d})2^{jd}$, but we will content ourselves with the crude estimate card$V_j \le 2^{(j+1)d}$ valid for each $j \ge 0$. It follows

$$\int_A \|E_K\xi - E_J\xi\|_\rho^{\mathrm{seq}}\, dP \le \frac{1}{2}P(A)\sum_{J\le j\le K}\frac{\sigma(2^{-j})}{\rho(2^{-j})}\phi^{-1}\Big(\frac{2^{(j+1)d}}{P(A)}\Big).$$

By comparing series and integral and using (3), we obtain

$$\int_A \|E_K\xi - E_J\xi\|_\rho^{\mathrm{seq}}\, dP \le c_1 P(A)\int_{2^{-K}}^{2^{-J}}\frac{\sigma(u)}{u\rho(u)}\phi^{-1}\Big(\frac{4^d}{P(A)u^d}\Big)\, du.$$

The proof of (10) is completed noting that ϕ^{-1} being concave and vanishing at 0, $\phi^{-1}(4^d r) \le 4^d\phi^{-1}(r)$ for each $r \in \mathbb{R}^+$.

THEOREM 9. *Assume that the continuous random field* $\xi = \{\xi(t),\ t \in T\}$ *satisfies condition* (9) *of the Theorem* 8. *Then* ξ *admits a version with almost all paths in* H_ρ^o.

Proof. From Theorem 8 it is easily seen that $(E_J\xi)$ is in probability Cauchy sequence in H_ρ^o and therefore $\lim_{J\to\infty} E_J\xi$ exists in probability. If this limit is denoted by $\widetilde{\xi}$ then it is easy to see that

$$\widetilde{\xi}(t) = \sum_{j=0}^{\infty}\sum_{v\in V_j}\lambda_{jv}(\xi)\Lambda_{jv}(t)$$

and is the version of ξ with paths in H_ρ^o.

Combining Theorem 9 and Lemma 7 gives a weak moments version of Ibragimov's result on Hölder regularity of random processes. Theorem 9 together with Lemma 6 lead to the same conclusion under control of Orlicz norms.

THEOREM 10. *Let* $p > 1$. *Assume that the continuous random field* $\xi = (\xi_t, t \in T)$ *satisfies the condition: for each* $t \in T$, $h \in C_t$,

$$P\{|\Delta_h^2\xi(t)| > \lambda\} \le \frac{c}{\lambda^p}\sigma^p(|h|),$$

where

$$\int_0^1 \frac{\sigma(u)}{u^{1+(d/p)}\rho(u)}\, du < \infty. \tag{12}$$

Then ξ *admits a version with almost all paths in the space* H_ρ^o.

THEOREM 11. *Let ϕ be a Young function. Assume that the continuous random field $\xi = (\xi_t, t \in T)$ satisfies the condition: for each $t \in T$ and $h \in C_t$,*

$$||\Delta_h^2 \xi(t)||_\phi \le c\sigma(|h|),$$

where

$$\int_0^1 \frac{\sigma(u)}{u\rho(u)} \phi^{-1}\Big(\frac{1}{u^d}\Big)\, du < \infty. \tag{13}$$

Then the random field ξ admits a version with almost all paths in the space H_ρ^o.

4. CLT IN H_ρ^o

We turn now to the central limit theorem for *random elements* in the separable space H_ρ^o. We shall study the cases of i.i.d. sequences and of triangular arrays of martingale differences. As a basic tool to handle this problem, we need the tightness criteria we are begining with.

4.1. Tightness conditions

THEOREM 12. *The sequence (ζ_n) of random elements in the Hölder space H_ρ^o is tight if and only if the following two conditions are satisfied:*

i) For each $t \in T$, $\lim_{A\to\infty} \sup_n P\{|\zeta_n(t)| > A\} = 0$;
ii) for each $\varepsilon > 0$ $\lim_{j\to\infty} \sup_n P\{\|\zeta_n - E_j\zeta_n\|_\rho^{\mathrm{seq}} > \varepsilon\} = 0$.

Proof. For fixed j, the space $E_j H_\rho^o$ is of finite dimension and we have on this space the equivalence of norms $\|y\|_\rho^{\mathrm{seq}}$ and $\max_{t\in W_j} |y(t)|$, $y \in E_j H_\rho^o$, of course with constants depending on j. Recall that for $t \in W_j$, $E_j x(t) = x(t)$. Now, using the flat concentration criterion (see Lemma 2.2 p. 40 in Ledoux Talagrand (1991)) it is easy to derive the tightness of (ζ_n) from i) and ii).

The necessity of i) for the tightness of (ζ_n) is obvious. For the necessity of ii), we invoke the following lemma, the proof of which can be found in Suquet (1996).

LEMMA 13. *Let $\mathcal{K}$ be a compact family (for the topology of weak convergence) of probability measures on the separable metric space H. Let $(F_j,\ j \ge 1)$ be a sequence of closed subsets of H decreasing to $\emptyset$. Define the functions $u_j : \mathcal{K} \to [0,1]$, $\mu \mapsto u_j(\mu) = \mu(F_j)$. Then the sequence (u_j) uniformly converges to zero on $\mathcal{K}$.*

Since the functionals $x \mapsto \|x - E_j x\|_\rho^{\mathrm{seq}}$ are continuous and decreasing to zero on H_ρ^o as j increase to infinity, the choice of the closed sets

$$F_j = \{x \in H_\rho^o;\ \|x - E_j x\|_\rho^{\mathrm{seq}} \ge \varepsilon\}, \quad j \uparrow \infty,$$

shows the necessity of ii) for the tightness of (ζ_n).

To obtain practical sufficient conditions for the tightness of (ζ_n), observe that if ξ is a random element in H_ρ^o, we have monotone convergence in (10) when K increases to infinity. Combining this limit version of (10) with Lemmas 2.2 or 2.3 gives then the following corollaries of Theorem 12.

COROLLARY 14. *The sequence (ζ_n) of random elements in the Hölder space H_ρ^o is tight if it satisfies i) of Theorem 12 together with*

ii) For each $t \in T$ and $h \in C_t$,

$$\sup_{n\geq 1} ||\Delta_h^2 \zeta_n(t)||_\phi \leq c\sigma(|h|),$$

where ϕ is a Young function and σ satisfies (13).

COROLLARY 15. *The sequence (ζ_n) of random elements in the Hölder space H_ρ^o is tight if it satisfies i) of Theorem 12 together with*

ii) For each $t \in T$ and $h \in C_t$,

$$\sup_{n\geq 1} P\left\{|\Delta_h^2 \zeta_n(t)| > \lambda\right\} \leq \frac{c}{\lambda^p}\sigma^p(|h|),$$

where σ satisfies (12).

4.2. The i.i.d. case

For a random element $\xi \in H_\rho^o$ we denote by $\xi_1, \dots, \xi_n$ independent copies of ξ and

$$\zeta_n := n^{-1/2} \sum_{k=1}^{n} \xi_k.$$

Recall, that a random element ξ satisfies the central limit theorem (denoted $\xi \in CLT(H_\rho^o)$), if the sequence (ζ_n) converges in distribution in H_ρ^o.

THEOREM 16. *Assume that the random element $\xi \in H_\rho^o$ satisfies the following conditions:*

i) $\mathbf{E}\,\xi(t) = 0$ and $\mathbf{E}\,\xi^2(t) < \infty$ for all $t \in T$;

ii) there exists a positive random variable M and an increasing function $\sigma : [0,\infty) \to R, \sigma(0) = 0$ such that $\mathbf{E}\,M^2 < \infty$ and

$$|\Delta_h^2 \xi(t)| \leq M\sigma(|h|), \quad \textit{for all} \ \ h \in C_t, t \in T,$$

where

$$\int_0^1 \frac{\sigma(s)}{s\rho(s)}\sqrt{\ln s^{-1}}\, ds < \infty.$$

Then $\xi \in CLT(H_\rho^o)$ *and* $\mathbf{E}\,\|\xi\|_\rho^2 < \infty$.

Proof. Consider the Rademacher sequence (ε_l) which is independent on (ξ_k) and may be constructed on another probability space, say, Ω'. By Theorem 10.14 in Ledoux-Talagrand (1991) it suffices to prove that for almost every ω of the probability space Ω supporting the ξ_k, the sequence of random elements $\widetilde{\zeta}_n^\omega$ of H_ρ^o defined on Ω' by

$$\widetilde{\zeta}_n^\omega(t) := n^{-1/2}\sum_{l=1}^n \varepsilon_l \xi_l(\omega, t), \quad t \in T$$

converges in distribution. For the convergence of its finite dimensional distributions, fix $t_1, \ldots, t_m$ in T and note that for any scalars $a_1, \ldots, a_m$

$$\mathbf{E}_\varepsilon \left(\sum_{i=1}^m a_i \widetilde{\zeta}_N^\omega(t_i)\right)^2 = \sum_{i,j=1}^m a_i a_j \frac{1}{n}\sum_{l=1}^n \xi_l(\omega, t_i)\xi_l(\omega, t_j).$$

Using i) and the strong law of large numbers we see that the factor of $a_i a_j$ in the above formula a.s. converges to $\mathbf{E}\,(\xi(t_i)\xi(t_j))$. Hence the convergence of finite dimensional distributions of $\widetilde{\zeta}_n^\omega$ holds for almost every ω by the finite-dimensional CLT. To check the tightness, we invoke Corrolary 14 whose Condition i) is a simple by product of the case $m = 1$ above. Next consider the function $\phi_2(t) = \exp\{t^2\} - 1, \quad t \geq 0$. The following is a simple corollary of the well known behavior of Rademacher sequence and Condition ii):

$$\begin{aligned} ||\Delta_h^2 \widetilde{\zeta}_n^\omega(t)||_{\phi_2}^2 = \frac{1}{n}\Big\|\sum_{l=1}^n \varepsilon_l \Delta_h^2 \xi_l(\omega, t)\Big\|_{\phi_2}^2 &\leq \frac{C}{n}\sum_{l=1}^n |\Delta_h^2 \xi_l(\omega, t)|^2 \\ &\leq \frac{C}{n}\sum_{l=1}^n M_l^2(\omega)\sigma^2(|h|), \end{aligned}$$

where $M_1, \ldots, M_n$ are independent copies of the random variable M. By the strong law of large numbers, for almost every $\omega \in \Omega$, $\sup_{n\geq 1} n^{-1}\sum_{l=1}^n M_l^2(\omega)$ is finite and so ζ_N^ω satisfies Condition ii) of Corollary 14. This ends the proof.

THEOREM 17. *Let* $\sigma : [0, \infty) \to R$ *be an increasing function with* $\sigma(0) = 0$. *Let* $p \geq 2$. *Assume that the random element* $\xi \in H_\rho^o$ *satisfies the following conditions:*

i) $\mathbf{E}\,\xi(t) = 0$ *and* $\mathbf{E}\,\xi^2(t) < \infty$ *for all* $t \in T$;

ii) for each $t \in T$ and $h \in C_t$

$$\mathbf{E}\,|\Delta_h^2 \xi(t)|^p \le \sigma^p(|h|),$$

where

$$\int_0^1 \frac{\sigma(u)}{u^{1+(d/p)}\rho(u)}\,du < \infty.$$

Then $\xi \in CLT(H_\rho^o)$.

Proof. From condition i) follows the convergence of finite dimensional distributions of (ζ_N) and the condition i) of Theorem 12. To complete the proof of the tightness, observe that by Rosenthal's L_p-inequality

$$\mathbf{E}\,|\Delta_h^2 \zeta_N(t)|^p \le c\mathbf{E}\,|\Delta_h^2 \xi(t)|^p \le c\sigma^p(|h|),$$

from which ii) of Corollary 15 is easily verified.

4.3. Martingale differences case

In this subsection we shall give a Hölder version of Brown's central limit theorem for martingale (see Brown (1971)). Let $\mathcal{X} = \{X_{n,k}, k = 1, \ldots, k_n, n \ge 1\}$ be an array of H_ρ^o-valued Bochner integrable random elements adapted to σ-fields $\mathbb{F} = \{\mathcal{F}_{n,k}; k = 0, 1, \ldots, k_n; n \ge 1\}$ such that for each $n, k \ge 1$ $\mathbf{E}\, X_{n,k}\big|\mathcal{F}_{n,k-1} = 0$ *a.s.* We shall assume that for each $t \in T$, $\mathbf{E}\, X_{n,k}^2(t) < \infty$. In this case we set

$$\sigma_n^2(t,s) := \sum_{k=1}^{k_n} \mathbf{E}\, X_{n,k}(t) X_{n,k}(s)\big|\mathcal{F}_{n,k-1}.$$

THEOREM 18. *Let $(\mathcal{X}, \mathbb{F})$ be a H_ρ^o-valued martingale difference array such that $\mathbf{E}\, X_{n,k}^2(t) < \infty$ for each $t \in T$. Assume also that the following conditions are satisfied:*

i) there exists a function $\psi : T \times T \to [0, \infty)$ such that for each $t, s \in T$

$$\sigma_n^2(t,s) \xrightarrow[n\to\infty]{P} \psi(t,s);$$

ii) for each $t \in T$ and $\varepsilon > 0$

$$\sum_{k=1}^{k_n} \mathbf{E}\, X_{n,k}^2(t) \mathbf{1}\{|X_{n,k}(t)| > \varepsilon\}\big|\mathcal{F}_{n,k-1} \xrightarrow[n\to\infty]{P} 0.$$

iii) there exists positive random variables $M_{n,1}, \dots, M_{n,k_n}$ and an increasing function $\sigma : [0,\infty) \to \mathbb{R}, \sigma(0) = 0$ such that $\sup_n \sum_{k=1}^{k_n} \mathbf{E}\, M_{n,k}^2 < \infty$ and

$$|\Delta_h^2 X_{n,k}(t)| \le M_{n,k}\, \sigma(|h|), \quad \text{for all } \ t \in T, h \in C_t,$$

where

$$\int_0^1 \frac{\sigma(s)}{s\rho(s)} \sqrt{\ln s^{-1}}\, ds < \infty. \tag{14}$$

Then $\sum_{k=1}^{k_n} X_{n,k}$ converges in distribution in the space H_ρ^o to zero mean Gaussian random field Y such that $\mathbf{E}\, Y(t)Y(s) = \psi(t,s)$ for $t,s \in T$.

Proof. Set

$$\zeta_n(t) = \sum_{k=1}^{k_n} X_{n,k}(t), \quad t \in T.$$

Fix $t_1, \dots, t_m \in T$. According to Cramer–Wold device $(\zeta_n(t_1), \dots, \zeta_n(t_m))$ converges in distribution provided any linear combination $\sum_{l=1}^m c_l \zeta_n(t_l)$ does, where $c_1, \dots, c_m \in \mathbb{R}$. Since

$$\sum_{l=1}^m c_l \zeta_n(t_l) = \sum_{k=1}^{k_n} \sum_{l=1}^m c_l X_{n,k}(t_l)$$

and $(\sum_{l=1}^m c_l X_{n,k}(t_l), k = 1, \dots, k_n)$ constitute the martingale difference array, the convergence in distribution of $\sum_{l=1}^m c_l \zeta_n(t_l)$ is an instant consequence of the Brown's central limit theorem.

To prove the tightness of the random field ζ_n we involve Corollary 14. Its first condition easily follows from i) and ii) and Rosenthal's inequality. To check the second condition of Corollary 14 note, that

$$\begin{aligned} P\left\{\|\zeta_n - E_J\zeta_n\|_\rho^{\mathrm{seq}} > \varepsilon\right\} &\le \varepsilon^{-1} \mathbf{E}\, \|\zeta_n - E_J\zeta_n\|_\rho^{\mathrm{seq}} \\ &\le \varepsilon^{-1} \sum_{j=J+1}^\infty \frac{1}{\rho(2^{-j})} \mathbf{E} \max_{v \in V_j} |\lambda_{j,v}(\zeta_n)|. \end{aligned} \tag{15}$$

Since $\lambda_{j,v}(\zeta_n) = \sum_{k=1}^{k_n} \lambda_{j,v}(X_{n,k})$, by Lemma 20 given in the appendix

$$\mathbf{E} \max_{v \in V_j} |\lambda_{j,v}(\zeta_n)| \le c\sqrt{\ln \operatorname{card}(V_j)} \Big(\sum_{k=1}^{k_n} \mathbf{E} \max_{v \in V_j} |\lambda_{j,v}(X_{n,k})|^2 \Big)^{1/2}. \tag{16}$$

Taking into account Condition iii) we obtain

$$\sum_{k=1}^{k_n} \mathbf{E} \max_{v \in V_j} |\lambda_{j,v}(X_{n,k})|^2 \le \sigma^2(2^{-j}) \sum_{k=1}^{k_n} \mathbf{E}\, M_{n,k}^2,$$

hence, since $\operatorname{card}(V_j) \le 2^{(j+1)d}$,

$$\mathbf{E} \max_{v \in V_j} |\lambda_{j,v}(\zeta_n)| \le c\sqrt{\ln 2^j}\,\sigma(2^{-j}).$$

Substituting this estimate into (15) and accounting (14) we complete the proof.

Let us remark, that the Condition iii) of the Theorem 18 can be replaced by the stronger condition

iii') $\sup_n \sum_{k=1}^{k_n} \mathbf{E}\,||X_{n,k}||_\sigma^2 < \infty$, *where* σ *satisfies (14).*

THEOREM 19. *Let* $(\mathcal{X}, \mathbb{F})$ *be a* H_ρ^o*-valued martingale difference array such that* $\mathbf{E}\,X_{n,k}^2(t) < \infty$ *for each* $t \in T$*. Assume also that the conditions i) and ii) of Theorem 18 are satisfied and*

iii) there exists an increasing function $\sigma : [0,\infty) \to \mathbb{R}, \sigma(0) = 0$ *such that*

$$\mathbf{E}\Big(\sum_{k=1}^{k_n} \mathbf{E}\left(\big(\Delta_h^2 X_{n,k}(t)\big)^2 \big| \mathcal{F}_{n,k-1}\right)\Big)^{p/2} + \sum_{k=1}^{k_n} \mathbf{E}\,|\Delta_h^2 X_{n,k}(t)|^p \le \sigma^p(|h|),$$

where $p \ge 2$ *and*

$$\int_0^1 \frac{\sigma(u)}{u^{1+(d/p)}\rho(u)}\,du < \infty.$$

Then $\sum_{k=1}^{k_n} X_{n,k}$ *converges in distribution in the space* H_ρ^o *to a zero mean Gaussian random field* Y *such that* $\mathbf{E}\,Y(t)Y(s) = \psi(t,s)$ *for* $t,s \in T$*.*

Proof. The proof is similar to that of Theorem 18. Only change to be made concerns the estimate (16). Instead of Lemma 20 one has to use Rosenthal's L_p inequality together with Lemma 6.

5. APPENDIX

Recall that we consider $\mathbb{R}^d$ endowed with the norm

$$|t| := \max_{1 \le i \le d} |t_i|\,, \quad t = (t_1, \ldots, t_d) \in \mathbb{R}^d.$$

Let $(X_n, n \ge 1)$ be a martingale difference sequence in $\mathbb{R}^d$ with respect to the increasing σ-algebras $(\mathcal{F}_n, n \ge 0)$. Set

$$S_0 = 0, \quad S_n = \sum_{k=1}^{n} X_k, \quad n \ge 1.$$

LEMMA 20. *There exists an absolute constant $c > 0$ such that for each $n \geq 1$*

$$\mathbf{E}\,|S_n| \leq c\sqrt{\ln d}\Big(\sum_{k=1}^n \mathbf{E}\,|X_k|^2\Big)^{1/2}.$$

Proof. It is proved by Bentkus (1990) that for each $\varepsilon > 0$ there exists an infinitely many times Frechet differentiable function $f_\varepsilon : \mathbb{R}^d \to \mathbb{R}^+$, such that $f_\varepsilon(0) = 0$ and

- $\sup_x \big||x| - f_\varepsilon(x)\big| \leq \varepsilon$;
- for each $k \in \mathbb{N}$ there exists a constant C_k independent on both n and d such that
$$\sup_x ||f_\varepsilon^{(k)}(x)|| \leq C_k \varepsilon^{-k+1} \ln^{k-1} d.$$

Hence
$$\mathbf{E}\,|S_n| \leq \varepsilon + \mathbf{E}\,f_\varepsilon(S_n) \tag{17}$$

Since
$$\mathbf{E}\,f_\varepsilon(S_n) = \sum_{k=1}^n \mathbf{E}\,(f_\varepsilon(S_k) - f_\varepsilon(S_{k-1}))$$

and $\mathbf{E}\,f'(S_{k-1})(X_k) = \mathbf{E}\,\mathbf{E}\,f'(S_{k-1})(X_k)\big|\mathcal{F}_{k-1} = 0$ we have by Taylor's formula
$$\mathbf{E}\,f_\varepsilon(S_n) = \sum_{k=1}^n \int_0^1 (1-\theta) f_\varepsilon''(S_{k-1} + \theta X_k)(X_k)^2 d\theta. \tag{18}$$

Now (17) and (18) yields

$$\mathbf{E}\,|S_n| \leq \varepsilon + \varepsilon^{-1} C_2 \ln d \sum_{k=1}^n \mathbf{E}\,|X_k|^2.$$

The proof is completed minimizing the right hand side with respect to $\varepsilon > 0$.

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