Hölderian invariance principles for triangular arrays of random variables*

Alfredas Račkauskas
Vilnius University and Institute of Mathematics and Informatics
Department of Mathematics, Vilnius University
Naugarduko 24, Lt-2006 Vilnius, Lithuania

Charles Suquet
Université des Sciences et Technologies de Lille
Mathématiques Appliquées, F.R.E. CNRS 2222
Bât. M2, U.F.R. de Mathématiques
F-59655 Villeneuve d'Ascq Cedex, France

Abstract

This paper extends Hölderian invariance principle by Lamperti (1962) to the case of partial sums processes based on a triangular array of row-wise independent random variables. As an application, necessary and sufficient conditions are obtained for the almost sure (resp. in probability) weak Hölder convergence of partial sums processes based on bootstrapped samples.

Keywords : bootstrap, functional central limit theorem, Hölder space, invariance principle, triangular array.

1 Introduction

The classical invariance principle or functional central limit theorem states the weak convergence of a partial sums process ξ_n to the Brownian motion W. When the paths of ξ_n are polygonal lines, the usual topological framework for this convergence is the space $C[0,1]$ of continuous functions on $[0,1]$, endowed with the supremum norm. Because the paths of the processes ξ_n and W have a better regularity than the sole continuity, one may be tempted to use some subspace of $C[0,1]$ with a stronger topology as a support for the paths. The first case to be considered in this prospect is undoubtedly the ladder of Hölder spaces $\mathcal{H}_\alpha^o$ with regularity exponent $0 < \alpha < 1/2$, where the bound $1/2$ is clearly induced by the regularity of W (for the definition of $\mathcal{H}_\alpha^o$, cf. Section 2). The

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question goes beyond the simple theoretical curiosity. Indeed this strengthening of the topology provides more continuous functionals of the paths, widening the applications area of the invariance principle.This proves useful in some statistical applications, particularly for the sharp detection of epidemic type changes [13, 14].

The first Hölderian invariance principle, which goes back to Lamperti [6], treats the case where ξ_n is the polygonal process with vertices $(k/n, n^{-1/2}S_k)$ with $S_k := X_1 + \cdots + X_k$ and the X_i's are centered i.i.d. Assuming the finiteness of $\mathbf{E}|X_1|^p$ for some $p > 2$, Lamperti establishes the weak convergence of ξ_n to W in every $\mathcal{H}_\alpha^o$ with $0 < \alpha < 1/2 - 1/p$. In order to test the optimality of his moment condition, Lamperti considers a symmetric random variable X_1 such that $\mathbf{P}\{X_1 \geq u\} = cu^{-p(\alpha)}$, $u \geq 1$ where $p(\alpha)$ is the critical exponent defined by

$$p(\alpha) := \frac{1}{1/2 - \alpha}. \tag{1}$$

Then he shows that the sequence (ξ_n) is not tight in $\mathcal{H}_\alpha^o$.

Lamperti's theorem was derived again by Kerkyacharian and Roynette [5] who used another method relying on Ciesielski's results [2] providing an analysis of Hölder spaces as sequence spaces, starting from their decomposition on the basis of triangular functions. Erickson [3] studied the case of partial sums processes indexed by $[0, 1]^d$. Hamadouche [4] investigated the case of some stationary weakly dependent sequences (X_n). In [9] the authors extended Lamperti's invariance principle to the case of an i.i.d. sequence (X_n) of random elements in a separable Banach space B, with Hölder spaces of functions $[0, 1] \to B$ related to the weight function $\rho(h) = h^\alpha \ln^\beta(1/h)$. Besides they showed in [10] how to relax considerably the moment assumption by combining self-normalization of the partial sums with some adaptive construction of the polygonal lines. Finally they gave in [12] a necessary and sufficient condition for the weak Hölder convergence of ξ_n to a Brownian motion, in the case where the X_i's have values in a separable Banach space B and where the Hölder space under consideration is related to the weight function $\rho(h) = h^\alpha L(1/h)$, L being a slowly varying function. In the setting of Lamperti's theorem, this necessary and sufficient condition reads :

$$\xi_n \xrightarrow[\mathcal{H}_\alpha^o]{\mathcal{D}} W \quad \text{if and only if} \quad \lim_{t\to\infty} t^{p(\alpha)} \mathbf{P}\big(|\xi_1| \geq t\big) = 0. \tag{2}$$

All these works concern the case of identically distributed X_k's. In this paper we tackle for the first time the case of triangular arrays

$$(X_{n,k}, k = 1, \dots, k_n), \quad n = 1, 2, \dots,$$

of row wise independent random variables. We assume in all the sequel that $\mathbf{E}X_{n,k} = 0$, $\sigma_{n,k}^2 := \operatorname{var}(X_{n,k}) = \mathbf{E}X_{n,k}^2 < \infty$ and $\sigma_{n,k} > 0$ for $k = 1, \dots, k_n$, $n = 1, 2, \dots$ Put $S_n(0) = 0$, $b_n(0) = 0$ and

$$S_n(k) = X_{n,1} + \cdots + X_{n,k}, \quad b_n(k) = \sigma_{n,1}^2 + \cdots + \sigma_{n,k}^2,$$

for $k = 1, 2, \dots, k_n$. We also assume that $b_n(k_n) = 1$. Consider the process $\Xi_n(t)$, $t \in [0, 1]$, built by affine interpolation between the points $(b_n(k), S_n(k))$, $k = 0, 1, \dots, k_n$:

$$\Xi_n(t) = S_n(k-1) + \sigma_{n,k}^{-2}(t - b_n(k))X_{n,k}, \quad b_n(k-1) \le t \le b_n(k),$$

$k = 1, \dots, k_n$. We study the convergence in distribution of this process in the functional framework of the Hölder spaces $\mathcal{H}_\alpha^o$ defined as follows. For $0 < \alpha < 1$, $\mathcal{H}_\alpha^o$ is the set of continuous functions $x : [0, 1] \to \mathbb{R}$ such that

$$\lim_{\delta \to 0} w_\alpha(x, \delta) = 0,$$

where

$$w_\alpha(x, \delta) := \sup_{0<|t-s|<\delta} \frac{|x(t) - x(s)|}{|s - t|^\alpha}.$$

This set $\mathcal{H}_\alpha^o$ becomes a separable Banach space when endowed with the norm

$$\|x\|_\alpha := |x(0)| + w_\alpha(x, 1).$$

In all the paper we consider only the ladder of spaces $\mathcal{H}_\alpha^o$ with $0 < \alpha < 1/2$.

The paper is organized as follows. Section 2 reminds the analysis of the Hölder spaces by the basis of Schauder triangular functions, leading to the tightness theorem in $\mathcal{H}_\alpha^o$. Section 3 is devoted to the general results whose statements are gathered in the subsection 3.1 while the proofs are deferred to the following subsections. Following the same way of exposition, Section 4 presents some applications to the partial sums processes built on bootstrapped samples.

2 Preliminaries

To establish Hölderian invariance principles, it is convenient to use some dyadic discretization procedure associated to the basis of Faber-Schauder triangular functions. Denote by D_j the set of dyadic numbers of level j in $[0, 1]$, *i. e.*

$$\mathrm{D}_0 = \{0, 1\}, \qquad \mathrm{D}_j = \left\{(2l-1)2^{-j};\ 1 \le l \le 2^{j-1}\right\}, \quad j \ge 1$$

and by D the set of all the dyadic numbers in $[0, 1]$, $\mathrm{D} = \cup_{j=0}^\infty \mathrm{D}_j$. Put for $r \in \mathrm{D}_j$, $j \ge 0$,

$$r^- := r - 2^{-j}, \quad r^+ := r + 2^{-j}.$$

We index each Faber-Schauder triangular function by the abscissa of its vertex. So for $r \in \mathrm{D}_j$, $j \ge 1$, Λ_r is the function defined by

$$\Lambda_r(t) := \begin{cases} 2^j(t - r^-) & \text{if } t \in (r^-, r]; \\ 2^j(r^+ - t) & \text{if } t \in (r, r^+]; \\ 0 & \text{elsewhere.} \end{cases}$$

In the special case $j = 0$, we take

$$\Lambda_0(t) := 1 - t, \quad \Lambda_1(t) := t, \quad t \in [0,1].$$

If x is a function $[0,1] \to \mathbb{R}$, let us define its Schauder coefficients $\lambda_r(x)$ by

$$\lambda_r(x) := x(r) - \frac{1}{2}\big(x(r^+) + x(r^-)\big), \quad r \in \mathrm{D}_j,\ j \geq 1$$

and $\lambda_r(x) := x(r)$ when $r \in \mathrm{D}_0$. It is well known that if x is continuous on $[0,1]$, it admits an expansion in Schauder series

$$x = \sum_{r \in \mathrm{D}} \lambda_r(x) \Lambda_r, \tag{3}$$

uniformly convergent on $[0,1]$. Ciesielski [2] proved that if x belongs to the separable Hölderian space $\mathcal{H}_\alpha^o$, the convergence (3) holds for the strong topology of this space. Moreover on $\mathcal{H}_\alpha^o$, the native Hölderian norm $\|x\|_\alpha$ is equivalent to a sequence Hölderian norm:

$$\|x\|_\alpha \sim \|x\|_\alpha^{\mathrm{seq}} := \sup_{j \geq 0} 2^{j\alpha} \max_{r \in \mathrm{D}_j} |\lambda_r(x)|. \tag{4}$$

It follows that $(\Lambda_r,\ r \in \mathrm{D})$ is a Schauder (unconditional) basis of $\mathcal{H}_\alpha^o$, which with the equivalence of norms (4) provides us the following tightness criterion.

Theorem 1. *The sequence $(Z_n)_{n \geq 1}$ of random elements in the Hölder space $\mathcal{H}_\alpha^o$ is tight if and only if it satisfies both following conditions.*

i) $\lim_{u \to \infty} \sup_{n \geq 1} \mathbf{P}\big\{|Z_n(0)| > u\big\} = 0$;

ii) for every $\varepsilon > 0$,

$$\lim_{J \to \infty} \limsup_{n \to \infty} \mathbf{P}\Big\{\sup_{j \geq J} 2^{j\alpha} \max_{r \in \mathrm{D}_j} |\lambda_r(Z_n)| > \varepsilon\Big\} = 0.$$

For a proof, we refer to [11].

3 General results

3.1 Statements

In the first Hölderian functional central limit theorem of this section, we assume the existence of moments of order $q > 2$ and generalize Lamperti's theorem to the case of triangular arrays.

Theorem 2. *If*

$$\lim_{n \to \infty} \max_{1 \leq k \leq k_n} \sigma_{n,k}^2 = 0 \tag{5}$$

and if for some $q > p(\alpha)$, where $p(\alpha)$ is defined by (1),

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \sigma_{n,k}^{-2\alpha q} \mathbf{E}|X_{n,k}|^q = 0, \tag{6}$$

then

$$\Xi_n \xrightarrow[n\to\infty]{\mathcal{D}} W \quad \text{in the space } \mathcal{H}_\alpha^o. \tag{7}$$

To check that Theorem 2 generalizes Lamperti's one, let us consider the special case where $k_n = n$, $X_{n,k} = n^{-1/2}X_k$, the sequence (X_k) being i.i.d. centered, with variance $\mathbf{E}X_k^2 = 1$ and $\mathbf{E}|X_1|^q < \infty$ for some $q > 2$. Since $\sigma_{n,k}^2 = n^{-1}$, (5) is trivially satisfied and (6) reduces to the convergence to 0 of $n^{1-q\alpha-q/2}$, which holds automatically because $q > 2$. So the only constraint is $q > p(\alpha)$. Hence Theorem 2 tells us in this case that Ξ_n converges in distribution to W in $\mathcal{H}_\alpha^o$ for each $\alpha < 1/2 - 1/q$, which is precisely Lamperti's result.

In order to relax the moment condition (6) in Theorem 2, let us introduce for each $\tau > 0$, the truncated random variables

$$X_{n,k,\tau} := X_{n,k}\mathbf{1}\{|X_{n,k}| \le \tau\sigma_{n,k}^{2\alpha}\}, \quad k = 1, \dots, k_n; n = 1, 2, \dots$$

Theorem 3. *Assume that the following conditions are satisfied.*

1. *For every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \mathbf{P}(|X_{n,k}| \ge \varepsilon\sigma_{n,k}^{2\alpha}) = 0. \tag{8}$$

2. *For every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \mathbf{E}X_{n,k}^2\mathbf{1}\{|X_{n,k}| \ge \varepsilon\} = 0. \tag{9}$$

3. *For some $q > p(\alpha)$,*

$$\lim_{\tau\to 0} \limsup_{n\to\infty} \sum_{k=1}^{k_n} \sigma_{n,k}^{-2\alpha q} \mathbf{E}|X_{n,k,\tau}|^q = 0. \tag{10}$$

Then the convergence in distribution (7) *holds true.*

Conditions (8) and (10) substitute to (6) and are evidently more general. Lindeberg's condition (9) insures that the triangular array $(X_{n,k})$ satisfies the central limit theorem. It implies also the infinitesimal negligibility condition (5).

Let us consider now the more specific case of a triangular array whose rows are linear combinations with deterministic coefficients of the same i.i.d. sequence. So let (X_k) be a sequence of i.i.d. centered random variables, with variance $\mathbf{E}X_k^2 = 1$. Let

$$\mathsf{a} := (a_{n,k},\ 1 \le k \le k_n,\ n \ge 1)$$

be a triangular array of *non null* real numbers satisfying

$$\forall n \geq 1, \quad \sum_{k=1}^{k_n} a_{n,k}^2 = 1. \tag{11}$$

$$\lim_{n\to\infty} \max_{1\leq k\leq k_n} a_{n,k}^2 = 0. \tag{12}$$

Let us denote by $\Xi_{\mathsf{a},n}$ the polygonal partial sums process built on the triangular array $(X_{n,k})$, where

$$X_{n,k} := a_{n,k}X_k, \quad k = 1, \dots, k_n,\ n \geq 1.$$

Theorem 4. *Assume that the following conditions are satisfied.*

$$\forall \varepsilon > 0, \quad \lim_{n\to\infty} \sum_{k=1}^{k_n} \mathbf{P}\big(|X_1| \geq \varepsilon |a_{n,k}|^{2\alpha-1}\big) = 0. \tag{13}$$

$$M := \sup_{t>0,\ n\geq 1} t^{p(\alpha)} \sum_{k=1}^{k_n} \mathbf{P}\big(|X_1| \geq t|a_{n,k}|^{2\alpha-1}\big) < \infty. \tag{14}$$

Then

$$\Xi_{\mathsf{a},n} \xrightarrow[n\to\infty]{\mathcal{D}} W \quad \text{in the space } \mathcal{H}_\alpha^o. \tag{15}$$

Besides, Condition (13) is necessary for the convergence (15).

Corollary 5. *Assume that the triangular array* a *satisfies (11), that the sequence* $(k_n)_{n\geq 1}$ *is non decreasing, such that*

$$\sup_{n\geq 1} \frac{k_{n+1}}{k_n} < \infty \tag{16}$$

and

$$\forall n \geq 1,\ \forall k = 1, \dots, k_n, \qquad \frac{\beta}{k_n} \leq a_{n,k}^2 \leq \frac{\gamma}{k_n}, \tag{17}$$

for some positive constants β, γ. *Then* $\Xi_{\mathsf{a},n}$ *converges in distribution to* W *in the space* $\mathcal{H}_\alpha^o$ *if and only if*

$$\lim_{t\to\infty} t^{p(\alpha)}\mathbf{P}(|X_1| \geq t) = 0. \tag{18}$$

This corollary extends the necessary and sufficient condition (2).

3.2 Proof of Theorem 2

First we note that Condition (5) gives the infinitesimal negligibility of $(X_{n,k})$. As $q > p(\alpha) > 2$, we get the elementary inequality

$$\sum_{k=1}^{k_n} \mathbf{E}X_{n,k}^2 \mathbf{1}\{|X_{n,k}| \geq \varepsilon\} \leq \frac{1}{\varepsilon^{q-2}} \sum_{k=1}^{k_n} \mathbf{E}|X_{n,k}|^q. \tag{19}$$

Accounting (5), Condition (6) gives immediatly the convergence to zero of $\sum_{k=1}^{k_n}\mathbf{E}|X_{n,k}|^q$, which in turn, implies together with (19) the Lindeberg's condition:

$$\forall\varepsilon>0,\quad \lim_{n\to\infty}\sum_{k=1}^{k_n}\mathbf{E}X_{n,k}^2\mathbf{1}\{|X_{n,k}|\geq\varepsilon\}=0.$$

Hence the triangular array $(X_{n,k})$ satisfies the central limit theorem and the classical functional central limit theorem (cf. [1]), namely

$$\Xi_n \xrightarrow[n\to\infty]{\mathcal{D}} W\quad\text{in the space } C[0,1].$$

As a result this provides the convergence of the finite dimensional distributions of the process Ξ_n.

To prove the tightness in $\mathcal{H}_\alpha^o$ of $(\Xi_n)_{n\geq 1}$, we use Theorem 1. Since $\Xi_n(0)=0$, we only have to check the second condition of this theorem. To this aim we have to prove that for every $\varepsilon>0$,

$$\lim_{J\to\infty}\limsup_{n\to\infty}P(J,n,\varepsilon)=0,\tag{20}$$

where

$$P(J,n,\varepsilon):=\mathbf{P}\left\{\sup_{j\geq J}2^{\alpha j}\max_{r\in \mathrm{D}_j}|\Xi_n(r)-\Xi_n(r^-)|\geq\varepsilon\right\}.$$

The handling of the differences $|\Xi_n(r^+)-\Xi_n(r)|$ is of course completely similar and will be omited.

Define for $t\in[0,1]$,

$$u(t):=\max\{k\geq 0: b_n(k)\leq t\},$$

which provides the inequalities

$$\forall t\in[0,1],\quad b_n(u(t))\leq t<b_n(u(t)+1).\tag{21}$$

For notational simplicity, we put

$$Y_n:=\max_{1\leq k\leq k_n}\sigma_{n,k}^{-2\alpha}|X_{n,k}|.$$

For $t\in[0,1]$, we have

$$\Xi_n(t)=S_n(u(t))+\frac{t-b_n(u(t))}{b_n(u(t)+1)-b_n(u(t))}X_{n,u(t)+1}.\tag{22}$$

Let $r\in\mathrm{D}_j$. To control $|\Xi_n(r)-\Xi_n(r^-)|$, we shall consider the three possible configurations of r^- and r, relatively to the intervals generated by the $b_n(k)$'s. The points r^- and r are either in the same interval (case 1), either in contiguous intervals (case 2), or separated by at least one interval (case 3).

Case 1, $u(r)=u(r^-)$. Clearly then $2^{-j}\leq\sigma_{n,u(r)+1}^2$ and

$$|\Xi_n(r)-\Xi_n(r^-)|=\sigma_{u(r)+1}^{-2}(r-r^-)|X_{n,u(r)+1}|\leq Y_n 2^{-j\alpha}.$$

Case 2, $u(r) = u(r^-) + 1$. Then $b_n(u(r^-)) \le r^- < b_n(u(r)) \le r < b_n(u(r)+1)$. In this case, (22) enables us to write :

$$\Xi_n(r) - \Xi_n(r^-) = \frac{r - b_n(u(r))}{\sigma^2_{n,u(r)+1}} X_{n,u(r)+1} + \frac{b_n(u(r)) - r^-}{\sigma^2_{n,u(r)}} X_{n,u(r)}. \tag{23}$$

Let us note that under this configuration, the ratio $\bigl(r - b_n(u(r))/\sigma^2_{n,u(r)+1}$ belongs to $[0,1]$ and its numerator $r - b_n(u(r))$ is bounded from above by $r - r^- = 2^{-j}$. As the exponent α is less than 1, it follows

$$\frac{r - b_n(u(r))}{\sigma^2_{n,u(r)+1}} \le \left(\frac{r - b_n(u(r))}{\sigma^2_{n,u(r)+1}}\right)^\alpha \le 2^{-j\alpha}\sigma^{-2\alpha}_{n,u(r)+1}. \tag{24}$$

Similarly we get

$$\frac{b_n(u(r)) - r^-}{\sigma^2_{n,u(r)}} \le 2^{-j\alpha}\sigma^{-2\alpha}_{n,u(r)}. \tag{25}$$

Gathering (23), (24) and (25) we obtain for the case 2 the upper bound :

$$|\Xi_n(r) - \Xi_n(r^-)| \le 2^{1-j\alpha} Y_n.$$

Case 3, $u(r) > u(r^-) + 1$. Using again (24) and (25) and triangular inequality gives

$$|\Xi_n(r) - \Xi_n(r^-)| \le |S_n(u(r)) - S_n(u(r^-)+1)| + 2^{1-j\alpha} Y_n.$$

This last upper bound being also valid in the cases 1 et 2, we finally get (with the convention $\sum_{k\in\emptyset} \equiv 0$):

$$\forall r \in \mathrm{D}_j, \quad |\Xi_n(r) - \Xi_n(r^-)| \le \Bigl|\sum_{k=u(r^-)+2}^{u(r)} X_k\Bigr| + 2^{1-j\alpha} Y_n. \tag{26}$$

This leads us to introduce the splitting

$$P(J,n,\varepsilon) \le P_1(J,n,\varepsilon) + P_2(n,\varepsilon),$$

where

$$\begin{aligned} P_1(J,n,\varepsilon) &:= \mathbf{P}\Bigl\{\sup_{j\ge J} 2^{\alpha j} \max_{r\in \mathrm{D}_j} \Bigl|\sum_{k=u(r^-)+2}^{u(r)} X_k\Bigr| > \varepsilon/2\Bigr\}, \quad (27)\\ P_2(n,\varepsilon) &:= \mathbf{P}\bigl\{|Y_n| \ge \varepsilon/4\bigr\}. \quad (28)\end{aligned}$$

So, noting that Condition (6) gives the convergence in probability of Y_n to zero, the proof of (20) reduces to that of the convergence

$$\lim_{J\to\infty} \limsup_{n\to\infty} P_1(J,n,\varepsilon) = 0. \tag{29}$$

By Rosenthal's inequality, we get with some constant $c = c(q)$,

$$\mathbf{E}\Big|\sum_{i=u(r^-)+2}^{u(r)} X_{n,k}\Big|^q \leq c\Big(\sum_{i=u(r^-)+2}^{u(r)} \mathbf{E}X_{n,k}^2\Big)^{q/2} + c\sum_{i=u(r^-)+2}^{u(r)} \mathbf{E}|X_{n,k}|^q.$$

Recalling that by definition of $u(r)$, we have

$$\sum_{k=u(r^-)+2}^{u(r)} \sigma_{n,k}^2 \leq r - r^- = 2^{-j},$$

we easily arrive to

$$P_1(J,n,\varepsilon) \leq \frac{c}{\varepsilon^q}\sum_{j\geq J} 2^{(q\alpha+1-q/2)j} + \frac{2^q c}{\varepsilon^q}\sum_{j\geq J} 2^{\alpha qj}\sum_{r\in \mathrm{D}_j}\sum_{i=u(r^-)+2}^{u(r)} \mathbf{E}|X_{n,k}|^q.$$

The first of these sums does not depend on n and goes to zero when J goes to infinity because $q > p(\alpha)$. Write $I(J,n,q)$ for the second sum in this upper bound (without the constant $2^q c\varepsilon^{-q}$). Exchanging summations leads to

$$I(J,n,q) = \sum_{k=1}^{k_n} \mathbf{E}|X_{n,k}|^q \sum_{j\geq J} 2^{\alpha qj}\sum_{r\in \mathrm{D}_j} \mathbf{1}\{u(r^-)+1 < k \leq u(r)\}.$$

Now consider for fixed k the inequalities

$$u(r^-)+1 < k \leq u(r). \tag{30}$$

Assume there is some $r \in \mathrm{D}_j$ satisfying (30) and let r' be another dyadic number in D_j. If $r' < r$, then $r' < r^-$ whence by increasingness of u, $u(r') < u(r^-)+1 < k$, which forbids r' to satisfy (30). If $r' > r$, then $r'^- > r$ whence $k \leq u(r) \leq u(r'^-) < u(r'^-)+1$, which forbids r' to satisfy (30). So for fixed k, there is *at most one* $r \in \mathrm{D}_j$ satisfying (30). When this r do exist, the definition of u provides the inequalities

$$r^- < \sum_{i=1}^{u(r^-)+1} \sigma_{n,i}^2 < \sum_{i=1}^{k} \sigma_{n,i}^2 \leq \sum_{i=1}^{u(r)} \sigma_{n,i}^2 \leq r,$$

from which we immediatly obtain $\sigma_{n,k}^2 \leq 2^{-j}$. Therefore

$$\forall k = 1,\dots,k_n, \quad \sum_{r\in \mathrm{D}_j} \mathbf{1}\{u(r^-)+1 < k \leq u(r)\} \leq \mathbf{1}\{2^{-j} \geq \sigma_{n,k}^2\},$$

which gives

$$\sum_{j\geq J} 2^{\alpha qj}\sum_{r\in \mathrm{D}_j} \mathbf{1}\{u(r^-)+1 < k \leq u(r)\} \leq \frac{2^{q\alpha}}{2^{q\alpha}-1}\sigma_{n,k}^{-2\alpha q}. \tag{31}$$

From this we finally deduce that

$$\forall J \geq 1,\ \forall n \geq 1, \quad I(J,n,q) \leq C \sum_{k=1}^{k_n} \sigma^{-2\alpha q} \mathbf{E}|X_{n,k}|^q,$$

with some constant $C = C(\alpha, q)$ and we conclude using Condition (6).

3.3 Proof of Theorem 3

The convergence of the finite dimensional distributions of Ξ_n follows from Lindeberg's condition as in the proof of Theorem 2. It remains to establish the tightness, whose proof reduces as previously to the asymptotic control of the quantities $P_1(J,n,\varepsilon)$ and $P_2(n,\varepsilon)$ defined by (27) and (28). First we note that

$$\mathbf{P}\Big(Y_n \geq \frac{\varepsilon}{4}\Big) \leq \sum_{k=1}^{k_n} \mathbf{P}\Big(|X_{n,k}| \geq \frac{\varepsilon}{4}\sigma_{n,k}^{2\alpha}\Big), \tag{32}$$

which due to (8), implies the convergence of $P_2(n,\varepsilon)$ to 0.

Write $A_n := \big\{\max_{1\leq k\leq k_n} |X_{n,k}| \leq \tau\sigma_{n,k}^{2\alpha}\big\} = \{Y_n \leq \tau\}$. Noting that on this event $X_{n,k} = X_{n,k,\tau}$ for $k = 1, \dots, k_n$, we may write

$$P_1(J,n,\varepsilon) \leq P_1'(J,n,\varepsilon,\tau) + \mathbf{P}(A_n^c),$$

where

$$P_1'(J,n,\varepsilon,\tau) := \mathbf{P}\Big\{\sup_{j\geq J} 2^{\alpha j} \max_{r\in \mathrm{D}_j} \Big|\sum_{k=u(r^-)+2}^{u(r)} X_{n,k,\tau}\Big| \geq \frac{\varepsilon}{2}\Big\}.$$

By (32) and (8), $\mathbf{P}(A_n^c)$ converges to 0. It remains to study the asymptotic behavior of $P_1'(J,n,\varepsilon,\tau)$. Before using Rosenthal's inequality as in the previous proof, it is worth noticing that unlike the $X_{n,k}$'s, the $X_{n,k,\tau}$'s are not centered.

To control the contribution of the $\mathbf{E}X_{n,k,\tau}$'s to P_1', we use repetedly Cauchy-Schwarz inequality to write

$$\begin{aligned}\Big|\sum_{k=u(r^-)+2}^{u(r)} \mathbf{E}X_{n,k,\tau}\Big| &\leq \sum_{k=u(r^-)+2}^{u(r)} \big(\mathbf{E}X_{n,k}^2\big)^{1/2} \mathbf{P}^{1/2}(|X_{n,k}| \geq \tau\sigma_{n,k}^{2\alpha}) \\ &\leq \Big[\sum_{k=u(r^-)+2}^{u(r)} \sigma_{n,k}^2\Big]^{1/2} \Big[\sum_{k=1}^{n} \mathbf{P}(|X_{n,k}| \geq \tau\sigma_{n,k}^{2\alpha})\Big]^{1/2}.\end{aligned}$$

From the definition of $u(r)$ we get

$$\sum_{k=u(r^-)+2}^{u(r)} \sigma_{n,k}^2 \leq r - r^- = 2^{-j}. \tag{33}$$

Therefore

$$\sup_{j\geq J} 2^{\alpha j} \max_{r\in \mathrm{D}_j} \Big| \sum_{k=u(r^-)+2}^{u(r)} \mathbf{E} X_{n,k,\tau} \Big| \leq 2^{-J(1/2-\alpha)} \Big[\sum_{k=1}^{n} \mathbf{P}(|X_{n,k}| \geq \tau \sigma_{n,k}^{2\alpha}) \Big]^{1/2}.$$

This upper bound is less than $\varepsilon/4$ for $n \geq n_0$, where n_0 depends on ε and τ but not on J. So for $n \geq n_0$, we have $P_1'(J,n,\varepsilon,\tau) \leq P_1''(J,n,\varepsilon,\tau)$, where

$$P_1''(J,n,\varepsilon,\tau) := \mathbf{P}\Big\{ \sup_{j\geq J} 2^{\alpha j} \max_{r\in \mathrm{D}_j} \Big| \sum_{k=u(r^-)+2}^{u(r)} (X_{n,k,\tau} - \mathbf{E} X_{n,k,\tau}) \Big| \geq \varepsilon/4 \Big\}.$$

We can now apply Rosenthal's inequality which, together with the elementary bound $\mathrm{var} X_{n,k,\tau} \leq \mathbf{E} X_{n,k}^2$ and accounting (33), gives :

$$\mathbf{E} \Big| \sum_{k=u(r^-)+2}^{u(r)} (X_{n,k,\tau} - \mathbf{E} X_{n,k,\tau}) \Big|^q \quad \leq \quad c 2^{-jq/2} + c \sum_{k=u(r^-)+2}^{u(r)} \mathbf{E} |X_{n,k,\tau}|^q.$$

It follows

$$P_1''(J,n,\varepsilon) \leq \frac{4^q c}{\varepsilon^q} \sum_{j\geq J} 2^{(q\alpha+1-q/2)j} + \frac{4^q c}{\varepsilon^q} \sum_{j\geq J} 2^{\alpha q j} \sum_{r\in \mathrm{D}_j} \sum_{k=u(r^-)+2}^{u(r)} \mathbf{E} |X_{n,k,\tau}|^q.$$

Recalling that $q > p(\alpha)$ and hence $q\alpha + 1 - q/2 < 0$ and using (31), we obtain

$$P_1''(J,n,\varepsilon) \leq C_1 2^{(q\alpha+1-q/2)J} + C_2 \sum_{k=1}^{n} \sigma_{n,k}^{-2\alpha q} \mathbf{E} |X_{n,k,\tau}|^q,$$

with some constants C_1 and C_2 depending α, q and ε. Let us observe that this upper bound holds for every $\tau > 0$. In view of the successive reductions in the study of $P(J,n,\varepsilon)$ we deduce that

$$\forall \tau > 0, \quad \limsup_{J\to\infty} \limsup_{n\to\infty} P(J,n,\varepsilon) \leq C_2 \limsup_{n\to\infty} \sum_{k=1}^{n} \sigma_{n,k}^{-2\alpha q} \mathbf{E} |X_{n,k,\tau}|^q.$$

Letting finally τ go to 0, we obtain due to (10),

$$\lim_{J\to\infty} \limsup_{n\to\infty} P(J,n,\varepsilon) = 0,$$

which ends the proof.

3.4 Proof du Theorem 4

To establish the convergence (15), we check that the random variables $X_{n,k} = a_{n,k} X_k$ fulfill the conditions of Theorem 3. Note that (13) is nothing else than the translation of (8) in the special case of the processes $\Xi_n = \Xi_{\mathsf{a},n}$.

Lindeberg's condition (9) follows from (11), (12) and of the finiteness of $\mathbf{E}X_1^2$ via the inequality

$$\sum_{k=1}^{k_n} a_{n,k}^2 \mathbf{E}X_1^2 \mathbf{1}\{|X_1| \geq \varepsilon |a_{n,k}|^{-1}\} \leq \mathbf{E}X_1^2 \mathbf{1}\{|X_1| \geq \varepsilon d_n^{1/2}\},$$

where

$$d_n := \min_{1\leq k\leq k_n} a_{n,k}^{-2} = \Big(\max_{1\leq k\leq k_n} a_{n,k}^2\Big)^{-1}. \tag{34}$$

To check (10), let us first note that for every positive random variable Y and every constant $c > 0$,

$$\mathbf{E}Y^q \mathbf{1}\{Y \leq c\} = \int_0^c qt^{q-1}\mathbf{P}(c \geq Y \geq t)\,\mathrm{d}t \leq \int_0^c qt^{q-1}\mathbf{P}(X \geq t)\,\mathrm{d}t. \tag{35}$$

The truncated variables are here $X_{n,k,\tau} = a_{n,k}X_k\mathbf{1}\{|X_k| \leq \tau|a_{n,k}|^{2\alpha-1}\}$. Applying (35) to the variables $|a_{n,k}|^{1-2\alpha}|X_k|$, we obtain with $q > p = p(\alpha)$,

$$\begin{aligned}
\sum_{k=1}^{k_n} \sigma_{n,k}^{-2\alpha q}\mathbf{E}|X_{n,k,\tau}|^q &= \sum_{k=1}^{k_n} \mathbf{E}\big(|a_{n,k}|^{1-2\alpha}|X_k|\big)^q \mathbf{1}\{|a_{n,k}|^{(1-2\alpha)}|X_k| \leq \tau\} \\
&\leq \int_0^\tau qt^{q-1}\sum_{k=1}^{k_n}\mathbf{P}\big(|X_k| \geq t|a_{n,k}|^{2\alpha-1}\big)\,\mathrm{d}t \\
&\leq \frac{Mq}{q-p}\tau^{q-p}.
\end{aligned}$$

As $q > p$, Condition (10) is satisfied and this completes the proof of the convergence (15).

Let us now show the necessity of (13) for this same convergence. By continuous mapping theorem, the convergence in distribution of $\Xi_{\mathsf{a},n}$ to W in $\mathcal{H}_\alpha^o$ gives for every $\delta > 0$:

$$w_\alpha(\Xi_{\mathsf{a},n},\delta) \xrightarrow[n\to\infty]{\mathcal{D}} w_\alpha(W,\delta). \tag{36}$$

Due to (12), we have $\max_{1\leq k\leq n} a_{n,k}^2 < \delta$ for $n \geq n_0(\delta)$, which enables us to bound the increments of $\Xi_{\mathsf{a},n}$ between two consecutive vertices by the modulus of Hölderian regularity $w_\alpha(\Xi_{\mathsf{a},n},\delta)$. More precisely, we get

$$\begin{aligned}
\mathbf{P}\Big\{\max_{1\leq k\leq n}|a_{n,k}X_k|a_{n,k}^{-2\alpha} > u\Big\} &= \mathbf{P}\Big\{\max_{0<k\leq n}\frac{|S_n(k)-S_n(k-1)|}{a_{n,k}^{2\alpha}} > u\Big\} \\
&\leq \mathbf{P}\Big\{\sup_{0<|t-s|<\delta}\frac{|\Xi_{\mathsf{a},n}(t)-\Xi_{\mathsf{a},n}(s)|}{|t-s|^\alpha} > u\Big\} \\
&= \mathbf{P}\big\{w_\alpha(\Xi_{\mathsf{a},n},\delta) > u\big\}.
\end{aligned}$$

By the convergence (36)

$$\limsup_{n\to\infty}\mathbf{P}(w_\alpha(\Xi_n,\delta) > u) \leq \mathbf{P}(w_\alpha(W,\delta) > u).$$

It follows that for every $\delta > 0$,

$$\limsup_{n\to\infty} \mathbf{P}\Big\{ \max_{1\le k\le n} |a_{n,k}|^{1-2\alpha}|X_k| > u \Big\} \le \mathbf{P}(w_\alpha(W,\delta) > u).$$

As $\mathbf{P}(w_\alpha(W,\delta) > u)$ goes to 0 when δ goes to 0 and accounting the independence of the X_k's we deduce that

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \mathbf{P}\big(|X_k| > u|a_{n,k}|^{2\alpha-1}\big) = 0,$$

which gives (13) since the X_k's have the same distribution.

3.5 Proof of Corollary 5

To establish the sufficiency of (18) for the convergence in distribution of $\Xi_{\mathsf{a},n}$ to W in $\mathcal{H}_\alpha^o$, we use Theorem 4. As a matter of fact, we check (13) under the asumption (12) which is more general than (17), by writing

$$\begin{aligned}
\sum_{k=1}^{k_n} \mathbf{P}\big\{|X_1| \ge \varepsilon|a_{n,k}|^{2\alpha-1}\big\} &= \sum_{k=1}^{k_n} a_{n,k}^2 a_{n,k}^{-2}\mathbf{P}\big\{|X_1| \ge \varepsilon(|a_{n,k}|^{-2})^{1/2-\alpha}\big\} \\
&\le \sum_{k=1}^{k_n} a_{n,k}^2 \sup_{t\ge|a_{n,k}|^{-2}} t\mathbf{P}\big\{|X_1| \ge \varepsilon t^{1/p(\alpha)}\big\} \\
&\le \sup_{t\ge d_n} t\mathbf{P}\big\{|X_1| \ge \varepsilon t^{1/p(\alpha)}\big\} \sum_{k=1}^{k_n} a_{n,k}^2,
\end{aligned}$$

where d_n is defined by (34). This upper bound goes to zero due to (11), (12) and (18). As ε is arbitrary, (13) is satisfied. The fulfilment of (14) is immediate due to Condition (17), indeed

$$\begin{aligned}
\sup_{t>0} t^{p(\alpha)} \sum_{k=1}^{k_n} \mathbf{P}\big(|X_1| \ge t|a_{n,k}|^{2\alpha-1}\big) &\le \sup_{t>0} t^{p(\alpha)} k_n \mathbf{P}\big(|X_1| \ge t\beta^{\alpha-1/2}k_n^{1/p(\alpha)}\big) \\
&= \beta^{(1/2-\alpha)\alpha} \sup_{u>0} u^{p(\alpha)}\mathbf{P}(|X_1| \ge u).
\end{aligned}$$

To show the necessity of (18) for the convergence in distribution in $\mathcal{H}_\alpha^o$ of $\Xi_{\mathsf{a},n}$ to W, we simply note that according to Theorem 4, this convergence implies Condition (13). This later together with the lower bound

$$\sum_{k=1}^{k_n} a_{n,k}^2 a_{n,k}^{-2}\mathbf{P}\big(|X_1| \ge \varepsilon|a_{n,k}|^{2\alpha-1}\big) \ge \frac{k_n}{\gamma}\mathbf{P}\big(|X_1| \ge \varepsilon\beta^{\alpha-1/2}k_n^{1/p(\alpha)}\big) \sum_{k=1}^{k_n} a_{n,k}^2,$$

valid for every $\varepsilon > 0$, give

$$\lim_{k_n\to\infty} k_n\mathbf{P}\big(|X_1| \ge k_n^{1/p(\alpha)}\big) = 0.$$

Due to (16) and the increasingness of k_n, we easily deduce (18).

4 Application to bootstrap

We consider now an application of the Hölderian invariance principles for triangular arrays to the Hölderian convergence in distribution of partial sums processes built on bootstraped samples. We obtain necessary and sufficient conditions for the convergence almost sure or in probability which are stated in the subsection 4.1 and proved in the following subsections.

4.1 Hölderian FCLT under bootstrap

Let $X_1, \dots, X_n$ be random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$, independent, with null expectation and having the same distribution function F. We write $\widehat{F}_n$ for the corresponding empirical distribution function. Let $X_1^*, \dots, X_n^*$ be a bootstraped sample of $\widehat{F}_n$, *i.e.* conditionally to $X_1, \dots, X_n$, the $X_1^*, \dots, X_n^*$ are independent random variables with the same distribution function $\widehat{F}_n$. We denote by E^* the conditional expectation given $X_1, \dots, X_n$.

Put $S_n^*(0) := 0$,

$$S_n^*(t) := \widehat{\sigma}_n^{-1} \sum_{j \le t} (X_j^* - \overline{X}_n), \quad 0 < t \le n,$$

with the classical notations

$$\overline{X}_n := n^{-1}(X_1 + \dots + X_n) \quad \text{and} \quad \widehat{\sigma}_n^2 := n^{-1} \sum_{k=1}^{n} (X_k - \overline{X}_n)^2.$$

Consider the polygonal random line $\xi_n^*(t)$, $t \in [0,1]$, with vertices the points $\big(k/n, n^{-1/2} S_n^*(k)\big)$, $k = 0, 1, \dots, n$. This polygonal line being doubly random, it is suitable to precise the convergence mode under consideration when studying its asymptotic behavior. Denote by d some metric inducing the convergence in distribution in $\mathcal{H}_\alpha^o$. The sequence ξ_n^* is said to converge in $\mathcal{D}$-probability to W in $\mathcal{H}_\alpha^o$ if $d(\xi_n^*, W)$ converges in probability to 0 (with an obvious abuse of notation since $d(\xi_n^*, W)$ denotes the distance between the *distribution* of ξ_n^* and that of W, both distributions being viewed as measures on the Borel σ-field of $\mathcal{H}_\alpha^o$). This convergence will be denoted by

$$\xi_n^* \xrightarrow[n\to\infty]{\mathcal{D}-\mathrm{Pr}} W \quad \text{in } \mathcal{H}_\alpha^o.$$

Similarly, if $d(\xi_n^*, W)$ converges $\mathbf{P}$-almost surely on Ω to 0, we say that ξ_n^* converges $\mathcal{D}$-almost surely to W in $\mathcal{H}_\alpha^o$, which we denote by

$$\xi_n^* \xrightarrow[n\to\infty]{\mathcal{D}-\mathrm{a.s.}} W \quad \text{in } \mathcal{H}_\alpha^o.$$

Theorem 6. *The moment condition*

$$\mathbf{E}|X_1|^{p(\alpha)} < \infty \tag{37}$$

is necessary and sufficient for the convergence

$$\xi_n^* \xrightarrow[n\to\infty]{\mathcal{D}-\text{a.s.}} W \quad \text{in } \mathcal{H}_\alpha^o. \tag{38}$$

Theorem 7. *The condition*

$$\lim_{t\to\infty} t^{p(\alpha)}\mathbf{P}(|X_1| > t) = 0 \tag{39}$$

is necessary and sufficient for the convergence

$$\xi_n^* \xrightarrow[n\to\infty]{\mathcal{D}-\text{Pr}} W \quad \text{in } \mathcal{H}_\alpha^o. \tag{40}$$

4.2 Proof of Theorem 6

To establish the sufficiency of (37), we use Theorem 2, conditionally to $X_1, \dots, X_n$, putting

$$X_{n,k} := n^{-1/2}\widehat{\sigma}_n^{-1}(X_k^* - \overline{X}_n), \quad k = 1, \dots, n.$$

Here $\sigma_{n,k}^2 = E^* X_{n,k}^{*2} = n^{-1}$, which reduces the proof of (38) to that of the convergence

$$n^{q\alpha-q/2}\widehat{\sigma}_n^{-q} nE^*|X_1^* - \overline{X}_n|^q \xrightarrow[n\to\infty]{\text{a.s.}} 0, \tag{41}$$

for some $q > p(\alpha) > 2$. As $\mathbf{E}X_1^2$ is finite by (37), the strong law of large numbers gives

$$\overline{X}_n \xrightarrow[n\to\infty]{\text{a.s.}} 0 \quad \text{and} \quad \widehat{\sigma}_n^2 \xrightarrow[n\to\infty]{\text{a.s.}} 1.$$

So (41) reduces to

$$n^{q\alpha-q/2} nE^*|X_1^*|^q = n^{-q/p(\alpha)} \sum_{k=1}^n |X_k|^q \xrightarrow[n\to\infty]{\text{a.s.}} 0. \tag{42}$$

According to Marcinkiewicz-Zygmund strong law of large numbers, if $(Y_k)_{k\geq 1}$ is an i.i.d. sequence such that $\mathbf{E}|Y_1|^\beta < \infty$ for some $\beta \in]0,1[$, then $n^{-1/\beta}(Y_1 + \cdots + Y_n)$ converges almost surely to 0. Choosing $Y_k = |X_k|^q$ and $\beta = p(\alpha)/q$ makes clearly (42) follow from (37).

Assuming now (38) satisfied, let us show the necessity of (37). From (38), the sequence of processes $(\xi_n^*)_{n\geq 1}$ is $\mathbf{P}$-almost surely tight in $\mathcal{H}_\alpha^o$. It follows [8, Th. 1.4] that

$$\text{a.s.,} \quad \forall\varepsilon > 0, \quad \lim_{\delta\to 0} \sup_{n\geq 1} P^*\big\{w_\alpha(\xi_n^*, \delta) > \varepsilon\big\} = 0,$$

where P^* is the probability conditional to the sequence (X_n). In particular

$$\text{a.s.,} \quad w_\alpha(\xi_n^*, 1/n) \xrightarrow[n\to\infty]{\text{P}^*} 0.$$

Considering the weighted increments of ξ_n^* between two consecutive vertices we deduce from the above convergence that

$$\text{a.s.,}\quad \max_{1\leq k\leq n} \frac{|\xi_n^*(k/n) - \xi_n^*((k-1)/n)|}{(1/n)^\alpha} \xrightarrow[n\to\infty]{\mathrm{P}^*} 0,$$

which may be recast with $p = p(\alpha)$ as

$$\text{a.s.,}\quad \max_{1\leq k\leq n} \frac{|X_k^* - \overline{X}_n|}{n^{1/p}\widehat{\sigma}_n} \xrightarrow[n\to\infty]{\mathrm{P}^*} 0,$$

or more explicitly

$$\forall\varepsilon > 0,\quad \mathbf{P}\Big(\max_{1\leq k\leq n} \frac{|X_k^* - \overline{X}_n|}{n^{1/p}\widehat{\sigma}_n} > \varepsilon \mid X_1, \ldots, X_n\Big) \xrightarrow[n\to\infty]{\text{a.s.}} 0. \tag{43}$$

As the X_k^* are conditionally to $(X_1, \ldots, X_n)$, independent identically distributed, (43) is equivalent to

$$\forall\varepsilon > 0,\quad n\mathbf{P}\big(|X_1^* - \overline{X}_n| > \varepsilon n^{1/p}\widehat{\sigma}_n \mid X_1, \ldots, X_n\big) \xrightarrow[n\to\infty]{\text{a.s.}} 0. \tag{44}$$

Now observe that

$$\mathbf{P}\big(|X_1^* - \overline{X}_n| > \varepsilon n^{1/p}\widehat{\sigma}_n \mid X_1, \ldots, X_n\big) = \frac{1}{n}\sum_{k=1}^n \mathbf{1}\big\{|X_k - \overline{X}_n| > \varepsilon n^{1/p}\widehat{\sigma}_n\big\},$$

whence going back to (44),

$$\forall\varepsilon > 0,\quad \sum_{k=1}^n \mathbf{1}\big\{|X_k - \overline{X}_n| > \varepsilon n^{1/p}\widehat{\sigma}_n\big\} \xrightarrow[n\to\infty]{\text{a.s.}} 0.$$

This sum of indicator functions cannot converge to zero unless it is null for every n bigger than some (random) n_0. This remark gives immediatly that

$$\max_{1\leq k\leq n} \frac{|X_k - \overline{X}_n|}{n^{1/p}\widehat{\sigma}_n} \xrightarrow[n\to\infty]{\text{a.s.}} 0. \tag{45}$$

Let us check now that

$$\frac{|\overline{X}_n|}{n^{1/p}\widehat{\sigma}_n} \xrightarrow[n\to\infty]{\text{a.s.}} 0. \tag{46}$$

Observing that $\widehat{\sigma}_n^2 = n^{-1}\sum_{i=1}^n X_i^2 - \overline{X}_n^2$, we can bound the denominator from below by $n^{1/p-1/2}(X_1^2 + \cdots + X_n^2)^{1/2}$. In the numerator, Cauchy-Schwarz inequality gives $|X_1 + \cdots + X_n| \leq n^{1/2}(X_1^2 + \cdots + X_n^2)^{1/2}$. So (46) follows from the upper bound

$$\frac{|\overline{X}_n|}{n^{1/p}\widehat{\sigma}_n} \leq \frac{1}{n^{1/p}}.$$

From (45) and (46) we immediatly obtain

$$\frac{\max_{1\le k\le n}|X_k|^2}{n^{2/p-1}(X_1^2+\cdots+X_n^2)} \xrightarrow[n\to\infty]{\text{a.s.}} 0. \tag{47}$$

Using the following Hölder inequality

$$\sum_{k=1}^n X_k^2 \le n^{1-2/p}\Big(\sum_{k=1}^n |X_k|^p\Big)^{2/p}$$

we finally deduce from (47) the convergence

$$\frac{\max_{1\le k\le n}|X_k|^p}{|X_1|^p+\cdots+|X_n|^p} \xrightarrow[n\to\infty]{\text{a.s.}} 0.$$

By Lemma 4.1 in Maller and Resnik [7] we can now conclude that $\mathbf{E}|X_1|^p$ is finite.

4.3 Proof of Theorem 7

Like in the proof of Theorem 6, we establish the sufficiency of (39) by checking that

$$n^{q\alpha-q/2}\widehat{\sigma}_n^{-q} nE^*|X_1^*-\overline{X}_n|^q \xrightarrow[n\to\infty]{\mathrm{P}} 0, \tag{48}$$

for some exponent $q>p=p(\alpha)$. Since $p(\alpha)>2$, Condition (39) implies the finiteness of $\mathbf{E}X_1^2$. The weak law of large numbers gives us then the convergences in probability of $\overline{X}_n$ to 0 and of $\widehat{\sigma}_n^2$ to 1, which reduce the proof of (48) to that of

$$n^{q\alpha-q/2}nE^*|X_1^*|^q = n^{-q/p}\sum_{k=1}^n |X_k|^q \xrightarrow[n\to\infty]{\mathrm{P}} 0.$$

This last convergence follows from (39) by Marcinkiewicz-Zygmund weak law of large numbers.

Assume now (40) satisfied, that is

$$d(\xi_n^*, W) \xrightarrow[n\to\infty]{\mathrm{P}} 0,$$

for some metric d inducing the convergence in distribution in $\mathcal{H}_\alpha^o$. Then from every subsequence of indexes $(n')\subset(n)$ we can extract a subsequence $(n'')\subset(n')$ such that

$$d(\xi_{n''}^*, W) \xrightarrow[n''\to\infty]{\text{a.s.}} 0.$$

In view of the proof of Theorem 7, the a.s. convergence (47) is then satisfied for the subsequence indexed by (n''). By arbitrariness of the initial subsequence (n') it follows

$$\frac{\max_{1\le k\le n}|X_k|^2}{n^{2/p-1}(X_1^2+\cdots+X_n^2)} \xrightarrow[n\to\infty]{\mathrm{P}} 0. \tag{49}$$

As $p > 2$, we immediatly deduce from (49) that

$$\frac{\max_{1\le k\le n}|X_k|^2}{(X_1^2+\cdots+X_n^2)} \xrightarrow[n\to\infty]{\text{P}} 0,$$

which due to Lemme 4.1 in Maller and Resnik [7], gives us the finiteness of $\mathbf{E}|X_1|^2$. From this together with (49), we deduce

$$\frac{\max_{1\le k\le n}|X_k|}{n^{1/p}} \xrightarrow[n\to\infty]{\text{P}} 0,$$

which is equivalent to (39). The proof is complete.

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