Operator fractional Brownian motion as limit of polygonal lines processes in Hilbert space*

ALFREDAS RAČKAUSKAS

Department of Mathematics and Informatics, Vilnius University, Naugarduko 24, LT-2006 Vilnius, Lithuania.

Institute of Mathematics and Informatics, Akademijos str. 4, LT-08663, Vilnius, Lithuania.

CHARLES SUQUET

Laboratoire P. Painlevé, UMR 8524 CNRS Université Lille I, Bât M2, Cité Scientifique, F-59655 Villeneuve d'Ascq Cedex, France.

In this paper we study long memory phenomenon of functional time series. We consider an operator fractional Brownian motion with values in a Hilbert space defined via operator valued Hurst coefficient. We prove that this process is a limiting one for polygonal lines constructed from partial sums of time series having space varying long memory.

Keywords: Fractional Brownian motion; Hilbert space; functional central limit theorem; long memory; linear processes.

Mathematics Subject Classifications (2000): 60F17; 60B12.

1. Introduction

Long memory phenomenon have played an important role since the 50's when discovered by Hurst in certain hydrologycal data sets. Historically this paradigm has been associated with slow decay of long-lag autocorrelation of a stochastic process and certain type of scaling properties embodied in a concept of self-similarity. Theoretical investigations go back to Mandelbrot and his co-authors (see [17], [18]). In the past two decades, the interest in long memory (equally named as long-range dependence) has increased especially in financial mathematics and econometrics, mostly due to the availability of precise empirical measurements such as tick-by-tick observations in stock markets for example.

For multivariate data, the theory of long range dependence and self-similarity of processes are studied through operator scaling almost in parallel with the theory of operator stable distributions and their generalized domain of attraction. We

*The research was partially supported by Lithuanian State Science and Studies Foundation, Grant No. T-68/09

refer to Doukhan *et al.* [8], Jurek and Mason [12], Marinucci and Robinson [19], Dolado and Marmol [7], Meerschaert and Scheffler [21] and references therein for a state-of-the-art of this field of research. This paper is devoted to the long memory phenomenon in connection with functional data analysis. We consider a stationary process (X_k), where each X_k takes values in a real separable Hilbert space (finite or infinite dimensional), say $\mathbb{H}$. Different criteria exist to define long memory of univariate time series. The most used are related to the asymptotic decay of the autocovariance function: (i) lack of summability of autocovariance function, (ii) regular variation of the autocovariance function at infinity with an exponent of variation $-1 < d \leq 0$. Following this classical scheme we consider a space varying decay of the autocovariance operators (Q_k) of (X_k) in a sense that there exist a nuclear operator Q and a self-adjoint operator D on $\mathbb{H}$ such that $Q_k \sim k^D Q$ and $-I < D \leq 0$, where I denotes the identity. In this case we say that the process (X_k) has *a space varying memory*. We discuss this phenomenon together with limiting properties of partial sums.

In Section 2 we introduce a continuous time model with space varying memory, namely an operator fractional Brownian motion which is defined via an operator valued Hurst exponent. In section 3 we consider linear processes in Hilbert space with regularly varying filters that are not summable. These processes have space varying long memory. Polygonal lines build from their partial sums converge in distribution to an operator fractional Brownian motion. The proof is exposed in Section 5, after providing in Section 4 an auxiliary result, which may be of independent interest, on the convergence of some Hilbert space valued stochastic processes build from linear processes whose coefficients are operators.

2. Operator fractional Brownian motion

Fractional Brownian motion is a Gaussian process with stationary but dependent increments. The dependence structure is modeled by its Hurst parameter $H \in (0,1)$ via the covariance function R_H defined by

$$R_H(s,t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad s,t \geq 0.$$

The fractional Brownian motion originated by Kolmogorov [13], has been studied in connection with many applications, e.g., financial time series, hydrology, telecommunications to name a few. Moreover a number of generalizations was suggested, from stable fractional motion to that one with time varying Hurst index. The existence of a fractional Brownian motion with values in a separable Hilbert space $\mathbb{H}$ is proved in [9]. Namely it is shown that for any self-adjoint nuclear operator Q on $\mathbb{H}$, and a Hurst index $H \in (1/2, 1)$ there exists a Gaussian $\mathbb{H}$-valued process $(B_{H,Q}(t), t \geq 0)$ on a probability space $(\Omega, \mathcal{F}, P)$ that satisfies

(i) $\mathbf{E}\, B_{H,Q}(t) = 0$ for all $t \geq 0$;
(ii) $R_{H,Q}(s,t) := \operatorname{cov}(B_{H,Q}(t), B_{H,Q}(t)) = 2^{-1}(t^{2H} + s^{2H} - |t-s|^{2H})Q$, for all $s,t \geq 0$;

(iii) $(B_{H,Q}(t), t \geq 0)$ has $\mathbb{H}$-valued continuous paths P-a.s.

In this section we consider a Hilbert space valued fractional Brownian motion with *space varying* Hurst parameter. The main aim of introducing such a process is to understand a space varying long memory phenomenon of infinite dimensional time series. Empirical evidences show a big interest in such models, see e.g., [4], [2], [22].

To be more precise we need to introduce some notations. Let $\mathbb{H}$ be a real separable Hilbert space of infinite or finite dimension with the inner product $\langle\, .\,, .\,\rangle$ and the corresponding norm $||\cdot||$, $||x||^2 = \langle x, x\rangle$. The space of bounded linear operators $u : \mathbb{H} \to \mathbb{H}$ is denoted by $L(\mathbb{H})$. We consider $L(\mathbb{H})$ as a Banach space with the usual uniform norm $||u|| = \sup_{||x||\leq 1} ||ux||$. The adjoint operator of an operator $u \in L(\mathbb{H})$ is u^* and $\mathrm{tr}(u)$ means the trace of u. Let $L_0(\mathbb{H})$ denote the space of compact operators $u : \mathbb{H} \to \mathbb{H}$, endowed with the usual operator norm. For the definition and the main algebraic and analytic properties of $L_0(\mathbb{H})$ we refer to Dunford-Schwartz [10]. For $T \in L_0(\mathbb{H})$ let $\lambda_k(T^*T)$ be the k'th positive eigenvalue of T^*T. Set $\mu_k = \sqrt{\lambda_k}$, the k'th singular value of operator T. Then define:

$$L_1(\mathbb{H}) = \Big\{T \in L_0(\mathbb{H}) : \sum_k \mu_k < \infty\Big\}.$$

The nuclear norm ν_1 on $L_1(\mathbb{H})$ is defined by $\nu_1(T) = \sum_{k=1}^{\infty} \mu_k$. Several properties of $L_1(\mathbb{H})$ are presented in [10].

For an operator $T \in L(\mathbb{H})$ we set $\mathrm{e}^T = \exp(T) = \sum_{k=0}^{\infty} T^k/k!$ provided the series converge in $L(\mathbb{H})$ and we set $\lambda^T = \exp(T \log \lambda)$, for $\lambda > 0$. We also denote

$$m_T = \inf_{||x||=1} \langle Tx, x\rangle, \quad M_T = \sup_{||x||=1} \langle Tx, x\rangle.$$

We refer to [1] for all information concerning the spectral theory of linear operators on Hilbert spaces. For a self-adjoint operator $T \in L(\mathbb{H})$ let $(E^T_\lambda, \lambda \in \mathbb{R})$ be a spectral decomposition of T, that is a family of orthoprojectors such that

(i) $E^T_\lambda = 0$ for $\lambda < m_T$, $E_\lambda = I$ for $\lambda \geq M_T$;
(ii) the function $\lambda \to E^T_\lambda$ is left continuous in strong topology;
(iii) $T = \int_{-\infty}^{\infty} \lambda \,\mathrm{d}E^T_\lambda$.

For any continuous function ϕ on $[m_T, M_T]$, we have $\phi(T) = \int \phi(\lambda)\,\mathrm{d}E^T_\lambda$ and for any $x, y \in \mathbb{H}$, $\langle\phi(T)x, y\rangle = \int \phi(\lambda)\,\mathrm{d}\langle E^T_\lambda x, y\rangle$. An operator $A \in L(\mathbb{H})$ is called non-negative (briefly $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{H}$. An operator $A \in L(\mathbb{H})$ is positive (denoted $A > 0$) provided $A \geq 0$ and $A \neq 0$. For operators $A, B \in L(\mathbb{H})$ the notation $A > B$ $(A < B)$ means that $A - B > 0$ $(B - A > 0)$.

If x and y are two vectors in $\mathbb{H}$, we denote by $x \otimes y$ the rank one operator defined for all $u \in \mathbb{H}$ by $(x \otimes y)(u) = \langle x, u\rangle y$. If X, Y are zero mean random elements in $\mathbb{H}$ with $\mathbf{E}\,||X||^2 < \infty, \mathbf{E}\,||Y||^2 < \infty$ then the covariance operator is $\mathrm{cov}(X, Y) := \mathbf{E}\,X \otimes Y \in L_1(\mathbb{H})$ (we refer to Vakhania, Tarieladze, Chobanyan [26] for probability distributions on Banach spaces).

Now we are prepared to define the operator fractional Brownian motion which is considered in this paper.

Definition 2.1. Let $Q \in L_1(\mathbb{H}), Q \geq 0$ and $H \in L(\mathbb{H}), H \geq 0$. A $\mathbb{H}$-valued Gaussian process $(B_{H,Q}(t),\, t \geq 0)$ on probability space $(\Omega, \mathcal{F}, P)$ is called an operator fractional Q-Brownian motion with Hurst parameter H (shortly ofBm with parameters (H,Q)), if this process satisfies

(i) $\mathbf{E}\, B_{H,Q}(t) = 0$ for all $t \geq 0$;
(ii) $R_{H,Q}(s,t) := \operatorname{cov}\big(B_{H,Q}(s), B_{H,Q}(t)\big) = 2^{-1}(t^{2H} + s^{2H} - |t-s|^{2H})Q$, for all $s,t \geq 0$.

In the rest of the paper we shall frequently use the notation

$$r_A(s,t) := \frac{1}{2}\left(t^{2A} + s^{2A} - |t-s|^{2A}\right)$$

for an operator A as well as for a real number A. The meaning of A in $r_A(s,t)$ should be everytime clear from the context.

The important question now is the existence of ofBm with parameters (H,Q). A partial answer is given in the following proposition.

Proposition 2.1. *Let the operator $H \in L(\mathbb{H})$, be self-adjoint and satisfy $\frac{1}{2}I < H < I$ (equivalently $1/2 < m_H, M_H < 1$). Let $Q \in L_1(\mathbb{H})$ be a non-negative operator commuting with H. Then there exists a fractional Q-Brownian motion with Hurst parameter H.*

Proof. For any $s,t \geq 0$, $R_{H,Q}(s,t)$ is a linear bounded operator. Since H and Q commute, it is a nonnegative operator. Indeed, consider the spectral measure E^H_λ, $\lambda \in \mathbb{R}$. As $H = \int_{-\infty}^{\infty} \lambda\, \mathrm{d}E^H_\lambda$ we have $\langle r_H(s,t)Qx, x\rangle = \int_{-\infty}^{\infty} r_\lambda(s,t)\, \mathrm{d}\langle E^H_\lambda Q^{1/2}x, Q^{1/2}x\rangle \geq 0$ for each $x \in \mathbb{H}$, since $r_\lambda(s,t) > [\min\{s,t\}]^{2\lambda} \geq 0$ for all $s,t \in [0,1]$ and $\lambda \in \mathbb{R}$. Moreover the operator $R_H(s,t)$ is nuclear as the product of the bounded operator $r_H(s,t)$ by the nuclear operator Q. Hence, for any $s,t \geq 0$, $R_{H,Q}(s,t)$ is the covariance operator of some mean zero $\mathbb{H}$ valued random element. As any non-negative definite function $(s,t) \mapsto T(s,t)$, with values in the set of self-adjoint nuclear operators on $\mathbb{H}$ defines uniquely the distribution of a zero mean Gaussian process, we have now to check that $(s,t) \mapsto R_{H,Q}(s,t)$ is positive definite, that is

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \langle R_{Q,H}(t_i,t_j)x_i, x_j\rangle \geq 0, \tag{2.1}$$

for any $t_1, \dots, t_n \geq 0$, $x_1, \dots, x_n \in \mathbb{H}$, $n \in \mathbb{N}$. Since $r_H(s,t) = \int_{-\infty}^{\infty} r_\lambda(s,t)\, \mathrm{d}E^H_\lambda$, we have

$$r_H(s,t) = \lim_{N\to\infty} \sum_{k=1}^{N} r_{\lambda_k}(s,t)(E^H_{\lambda_k} - E^H_{\lambda_{k-1}})$$

for any partition (λ_k) of the interval $[m_H - \varepsilon, M_H]$ with diameter tending to zero. Then (2.1) follows from

$$\sum_{i=1}^{n}\sum_{j=1}^{n} r_{\lambda_k}(t_i,t_j)\langle (E^H_{\lambda_k} - E^H_{\lambda_{k-1}})Qx_i, x_j\rangle \geq 0, \tag{2.2}$$

for any $t_1,\ldots,t_n \geq 0$, $x_1,\ldots,x_n \in \mathbb{H}$, $n \in \mathbb{N}$. Since the operator $E^H_{\lambda_k} - E^H_{\lambda_{k-1}} : \mathbb{H} \to \mathbb{H}$ is a projector and commutes with Q, the left-hand side of (2.2) is equal to

$$\sum_{i=1}^{n}\sum_{j=1}^{n} r_{\lambda_k}(t_i,t_j)\langle Qy_{k,i}, y_{k,j}\rangle,$$

where $y_{k,j} = (E^H_{\lambda_k} - E^H_{\lambda_{k-1}})x_j$. From the existence of the fractional Q-Brownian motion with values in Hilbert space and Hurst index $\lambda_k \in (1/2, 1)$ as proved in [9], Prop. 2.1, it follows that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} r_{\lambda_k}(t_i,t_j)\langle Qz_i, z_j\rangle \geq 0, \tag{2.3}$$

for any $t_1,\ldots,t_n \geq 0$, $z_1,\ldots,z_n \in \mathbb{H}$, $n \in \mathbb{N}$. Hence, (2.1) is proved. □

Throughout we consider only operator fractional Brownian motions $B_{H,Q}$ with commuting operators H and Q and we shall assume that $\frac{1}{2}I < H < I$.

As mentioned in the introduction, self-similarity of processes introduced by Lamperti [16] is one of the sources of long memory. Operator self-similar processes appeared later in the paper by Laha and Rohatgi [15] and were investigated by Matache and Matache [20]. Let us recall that a stochastic process $\{X(t), t > 0\}$ with values in a Banach space E is operator self-similar if there exists a family of linear bounded operators $\{A(a), a > 0\}$ on E such that for each $a > 0$,

$$\{X(at), t > 0\} \stackrel{\mathcal{D}}{=} \{A(a)X(t), a > 0\},$$

where $\stackrel{\mathcal{D}}{=}$ means equality in distribution. The family $\{A(a), a > 0\}$ is refered to as the scaling family of operators.

Proposition 2.2. *The ofBm with parameters (H, Q) is operator self-similar with the scaling family $\{a^H, a > 0\}$.*

Proof. Since $\{a^H, a > 0\}$ is a multiplicative group of operators and Q commutes with H we have

$$\begin{aligned}\operatorname{cov}(B_{H,Q}(as), B_{H,Q}(at)) &= 2^{-1}((at)^{2H} + (as)^{2H} - |at - as|^{2H})Q \\ &= a^{2H}2^{-1}((t)^{2H} + (s)^{2H} - |t-s|^{2H})Q \\ &= \operatorname{cov}(a^H B_{H,Q}, a^H B_{H,Q}).\end{aligned}$$

This yields self-similarity of the Gaussian process $(B_{H,Q}(t), t \geq 0)$. □

To investigate the smoothness of the paths of the Gaussian process $(B_{H,Q}(t))_{t\geq 0}$, we use the following estimate of its increments where the parameter m_H plays the main role.

Proposition 2.3. *For any $s, t > 0$ such that $|t-s| \leq 1$ it holds*

$$\mathbf{E}\,||B_{H,Q}(t) - B_{H,Q}(s)||^2 \leq |t-s|^{2m_H}\operatorname{tr}(Q). \tag{2.4}$$

Proof. For any $x \in \mathbb{H}$ we have

$$\begin{aligned}\mathbf{E}\,\langle B_{H,Q}(t) - B_{H,Q}(s), x\rangle^2 &= \langle R_{H,Q}(t,t)x, x\rangle - 2\langle R_{H,Q}(t,s)x, x\rangle + \langle R_{H,Q}(s,s)x, x\rangle \\ &= \langle |t-s|^{2H} Qx, x\rangle \qquad (2.5)\\ &= \int |t-s|^{2\lambda}\,\mathrm{d}\langle E^H_\lambda Qx, x\rangle. \qquad (2.6)\end{aligned}$$

As $|t-s|^{2H}Q$ is a nuclear operator, choosing any orthonormal basis $(x_j)_{j\geq 1}$ in $\mathbb{H}$, we deduce from (2.5) that

$$\mathbf{E}\,||B_{H,Q}(t) - B_{H,Q}(s)||^2 = \sum_{j=1}^{\infty}\langle |t-s|^{2H}Qx_j, x_j\rangle = \operatorname{tr}(|t-s|^{2H}Q).$$

Now recalling that $|t-s| \leq 1$ and that $E^H_\lambda = 0$ for $\lambda < m_H$ we deduce from (2.6) that

$$\begin{aligned}\operatorname{tr}(|t-s|^{2H}Q) &= \sum_{j=1}^{\infty}\int |t-s|^{2\lambda}\,\mathrm{d}\langle E^H_\lambda Qx_j, x_j\rangle \leq |t-s|^{2m_H}\sum_{j=1}^{\infty}\int \mathrm{d}\langle E^H_\lambda Qx_j, x_j\rangle \\ &= |t-s|^{2m_H}\operatorname{tr}(Q),\end{aligned}$$

so (2.4) is established. □

The estimate (2.4) enables us to obtain the following.

Theorem 2.1. *The space fractional Q-Brownian motion with Hurst index H has a continuous version which satisfies on every bounded interval $[a, b] \subset [0, \infty)$*

$$\sup_{a\leq s<t\leq b}\frac{||B_{H,Q}(t) - B_{H,Q}(s)||}{(t-s)^{m_H}|\ln(t-s)|^{1/2}} < \infty, \quad \text{a.s.} \tag{2.7}$$

Proof. As $B_{H,Q}$ is a Gaussian process with values in the Banach space $\mathbb{H}$ and satisfying by (2.4) an estimate of the form $\mathbf{E}\,\|B_{H,Q}(t+h) - B_{H,Q}(t)\|^2 \leq \sigma(h)^2$, we have for some version of $B_{H,Q}$:

$$\sup_{\substack{0<h\leq b-a\\ a\leq t\leq b}}\frac{\|B_{H,Q}(t+h) - B_{H,Q}(t)\|}{\rho(h)} < \infty, \quad \text{a.s.,}$$

for any $\rho(h) = h^\alpha\ell(h)$ (with $0 < \alpha < 1$ and ℓ slowly varying) such that $\liminf_{h\to 0}\rho(h)(|\ln h|^{1/2}\sigma(h))^{-1} > 0$, see e.g. Corollary 4 (i) in [24]. Then (2.7) follows from the choice $\rho(h) = \sigma(h)|\ln h|^{1/2}$ with $\sigma(h)$ given by (2.4). □

3. Long memory linear processes

Consider a $\mathbb{H}$-valued linear process (X_k) defined by

$$X_k = \sum_{j=0}^{\infty} u_j \varepsilon_{k-j}, \tag{3.1}$$

where $u_0 = I$ is the identity map, $(u_j, j \geq 1) \subset L(\mathbb{H})$ is a given sequence of operators such that $\sum_j ||u_j||^2 < \infty$ and $(\varepsilon_j, j \in \mathbb{Z})$ is a sequence of independent identically distributed (i.i.d.) random elements in $\mathbb{H}$ with mean zero and finite second moment $\sigma_0^2 = \mathbf{E}\,||\varepsilon_0||^2$. For simplicity we assume $\sigma_0^2 = 1$. Let Q denotes the covariance operator of ε_0. From the theoretical point of view, one of the most interesting features of difference between short and long memory of the linear process (X_k) is in the limit behavior of the corresponding partial sums process. In this section we shall consider a polygonal line process. Set $S_0 = 0$ and

$$S_n = \sum_{k=1}^{n} X_k, \quad n \geq 1. \tag{3.2}$$

The polygonal line process based on partial sums S_k, $k \geq 1$, is defined by

$$\zeta_n(t) = S_{[nt]} + (nt - [nt])X_{[nt]+1}, \quad t \in [0,1].$$

We consider this process in the space $C([0,1];\mathbb{H})$, the Banach space of continuous functions $x : [0,1] \to \mathbb{H}$ endowed with the norm $||x|| = \sup_{0\leq t\leq 1} ||x(t)||$.

The following result is proved in [5], see also [23] for a more general approach.

Proposition 3.1. *Assume that the filter (u_k) is summable, that is $\sum_k ||u_k|| < \infty$. Let $A = \sum_{k=1}^{\infty} u_k$. Then*

$$n^{-1/2}\zeta_n \xrightarrow[n\to\infty]{\mathcal{D}} W_{AQA^*} \quad in \quad C([0,1];\mathbb{H}),$$

where $W_{AQA^*} = B_{1/2,AQA^*}$ is a $\mathbb{H}$ valued Brownian motion.

Autocovariance operator of lag k of time series (X_j) is

$$Q_k = \mathbf{E}\,X_0 \otimes X_k = \sum_{j=0}^{\infty} u_j^* Q u_{j+k}.$$

Since $||Q_k|| \leq ||Q|| \sum_{j=0}^{\infty} ||u_j|| \cdot ||u_{j+k}||$ absolute summability of the linear filter (u_k) generates short memory of the process (X_k) in a sense that the sequence of autocovariance operators (Q_k) is absolutely summable, $\sum_{k=1}^{\infty} ||Q_k|| < \infty$. Even more, it is summable in the nuclear norm. Indeed, since $\nu_1(uTv) \leq ||u||\nu_1(T)||v||$ for any nuclear operator T and any $u, v \in L(\mathbb{H})$, we have

$$\sum_k \nu_1\Big(\sum_{j=0}^{\infty} u_j^* Q u_{j+k}\Big) \leq \sum_k \sum_{j=1}^{\infty} ||u_j||\nu_1(Q)||u_{j+k}||.$$

If sumability of autocovariance sequence fails, then the limiting distribution needs not to be a Wiener process and norming needs not to be a classical $\sqrt{n}$. This phenomenon has been understood long ago. We refer to a survey paper by Samorodnitsky [25] for more information on this phenomenon of long memory.

In this section we consider linear process (X_k) for which summability of the filter fails but operators (u_k) are regularly varying. More precisely we restrict ourselves to the case

$$u_k = k^{-D}, \quad k \geq 1, \tag{3.3}$$

where $D \in L(\mathbb{H})$ satisfies $\frac{1}{2}I < D < I$. Moreover we assume that the operators Q and D commute. Then the condition $\sum_k ||u_k||^2 < \infty$ is satisfied but the absolute summability of autocoavariances operators $(Q_k, k \geq 0)$ fails since for an eigenvector x_0 corresponding to the eigenvalue M_D we have

$$\sum_{k=1}^{N} ||Q_k|| = \sum_{k=1}^{N} \Big\| \sum_{j=1}^{\infty} j^{-D} Q (j+k)^{-D} \Big\| \geq \sum_{k=1}^{N} \sum_{j=1}^{\infty} j^{-M_D} (j+k)^{-M_D} \langle Q x_0, x_0 \rangle \to \infty$$

as $N \to \infty$ since $M_D < 1$. Moreover, since Q and D commute, one can show that $Q_k \sim k^{-2D+I} Q$ that is $k^{2D-I} Q_k$ tends to cQ when k tends to infinity.

The main result of this paper is the following theorem.

Theorem 3.1. *Assume that the linear filter (u_k) satisfies* (3.3) *and covariance operator Q commutes with D. Set $H = \frac{3}{2}I - D$. Then*

$$c(D) n^{-H} \zeta_n \xrightarrow[n\to\infty]{\mathcal{D}} B_{H,Q} \quad in \quad C([0,1]; \mathbb{H}), \tag{3.4}$$

where $B_{H,Q}$ is an operator fractional Q-Brownian motion with operator Hurst index H and the operator $c(D) \in L(\mathbb{H})$ is defined by

$$c^2(D) = \int_{-\infty}^{\infty} \frac{(3-2\lambda)(1-\lambda)}{\beta((2\lambda-1)(1-\lambda))} \, \mathrm{d}E_\lambda^D.$$

The proof of this result is given in Section 5. It is deduced from more general functional central limit theorem stated and proved in the next section.

Remark 3.1. In the scalar case ($\mathbb{H} = \mathbb{R}$) where the u_k's are real numbers, more general results are known. Konstantopoulos and Sakhanenko [14] proved the weak convergence of a step partial sums process build on the X_k's to a fractional Brownian motion with Hurst index $H \in (1/2, 1]$, assuming that $\operatorname{Var} S_n$ is regularly varying with exponent $2H$, a condition which is also necessary. Recently, Dedecker, Merlevède and Peligrad [6] extended this result to a large class of linear processes with dependent innovations.

4. Auxiliary result

Assume that the sequence $(Z_n)_{n\geq 1}$ of random elements in the space $C([0,1];\mathbb{H})$, has a representation

$$Z_n(t) = \sum_{k\in\mathbb{Z}} a_{nk}(t)\varepsilon_k, \quad t\in[0,1], n\geq 1, \tag{4.1}$$

where $(\varepsilon_j, j\in\mathbb{Z})$ is a sequence of i.i.d. random elements in $\mathbb{H}$ with mean zero, finite second moment $\sigma_0^2 = \mathbf{E}\,||\varepsilon_0||^2 = 1$ and covariance operator Q and for each $t\in[0,1]$ and each $n\geq 1$, $(a_{nk}(t), k\in\mathbb{Z})$ is a sequence in $L(\mathbb{H})$.

Let $(Z_Q(t),\, t\in[0,1])$ be a $C([0,1];\mathbb{H})$-valued mean zero Gaussian random process with covariance kernel $K_Q(s,t)$:

$$\mathbf{E}\, Z_Q(s)\otimes Z_Q(t) = K_Q(s,t), \quad s,t\in[0,1].$$

Set

$$K_n(s,t) := \sum_{k\in\mathbb{Z}} a_{nk}(t)Qa_{nk}^*(s), \quad s,t\in[0,1].$$

We are interested in the convergence in distribution of the sequence $(Z_n, n\geq 1)$ to the process Z_Q.

Theorem 4.1. *Assume that the following conditions are satisfied:*

(C0) $\lim_{n\to\infty}\sup_{j\in\mathbb{Z}}||a_{nj}(t)|| = 0$ *for each* $t\in[0,1]$;

(C1) $\limsup_{n\to\infty}\sum_{j\in\mathbb{Z}}||a_{nj}(t)||^2 < \infty$ *for each* $t\in[0,1]$;

(C2) *there are constants* $\beta\in(1/2,1]$ *and* $c>0$ *such that*

$$\limsup_{n\to\infty}\sum_{k\in\mathbb{Z}}||a_{nk}(t)-a_{nk}(s)||^2 \leq c|t-s|^{2\beta} \quad \textit{for all} \quad s,t\in[0,1];$$

(C3) $\lim_{n\to\infty}\nu_1(K_n(s,t)-K_Q(s,t)) = 0$ *for all* $s,t\in[0,1]$.

Then

$$Z_n \xrightarrow[n\to\infty]{\mathcal{D}} Z_Q \quad \textit{in} \quad C([0,1];\mathbb{H}). \tag{4.2}$$

In the next section we shall apply this theorem with a_{nj} satisfying $a_{nj}(0)=0$, $n\geq 1$, $j\in\mathbb{Z}$, in which special case Condition (C1) is an immediate consequence of (C2).

Classically Z_n converges weakly to Z_Q in $C([0,1],\mathbb{H})$ if and only if

a) the "finite dimensional" distributions of Z_n converge to those of Z_Q;
b) the sequence $(Z_n)_{n\geq 1}$ is tight in $C([0,1];\mathbb{H})$.

It is worth noticing here that the expression "finite dimensional" used to keep the analogy with the classical setting $\mathbb{H}=\mathbb{R}$, may be misleading. The meaning of a) is that the following convergence holds true for any $d\geq 1$ and any choice of d different real $t_1,\dots,t_d\in[0,1]$:

$$(Z_n(t_1),\dots,Z_n(t_d)) \xrightarrow[n\to\infty]{\mathcal{D}} (Z_Q(t_1),\dots,Z_Q(t_d)) \quad \text{in } \mathbb{H}^d,$$

where $\mathbb{H}^d$ is possibly infinite dimensional like $\mathbb{H}$. So to check a), we need a preparatory investigation of some special central limit result in $\mathbb{H}$. Namely, let us consider an array $(b_{nk}, k \in \mathbb{Z}) \subset L(\mathbb{H})$ and define

$$X_n = \sum_{k\in\mathbb{Z}} b_{nk}\varepsilon_k, \quad Y_n = \sum_{k\in\mathbb{Z}} b_{nk}\gamma_k,$$

where (γ_k) is a sequence of i.i.d. Gaussian random elements in $\mathbb{H}$ with mean zero and covariance operator Q. We shall establish conditions on (b_{nk}) under which the sequences (X_n) and (Y_n) have the same limiting behavior in the sense, that if one converge in distribution then another does the same and their limits coincide.

To this aim we consider for probability measures μ, ν on Hilbert space $\mathbb{H}$ the distance

$$\zeta_3(\mu,\nu) = \sup_{f\in\mathcal{F}_3} \Big| \int f d\mu - \int f d\nu \Big|,$$

where $\mathcal{F}_3$ is the set of three times Frechet differentiable functions $f : \mathbb{H} \to \mathbb{R}$ such that

$$\sup_{x\in\mathbb{H}} |f^{(j)}(x)| \le 1, \quad \text{for} \quad j = 0, \dots, 3.$$

For $\mathbb{H}$-valued random elements X, Y we write $\zeta_3(X, Y)$ for $\zeta_3(P_X, P_Y)$, where P_X is the distribution of X.

Proposition 4.1. *If the following two conditions*

$$\lim_{n\to\infty} d_n := \lim_{n\to\infty} \sup_{j\in\mathbb{Z}} \left\|b_{nj}\right\| = 0 \tag{4.3}$$

and

$$b := \limsup_{n\to\infty} \sum_{j\in\mathbb{Z}} \left\|b_{nj}\right\|^2 < \infty \tag{4.4}$$

are satisfied, then

$$\lim_{n\to\infty} \zeta_3(X_n, Y_n) = 0. \tag{4.5}$$

Proof. Since the class $\mathcal{F}_3$ is invariant with respect to translations and due to independence of ε_i's and γ_i's, we have

$$\zeta_3(X_n, Y_n) \le \sum_{k\in\mathbb{Z}} \zeta_3(b_{nk}\varepsilon_k, b_{nk}\gamma_k).$$

Here we need the following lemma.

Lemma 4.1. *Let X, Y be $\mathbb{H}$-valued random elements. If* $\mathbf{E}\,X = \mathbf{E}\,Y = 0$, *and* $\operatorname{cov}(X) = \operatorname{cov}(Y)$ *then for each* $a > 0$

$$\zeta_3(X,Y) \le \left(1 + \frac{1}{a} + \frac{2}{a^2}\right) \left(\mathbf{E}\,||X||^2 \mathbf{1}_{\{||X||>a\}} + \mathbf{E}\,||Y||^2 \mathbf{1}_{\{||Y||>a\}}\right) + a\left(\mathbf{E}\,||X||^2 + \mathbf{E}\,||Y||^2\right). \tag{4.6}$$

Proof of Lemma 4.1. Let f be an arbitrary element of $\mathcal{F}_3$. Putting $X' = X\mathbf{1}_{\{||X||\le a\}}$, we note that $f(X) - f(X') = (f(X) - f(0))\,\mathbf{1}_{\{||X||>a\}}$, whence recalling the uniform boundedness of the class $\mathcal{F}_3$, we obtain

$$\mathbf{E}\,|f(X) - f(X')| \le \frac{2}{a^2}\mathbf{E}\,||X||^2\mathbf{1}_{\{||X||>a\}}.$$

Now applying Taylor's expansion to $f(X')$ gives

$$\mathbf{E}\,f(X') = \mathbf{E}\,f(0) + \mathbf{E}\,f^{(1)}(0)X' + \frac{1}{2}\mathbf{E}\,f^{(2)}(0)(X', X') + \frac{1}{6}\mathbf{E}\,f^{(3)}(\theta X')(X', X', X'),$$

where θ is a random variable uniformly distributed on $[0, 1]$ independent of X'. As the same holds for $\mathbf{E}\,f(Y)$, with another θ say $\tilde{\theta}$, we have

$$|\mathbf{E}\,f(X) - \mathbf{E}\,f(Y)| \le I_1 + I_2 + \frac{1}{2}I_3 + \frac{1}{6}I_4,$$

where

$$\begin{aligned}
I_1 &= \frac{2}{a^2}\left(\mathbf{E}\,||X||^2\mathbf{1}_{\{||X||>a\}} + \mathbf{E}\,||Y||^2\mathbf{1}_{\{||Y||>a\}}\right),\\
I_2 &= \left|\mathbf{E}\,f^{(1)}(0)X' - \mathbf{E}\,f^{(1)}(0)Y'\right|,\\
I_3 &= \left|\mathbf{E}\,f^{(2)}(0)(X', X') - \mathbf{E}\,f^{(2)}(0)(Y', Y')\right|,\\
I_4 &= \left|\mathbf{E}\,f^{(3)}(\theta X')(X', X', X') - \mathbf{E}\,f^{(3)}(\tilde{\theta}Y')(Y', Y', Y')\right|.
\end{aligned}$$

Since $\mathbf{E}\,X = \mathbf{E}\,Y$ we have with $X'' := X - X'$,

$$I_2 = \left|\mathbf{E}\,f^{(1)}(0)X'' - \mathbf{E}\,f(0)Y''\right| \le \mathbf{E}\,||X''|| + \mathbf{E}\,||Y''|| \le \frac{1}{a}\left(\mathbf{E}\,||X''||^2 + \mathbf{E}\,||Y''||^2\right).$$

Using bilinearity of $f^{(2)}(0)$ and the definition of X', X'', it is easily seen that $f^{(2)}(0)(X', X') = f^{(2)}(0)(X, X) - f^{(2)}(0)(X'', X'')$ and the same holds true for Y. Moreover from $\mathbf{E}\,X = \mathbf{E}\,Y$ and $\mathrm{cov}(X) = \mathrm{cov}(Y)$, it follows that $\mathbf{E}\,f^{(2)}(0)(X, X) = \mathbf{E}\,f^{(2)}(0)(Y, Y)$. All this leads to

$$I_3 = \left|\mathbf{E}\,f^{(2)}(0)(X'', X'') - \mathbf{E}\,f^{(2)}(0)(Y'', Y'')\right| \le \mathbf{E}\,||X''||^2 + \mathbf{E}\,||Y''||^2.$$

Finally

$$I_4 \le \mathbf{E}\,||X'||^3 + \mathbf{E}\,||Y'||^3 \le a[\mathbf{E}\,||X||^2 + \mathbf{E}\,||Y||^2].$$

Collecting the estimates for $I_1, \dots, I_4$, we obtain the result. □

Now Lemma 4.1 yields for any $a > 0$,

$$\zeta_3(b_{nk}\varepsilon_k, b_{nk}\gamma_k) \le (1 + a^{-1} + 2a^{-2})||b_{nk}||^2\nu_n + 2a||b_{nk}||^2$$

where

$$\nu_n = \mathbf{E}\,||\varepsilon_1||^2\mathbf{1}_{\{||\varepsilon_1||>a/d_n\}} + \mathbf{E}\,||\gamma_1||^2\mathbf{1}_{\{||\gamma_1||>a/d_n\}}.$$

Summing these estimates we get

$$\limsup_{n\to\infty}\zeta_3(X_n, Y_n) \le \limsup_{n\to\infty}[(1 + a^{-1} + a^{-2})b\nu_n + 2ab] = 2ab.$$

Since $a > 0$ is arbitrary the limit of $\zeta_3(X_n, Y_n)$ exists and is indeed zero. $\square$

Now we are ready to prove the "finite dimensional" convergence in Theorem 4.1. Let $t_1, \dots, t_d \in [0,1]$. In order to prove that

$$(Z_n(t_1), \dots, Z_n(t_d)) \xrightarrow[n\to\infty]{\mathcal{D}} (Z_Q(t_1), \dots, Z_Q(t_d)), \quad \text{in } \mathbb{H}^d, \tag{4.7}$$

we claim that it is enough to check that

$$V_n := \sum_{k=1}^{d} c_k Z_n(t_k) \xrightarrow[n\to\infty]{\mathcal{D}} V := \sum_{k=1}^{d} c_k Z_Q(t_k), \quad \text{in } \mathbb{H}, \tag{4.8}$$

for any collection of *operators* $c_1, \dots, c_d \in L(\mathbb{H})$. Indeed the choice of all c_j null in (4.8) except for one equal to the identity gives the weak-convergence in $\mathbb{H}$ of each $Z_n(t_k)$ to the corresponding $Z_Q(t_k)$, which in turn, implies the tightness of $(Z_n(t_1), \dots, Z_n(t_d))$ in $\mathbb{H}^d$. Next choosing all c_k's with the same one dimensional range leads easily to the convergence

$$\sum_{k=1}^{d} \langle h_k, Z_n(t_k)\rangle \xrightarrow[n\to\infty]{\mathcal{D}} \sum_{k=1}^{d} \langle h_k, Z_Q(t_k)\rangle, \quad \text{in } \mathbb{R}, \quad (h_1, \dots, h_d) \in \mathbb{H}^d.$$

Since the left hand side is the image of $((Z_n(t_1), \dots, Z_n(t_d))$ by a general continuous linear functional on $\mathbb{H}^d$, (4.7) follows.

As

$$\sum_{k=1}^{d} c_k Z_n(t_k) = \sum_i \Big(\sum_{k=1}^{d} c_k a_{ni}(t_k) \Big) \varepsilon_i = \sum_i b_{ni} \varepsilon_i,$$

denoting

$$b_{ni} = \sum_{k=1}^{d} c_k a_{ni}(t_k)$$

we are in a position to apply Proposition 4.1. As the conditions (4.3) and (4.4) easily follows from (C0) and (C1), $\zeta_3\Big(\sum_i b_{ni}\varepsilon_i, \sum_i b_{ni}\gamma_i \Big) \to 0$ as $n \to \infty$. It is proved by Giné and León [11], that the distance ζ_3 induces the weak topology on the set of probability measures in any separable Hilbert space. Hence $\sum_i b_{ni}\varepsilon_i$ has the same limit in distribution as $\sum_i b_{ni}\gamma_i$ whenever this later converges. Since $\sum_i b_{ni}\gamma_i$ is a sequence of Gaussian random elements, it is well known that this sequence converge to a Gaussian random element V if and only if

$$\lim_{n\to\infty} \nu_1 \Big(\operatorname{cov}\Big(\sum_i b_{ni}\gamma_i \Big) - \operatorname{cov}(V) \Big) = 0. \tag{4.9}$$

Hence, we have to identify the covariance operator V. Since

$$\operatorname{cov}\Big(\sum_i b_{ni}\gamma_i \Big) = \sum_i b_{ni} Q b_{ni}^* = \sum_i \sum_{k,j=1}^{d} c_k a_{ni}(t_k) Q a_{ni}(t_j) c_j^* = \sum_{j,k=1}^{d} c_k K_n(t_j, t_k) c_j^*$$

and

$$\nu_1\big(c_k K_n(t_j,t_k)c_j^* - c_k K_Q(t_j,t_k)c_j^*\big) \le \|c_k\|\nu_1\big(K_n(t_j,t_k) - K_Q(t_j,t_k)\big)\|c_j^*\|$$

we have (4.9) with $\operatorname{cov}(V) = \sum_{j,k=1}^d c_k K_Q(t_j,t_k)c_j^*$.

Next we investigate tightness. General conditions implying the tightness of a sequence of random elements in $C[0,1] = C([0,1];\mathbb{R})$ may be found in Billingsley [3]. As Arzela-Ascoli theorem is known to hold in $C([0,1],\mathbb{H})$ the following tightness criterion in $C([0,1],\mathbb{H})$ is a simple adaptation of Th. 12.3 in [3].

Proposition 4.2. *The tightness of* $(Z_n)_{n\ge1}$ *in* $C([0,1],\mathbb{H})$ *takes place provided that*

i) for every $t \in [0,1]$, $(Z_n(t))_{n\ge1}$ *is tight on* $\mathbb{H}$*;*
ii) there exist $\gamma \ge 0$, $a > 1$ *and a continuous increasing function* $F : [0,1] \to \mathbb{R}$ *such that*

$$P(\|Z_n(s) - Z_n(t)\| > \lambda) \le \lambda^{-\gamma}|F(s) - F(t)|^a.$$

We have by condition (C2)

$$\mathbf{E}\,\|Z_n(t) - Z_n(s)\|^2 \le c_2\mathbf{E}\,||\varepsilon_0||^2\Big(\sum_j \|a_{nj}(t) - a_{nj}(s)\|^2\Big) \le c_2\mathbf{E}\,||\varepsilon_0||^2|t-s|^{2\beta}.$$

so ii) is satisfied with $F(t) = t$, since $1/2 < \beta$. The proof is completed.

5. Proof of Theorem 3.1

Denote $b_n = c(D)n^{-H}$. Setting $u_j = 0$ for $j < 0$ we have

$$\zeta_n(t) = \sum_{j\in\mathbb{Z}} a_{nj}(t)\varepsilon_j,$$

with

$$a_{nj}(t) = b_n\left(\sum_{k=1}^{[nt]} u_{k-j} + \{nt\}u_{[nt]+1-j}\right), \quad t \in [0,1], j \in \mathbb{Z},$$

where for any non negative real number x, $[x]$ denotes its integer part $[x] \le x < [x]+1$, $[x] \in \mathbb{N}$ and $\{x\} = x - [x]$ its fractional part.

Hence, for each $j \in \mathbb{Z}$, a_{nj} is a $L(\mathbb{H})$-valued determistic polygonal line function. We shall check the assumptions (C0)–(C3) of Theorem 4.1 in order to establish the claimed convergence of $b_n\zeta_n$.

First we note that

$$n\|b_n\|^2 = o(1). \tag{5.1}$$

Indeed we recall that $\frac{1}{2}I < H < I$, whence $\|n^{-H}\| \le n^{-m_H}$ with $m_H > \frac{1}{2}$.

Now it is easy to see that $\|a_{nj}(t)\|^2 = O(n\|b_n\|^2)$ uniformly in $j \in \mathbb{Z}$, since

$$\|a_{nj}(t)\|^2 \le \|b_n\|^2\left(\sum_{k=1}^{[nt]}\|u_{k-j}\|\right)^2 \le \|b_n\|^2 n\sum_{k=1}^{n}\|u_{k-j}\|^2 \le n\|b_n\|^2\sum_{i\in\mathbb{Z}}\|u_i\|^2.$$

Hence (C0) is satisfied.

Noting that $a_{nj}(0) = 0$, it is enough to check (C2) to have (C1).

To check (C2), we use the simple fact that the norm of any increment of the operator valued polygonal line a_{nj} is dominated by the corresponding increment of the real valued polygonal line obtained by replacing each u_i by its operator norm in the definition of a_{nj}. This way we obtain an increasing polygonal line with maximal slope $\|b_n\| \max_{1\le k\le n} \|u_{k-j}\|$ and accounting (5.1), this gives the estimate

$$\|a_{nj}(t) - a_{nj}(s)\|^2 \le \frac{c}{n} \max_{1\le k\le n} \|u_{k-j}\|^2 |t-s|^2,$$

with some positive constant c depending only on B. Summing over j, we obtain

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \|a_{nj}(t) - a_{nj}(s)\|^2 &\le c|t-s|^2 \sum_{j\in\mathbb{Z}} \frac{1}{n} \max_{1\le k\le n} \|u_{k-j}\|^2 \\
&\le c|t-s|^2 \sum_{j\in\mathbb{Z}} \frac{1}{n} \sum_{1\le k\le n} \|u_{k-j}\|^2 \\
&= c|t-s|^2 \frac{1}{n} \sum_{1\le k\le n} \sum_{j\in\mathbb{Z}} \|u_{k-j}\|^2 \\
&= c|t-s|^2 \sum_{i\in\mathbb{Z}} \|u_i\|^2,
\end{aligned}$$

so (C2) is fulfilled with $\beta = 1$.

To check (C3), we first note that (recalling that D commutes with Q and is self-adjoint):

$$\begin{aligned}
K_n(s,t) &= \sum_{i\in\mathbb{Z}} a_{ni}(t) Q a_{ni}(s)^* \\
&= \sum_{i\in\mathbb{Z}} b_n \left(\sum_{k=1}^{[nt]} u_{k-i} + \{nt\} u_{[nt]-i+1} \right) Q \left(\sum_{l=1}^{[ns]} u^*_{l-i} + \{ns\} u^*_{[ns]-i+1} \right) b_n^* \\
&= \left(b_n b_n^* \sum_{k=1}^{[nt]} \sum_{l=1}^{[ns]} \sum_{i\in\mathbb{Z}} u_{k-i} u^*_{l-i} \right) Q + R_n \\
&= A_n(s,t) Q + R_n.
\end{aligned}$$

Let us check that R_n goes to zero in nuclear norm. From the explicit expression

$$\begin{aligned}
R_n = b_n^2 \sum_{i\in\mathbb{Z}} \Bigg(\{nt\} \sum_{l=1}^{[ns]} u_{[nt]-i+1} u^*_{l-i} + \{ns\} \sum_{k=1}^{[nt]} u_{k-i} u^*_{[ns]-i+1} \\
+ \{nt\}\{ns\} u_{[nt]-i+1} u^*_{[ns]-i+1} \Bigg) Q,
\end{aligned}$$

it is easy to deduce the estimate

$$\nu_1(R_n) \leq 3n\|b_n\|^2 \sum_{i\in\mathbb{Z}} \|u_i\|^2 \nu_1(Q) = o(1),$$

according to (5.1). Now as

$$\nu_1\big(A_n(s,t)Q - r_H(s,t)Q\big) \leq \|A_n(s,t) - r_H(s,t)\|\nu_1(Q),$$

the problem is reduced to the convergence of $A_n(s,t)$ to $r_H(s,t)$ in operator norm.

The following Laplace transform formula is convenient to compute explicitly the sum indexed by $\mathbb{Z}$ in the expression of $A_n(s,t)$:

$$a^{-D} = \Gamma^{-1}(D) \int_0^\infty \mathrm{e}^{-ax} x^{D-I}\,\mathrm{d}x, \quad a > 0, \tag{5.2}$$

Concerning the definition of $f(D)$ for various functions f, we refer to section VII.3 in Dunford and Schwartz [10]. As $u_k = k^{-D}$ for $k \geq 1$, $u(0) = I$ and $u_k = 0$ for $k < 0$, this formula enables us to write for $k \neq l$:

$$\begin{aligned}\sum_{j\in\mathbb{Z}} u_{k-j}u_{l-j}^* = &\sum_{j<\min(k,l)} (k-j)^{-D}(l-j)^{-D} + |k-l|^{-D} \\ =&\Gamma^{-2}(D)\int_0^\infty\int_0^\infty F_{k,l}(x,y)x^{D-I}y^{D-I}\,\mathrm{d}x\,\mathrm{d}y \\ &+\Gamma^{-1}(D)\int_0^\infty \mathrm{e}^{-|k-l|x}x^{D-I}\,\mathrm{d}x,\end{aligned} \tag{5.3}$$

where

$$F_{k,l}(x,y) = \sum_{j<\min(k,l)} \mathrm{e}^{-kx-ly+j(x+y)} = \frac{\exp(-kx-ly+\min(k,l)(x+y))}{\exp(x+y)-1}.$$

In the following computation of $A_n(s,t)$, we can forget the special case $k = l$ since the corresponding contribution satisfies

$$\left\| b_n b_n^* \sum_{k=1}^{\min([ns],[nt])} \sum_{j\in\mathbb{Z}} u_{k-j}u_{k-j}^* \right\| \leq \|b_n\|^2 n \sum_{i\in\mathbb{Z}} \|u_i\|^2,$$

which tends to zero when n goes to infinity due to (5.1) and the square summability of $(u_i)_{i\in\mathbb{Z}}$.

Now we have to sum over k and l inside the integrals of (5.3). We shall explicit the computation assuming that $s \leq t$. It is convenient to introduce the sets

$$\begin{aligned}\Delta_1 &:= \{(k,l)\in\mathbb{N}^2;\ 1\leq l\leq[ns], 1\leq k-l\leq[nt]-[ns]\};\\ \Delta_2 &:= \{(k,l)\in\mathbb{N}^2;\ 1\leq l\leq[ns], k\leq[nt], [nt]-[ns]<k-l\};\\ \Delta_3 &:= \{(k,l)\in\mathbb{N}^2;\ k\geq 1, l\leq[ns], l-k\geq 1\};\\ \Delta &:= \Delta_1\cup\Delta_2\cup\Delta_3 = \{(k,l)\in\mathbb{N}^2;\ 1\leq k\leq[nt], 1\leq l\leq[ns], k\neq l\}.\end{aligned}$$

To simplify the writing of forthcoming computations it is worth to have at hand the following elementary formula where a, b, c denote integers.

$$\sum_{j=a}^{b}(c-j)\mathrm{e}^{-jx} = \frac{(c-a)\mathrm{e}^{(-a+2)x}-(c-b-1)\mathrm{e}^{(-b+1)x}-(c-a+1)\mathrm{e}^{(-a+1)x}+(c-b)\mathrm{e}^{-bx}}{(\mathrm{e}^{x}-1)^2}. \tag{5.4}$$

Now noting that on $\Delta_1 \cup \Delta_2$,

$$F_{k,l}(x,y) = \frac{\exp\big(-(k-l)x\big)}{\exp(x+y)-1},$$

we obtain

$$\sum_{(k,l)\in\Delta_1} F_{k,l}(x,y) = \sum_{j=1}^{[nt]-[ns]}[ns]\frac{\mathrm{e}^{-jx}}{\mathrm{e}^{x+y}-1} = \frac{[ns]\big(1-\mathrm{e}^{-([nt]-[ns])x}\big)}{(\mathrm{e}^{x}-1)(\mathrm{e}^{x+y}-1)} \tag{5.5}$$

and using (5.4),

$$\begin{aligned}\sum_{(k,l)\in\Delta_2} F_{k,l}(x,y) &= \sum_{j=[nt]-[ns]+1}^{[nt]-1}([nt]-j)\frac{\mathrm{e}^{-jx}}{\mathrm{e}^{x+y}-1}\\ &= \frac{([ns]-1)\mathrm{e}^{-([nt]-[ns]-1)x}-[ns]\mathrm{e}^{-([nt]-[ns])x}+\mathrm{e}^{-([nt]-1)x}}{(\mathrm{e}^{x}-1)^2(\mathrm{e}^{x+y}-1)}.\end{aligned} \tag{5.6}$$

Gathering (5.5) and (5.6) gives

$$\sum_{(k,l)\in\Delta_1\cup\Delta_2} F_{k,l}(x,y) = \frac{[ns](\mathrm{e}^{x}-1)-\mathrm{e}^{-([nt]-[ns]-1)x}+\mathrm{e}^{-([nt]-1)x}}{(\mathrm{e}^{x}-1)^2(\mathrm{e}^{x+y}-1)} =: f_n(x,y). \tag{5.7}$$

As on Δ_3,

$$F_{k,l}(x,y) = \frac{\exp\big(-(l-k)y\big)}{\exp(x+y)-1},$$

it is clear that the sum over Δ_3 is obtained by exchanging x and y and puting $t=s$ in (5.6), that is

$$\sum_{(k,l)\in\Delta_3} F_{k,l}(x,y) = \frac{([ns]-1)\mathrm{e}^{y}-[ns]+\mathrm{e}^{-([ns]-1)y}}{(\mathrm{e}^{y}-1)^2(\mathrm{e}^{x+y}-1)} =: g_n(x,y). \tag{5.8}$$

Finally, noting that on $\Delta_1\cup\Delta_2\cup\Delta_3$, $\exp(-|k-l|x) = (\exp(2x)-1)F_{k,l}(x,x)$ we can exploit the above results to obtain:

$$\begin{aligned}\sum_{(k,l)\in\Delta} \mathrm{e}^{-|k-l|x} &= \frac{2[ns]\mathrm{e}^{x}-2[ns]-\mathrm{e}^{x}-\mathrm{e}^{-([nt]-[ns]-1)x}+\mathrm{e}^{-([ns]-1)x}+\mathrm{e}^{-([nt]-1)x}}{(\mathrm{e}^{x}-1)^2}\\ &=: h_n(x).\end{aligned} \tag{5.9}$$

Going back to (5.3) and recalling that $b_n b_n^* = c^2(D)n^{-3I+2D}$, we have now the integral representation

$$\begin{aligned} b_n^2 \sum_{(k,l)\in\Delta} \sum_{j\in\mathbb{Z}} u_{k-j}u_{l-j}^* =& c^2(D)\Gamma^{-2}(D) \int_0^\infty \int_0^\infty \frac{1}{n} f_n(x,y)(nx)^{D-I}(ny)^{D-I}\,\mathrm{d}x\mathrm{d}y \\ &+ c^2(D)\Gamma^{-2}(D) \int_0^\infty \int_0^\infty \frac{1}{n} g_n(x,y)(nx)^{D-I}(ny)^{D-I}\,\mathrm{d}x\mathrm{d}y \\ &+ c^2(D)\Gamma^{-2}(D) \int_0^\infty n^{-2I+D} h_n(x)(nx)^{D-I}\,\mathrm{d}x. \end{aligned}$$

Let us denote the three above integrals by $I_1(n)$, $I_2(n)$, $I_3(n)$. The change of variable $(u,v) = (nx, ny)$ in $I_1(n)$ gives

$$I_1(n) = \int_0^\infty \int_0^\infty \frac{[ns](\mathrm{e}^{\frac{u}{n}} - 1) - \mathrm{e}^{-\frac{[nt]-[ns]-1}{n}u} + \mathrm{e}^{-\frac{[nt]-1}{n}u}}{n^3(\mathrm{e}^{\frac{u}{n}} - 1)^2(\mathrm{e}^{\frac{u+v}{n}} - 1)} u^{D-I}v^{D-I}\,\mathrm{d}u\mathrm{d}v$$

Postponing the justification of interversion between integral and limit we obtain then:

$$\lim_{n\to\infty} I_1(n) = \int_0^\infty \int_0^\infty \frac{su - \mathrm{e}^{-(t-s)u} + \mathrm{e}^{-tu}}{u^2(u+v)} u^{D-I}v^{D-I}\,\mathrm{d}u\ \mathrm{d}v =: I_1,$$

where the limit is in the operator norm. The change of variables $(u,v) \to (u,w)$ where $w = u+v$ gives

$$I_1 = \int_0^\infty \frac{su - \mathrm{e}^{-(t-s)u} + \mathrm{e}^{-tu}}{u^2} u^{D-I} \left\{ \int_u^\infty (w-u)^{D-I}w^{-I}\,\mathrm{d}w \right\} \mathrm{d}u$$

and putting $w = \frac{u}{x}$, $0 < x \le 1$ in the inside integral leads to

$$\begin{aligned} I_1 &= \int_0^\infty (su - \mathrm{e}^{-(t-s)u} + \mathrm{e}^{-tu})u^{2D-4I} \left\{ \int_0^1 (1-x)^{D-I}x^{-D}\,\mathrm{d}x \right\} \mathrm{d}u \\ &= \beta(D, I-D) \int_0^\infty (su - \mathrm{e}^{-(t-s)u} + \mathrm{e}^{-tu})u^{2D-4I}\,\mathrm{d}u =: \beta(D, I-D)I_1'. \end{aligned}$$

The integral I_1' can be computed integrating twice by parts, noting that $su - \mathrm{e}^{-(t-s)u} + \mathrm{e}^{-tu} = O(u^2)$ when u goes to zero. This leads to

$$I_1' = \int_0^\infty (-(t-s)^2\mathrm{e}^{-(t-s)u} + t^2\mathrm{e}^{-tu})u^{2D-2I}(2D-3I)^{-1}(2D-2I)^{-1}\,\mathrm{d}u.$$

Now it is easy to check that for every positive number a,

$$\int_0^\infty a^2\mathrm{e}^{-ax}x^{2D-2I}\,\mathrm{d}x = \Gamma(2D-I)a^{2H},$$

which enables us to see that

$$I_1 = \beta(D, I-D)\Gamma(2D-I)(2D-3I)^{-1}(2D-2I)^{-1}(t^{2H} - (t-s)^{2H}),$$

which can be recast (using the relation between β and Γ functions) as

$$I_1 = 2^{-1}\Gamma(\frac{3}{2}I - H)\Gamma(2I-2H)\Gamma(H - \frac{1}{2}I)H^{-1}(2H-I)^{-1}(t^{2H} - (t-s)^{2H}).$$

As already observed for the sumation over Δ_3, we have a similar result for the limit of $I_2(n)$, just by putting $t = s$ and exchanging the roles of x and y. Then clearly $I_2 = s^{2H}$ up to the same operator constant as I_1. Postponing the verification of the convergence to zero of $I_3(n)$, we finally arrive at:

$$\lim_{n\to\infty} A_n(s,t) = c^2(D)\Gamma^{-2}(D)\Gamma(\frac{3}{2}I-H)\Gamma(2I-2H)\Gamma(H-\frac{1}{2}I)H^{-1}(2H-I)^{-1}r_H(t,s).$$

As $D = \frac{3}{2}I - H$, the above operator constant may be rewritten as

$$2^{-1}c^2(D)\Gamma(2I-2H)\Gamma(H-\frac{1}{2}I)H^{-1}(2H-I)^{-1}\Gamma^{-1}(\frac{3}{2}I-H) =$$
$$2^{-1}c^2(D)\beta(2I-2H, H-\frac{1}{2}I)H^{-1}(2H-I)^{-1} = 2^{-1}I$$

by the definition of the operator $c(D)$. Hence we obtain the desired limit, that is

$$\lim_{n\to\infty} A_n(s,t) = r_H(t,s).$$

To complete the proof, it remains to justify the convergences in operator norm of $I_1(n)$, $I_2(n)$, $I_3(n)$ respectively to I_1, I_2 and 0. In fact the problem is reduced to a problem of convergence of integrals of real valued functions, since the three integrals can be viewed as Bochner integrals of $L(\mathbb{H})$ valued functions of the form $\int \psi_n(z)T(z)\,\mathrm{d}z$ where the function $\psi_n(z)$ is real valued and the function $T(z)$ is operator valued. Then to prove the convergence $\int \psi_n T$ to $\int \psi T$ in operator norm, we can just write

$$\left\| \int \psi_n(z)T(z)\,\mathrm{d}z - \int \psi(z)T(z)\,\mathrm{d}z \right\| \le \int |\psi_n(z) - \psi(z)| \|T(z)\|\,\mathrm{d}z$$

and then prove the convergence to zero of the right hand side.

Let us start with $I_1(n)$ which can be recast for convenience as:

$$I_1(n) = \int_{\mathbb{R}_+^2} \frac{[ns](1-\mathrm{e}^{-\frac{u}{n}}) - \mathrm{e}^{-\frac{[nt]-[ns]}{n}u} + \mathrm{e}^{-\frac{[nt]}{n}u}}{4n^3\sinh^2(\frac{u}{2n})(\mathrm{e}^{\frac{u+v}{n}}-1)} u^{D-I}v^{D-I}\,\mathrm{d}u\,\mathrm{d}v.$$

Denoting by $\psi_n(u,v)$ the fraction inside this integral, the problem is clearly reduced via Lebesgue bounded convergence theorem to finding a function $\phi(u,v)$ such that $\psi_n(u,v) \le \phi(u,v)$ (note that ψ_n is non negative) and

$$\int_{\mathbb{R}_+^2} \phi(u,v)u^{m(u)-1}v^{m(v)-1}\,\mathrm{d}u\,\mathrm{d}v < \infty, \tag{5.10}$$

where $m(u) = m_D$ for $0 < u < 1$ and $m(u) = M_D$ for $u \ge 1$. Looking at their power series expansions, it is obvious that the positive functions $t \mapsto t(\exp(c/t) - 1)$ and $t \mapsto t\sinh(c/t)$ are decreasing (c being any positive constant), from which we see that the denominator in ψ_n is decreasing in n and hence we can replace it by its limit to obtain an upper bound for $\psi_n(u,v)$:

$$\psi_n(u,v) \le \frac{[ns](1-\mathrm{e}^{-\frac{u}{n}}) - \mathrm{e}^{-\frac{[nt]-[ns]}{n}u} + \mathrm{e}^{-\frac{[nt]}{n}u}}{u^2(u+v)}.$$

Now, using the bound $1-\mathrm{e}^{-x} \le x$ for $x \ge 0$, writing $[ns] = s - \{ns\}$ and introducing the function

$$r(s,t,u) := su - \mathrm{e}^{-(t-s)u} + \mathrm{e}^{-tu},$$

we obtain

$$\psi_n(u,v) \le \frac{r(s - \frac{\{ns\}}{n}, t - \frac{\{nt\}}{n}, u)}{u^2(u+v)}.$$

Then it is elementary to check that r is non decreasing in each variable s and t (when the two others are fixed), which leads finally to

$$\psi_n(u,v) \le \frac{r(s,t,u)}{u^2(u+v)} = \frac{su - \mathrm{e}^{-(t-s)u} + \mathrm{e}^{-tu}}{u^2(u+v)} =: \phi(u,v)$$

and with this choice of ϕ it is easy to see that (5.10) is satisfied. Hence the convergence in operator norm of $I_1(n)$ to I_1 is established. Clearly the corresponding result for I_2 holds by choosing $t = s$ and exchanging the roles of u and v.

Concerning $I_3(n)$, we have

$$I_3(n) = n^{D-I} \int_0^\infty \frac{2[ns](1-\mathrm{e}^{-\frac{u}{n}}) - 1 - \mathrm{e}^{-\frac{[nt]-[ns]}{n}u} + \mathrm{e}^{-\frac{[nt]}{n}u} + \mathrm{e}^{-\frac{[ns]}{n}u}}{4n^2 \sinh^2(\frac{u}{2n})} u^{D-I} \,\mathrm{d}u \tag{5.11}$$

and as $\|n^{D-I}\| \le n^{M_D - 1}$ and $M_D < 1$, we just have to show the convergence

$$n^{I-D} I_3(n) \xrightarrow[n\to\infty]{} \int_0^\infty \frac{2su - 1 - \mathrm{e}^{-(t-s)u} + \mathrm{e}^{-tu} + \mathrm{e}^{-su}}{u^2} u^{D-I} \,\mathrm{d}u.$$

Using the same method as above, this reduce to finding an upper bound $\Psi(u)$ for the fraction in (5.11) such that

$$\int_0^\infty \Psi(u) u^{m(u)-1} \,\mathrm{d}u < \infty. \tag{5.12}$$

Exploiting the work already done with $I_1(n)$, it is immediate to see that

$$\Psi(u) = \frac{r(s,t,u) + r(s,s,u)}{u^2}$$

is a suitable choice. The proof is complete.

Bibliography

1. N.I. Akhiezer and I.M. Glazman. *Theory of Linear Operators in Hilbert Space*, volume 1. Pi, Boston-London-Melbourne, 1981.
2. J. Alvarez-Ramirez, J. Alvarez, E. Rodriguez, and G. Fernandez-Anaya. Time-varying Hurst exponent for US stock markets. *Physica A: Statistical Mechanics and its Applications*, 387:6159–6169, 2008.
3. P. Billingsley. *Convergence of probability measures*. Wiley, New York, 1968.
4. D.O. Cajueiro and B.M. Tabak. Testing for time-varying long-range dependence in volatility for emerging market. *Physica A: Statistical Mechanics and its Applications*, 346:577–588, 2005.

5. J. Dedecker and F. Merlevède. The conditional central limit theorem in Hilbert spaces. *Stochastic Process. Appl.*, 108:229–262, 2003.
6. J. Dedecker, F. Merlevède, and M. Peligrad. Invariance principles for linear processes. application to isotonic regression. arXiv:0903.1951v1, 2009.
7. J.J. Dolado and F. Marmol. Asymptotic inference results for multivariate long-memory processes. *Econometrics Journal*, 7(1):168–190, June 2004.
8. P. Doukhan, G. Oppenheim, and M. Taqqu, editors. *Theory and Applications of Long-Range Dependence*. Birkhäuser, Boston, Basel, Berlin, 2003.
9. T.E. Duncan, B. Pasik-Duncan, and B. Maslowski. Fractional Brownian motion and stochastic equations in Hilbert spaces. *Stoch. Dyn.*, 2(2):225–250, 2002.
10. N. Dunford and J.T. Schwartz. *Linear Operators*, volume 1. Wiley, 1988.
11. E. Giné and J.R. León. On the central limit theorem in Hilbert space. *Stochastica*, 4(1):43–71, 1980.
12. Z. Jurek and J. Mason. *Operator-Limit Distributions in Probability Theory*. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 1993.
13. A.N. Kolmogorov. Wienersche spiralen und einige andere interessante kurven in Hilbertschen raum. *Doklady Akademii Nauk URSS*, 26:115–118, 1940.
14. T. Konstantopoulos and A. Sakhanenko. Convergence and convergence rate to fractional Brownian motion for weighted random sums. *Siberian Electron. Math. Rep.*, 1:47–63, 2004.
15. R.G. Laha and V.K. Rohatgi. Operator self-similar processes in $\mathbf{R}^d$. *Stochastic Processes Appl.*, 12:73–84, 1982.
16. J. Lamperti. Semi-stable stochastic processes. *Trans. Amer. Math. Soc.*, 104:62–78, 1962.
17. B.B. Mandelbrot and J.W. Van Ness. Fractional Brownian motion, fractional noises and applications. *SIAM Review*, 10:422–437, 1968.
18. B.B. Mandelbrot and J.R. Wallis. Some long-run properties of geophysical records. *Water Resour. Res.*, 5(2):231–340, 1969.
19. D. Marinucci and P.M. Robinson. Weak convergence of multivariate fractional processes. *Stochastic Processes Appl.*, 86(1):103–120, 2000.
20. M.T. Matache and V. Matache. Operator self-similar processes on Banach spaces. *Journal of Applied Mathematics and Stochastic Analysis*, 2006, 2006. Article ID 82838.
21. M.M. Meerschaert and H.P. Scheffler. *Limit Distributions for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice*. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 2001.
22. J. Preciado and H. Morris. The varying behavior of U.S. market persistence. In *Proceedings of the World Congress on Engineering and Computer Science*, San Francisco, USA, October 2008. WCECS 2008.
23. A. Račkauskas and Ch. Suquet. On limit theorems for Banach space valued linear processes. *Lithuanian Math. J.*, 2010. To appear.
24. A. Račkauskas and Ch. Suquet. Hölder versions of Banach spaces valued random fields. *Georgian Mathematical Journal*, 8(2):347–362, 2001.
25. G. Samorodnitsky. Long range dependence. *Foundations and Trends in Stochastic Systems*, 1(3):163–257, 2006.
26. N.N. Vakhania, V.I. Tarieladze, and S.A. Chobanyan. *Probability distributions on Banach spaces*, volume 14 of *Mathematics and Its Applications (Soviet Series)*. D. Reidel Publishing Company, Dordrecht, Tokyo, 1987. Transl. from the Russian by Wojbor A. Woyczynski.