# Normal approximation for quasi associated random fields

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**Abstract**

For quasi associated random fields (comprising negatively and positively dependent fields) on $\mathbb{Z}^d$ we use Stein's method to establish the rate of normal approximation for partial sums taken over arbitrary finite subsets $\mathcal{U}$ of $\mathbb{Z}^d$.

*Key words:* Random fields; Dependence conditions; Positive and negative association; Lindeberg function; CLT; Convergence rates; Maximum of partial sums.

## 1 Introduction

There are a number of interesting stochastic models described by means of families of random variables possessing properties of positive or negative dependence or their modifications. One can refer to the pioneering papers by Harris (1960), Lehman (1966), Esary *et al.* (1967), Fortuin *et al.* (1971), Joag-Dev and Proschan (1983).

**Definition 1 (Esary et al. (1967))** *A finite collection $Y = (Y_1, \ldots, Y_n)$ of real valued random variables $Y_k$, $k = 1, \ldots, n$, is called associated or positively dependent if $\operatorname{Cov}\big(f(Y), g(Y)\big) \geq 0$ for any coordinate-wise nondecreasing functions $f$, $g : \mathbb{R}^n \to \mathbb{R}$, whenever the covariance exists. An infinite family of random variables is associated if this is valid for every finite sub-family.*

The association and related concepts (e.g. positive quadrant dependence, etc.) were initially connected with reliability theory and mathematical statistics only. Percolation

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theory and statistical mechanics, where one considers random variables "satisfying the FKG inequalities" (Fortuin *et al.* (1971)) implying the association, provide a different domain of applications of this notion.

**Definition 2 (Joag-Dev and Proschan (1983))** *Real valued random variables $Y_k$, $k = 1,\dots,n$, and a family thereof, are called negatively dependent if, each time the covariance exists,*

$$\operatorname{Cov}\Big(f(Y_i, i\in I), g(Y_j, j\in J)\Big) \le 0, \tag{1}$$

*for every pair of* disjoint *subsets $I$, $J$ of $\{1,\dots,n\}$ and for any coordinate-wise nondecreasing functions $f:\mathbb{R}^I\to\mathbb{R}$, $g:\mathbb{R}^J\to\mathbb{R}$.*

Sampling without replacement provides an example of negatively dependent random variables (see Joag-Dev and Proschan (1983) for this and other examples).

Newman (1984) calls a family of random variables *weakly associated* if it satisfies the requirement of Definition 2, but with reversed sign in the inequality (1).

Evidently, associated families of random variables are weakly associated. Note also that any family of independent random variables is automatically associated and negatively dependent. Instead of the terms negative dependence and weak association one uses also negative association (NA) and positive association (PA). It is worth mentioning that concepts of mixing or positive (negative) dependence offer complementary approaches to analysis of dependent random variables. The main advantage of dealing with positively or negatively dependent random fields is due to the fact that most of their properties are determined by the covariance structure whereas the calculation of mixing coefficients is in general a nontrivial problem (we refer to the book on mixing by Doukhan (1994)).

Starting from the seminal paper by Newman (1980), during the last two decades, various classical limit theorems of probability theory (CLT, SLLN, weak and strong invariance principles, LIL and FLIL, Glivenko-Cantelli type theorems, etc.) were established for stochastic processes and random fields under the positive or negative dependence conditions mentioned above.

In this paper, we prove more general variants of CLT for random fields involving wider classes of random variables (Theorems 4-13 of Section 2). Section 3 is devoted to an important new result by Shao (2000) showing that the expectations of convex increasing functions of maxima of partial sums of negatively dependent random variables can be estimated by means of the independent copies of summands. We demonstrate that it is impossible to get an exact analogue of this result for negatively dependent random fields $\{X_j, j\in\mathbb{Z}^d\}$ with $d>1$.

**Definition 3** *Call a collection of real valued random variables $Y=\{Y_t, t\in T\}$ with $\mathbb{E}\,Y_t^2<\infty$ $(t\in T)$ quasi associated if, for all finite disjoint subsets $I$, $J$ of $T$ and any Lipschitz functions $f:\mathbb{R}^I\to\mathbb{R}$, $g:\mathbb{R}^J\to\mathbb{R}$, one has*

$$|\operatorname{Cov}\Big(f(Y_i, i\in I), g(Y_j, j\in J)\Big)| \le \sum_{i\in I}\sum_{j\in J} L_i(f)L_j(g)|\operatorname{Cov}(X_i,X_j)|, \tag{2}$$

*where the coordinate-wise Lipschitz constants $L_i(f)$ are such that, for all $x=(x_i, i\in I)$, $y=(y_i, i\in I)$ in $\mathbb{R}^I$,*

$$|f(x)-f(y)| \le \sum_{i\in I} L_i(f)|x_i-y_i|.$$

Obviously, if a random field $\{Y_t, t \in T\}$ is quasi associated, the same is true for the centered field $\{Y_t - \mathbb{E}\, Y_t, t \in T\}$. Inequality (2) is satisfied for PA or NA random fields, see Bulinski and Shabanovich (1998). An analogue of (2) for smooth functions $f$ and $g$ was firstly proved in Birkel (1988) for associated random variables (related results appeared in Newman (1984), Roussas (1994), Peligrad and Shao (1995) and Bulinski (1996)). Note that covariance inequalities are powerful tools in establishing moment inequalities and limit theorems for sums of dependent r.v.'s. We refer, e.g., to Ibragimov and Linnik (1971), Withers (1981), Bradley and Bryc (1985), Doukhan (1994), Bakhtin and Bulinski (1997), Louhichi (1998), Rio (2000). Interesting examples of application of covariance inequalities in statistical problems are provided by Doukhan and Louhichi (1999).

## 2 Normal Approximation

Let $X = \{X_j, j \in \mathbb{Z}^d\}$, $d \geq 1$, be a centered random field such that $\mathbb{E}\,|X_j|^2 < \infty$ for all $j \in \mathbb{Z}^d$. For a finite subset $\mathcal{U}$ of $\mathbb{Z}^d$ denote

$$W = B^{-1} \sum_{j \in \mathcal{U}} X_j, \qquad B^2 = \sum_{j \in \mathcal{U}} \mathbb{E}\, X_j^2, \tag{3}$$

where the trivial case $B^2 = 0$ is excluded. Introduce further

$$R = B^{-2} \sum_{\substack{j,q \in \mathcal{U} \\ j \neq q}} |\operatorname{Cov}(X_j, X_q)|$$

and, for $\varepsilon > 0$, the Lindeberg function

$$\mathcal{L}_\varepsilon = B^{-2} \sum_{j \in \mathcal{U}} \mathbb{E}\, X_j^2 \mathbf{1}\{|X_j| > \varepsilon B\}.$$

Evidently, $B^2$, $W$, $R$ and $\mathcal{L}_\varepsilon$ are functions of $X_j$, $j \in \mathcal{U}$, and we use also notations $B^2(X,\mathcal{U})$, $W(X,\mathcal{U})$, $R(X,\mathcal{U})$ and $\mathcal{L}_\varepsilon(X,\mathcal{U})$.
**Theorem 4** *If $X = \{X_j, j \in \mathbb{Z}^d\}$, $d \geq 1$, is a quasi associated centered random field, then, for any finite subset $\mathcal{U}$ of $\mathbb{Z}^d$, every $x \in \mathbb{R}$ and arbitrary positive $\varepsilon$, $\gamma$,*

$$|P(W \geq x) - P(Z \geq x)| \leq P(x - \gamma \leq Z \leq x + \gamma) + C\Big\{(3/2)\varepsilon + (4+\varepsilon)\mathcal{L}_\varepsilon + (1+2\varepsilon)R\Big\}, \tag{4}$$

*where $Z$ is a standard normal random variable, and one can take*

$$C = C(\gamma) = 2 + 2/\gamma. \tag{5}$$

**PROOF.** It is well known (see Stein (1986)) that, for any bounded continuous function $g : \mathbb{R} \to \mathbb{R}$, the unique bounded solution of the equation

$$f'(w) - w f(w) = g(w) - \mathbb{E}\, g(Z) \tag{6}$$

is determined by the formula

$$f(w) = \exp(w^2/2) \int_{-\infty}^{w} \Big(g(t) - \mathbb{E}\, g(Z)\Big) \exp(-t^2/2)\, dt. \tag{7}$$

For fixed $x \in \mathbb{R}$ and $\gamma > 0$ define a smooth nondecreasing function $g(w) = g_{x,\gamma}(w)$ in such a way that $g(w) = 0$ if $w < x$, $g(w) = 1$ if $w > x + \gamma$ and $g'(w) \le 2/\gamma$ for all $x \in \mathbb{R}$. Hence, for any $x, t \in \mathbb{R}$, $\gamma > 0$,

$$\mathbf{1}_{[x,\infty)}(t) \ge g(t) \ge \mathbf{1}_{[x+\gamma,\infty)}(t). \tag{8}$$

Then one can prove that for the function $f(w)$ (depending on $x$ and $\gamma$) given by (7) there exists $f''(w)$ for all $w \in \mathbb{R}$ and, for every $x \in \mathbb{R}$, $\gamma > 0$,

$$\| f \|_\infty \le \sqrt{\pi/2}, \quad \| f' \|_\infty \le 2, \quad \| f'' \|_\infty \le \sqrt{\pi/2} + \| g' \|_\infty.$$

Consequently $f$, $f'$ are Lipschitz functions and, for the corresponding Lipschitz constants,

$$\max\{\mathrm{Lip}(f), \mathrm{Lip}(f')\} \le C \tag{9}$$

the bound $C$ being given in (5).

The ingenious Stein's idea is that substituting in (6) instead of $w$ a random variable $W$ and taking the expectation one gets

$$\mathbb{E}\, f'(W) - \mathbb{E}\, W f(W) = \mathbb{E}\, g(W) - \mathbb{E}\, g(Z). \tag{10}$$

So the left hand side of (10) gives an accuracy of normal approximation for $\mathbb{E}\, g(W)$. Note that we can not use here the discontinuous indicator function $g(w) = \mathbf{1}_{(-\infty,x]}(w)$ to measure the Kolmogorov distance between distribution functions of $W$ and $Z$.

Our further steps consist of evaluating $\mathbb{E}\, f'(W) - \mathbb{E}\, W f(W)$. For a given $\varepsilon > 0$, define the Lipschitz function

$$h(t) = h_\varepsilon(t) = \begin{cases} -\varepsilon \text{ if } t < -\varepsilon, \\ t \quad \text{if } -\varepsilon \le t \le \varepsilon, \\ \varepsilon \quad \text{if } t > \varepsilon. \end{cases} \tag{11}$$

Set for $j \in \mathcal{U}$

$$\xi_j = X_j/B, \quad \xi_{j,1} = h(\xi_j), \quad \xi_{j,2} = \xi_j - \xi_{j,1}, \quad W^{(j)} = W - \xi_j. \tag{12}$$

Clearly $\xi_{j,1}$ and $\xi_{j,2}$ depend on $\varepsilon$. Using the obvious relations

$$\xi_j = \xi_{j,1} + \xi_{j,2}, \quad W = W^{(j)} + \xi_j = W^{(j)} + \xi_{j,1} + \xi_{j,2},$$

one can write

$$\mathbb{E}\, W f(W) = \sum_{j \in \mathcal{U}} \mathbb{E}\, \xi_j f(W) = R_1 + R_2 + R_3 + R_4, \tag{13}$$

where

$$
\begin{aligned}
R_1 &= \sum_{j\in\mathcal{U}} \mathbb{E}\,\xi_j f(W^{(j)}),\\
R_2 &= \sum_{j\in\mathcal{U}} \mathbb{E}\Big\{\xi_{j,2}\big[f(W) - f(W^{(j)})\big]\Big\},\\
R_3 &= \sum_{j\in\mathcal{U}} \mathbb{E}\Big\{\xi_{j,1}\big[f(W^{(j)} + \xi_{j,1} + \xi_{j,2}) - f(W^{(j)} + \xi_{j,1})\big]\Big\},\\
R_4 &= \sum_{j\in\mathcal{U}} \mathbb{E}\Big\{\xi_{j,1}\big[f(W^{(j)} + \xi_{j,1}) - f(W^{(j)})\big]\Big\}.
\end{aligned}
$$

Here $R_2$, $R_3$ and $R_4$ depend also on $\varepsilon$.

Note that if $F:\mathbb{R}\to\mathbb{R}$, $G:\mathbb{R}^I\to\mathbb{R}$ are Lipschitz functions, then $F(G(.))$ is Lipschitz as well and its coordinate-wise Lipschitz constants $L_i$ can be chosen so that $L_i\big(F(G(.))\big)\le \operatorname{Lip}F\cdot L_i(G)$, $i\in I$. Therefore, (2) and (9) imply

$$
|R_1| \le \sum_{j\in\mathcal{U}} \big|\operatorname{Cov}\big(\xi_j, f(W^{(j)})\big)\big| \le \frac{C}{B^2}\sum_{j\in\mathcal{U}}\sum_{\substack{j,q\in\mathcal{U}\\ j\ne q}} |\operatorname{Cov}(X_j, X_q)| = CR. \tag{14}
$$

Taking into account (9), we observe that

$$
\begin{aligned}
|R_2| \le\ & \sum_{j\in\mathcal{U}} \Big|\mathbb{E}\Big\{\xi_{j,2}\big[f(W^{(j)} + \xi_{j,1} + \xi_{j,2}) - f(W^{(j)} + \xi_{j,1})\big]\Big\}\Big|\\
&+ \sum_{j\in\mathcal{U}} \Big|\mathbb{E}\Big\{\xi_{j,2}\big[f(W^{(j)} + \xi_{j,1}) - f(W^{(j)})\big]\Big\}\Big|\\
&\le C\sum_{j\in\mathcal{U}}\Big(\mathbb{E}\,|\xi_{j,2}|^2 + \mathbb{E}\,|\xi_{j,2}||\xi_{j,1}|\Big)\\
&\le C\left(\sum_{j\in\mathcal{U}} \mathbb{E}\,|\xi_j|^2\mathbf{1}_{\{|\xi_j|>\varepsilon\}} + \varepsilon\sum_{j\in\mathcal{U}} \mathbb{E}\,|\xi_j|\mathbf{1}_{\{|\xi_j|>\varepsilon\}}\right)\\
&\le 2C\mathcal{L}_\varepsilon,
\end{aligned} \tag{15}
$$

in view of the following estimates:

$$
|\xi_{j,1}| \le \varepsilon, \quad |\xi_{j,2}| \le |\xi_j|\mathbf{1}_{\{|\xi_j|>\varepsilon\}} \le (\xi_j^2/\varepsilon)\mathbf{1}_{\{|\xi_j|>\varepsilon\}}. \tag{16}
$$

Analogously, we get

$$
|R_3| \le C\sum_{j\in\mathcal{U}} \mathbb{E}\,|\xi_{j,1}||\xi_{j,2}| \le C\mathcal{L}_\varepsilon. \tag{17}
$$

The Taylor formula yields

$$
f(W^{(j)} + \xi_{j,1}) - f(W^{(j)}) = f'(W^{(j)})\xi_{j,1} + \frac{1}{2}f''(\eta_j)\xi_{j,1}^2, \tag{18}
$$

where $\eta_j = \eta_j(\omega)$ is a point between $W^{(j)}(\omega)$ and $W^{(j)}(\omega) + \xi_{j,1}(\omega)$ ($\omega\in\Omega$, all random fields under consideration being defined on the same probability space $(\Omega,\mathcal{F},P)$). Thus,

$$
R_4 = \sum_{j\in\mathcal{U}} \mathbb{E}\,\xi_{j,1}^2 f'(W^{(j)}) + \Delta_1,
$$

where, because of relations $|\xi_{j,1}| \le |\xi_j|$ for $j \in \mathcal{U}$ and $\sum_{j\in\mathcal{U}} \mathbb{E}\,\xi_j^2 = 1$, we have

$$|\Delta_1| \le \frac{1}{2}\sum_{j\in\mathcal{U}} \mathbb{E}\,|\xi_{j,1}|^3 |f''(\eta_j)| \le \frac{1}{2}\| \, f'' \, \|_\infty \sum_{j\in\mathcal{U}} \mathbb{E}\,|\xi_{j,1}|^3 \le \frac{1}{2} C\varepsilon. \tag{19}$$

Further on,

$$\sum_{j\in\mathcal{U}} \mathbb{E}\Big\{\xi_{j,1}^2 f'(W^{(j)})\Big\} = \sum_{j\in\mathcal{U}} \mathrm{Cov}\Big(\xi_{j,1}^2, f'(W^{(j)})\Big) + \sum_{j\in\mathcal{U}} \mathbb{E}\,\xi_{j,1}^2\,\mathbb{E}\,f'(W^{(j)}). \tag{20}$$

Note that $h^2$ is a Lipschitz function with $\mathrm{Lip}(h^2) = 2\varepsilon$. By (2) and (9),

$$\Big|\sum_{j\in\mathcal{U}} \mathrm{Cov}\Big(\xi_{j,1}^2, f'(W^{(j)})\Big)\Big| \le \frac{2C\varepsilon}{B^2} \sum_{\substack{j,q\in\mathcal{U}\\ j\ne q}} |\,\mathrm{Cov}(X_j, X_q)| = 2C\varepsilon R. \tag{21}$$

Now,

$$\sum_{j\in\mathcal{U}} \mathbb{E}\,\xi_{j,1}^2\,\mathbb{E}\,f'(W^{(j)}) = \sum_{j\in\mathcal{U}} \mathbb{E}\,\xi_{j,1}^2\,\mathbb{E}\,f'(W) + \sum_{j\in\mathcal{U}} \mathbb{E}\,\xi_{j,1}^2\Big(\mathbb{E}\,f'(W^{(j)}) - \mathbb{E}\,f'(W)\Big) \tag{22}$$

and

$$\Big|\sum_{j\in\mathcal{U}} \mathbb{E}\,\xi_{j,1}^2\Big(\mathbb{E}\,f'(W^{(j)}) - \mathbb{E}\,f'(W)\Big)\Big| \le \Delta_2 + \Delta_3, \tag{23}$$

where

$$\begin{aligned}
\Delta_2 &= \sum_{j\in\mathcal{U}} \mathbb{E}\,\xi_{j,1}^2 |\,\mathbb{E}\{f'(W^{(j)} + \xi_{j,1} + \xi_{j,2}) - f'(W^{(j)} + \xi_{j,1})\}|,\\
\Delta_3 &= \sum_{j\in\mathcal{U}} \mathbb{E}\,\xi_{j,1}^2 |\,\mathbb{E}\{f'(W^{(j)} + \xi_{j,1}) - f'(W^{(j)})\}|.
\end{aligned}$$

Relations (9) and (16) yield

$$\Delta_2 \le C\sum_{j\in\mathcal{U}} \mathbb{E}\,\xi_{j,1}^2\,\mathbb{E}\,|\xi_{j,2}| \le C\varepsilon^2 \sum_{j\in\mathcal{U}} \mathbb{E}\,|\xi_{j,2}| \le C\varepsilon\mathcal{L}_\varepsilon, \tag{24}$$

$$\Delta_3 \le C\sum_{j\in\mathcal{U}} \mathbb{E}\,\xi_{j,1}^2\,\mathbb{E}\,|\xi_{j,1}| \le C\varepsilon. \tag{25}$$

Using again the relation $\sum_{j\in\mathcal{U}} \mathbb{E}\,\xi_j^2 = 1$, we have

$$\mathbb{E}\,f'(W)\sum_{j\in\mathcal{U}} \mathbb{E}\,\xi_{j,1}^2 = \mathbb{E}\,f'(W) + \mathbb{E}\,f'(W)\sum_{j\in\mathcal{U}}\Big(\mathbb{E}\,\xi_{j,1}^2 - \mathbb{E}\,\xi_j^2\Big). \tag{26}$$

On account of (9) and (12) we obtain

$$\Big|\mathbb{E}\,f'(W)\sum_{j\in\mathcal{U}}\Big(\mathbb{E}\,\xi_{j,1}^2 - \mathbb{E}\,\xi_j^2\Big)\Big| \le \|\, f' \,\|_\infty \sum_{j\in\mathcal{U}} \mathbb{E}\,|\xi_j^2 - \xi_{j,1}^2| \le C\sum_{j\in\mathcal{U}} \mathbb{E}\,|\xi_j^2| \mathbf{1}_{\{|\xi_j|>\varepsilon\}} = C\mathcal{L}_\varepsilon. \tag{27}$$

Hence (18)–(27) imply that

$$R_4 = \mathbb{E}\,f'(W) + \Delta_4,$$

where

$$|\Delta_4| \le (3/2)C\varepsilon + C(1+\varepsilon)\mathcal{L}_\varepsilon + 2C\varepsilon R. \tag{28}$$

Finally, due to (8), for any $x \in \mathbb{R}$, $\gamma > 0$,

$$\begin{aligned} P(W \ge x) - P(Z \ge x+\gamma) &\ge \mathbb{E}\, g(W) - \mathbb{E}\, g(Z) = \mathbb{E}\, f'(W) - \mathbb{E}\, W f(W) \\ &= \mathbb{E}\, f'(W) - R_1 - R_2 - R_3 - \mathbb{E}\, f'(W) - \Delta_4 \\ &= -R_1 - R_2 - R_3 - \Delta_4. \end{aligned} \tag{29}$$

According to (14)–(17) and (28),

$$\Big|\sum_{i=1}^{3} R_i + \Delta_4\Big| \le (3/2)C\varepsilon + C(4+\varepsilon)\mathcal{L}_\varepsilon + C(1+2\varepsilon)R. \tag{30}$$

For the function $\tilde{g}(t) = g_{x-\gamma,\gamma}(t)$, $t \in \mathbb{R}$, we get in a similar way

$$P(W \ge x) - P(Z \ge x-\gamma) \le \mathbb{E}\, \tilde{g}(W) - \mathbb{E}\, \tilde{g}(Z). \tag{31}$$

So we come to the same bounds as in (29) and (30). Thus, the estimate (4) is established. □

**Remark 5** To prove Theorem 4 and other results concerning quasi associated random fields we actually need only the property (2) where the cardinality of the set $I$ is equal to 1.

**Corollary 6** *For a family of quasi associated centered random fields $X^{(n)} = \{X_j^{(n)}, j \in \mathbb{Z}^d\}$, $n \in \mathbb{N}$, and a family of finite subsets $\mathcal{U}_n$ of $\mathbb{Z}^d$, the CLT holds, i.e.*

$$W\Big(X^{(n)}; \mathcal{U}_n\Big) \xrightarrow{law} Z \quad as \quad n \to \infty,$$

*whenever, for every $\varepsilon > 0$,*

$$\mathcal{L}_\varepsilon\Big(X^{(n)}; \mathcal{U}_n\Big) \to 0 \quad and \quad R\Big(X^{(n)}; \mathcal{U}_n\Big) \to 0 \quad as\ n \to \infty. \tag{32}$$

**Remark 7** If $X^{(n)}$ are positively or negatively dependent random fields, the condition on $R$ in (32) means that asymptotically the behaviour of sums is similar to the case of independent r.v.'s, since

$$\sum_{j \in \mathcal{U}_n} \operatorname{Var} X_j^{(n)} \Big/ \operatorname{Var}\Big(\sum_{j \in \mathcal{U}_n} X_j^{(n)}\Big) \to 1, \qquad \text{as } n \to \infty.$$

**Remark 8** Our Theorem 4 comprises Theorem 1 by Bulinski and Vronski (1996) where a strictly stationary associated random field $X = \{X_j, j \in \mathbb{Z}^d\}$ was studied under the condition

$$\sigma^2 = \sum_{j \in \mathbb{Z}^d} \operatorname{Cov}(X_0, X_j) < \infty$$

and summation was carried over finite sets $\mathcal{U}_n \subset \mathbb{Z}^d$, growing in the Van Hove sense (appropriate to the discrete case). The latter result generalized the classical Newman's

CLT where partial sums were taken over blocks. We use the renorm group approach, considering for $m = (m_1, \dots, m_d) \in \mathbb{N}^d$ auxiliary random fields

$$Y_q^{(m)} = \sum_{j \in \Pi_q^{(m)}} X_j, \qquad q = (q_1, \dots, q_d) \in \mathbb{Z}^d,$$

where $\Pi_q^{(m)} = \{i = (i_1, \dots, i_d) \in \mathbb{Z}^d : (q_k - 1)m_k < i_k \le q_k m_k\}$. Note that here we consider more general dependence conditions and neither stationarity nor the existence of absolute moments of summands of order higher than two is required. Moreover, no conditions are imposed on the growth of the sets $\mathcal{U}_n$ used to form the partial sums and there are no hypotheses concerning the rates of decrease of the covariance functions of random fields under consideration (cf. Cox and Grimmett (1984), Roussas (1994), Bulinski and Keane (1996), Bulinski and Vronski (1996)).

Let now $X = \{X_j, j \in \mathbb{Z}^d\}$, $d \ge 1$, be a random field such that

$$\mathbb{E}\, X_j = 0, \quad \mathbb{E}\,|X_j|^s < \infty \quad \text{for some } s \in (2, 3] \text{ and all } j \in \mathbb{Z}^d. \tag{33}$$

For a finite subset $\mathcal{U}$ of $\mathbb{Z}^d$, denote by $L_s$ the Lyapounov fraction

$$L_s = B^{-s} \sum_{j \in \mathcal{U}} \mathbb{E}\,|X_j|^s,$$

where $B^2 > 0$ is defined in (3).

**Theorem 9** *If $X$ is a quasi associated centered random field satisfying* (33)*, then, for any finite subset $\mathcal{U}$ of $\mathbb{Z}^d$, every $x \in \mathbb{R}$ and arbitrary positive $\gamma$, one has*

$$|P(W \ge x) - P(Z \ge x)| \le P(x - \gamma \le Z \le x + \gamma) + 3CR + (13/2)CL_s, \tag{34}$$

*where $W$, $Z$ and $C$ are the same as in* (3) *and* (5)*.*

**PROOF.** The scheme of the proof follows that of Theorem 4. Take $\varepsilon = 1$ in the definition of the function $h$ in (11) and note that $\mathcal{L}_1 \le L_s$. Instead of (25) one has now

$$\Delta_3 \le C \sum_{j \in \mathcal{U}} \mathbb{E}\,\xi_{j,1}^2\, \mathbb{E}\,|\xi_{j,1}| \le C \sum_{j \in \mathcal{U}} \mathbb{E}\,|\xi_{j,1}|^3 \le CL_s,$$

where the Lyapunov inequality and the bounds $|\xi_{j,1}| \le 1$, $|\xi_{j,1}| \le |\xi_j|$ $(j \in \mathcal{U})$ are used. To estimate $|\Delta_1|$ we modify (19) in the same way. $\square$

**Remark 10** The bound (34) is similar to Lemma 3 in Shao and Su (1999). Unfortunately in that paper, there is a gap in the proof (see p. 143) due to the application of the Hoeffding formula to discontinuous functions. Although in both papers Stein's method is used with a similar splitting $(R_1, R_2, R_3, R_4)$, our treatment of $R_4$ is different.

Now we turn to the general dependence conditions for random fields proposed initially by Doukhan and Louhichi (1999) for stochastic processes. Note that for random fields there are no "future" and "past", but it is natural to measure the dependence between a single random variable $X_j$ and other $X_q$, $q \in \mathbb{Z}^d$, when $\|q - j\|$ is "large" $(\|j\| = \max_{1 \le k \le d} |j_k|$

for $j = (j_1, \dots, j_k) \in \mathbb{Z}^d$), cf. Coulon-Prieur and Doukhan (2000), where for a process ($d = 1$) the dependence between pairs $X_i$, $X_j$ and all $X_q$ from the "past" was measured.

To simplify further exposition we shall from now on consider Lipschitz functions $f : \mathbb{S} \to \mathbb{T}$, where $\mathbb{S}$ and $\mathbb{T}$ are some metric spaces with distances $\rho$ and $\tau$ respectively, which means that

$$\mathrm{Lip}(f) = \sup_{\substack{x,y\in\mathbb{S}\\ x\neq y}} \frac{\tau\Big(f(x), f(y)\Big)}{\rho(x,y)} < \infty.$$

Thus in the special case of Euclidean spaces we drop the distinction between the possibly different behavior of $f$ in various directions.

For a finite subset $\mathcal{U}$ of $\mathbb{N}^d$, denote by $\mathrm{BL}(\mathbb{S}^{\mathcal{U}})$ the class of bounded Lipschitz functions $f : \mathbb{S}^{\mathcal{U}} \to \mathbb{R}$. In a space $\mathbb{S}^{\mathcal{U}}$ we apply the metric $\sum_{j\in\mathcal{U}} \rho(x_j, y_j)$ for $x = (x_j, j \in \mathcal{U}) \in \mathbb{S}^{\mathcal{U}}$, $y = (y_j, j \in \mathcal{U}) \in \mathbb{S}^{\mathcal{U}}$ (if $\mathbb{S} = \mathbb{R}$ , the Euclidean distance is used). Let $\theta = (\theta_r)_{r\in\mathbb{N}}$ be a sequence of nonnegative numbers tending to 0 as $r \to \infty$.

**Definition 11** *A random field $X = \{X_j, j \in \mathbb{Z}^d\}$, $d \geq 1$, with values in a metric space $\mathbb{S}$ is called $(\mathrm{BL}, \theta)$-dependent if, for each $j \in \mathbb{Z}^d$ and any $\mathcal{J} \subset \mathbb{Z}^d$ such that $j \notin \mathcal{J}$,*

$$\Big|\mathrm{Cov}\Big(f(X_j), g(X_q, q \in \mathcal{J})\Big)\Big| \leq \theta_r \,\mathrm{Lip}(f) \cdot \mathrm{Lip}(g) \qquad (35)$$

*for all $f \in \mathrm{BL}(\mathbb{S})$, $g \in \mathrm{BL}(\mathbb{S}^{\mathcal{J}})$, where $r = \inf\{\|j - q\|;\, q \in \mathcal{J}\}$.*

**Remark 12** Clearly a quasi associated random field $X$ satisfies (35) with

$$\theta_r = \sup_{j\in\mathbb{Z}^d} \sum_{\substack{q\in\mathbb{Z}^d\\ \|j-q\|\geq r}} |\,\mathrm{Cov}(X_j, X_q)|, \qquad (36)$$

whenever $\theta_1 < \infty$. The coefficient (36) was used in Bulinski (1995) to prove FLIL for associated random fields and in Bulinski (1996) to obtain the Berry-Esséen type estimates for PA and NA random fields. For positively dependent random fields (36) is the well known Cox-Grimmett coefficient (see Cox and Grimmett (1984)). A different coefficient was employed in Oliveira and Suquet (1995), Morel and Suquet (2000). It is worth mentioning that in Definition 11 no moment conditions are imposed on the field $X$. Furthermore, if $\mathbb{S}$ is a normed vector space (with the corresponding distance), $\mathbb{E}\,|X|_j < \infty$ for all $j \in \mathbb{Z}^d$ and (35) is valid, the centered random field $\{X_j - \mathbb{E}\,X_j, j \in \mathbb{Z}^d\}$ is also $(\mathrm{BL}, \theta)$-dependent with the same sequence $\theta$ as for the field $X$.

**Theorem 13** *Let $X$ be a centered $(\mathrm{BL}, \theta)$-dependent random field such that $\mathbb{E}\,X_j^2 < \infty$ for all $j \in \mathbb{Z}^d$. Then, for any finite subset $\mathcal{U}$ of $\mathbb{Z}^d$, the estimate of the type (4) holds with $R$ replaced by $|\mathcal{U}|B_n^{-2}\theta_1$ and $(4+\varepsilon)\mathcal{L}_\varepsilon$ by $(4+\varepsilon+2\varepsilon^{-1})\mathcal{L}_\varepsilon$. If, moreover, (33) is satisfied, then one can use in the right-hand side of (34) the same replacement for $R$ and $(17/2)CL_s$ instead of $(13/2)CL_s$.*

**PROOF.** In view of (35) we have, instead of (14) and (21),

$$|R_1| \leq \sum_{j\in\mathcal{U}} |\,\mathrm{Cov}\Big(\xi_{j,1}, f(W^{(j)})\Big)| + 2\|\, f\,\|_\infty \sum_{j\in\mathcal{U}} \mathbb{E}\,|\xi_{j,2}| \leq C|\mathcal{U}|B^{-2}\theta_1 + 2C\varepsilon^{-1}\mathcal{L}_\varepsilon,$$

$$\Big|\sum_{j\in\mathcal{U}} \mathrm{Cov}\Big(\xi_{j,1}^2, f'(W^{(j)})\Big)\Big| \leq 2\varepsilon C|\mathcal{U}|B^{-2}\theta_1.$$

To prove the second statement of Theorem 13 take $\varepsilon = 1$ as in the proof of Theorem 9 and use again the estimate $\mathcal{L}_1 \le L_s$. $\square$

If the function $B^2(X,\mathcal{U})$ behaves "regularly" (as in stationary case) then the term $|\mathcal{U}|B^{-2}\theta_1$ is not vanishing for growing $\mathcal{U}$. So (cf. Remark 8) it is reasonable to use the Bernstein blocks technique to invoke $\theta_r$ instead of $\theta_1$ where $r$ is a distance between "large" blocks. To this end, some modifications of the Definition 11 are appropriate. We provide here only the following one. Let $\Theta$ be a family of nonnegative functions $\theta(I,J)$ of finite sets $I,J \subset \mathbb{Z}^d$ such that $\theta(I,J) \to 0$ when $I$ and $J$ are shifted so that $\inf\{\|i-j\|;\ i \in I, j \in J\} \to \infty$.

**Definition 14** *A random field $X = \{X_j, j \in \mathbb{Z}^d\}$, $d \ge 1$, with values in a metric space $\mathbb{S}$ is called* $(\mathrm{BL},\Theta)$*-dependent, if there exists a function $\theta \in \Theta$ such that, for each pair of disjoint finite sets $I, J \subset \mathbb{Z}^d$,*

$$|\operatorname{Cov}(f(X_i, i \in I), g(X_j, j \in J))| \le \theta(I,J)\operatorname{Lip}(f)\operatorname{Lip}(g)$$

*for all $f \in \mathrm{BL}(\mathbb{S}^I)$, $g \in \mathrm{BL}(\mathbb{S}^J)$.*

It is natural to restrict the class $\Theta$ and suppose that $\theta(I,J) \le u(|I|,|J|;r)$ for some function $u : \mathbb{N}\times\mathbb{N}\times\mathbb{R}_+ \to \mathbb{R}_+$, where $|I|$ stands for cardinality of $I$ and $r = \inf\{\|i-j\|;\ i \in I, j \in J\}$. For quasi associated random fields $\theta(I,J) = \sum_{i\in I}\sum_{j\in J}|\operatorname{Cov}(X_i,X_j)| \le \min\{|I|,|J|\}\theta_r$ where $\theta_r$ is given by (36), whenever $\theta_1 < \infty$.

**Remark 15** Let $\zeta^{(n)} = \{\zeta_j^{(n)}, j \in \mathbb{Z}^d\}$ be $(\mathrm{BL},\theta^{(n)})$-dependent random field with values in a metric space $\mathbb{S}$, and $\theta^{(n)} = \{\theta_r^{(n)}, r \ge 1\}$, $n \in \mathbb{N}$. Define $X_j^{(n)} = F_j^{(n)}(\zeta_j^{(n)})$, $j \in \mathbb{Z}^d$, $n \in \mathbb{N}$, taking Lipschitz functions $F_j^{(n)} : S \to \mathbb{R}$ ($n \in \mathbb{N}$, $j \in \mathbb{Z}^d$). Define for a family of finite sets $\mathcal{U}_n \subset \mathbb{Z}^d$

$$\lambda_n = \sup_{j\in\mathcal{U}_n}\operatorname{Lip}\Big(F_j^{(n)}\Big), \quad n \in \mathbb{N}.$$

Then it is easily seen that $X^{(n)} = \{X_j^{(n)}, j \in \mathcal{U}_n\}$ is $(\mathrm{BL},\lambda_n^2\theta^{(n)})$-dependent random field where $\lambda_n^2\theta^{(n)} = \{\lambda_n^2\theta_r^{(n)}, r \ge 1\}$, $n \in \mathbb{N}$ (to use the definitions for random fields on $\mathbb{Z}^d$ we set $X_j^{(n)} = 0$ for $j \in \mathbb{Z}^d \setminus \mathcal{U}_n$). Using Theorem 13 we come to analogues of the results by Coulon-Prieur and Doukhan (2000) but we do not assume that $F_j^{(n)}$ are bounded, we need not impose conditions on variances of partial sums (see (6) in the cited paper) and the finite sets $\mathcal{U}_n$ of arbitrary configuration can be used as well.

**Remark 16** Recently Stein's technique was used by Pruss and Szynal (2000) to establish the CLT for negatively correlated random variables with negatively correlated squares. Note that this interesting result does not comprise quasi associated random variables and is aimed at generalization of CLT for pairwise dependent families of r.v.'s.

## 3 On partial sums of negatively associated random fields

For a sequence of NA random variables $X_1, X_2, \ldots$, let $X_1^*, X_2^*, \ldots$ be their independent copies, that is $\{X_i^*, i \ge 1\}$ is a sequence of independent random variables and, for each

$i \geq 1$, $X_i^*$ has the same distribution as $X_i$. Set

$$S_j = \sum_{i=1}^{j} X_i, \quad S_j^* = \sum_{i=1}^{j} X_i^*, \qquad M_n = \max_{1 \leq i \leq n} S_i, \quad M_n^* = \max_{1 \leq i \leq n} S_i^*.$$

Recently Shao established the following useful result.

**Theorem 17 (Shao (2000))** *Let $X_1, X_2, \ldots$ be a sequence of negatively dependent random variables. Then, for any convex function $f : \mathbb{R} \to \mathbb{R}$ and each $n \in \mathbb{N}$,*

$$\mathbb{E}\, f(S_n) \leq \mathbb{E}\, f(S_n^*). \tag{37}$$

*If, moreover, $f$ is nondecreasing, then*

$$\mathbb{E}\, f(M_n) \leq \mathbb{E}\, f(M_n^*). \tag{38}$$

It would be desirable to have an analogue of this theorem for negatively associated random fields. It is clear that (37) does not involve the partial order of the parameter set, so its counterpart is valid also for sums $S(\mathcal{U})$ taken over arbitrary finite subsets $\mathcal{U}$ of $\mathbb{Z}^d$. As for the maximum of partial sums, the situation is quite different in the case of random fields.

Let $X = \{X_j, j \in \mathbb{Z}^d\}$ be a NA random field. For $j$, $q \in \mathbb{Z}^d$, the notation $j \leq q$ means that $j_k \leq q_k$ for all $k = 1, \ldots, d$. Set

$$S_q = \sum_{\mathbf{1} \leq j \leq q} X_j, \qquad M_n = \max_{\mathbf{1} \leq q \leq n} S_q,$$

where $q, n \in \mathbb{N}^d$, $\mathbf{1} = (1, \ldots, 1) \in \mathbb{N}^d$. Denote by $\{X_j^*, j \in \mathbb{Z}^d\}$ a random field consisting of independent random variables where each $X_j^*$ has the same distribution as the corresponding $X_j$. Set

$$S_q^* = \sum_{\mathbf{1} \leq j \leq q} X_j^*, \qquad \widehat{M_n} = \max_{\mathbf{1} \leq q \leq n} S_q^*, \qquad q, n \in \mathbb{N}^d.$$

**Proposition 18** *For every $d > 1$ and any $n \in \mathbb{N}^d$, one can construct a NA random field $X = \{X_j, j \in \mathbb{Z}^d\}$ such that, for all nondecreasing (not necessarily convex) functions $f : \mathbb{R} \to \mathbb{R}$,*

$$\mathbb{E}\, f(M_n) \geq \mathbb{E}\, f(\widehat{M_n}).$$

**PROOF.** Clearly, it is sufficient to provide an example for $d = 2$ and $n = (2, 2)$ (we can take other variables $X_j = 0$).

Let $X_{1,1} = 0$, $X_{1,2} = -X_{2,1}$ where $P(X_{1,2} = 1) = P(X_{1,2} = -1) = 1/2$ and $X_{2,2} = c < -2$.

Recall (see, e.g., Joag-Dev and Proschan (1983)) that if $\{X_i, i \in I_k\}$, $k = 1, \ldots, N$, are independent collections of NA random variables, then the whole collection $\{X_i, i \in I_k, k = 1, \ldots, N\}$ is NA. Since $X_{1,2}$ and $X_{2,1}$ are NA, the same is true for $X_{1,1}, X_{1,2}, X_{2,1}, X_{2,2}$.

For $n = (2, 2)$ we see that

$$\begin{aligned} M_n &= \max\{X_{1,1}; X_{1,1} + X_{1,2}; X_{1,1} + X_{2,1}; X_{1,1} + X_{1,2} + X_{2,1} + X_{2,2}\} \\ &= \max\{0; X_{1,2}; -X_{1,2}; c\} = |X_{1,2}| = 1. \end{aligned}$$

Because of $X^*_{1,1} = 0$ and $X^*_{1,2} + X^*_{2,1} + X^*_{2,2} < 0$, one has, for the same $n$,

$$\begin{aligned} \widehat{M_n} &= \max\{X^*_{1,1}; X^*_{1,1} + X^*_{1,2}; X^*_{1,1} + X^*_{2,1}; X^*_{1,1} + X^*_{1,2} + X^*_{2,1} + X^*_{2,2}\} \\ &= \max\{0; X^*_{1,2}; X^*_{2,1}\}. \end{aligned}$$

It follows clearly that

$$\mathbb{E} f(\widehat{M_n}) = \mathbb{E} f(\max\{0; X^*_{1,2}; X^*_{1,2}\}) = \frac{1}{4} f(0) + \frac{3}{4} f(1).$$

Then from $f(1) \geq f(0)$ and $\mathbb{E} f(M_n) = f(1)$ we obtain $\mathbb{E} f(M_n) \geq \mathbb{E} f(\widehat{M_n})$. $\square$

**Corollary 19** *For $a < b$ ($a$, $b \in \mathbb{R}$) let $\mathcal{F}_{a,b}$ be the class of functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(a) < f(b)$. Then for every $d > 1$, each $n \in \mathbb{N}^d$, one can construct a NA random field $Y = \{Y_j, j \in \mathbb{Z}^d\}$ such that*

$$\mathbb{E} f(M_n) > \mathbb{E} f(\widehat{M_n}) \quad \text{for any } f \in \mathcal{F}_{a,b}.$$

**PROOF.** Having observed, that the negative dependence of the random field used in the proof of Theorem 18 is preserved by the transformation $g : x \mapsto Ax + B$ ($A > 0$), which maps $[0, 1]$ onto $[a, b]$, it suffices to consider $Y = \{Y_j = g(X_j), j \in \mathbb{Z}^d\}$. $\square$


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